Arnold Math J. (2015) 1:233–242
DOI 10.1007/s40598-015-0018-2

RESEARCH CONTRIBUTION

Homology of Spaces of Non-Resultant Homogeneous Polynomial Systems in $\mathbb{R}^2$ and $\mathbb{C}^2$

V. A. Vassiliev[1]

Received: 7 November 2014 / Accepted: 3 June 2015 / Published online: 11 August 2015

Abstract The *resultant* variety in the space of systems of homogeneous polynomials of some given degrees consists of such systems having non-trivial solutions. We calculate the integer cohomology groups of all spaces of non-resultant systems of polynomials $\mathbb{R}^2 \to \mathbb{R}$, and also the rational cohomology rings of spaces of non-resultant systems and non-m-discriminant polynomials in $\mathbb{C}^2$.

Keywords Resultant variety · Simplicial resolution · Spectral sequence · Configuration space · Caratheodory theorem

1 Introduction

Given n natural numbers $d_1 \geq d_2 \geq \cdots \geq d_n$, consider the space of all real homogeneous polynomial systems

$$\begin{cases} a_{1,0}x^{d_1} + a_{1,1}x^{d_1-1}y + \cdots + a_{1,d_1}y^{d_1} \\ \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n,0}x^{d_n} + a_{n,1}x^{d_n-1}y + \cdots + a_{n,d_n}y^{d_n} \end{cases} \tag{1}$$

in two real variables x, y.

Supported by Program "Leading scientific schools", Grant No. NSh-5138.2014.1 and RFBR Grant 13-01-00383.

✉ V. A. Vassiliev
vva@mi.ras.ru

[1] Steklov Mathematical Institute of Russian Academy of Sciences, National Research University Higher School of Economics, Moscow, Russia

We will refer to this space as $\mathbb{R}^D$, $D = \sum_1^n (d_i + 1)$. The *resultant variety* $\Sigma \subset \mathbb{R}^D$ is the space of all systems having non-zero solutions. Σ is a semialgebraic subvariety of codimension $n-1$ in $\mathbb{R}^D$.

Below we calculate the cohomology group of its complement, $H^*(\mathbb{R}^D \setminus \Sigma)$. Also, we calculate the rational cohomology rings of the complex analogs $\mathbb{C}^D \setminus \Sigma_{\mathbb{C}}$ of all spaces $\mathbb{R}^D \setminus \Sigma$.

For the "affine" version of the "real" problem (concerning the space of non-resultant systems of polynomials $\mathbb{R}^1 \to \mathbb{R}^1$ with leading terms x^{d_i}), see, e.g., Vassiliev (1994, 1997) and Kozlowski and Yamaguchi (2000); for the "complex" problem with $n = 2$ see also Cohen et al. (1991). A similar calculation for spaces of real homogeneous polynomials in $\mathbb{R}^2$ without zeros of multiplicity $\geq m$ was done in Vassiliev (1998).

The entire study of homology groups of spaces of non-singular (in appropriate sense) objects goes back to the Arnold's works (1970, 1989), as well as the idea of using the Alexander duality in this problem.

2 Main Results

2.1 Notation

For any natural p, denote by $N(p)$ the sum of all numbers $d_i + 1$, $i = 1, \ldots, n$, which are less than or equal to p, plus p times the number of those d_i which are equal to or greater than p. [In other words, $N(p)$ is the area of the part of Young diagram $(d_1 + 1, \ldots, d_n + 1)$ strictly to the left from the $(p+1)$-th column.] Let the index $\Upsilon(p)$ be equal to the number of even numbers $d_i \geq p$ if p is even, and to the number of odd numbers $d_i \geq p$ if p is odd. By $\tilde{H}^*(X)$ we denote the cohomology group reduced modulo a point. $\overline{H}_*(X)$ denotes the Borel–Moore homology group, i.e. the homology group of the complex of locally finite singular chains of X.

Theorem 1 *If the space $\mathbb{R}^D \setminus \Sigma$ is non-empty (i.e. either $n > 1$ or d_1 is even), then the group $\tilde{H}^*(\mathbb{R}^D \setminus \Sigma, \mathbb{Z})$ is equal to the direct sum of following groups:*

A) For any $p = 1, \ldots, d_3$,
if $\Upsilon(p)$ is even, then $\mathbb{Z}$ in dimension $N(p) - 2p$ and $\mathbb{Z}$ in dimension $N(p) - 2p + 1$,
if $\Upsilon(p)$ is odd, then only one group $\mathbb{Z}_2$ in dimension $N(p) - 2p + 1$;

B) If $d_1 - d_2$ is odd, then an additional summand $\mathbb{Z}$ in dimension $D - d_1 - d_2 - 2$. If $d_1 - d_2$ is even, then an additional summand $\mathbb{Z}^{d_2 - d_3 + 1}$ in dimension $D - d_1 - d_2 - 1$ and (if $d_2 \neq d_3$) a summand $\mathbb{Z}^{d_2 - d_3}$ in dimension $D - d_1 - d_2 - 2$.

Example 1 Let $n = 2$ [so that part (A) in the statement of Theorem 1 is void]. If d_1 and d_2 are of the same parity, then $\mathbb{R}^D \setminus \Sigma$ consists of $d_2 + 1$ connected components, each of which is homotopy equivalent to a circle. For an invariant, which separates systems belonging to different components, we can take the index of the induced map of the unit circle $S^1 \subset \mathbb{R}^2$ into $\mathbb{R}^2 \setminus 0$. This index can take all values of the same parity as d_1 and d_2 from the segment $[-d_2, d_2]$. The 1-dimensional cohomology class inside any component is just the rotation number of the image of a fixed point [say, $(1, 0)$] around the origin. Moreover, the images of this point under our non-resultant systems define a map $\mathbb{R}^D \setminus \Sigma \to \mathbb{R}^2 \setminus 0$; it is easy to see that any fiber of this map consists of $d_2 + 1$ contractible components.

If d_1 and d_2 are of different parities, then the space $\mathbb{R}^D \setminus \Sigma$ has the homology of a two-point set. The invariant separating its two connected components can be calculated as the parity of the number of zeros of the odd-degree polynomial of our non-resultant system, which lie in the (well-defined) domain in $\mathbb{RP}^1$ where the even-degree polynomial is positive.

Now, let $\mathbb{C}^D$ be the space of all polynomial systems (1) with *complex* coefficients $a_{i,j}$, and $\Sigma_{\mathbb{C}} \subset \mathbb{C}^D$ the set of systems having solutions in $\mathbb{C}^2 \setminus 0$.

Theorem 2 *For any $n > 1$, the ring $H^*(\mathbb{C}^D \setminus \Sigma_{\mathbb{C}}, \mathbb{Q})$ is an exterior algebra over $\mathbb{Q}$ with two generators of dimensions $2n-3$ and $2n-1$. Namely, these generators are the linking number with the Borel–Moore fundamental class of entire resultant variety and the pull-back of the basic cohomology class under the map $\mathbb{C}^D \setminus \Sigma_{\mathbb{C}} \to \mathbb{C}^n \setminus 0$ defined by restrictions of non-resultant systems $(f_1, \ldots, f_n)$ to the point $(1, 0)$. The weight filtrations of these two generators and their product in the mixed Hodge structure of $\mathbb{C}^D \setminus \Sigma_{\mathbb{C}}$ are equal to $2n-2$, $2n$ and $4n-2$ respectively.*

Consider also the space $\mathbb{C}^{d+1}$ of all complex homogeneous polynomials

$$a_0 x^d + a_1 x^{d-1} y + \cdots + a_d y^d,$$

and *m-discriminant* Σ_m in it consisting of all polynomials vanishing on some line with multiplicity $\geq m$.

Theorem 3 *For any $m > 1$ and $d \geq 2m-1$, the ring $H^*(\mathbb{C}^{d+1} \setminus \Sigma_m, \mathbb{Q})$ is isomorphic to an exterior algebra over $\mathbb{Q}$ with two generators of dimensions $2m-3$ and $2m-1$. The weight filtrations of these two generators and of their product are equal to $2m-2$, $2m$ and $4m-2$ respectively. For any $m > 1$ and $d \in [m+1, 2m-2]$, this ring is isomorphic to $\mathbb{Q}$ in dimensions $0, 2m-3, 2m-1$ and $2d-2$, and is trivial in all other dimensions; the multiplication is obviously trivial. For $d = m > 1$ this ring is isomorphic to $\mathbb{Q}$ in dimensions 0 and $2m-3$, and is trivial in all other dimensions.*

3 Some Preliminary Facts

Denote by $B(M, p)$ the *configuration space* of subsets of cardinality p of a topological space M.

Lemma 1 *For any natural p, there is a locally trivial fiber bundle $B(S^1, p) \to S^1$ whose fiber is homeomorphic to $\mathbb{R}^{p-1}$. This fiber bundle is non-orientable if p is even, and is orientable (and hence trivial) if p is odd.*

Indeed, the projection of this fiber bundle can be realised as the product of p points of the unit circle in $\mathbb{C}^1$. The fiber of this bundle can be identified in terms of the universal covering $\mathbb{R}^p \to T^p$ with any connected component of some hyperplane $\{x_1 + \cdots + x_p = \text{const}\}$, from which all affine planes given by $x_i = x_j + 2\pi k, i \neq j$, $k \in \mathbb{Z}$, are removed. Such a component is convex and hence diffeomorphic to $\mathbb{R}^{p-1}$. The assertion on orientability can be checked immediately. □

Let us embed a manifold M generically into the space $\mathbb{R}^T$ of a very large dimension, and denote by M^{*r} the union of all $(r-1)$-dimensional simplices in $\mathbb{R}^T$, whose vertices lie in this embedded manifold (and the "genericity" of the embedding means that if two such simplices have a common point in $\mathbb{R}^T$, then their minimal faces containing this point coincide).

Proposition 1 (C. Caratheodory theorem: see also Vassiliev 1997; Kallel and Karoui 2011) *For any* $r \geq 1$*, the space* $(S^1)^{*r}$ *is homeomorphic to* S^{2r-1}.

Remark 1 This homeomorphism can be realized as follows. Consider the space $\mathbb{R}^{2r+1}$ of all real homogeneous polynomials $\mathbb{R}^2 \to \mathbb{R}^1$ of degree $2r$, the convex cone in this space consisting of everywhere non-negative polynomials, and (also convex) dual cone in the dual space $\widehat{\mathbb{R}}^{2r+1}$ consisting of linear forms taking only positive values inside the previous cone. The intersection of the boundary of this dual cone with the unit sphere in $\widehat{\mathbb{R}}^{2r+1}$ is naturally homeomorphic to $(S^1)^{*r}$; on the other hand it is homeomorphic to the boundary of a convex $2r$-dimensional domain.

Lemma 2 (see Vassiliev 1999, Lemma 3) *For any* $r > 1$*, the group* $H_*((S^2)^{*r}, \mathbb{Q})$ *is trivial in all positive dimensions.* □

Consider the "sign local system" $\pm\mathbb{Q}$ over $B(\mathbb{CP}^1, p)$, i.e. the local system of groups with fiber $\mathbb{Q}$ such that the elements of $\pi_1(B(\mathbb{CP}^1, p))$ defining odd (respectively, even) permutations of p points in $\mathbb{CP}^1$ act in the fiber as multiplication by -1 (respectively, by 1).

Lemma 3 (see Vassiliev 1999, Lemma 2) *All Borel–Moore homology groups* $\overline{H}_i(B(\mathbb{CP}^1, p); \pm\mathbb{Q})$ *with* $p \geq 1$ *are trivial except*

$$\overline{H}_0(B(\mathbb{CP}^1, 1), \pm\mathbb{Q}) \cong \overline{H}_2(B(\mathbb{CP}^1, 1), \pm\mathbb{Q}) \cong \overline{H}_2(B(\mathbb{CP}^1, 2), \pm\mathbb{Q}) \cong \mathbb{Q}.$$

□

4 Proof of Theorem 1

Following Arnold (1970), we use the Alexander duality

$$\tilde{H}^i(\mathbb{R}^D \setminus \Sigma) \simeq \overline{H}_{D-i-1}(\Sigma). \tag{2}$$

4.1 Simplicial Resolution of Σ

To calculate the right-hand group in (2), we construct a *resolution* of the space Σ. Let $\chi : \mathbb{RP}^1 \to \mathbb{R}^T$ be a generic embedding, $T \gg d_1$. For any system $\Phi = (f_1, \ldots, f_n) \in \Sigma$ not equal identically to zero, consider the simplex $\Delta(\Phi)$ in $\mathbb{R}^T$ spanned by the images $\chi(x_i)$ of all points $x_i \in \mathbb{RP}^1$ corresponding to all lines, on which the system f has a common root. (The maximal possible number of such lines is obviously equal to d_1.)

Furthermore, consider a subset in the direct product $\mathbb{R}^D \times \mathbb{R}^T$, namely, the union of all simplices of the form $\Phi \times \Delta(\Phi)$, $\Phi \in \Sigma \setminus 0$. This union is not closed: the set of its limit points not belonging to it is the product of the point $0 \in \mathbb{R}^D$ (corresponding to the zero system) and the union of all simplices in $\mathbb{R}^T$ spanned by the images of no more than d_1 different points of the line $\mathbb{RP}^1$. By the Caratheodory theorem, the latter union is homeomorphic to the sphere S^{2d_1-1}. We can assume that our embedding $\chi : \mathbb{RP}^1 \to \mathbb{R}^T$ is algebraic, and hence this sphere is semialgebraic. Take a generic $2d_1$-dimensional semialgebraic disc in $\mathbb{R}^T$ bounded by this sphere (e.g., the union of segments connecting the points of this sphere with a generic point in $\mathbb{R}^T$), and add the product of the point $0 \in \mathbb{R}^D$ and this disc to the previous union of simplices $\Phi \times \Delta(\Phi) \subset \mathbb{R}^D \times \mathbb{R}^T$. The resulting closed subset in $\mathbb{R}^D \times \mathbb{R}^T$ will be denoted by σ and called a *simplicial resolution* of Σ.

Lemma 4 *The obvious projection $\sigma \to \Sigma$ (induced by the projection of $\mathbb{R}^D \times \mathbb{R}^T$ onto the first factor) is proper, and the induced map between one-point compactifications of these spaces is a homotopy equivalence.*

This follows easily from the fact that this projection is a stratified map of semialgebraic spaces, and the preimage of any point of Σ is contractible: see Vassiliev (1994, 1997). □

So, we can (and will) calculate the group $\overline{H}_*(\sigma)$ instead of $\overline{H}_*(\Sigma)$.

Remark 2 There is a different construction of a simplicial resolution of Σ in terms of "Hilbert schemes". Namely, let I_p be the space of all ideals of codimension p in the space of smooth functions $\mathbb{RP}^1 \to \mathbb{R}^1$ equipped with the natural "Grassmannian" topology. It is easy to see that I_p is homeomorphic to the p-th symmetric power $S^p(\mathbb{RP}^1) = (\mathbb{RP}^1)^p/S(p)$; in particular, it contains the configuration space $B(\mathbb{RP}^1, p)$ as an open dense subset. Consider the disjoint union of these d_1 spaces $I_1, \dots, I_{d_1}$ augmented with the one-point set I_∞ symbolizing the zero ideal. The incidence of ideals makes this union a partially ordered set. Consider the continuous order complex Ξ_{d_1} of this poset, i.e. the subset in the join $I_1 * \cdots * I_{d_1} * I_\infty$ consisting of simplices, whose all vertices are incident to one another. For any polynomial system $\Phi = (f_1, \dots, f_n) \in \mathbb{R}^D$, denote by $\Xi(\Phi)$ the subcomplex in Ξ_{d_1} consisting of all simplices, whose all vertices correspond to ideals containing all polynomials $f_1, \dots, f_n$. The simplicial resolution $\tilde{\sigma} \subset \Sigma \times \Xi_{d_1}$ is defined as the union of simplices $\Phi \times \Xi(\Phi)$ over all $\Phi \in \Sigma$.

This construction is homotopy equivalent to the previous one. In particular, the Caratheodory theorem has the following version (see Kallel and Karoui (2011)): the continuous order complex of the poset of all ideals of codimension $\leq r$ in the space of functions $S^1 \to \mathbb{R}^1$ is *homotopy equivalent* to S^{2r-1}.

However, this construction is less convenient for our practical calculations than the one described above and used previously in Vassiliev (1994, 1999) [and extended to some more complicated situations in Gorinov (2005)].

The space σ has a natural increasing filtration $F_1 \subset \cdots \subset F_{d_1+1} = \sigma$: its term F_p, $p \leq d_1$, is the closure of the union of all simplices of the form $\Phi \times \Delta(\Phi)$ over all polynomial systems Φ having no more than p lines of common zeros. Alternatively, it can be described as the union of all no more than $(p-1)$-dimensional faces of all simplices $\Phi \times \Delta(\Phi)$ over all systems $\Phi \in \Sigma \setminus 0$, completed with all no more than $(p-1)$-dimensional simplices spanning some $\leq p$ points of the manifold $\{0\} \times \chi(\mathbb{RP}^1)$.

Lemma 5 *For any $p = 1, \dots, d_1$, the term $F_p \setminus F_{p-1}$ of our filtration is the space of a locally trivial fiber bundle over the configuration space $B(\mathbb{RP}^1, p)$, with fibers equal to the direct product of a $(p-1)$-dimensional open simplex and a $(D - N(p))$-dimensional real space. The corresponding bundle of open simplices is orientable if and only if p is odd (i.e. exactly when the base configuration space is orientable), and the bundle of $(D - N(p))$-dimensional spaces is orientable if and only if the index $\Upsilon(p)$ is even.*

The last term $F_{d_1+1} \setminus F_{d_1}$ of this filtration is homeomorphic to an open $2d_1$-dimensional disc.

Indeed, to any configuration $(x_1, \dots, x_p) \in B(\mathbb{RP}^1, p)$, $p \le d_1$, there corresponds the direct product of the interior part of the simplex in $\mathbb{R}^T$ spanned by the images $\chi(x_i)$ of points of this configuration, and the subspace of $\mathbb{R}^D$ consisting of polynomial systems that have solutions on corresponding p lines in $\mathbb{R}^2$. The codimension of the latter subspace is equal exactly to $N(p)$. The assertion concerning the orientations can be checked in a straightforward way. The description of $F_{d_1+1} \setminus F_{d_1}$ follows immediately from the construction and the Caratheodory theorem. □

Consider the spectral sequence $E^r_{p,q}$, calculating the group $\overline{H}_*(\Sigma)$ and generated by this filtration. Its term $E^1_{p,q}$ is canonically isomorphic to the group $\overline{H}_{p+q}(F_p \setminus F_{p-1})$. By Lemma 5, its column $E^1_{p,*}$, $p \le d_1$, is as follows. If $\Upsilon(p)$ is even, then this column contains exactly two non-trivial terms $E^1_{p,q}$, both isomorphic to $\mathbb{Z}$, for q equal to $D - N(p) + p - 1$ and $D - N(p) + p - 2$. If $\Upsilon(p)$ is odd, then this column contains only one non-trivial term $E^1_{p,q}$ isomorphic to $\mathbb{Z}_2$, for $q = D - N(p) + p - 2$. Finally, the column $E^1_{d_1+1,*}$ contains only one non-trivial element $E^1_{d_1+1,d_1-1} \cong \mathbb{Z}$.

Before calculating the differentials and further terms E^r, $r > 1$, let us consider several basic examples.

4.2 The Case $n = 1$

If our system consists of only one polynomial of degree d_1, then the term E^1 of our spectral sequence looks as in Fig. 1; in particular, all non-trivial groups $E^1_{p,q}$ lie in two rows $q = d_1$ and $q = d_1 - 1$.

Lemma 6 *If $n = 1$, then in both cases of even or odd d_1, all possible horizontal differentials $\partial_1 : E^1_{p,d_1-1} \to E^1_{p-1,d_1-1}$ of the form $\mathbb{Z} \to \mathbb{Z}_2$, $p = d_1 + 1, d_1 -$*

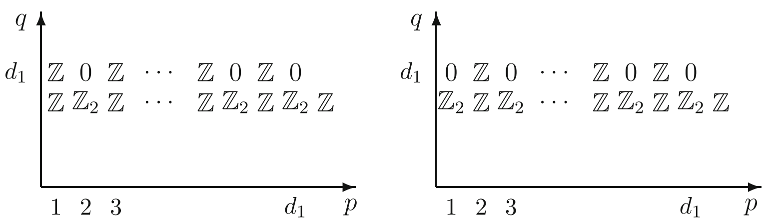

Fig. 1 E^1 for $n = 1$, d_1 even (*left*) and $n = 1$, d_1 odd (*right*)

1, $d_1 - 3, \dots$ are epimorphisms, and all differentials $\partial_2 : E^2_{p,d_1-1} \to E^2_{p-2,d_1}$ of the form $\mathbb{Z} \to \mathbb{Z}$, $p = d_1 + 1, d_1 - 1, d_1 - 3, \dots$ are isomorphisms. In particular, the unique surviving term $E^3_{p,q}$ for the "even" spectral sequence is $E^3_{1,d_1-1} \cong \mathbb{Z}$, and for the "odd" one it is $E^3_{2,d_1-1} \cong \mathbb{Z}$.

Indeed, in both cases we know the answer. In the "odd" case, the discriminant coincides with entire $\mathbb{R}^D = \mathbb{R}^{d_1+1}$. In the "even" case, its complement consists of two contractible components, so that $\overline{H}_*(\Sigma) = \mathbb{Z}$ in dimension d_1 and is trivial in all other dimensions. Therefore, all terms $E_{p,q}$ with $p+q$ not equal to $d_1 + 1$ (respectively, to d_1) in the odd- (respectively, even-) dimensional case should die at some stage; this is possible only if all assertions of our lemma hold. □

4.3 The Case $n = 2$

There are two very different situations depending on the parity of $d_1 - d_2$. In Fig. 2, we demonstrate these situations in two particular cases: $(d_1, d_2) = (6, 3)$ and $(7, 3)$. However, the general situation is essentially the same; namely, the following is true.

If $n = 2$ and $d_1 - d_2$ is odd, then all indices $\Upsilon(p)$, $p = 1, \dots, d_2 + 1$, are odd, and hence all non-trivial groups $E^1_{p,q}$ with such p lie on the line $\{p + q = d_1 + d_2\}$ only and are equal to $\mathbb{Z}_2$.

If $n = 2$ and $d_1 - d_2$ is even, then all indices $\Upsilon(p)$, $p = 1, \dots, d_2 + 1$, are even, and hence all non-trivial groups $E^1_{p,q}$ with such p lie on two lines $\{p+q = d_1 + d_2\}$, $\{p + q = d_1 + d_2 + 1\}$, and are all equal to $\mathbb{Z}$.

In both cases, all groups $E^1_{p,q}$ with $p > d_2$ are the same as in the case $n = 1$ with the same d_1. Moreover, the differentials ∂_1 and ∂_2 between these groups are also the same as for $n = 1$; therefore, all of these groups die at E^3 except for $E^3_{d_2+1,d_1-1} \cong \mathbb{Z}$ for even $d_1 - d_2$, and $E^3_{d_2+2,d_1-1} \cong \mathbb{Z}$ for odd $d_1 - d_2$.

In the case of even $d_1 - d_2$, all other differentials between the groups $E^r_{p,q}$ are trivial, because otherwise the group $\tilde{H}^0(\mathbb{R}^D \setminus \Sigma)$ would be smaller than $\mathbb{Z}^{d_2}$, in contradiction to $d_2 + 1$ different components of this space indicated in Example 1.

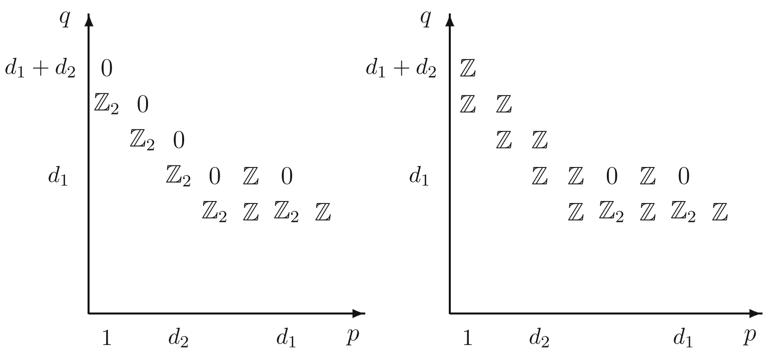

Fig. 2 E^1 for $n = 2$, $d_1 - d_2$ odd (*left*) and $n = 2$, $d_1 - d_2$ even (*right*)

On the contrary, if $d_1 - d_2$ is odd, then all the differentials $d_r : E^r_{d_2+2,d_1-1} \to E^r_{d_2+2-r,d_1-2+r}$, $r = 1, \ldots, d_1 - d_2 + 1$, are epimorphic just because the integer cohomology group of the topological space $\mathbb{R}^D \setminus \Sigma$ cannot have non-trivial torsion subgroup in dimension 1. Therefore, the unique nontrivial group $E^\infty_{p,q}$ in this case is $E^\infty_{d_2+2,d_1-1} \cong \mathbb{Z}$.

This proves Theorem 1 for $n = 2$.

4.4 The General Case

Now suppose that our systems (1) consist of $n \geq 3$ polynomials. Let again σ be the simplicial resolution of the corresponding resultant variety constructed in Sect. 4.1, and σ' be the simplicial resolution of the resultant variety for $n = 2$ and the same d_1 and d_2. The parts $\sigma \setminus F_{d_3}(\sigma)$ and $\sigma' \setminus F_{d_3}(\sigma')$ of these resolutions are canonically homeomorphic to one another as filtered spaces. In particular, $E^1_{p,q}(\sigma) = E^1_{p,q}(\sigma')$ if $p > d_3$, and $E^r_{p,q}(\sigma) = E^r_{p,q}$ if $p \geq d_3 + r$. All non-trivial terms $E^r_{p,q}(\sigma)$ with $p \leq d_3$ are placed in such a way that no non-trivial differentials ∂_r can act between these terms, as well as no differentials can act to these terms from the cells $E^r_{p,q}$ with $p > d_3$, which have survived the differentials between these cells described in the previous subsection.

Therefore, the final term $E^\infty_{p,q}(\sigma)$ coincides with $E^1_{p,q}(\sigma)$ in the domain $\{p \leq d_3\}$, and coincides with the term $E^\infty_{p,q}(\sigma')$ of the truncated spectral sequence calculating the Borel–Moore homology of $\sigma' \setminus F_{d_3}(\sigma')$ in the domain $\{p > d_3\}$. This completes the proof of Theorem 1. □

5 Proof of Theorem 2

The simplicial resolution $\sigma_{\mathbb{C}}$ of $\Sigma_{\mathbb{C}}$ appears in the same way as its real analog σ in the previous section. It also has a natural filtration $F_1 \subset \cdots \subset F_{d_1+1} = \sigma_{\mathbb{C}}$. For $p \in [1, d_1]$, its term $F_p \setminus F_{p-1}$ is fibered over the configuration space $B(\mathbb{CP}^1, p)$; its fiber over a configuration $(x_1, \ldots x_p)$ is equal to the product of the space $\mathbb{C}^{D-N(p)}$ (consisting of all complex systems (1) vanishing at all lines corresponding to the points of this configuration) and the $(p-1)$-dimensional simplex whose vertices correspond to the points of the configuration. In particular, our spectral sequence calculating rational Borel-Moore homology of $\sigma_{\mathbb{C}}$ has $E^1_{p,q} \cong \overline{H}_{q-2(D-N(p))+1}(B(\mathbb{CP}^1, p); \pm\mathbb{Q})$ for such p. By Lemma 3, only the following such groups are non-trivial: $E^1_{1,2(D-n)-1} \cong \mathbb{Q}$, $E^1_{1,2(D-n)+1} \cong \mathbb{Q}$, and (if $d_1 > 1$) $E^1_{2,2(D-2n)+1} \cong \mathbb{Q}$.

The last term $F_{d_1+1} \setminus F_{d_1}$ is homeomorphic to the cone over the d_1-th self-join $(\mathbb{CP}^1)^{*d_1}$ with the base of this cone removed (as it belongs to F_{d_1}). Therefore, by Lemma 2, the column $E^1_{d_1+1,*}$ is trivial if $d_1 > 1$, and contains a unique non-trivial group $E^1_{2,1} \cong \mathbb{Q}$ if $d_1 = 1$.

So, in any case, the first sheet E^1 of our spectral sequence has only three non-trivial terms $E^1_{1,2(D-n)-1}$, $E^1_{1,2(D-n)+1}$, and $E^1_{2,2(D-2n)+1}$, all of which are isomorphic to $\mathbb{Q}$. The differentials in it are obviously trivial; therefore, the group $\overline{H}_*(\sigma)$ has three non-trivial terms in dimensions $2(D-n)$, $2(D-n)+2$, and $2(D-2n)+3$. By Alexander duality in the space $\mathbb{C}^D$, this gives us three groups $\tilde{H}^{2n-3} \cong \mathbb{Q}$, $\tilde{H}^{2n-1} \cong \mathbb{Q}$, and $\tilde{H}^{4n-4} \cong \mathbb{Q}$, and zero in all other dimensions.

All assertions of Theorem 2 concerning the ring structure, realization of cohomology classes, and the weight filtration are well-known or obvious in the case $d_1 = 1$ (when $D = 2n$ and $\mathbb{C}^{2n} \setminus \Sigma_{\mathbb{C}}$ is the space of pairs of linearly independent vectors in $\mathbb{C}^n$, and is homotopy equivalent to the Stiefel manifold $V_2(\mathbb{C}^n)$). The general case can be deduced from this one by the map $P : \mathbb{C}^{2n} \setminus \Sigma_{\mathbb{C}} \to \mathbb{C}^D \setminus \Sigma_{\mathbb{C}}$ sending any collection of linear functions $(f_1, \dots, f_n)$ to $(f_1^{d_1}, \dots, f_n^{d_n})$. Indeed, the realization of $(2n-1)$-dimensional classes follows from the commutative diagram

$$\begin{array}{ccc} \mathbb{C}^{2n} \setminus \Sigma_{\mathbb{C}} & \xrightarrow{P} & \mathbb{C}^D \setminus \Sigma_{\mathbb{C}} \\ \downarrow & & \downarrow \\ \mathbb{C}^n \setminus 0 & \longrightarrow & \mathbb{C}^n \setminus 0 \end{array},$$

where the lower horizontal arrow is defined by

$$(z_1, \dots, z_n) \mapsto (z_1^{d_1}, \dots, z_n^{d_n})$$

and induces an isomorphism of $(2n-1)$-dimensional rational homology groups. The assertion on the realization of $(2n-3)$-dimensional classes is obvious. The statements on the multiplication and the weight filtration follow from the naturality of these structures. $\square$

6 Proof of Theorem 3

The additive part of this theorem can be proved in almost the same way as that of Theorem 1: see Vassiliev (1998). In particular, we construct a simplicial resolution σ_m of the m-discriminant variety Σ_m. It has a natural filtration $\Phi_1 \subset \dots \subset \Phi_{[d/m]} \subset \Phi_{[d/m]+1} = \sigma_m$. The term $\Phi_p \setminus \Phi_{p-1}$, $p \leq [d/m]$, of this filtration is the space of a fiber bundle with the base $B(\mathbb{CP}^1, p)$. Its fiber over the collection of points $(z_1, \dots, z_p) \subset \mathbb{CP}^1$ is the product of an open $(p-1)$-dimensional simplex whose vertices are related with these p points, and the subspace of codimension mp in $\mathbb{C}^{d+1}$ consisting of all polynomials having m-fold zeros on the corresponding p lines. The term $\Phi_{[d/m]+1} \setminus \Phi_{[d/m]}$ appears from the zero polynomial and is the cone over the space $(\mathbb{CP}^1)^{*[d/m]}$ with the base of this cone removed. The term E^1 of the corresponding spectral sequence can be calculated immediately with the help of Lemmas 2 and 3. Its shape implies that all further differentials of the spectral sequence are trivial, with unique exception in the case $d = m$, when all non-zero (isomorphic to $\mathbb{Q}$) groups of E^1 are $E^1_{1,3}$, $E^1_{1,1}$, and $E^1_{2,1}$. In this case, the differential $\partial_1 : E^1_{2,1} \to E^1_{1,1}$ is an isomorphism, because the zero section of the tautological bundle over $\mathbb{CP}^1$ defines a non-zero element of the 2-dimensional Borel–Moore homology group of the space of this bundle. Therefore, the only surviving term is $E^2_{1,3} \cong \mathbb{Q}$; by Alexander duality, it gives us a $(2m-3)$-dimensional cohomology class.

The remaining statements of Theorem 3 are based on the following comparison lemma. Consider the map $J : \mathbb{C}^{d+1} \to \mathbb{C}^D$, $D = m(d+2-m)$, sending any homogeneous polynomial $\mathbb{C}^2 \to \mathbb{C}^1$ of degree d to the collection of all its partial derivatives of order $m-1$.

Lemma 7 *For any $d \geq m > 1$, $\Sigma_m = J^{-1}(\Sigma_{\mathbb{C}})$. For any $d \geq 2m-1$, the induced map of cohomology groups, $J^* : H^*(\mathbb{C}^D \setminus \Sigma_{\mathbb{C}}, \mathbb{Q}) \to H^*(\mathbb{C}^{d+1} \setminus \Sigma_m, \mathbb{Q})$, is an isomorphism.*

This is a standard comparison theorem of our spectral sequences: see especially Section IV.7 in Vassiliev (1994, 1997). □

Now the assertions of Theorem 3 on the multiplication and weight filtrations follow from the similar assertions of Theorem 2 by the naturality of these structures. □

Acknowledgments I thank O. A. Malinovskaya for a useful discussion, and also the referee for suggesting to include the assertions on Hodge structure in Theorems 2 and 3.

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