

Constructive Geometrization of Thurston Maps and Decidability of Thurston Equivalence

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Received: 14 November 2014 / Revised: 3 June 2015 / Accepted: 4 August 2015 /
Published online: 7 September 2015
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Abstract The key result in the present paper is a direct analogue of the celebrated Thurston’s Theorem Douady and Hubbard (Acta Math 171:263–297, 1993) for marked Thurston maps with parabolic orbifolds. Combining this result with previously developed techniques, we prove that every Thurston map can be constructively geometrized in a canonical fashion. As a consequence, we give a partial resolution of the general problem of decidability of Thurston equivalence of two postcritically finite branched covers of S^2 (cf. Bonnot et al. Moscow Math J 12:747–763, 2012).

Keywords Thurston equivalence · Thurston obstruction · Lattès map · Geometrization

1 Introduction

A Thurston map is a basic object of study in one-dimensional dynamics: a branched covering f of the 2-sphere with finite critical orbits. Such a map can be described in a purely combinatorial language by introducing a suitable triangulation of S^2 whose

M. Yampolsky’s research was partially supported by NSERC Discovery Grant and Simons Fellowship.

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set of vertices includes the critical orbits of f . Different combinatorial descriptions of the map lead to a natural *combinatorial* or *Thurston equivalence* relation. A natural question arises whether given two such combinatorial objects, it can be decided if they are equivalent or not in some systematic, i.e. algorithmic, fashion.

We briefly outline the history of the problem. A central theorem in the subject is the result of Thurston (Douady and Hubbard 1993) that describes, in a topological language, which Thurston maps are combinatorially equivalent to rational mappings of $\hat{\mathbb{C}}$. In the case when an equivalent rational mapping exists, it is essentially unique, and the proof of the theorem (Douady and Hubbard 1993) supplies an iterative algorithm for approximating its coefficients. The only obstacle for the existence of a Thurston equivalent rational map is the presence of a *Thurston obstruction* which is a finite collection of curves in S^2 that satisfies a certain combinatorial inequality. Equivalence to a rational mapping can thus be seen as a *geometrization* of the branched covering: equipping the topological object with a canonical geometric description.

In Bonnot et al. (2012) it was shown that, outside of some exceptional cases, the question of Thurston equivalence to a rational mapping is *algorithmically decidable*. Namely, there exists an algorithm \mathcal{A}_1 which, given a combinatorial description of f , outputs 1 if f is equivalent to a rational mapping and 0 otherwise. Moreover, in the former case, \mathcal{A}_1 identifies the rational mapping. Since two different rational mappings are easy to distinguish—for instance, by comparing their coefficients after some normalization—this implies that in the case when either f or g has no Thurston obstruction, the statement of the Main Theorem I can be deduced from the existence of \mathcal{A}_1 (Bonnot et al. 2012).

Our work concentrates on the situation when Thurston maps are obstructed. In this case, geometrization may be achieved by *decomposition* into geometrizable components (Pilgrim 1827). We show:

Main Theorem I *Every Thurston map admits a constructive canonical geometrization.*

The main step in the proof is a direct analogue of Thurston's Theorem for the exceptional cases, Thurston maps with parabolic orbifolds:

Main Theorem II *A marked Thurston map with parabolic orbifold is geometrizable if and only if it has no degenerate Levy cycles.*

Detailed versions of both statements will be given below (see Theorems 6.1 and 4.1), after some preliminaries. As a consequence we obtain a partial resolution of the general question of decidability of Thurston equivalence:

Main Theorem III *There exists an algorithm \mathcal{A} which does the following. Let f and g be marked Thurston maps and assume that every element of the canonical geometrization of f has hyperbolic orbifold. The algorithm \mathcal{A} , given the combinatorial descriptions of f and g , outputs 1 if f and g are Thurston equivalent and 0 otherwise.*

2 Geometric Preliminaries

Mapping Class Groups

When we talk about a surface with holes, we will always mean a surface S with boundary, which is obtained from a surface without holes by removing a collection of disjoint Jordan disks. A surface S is of *finite topological type* if it is a genus g surface with m holes and n punctures, where $g, m, n < \infty$. The *Mapping Class Group* $MCG(S)$ is defined as the group of homeomorphisms $S \rightarrow S$ which restrict to the identity on ∂S , up to isotopy relative ∂S .

The elements of $MCG(S)$ are allowed to interchange the punctures of S ; if we further restrict to homeomorphisms which fix each puncture individually, we obtain the *pure Mapping Class Group* $PMCG(S)$. If we denote by Σ_n the group of permutations of n elements (punctures, in our case), then we have a short exact sequence

$$1 \longrightarrow PMCG(S) \longrightarrow MCG(S) \longrightarrow \Sigma_n \longrightarrow 1.$$

We refer the reader to [Farb and Margalit \(2011\)](#) for a detailed discussion of Mapping Class Groups. Replacing homeomorphisms with diffeomorphisms, and/or isotopy with homotopy leads to an equivalent definition of $MCG(S)$.

Throughout the article, we denote by T_γ the Dehn twist around a curve γ . Let $PMCG(S^2, P)$ be the pure Mapping Class Group of the two-dimensional sphere, where $P \subset S^2$ is a finite set of punctures. We use the following fact:

Proposition 2.1 *The group $PMCG(S^2, P)$ is generated by a finite number of explicit Dehn twists.*

The finiteness of the number of generating twists is a classical result of Dehn; [Lickorish \(1964\)](#) has made the construction explicit. See, for example, [Farb and Margalit \(2011\)](#) for an exposition.

3 Thurston Maps

In this section we recall the basic setting of Thurston’s characterization of rational functions.

3.1 Branched Covering Maps

Let $f : S^2 \rightarrow S^2$ be an orientation-preserving branched covering self-map of the two-sphere. We define the *postcritical set* P_f by

$$P_f := \bigcup_{n>0} f^{on}(\Omega_f),$$

where Ω_f is the set of critical points of f . When the postcritical set P_f is finite, we say that f is *postcritically finite*.

A (marked) Thurston map is a pair (f, Q_f) where $f : S^2 \rightarrow S^2$ is a postcritically finite ramified covering of degree at least 2 and Q_f is a finite collection of marked points $Q_f \subset S^2$ which contains P_f and is f -invariant: $f(Q_f) \subset Q_f$. Thus, all elements of Q_f are pre-periodic for f .

Thurston Equivalence

Two marked Thurston maps (f, Q_f) and (g, Q_g) are Thurston (or combinatorially) equivalent if there are homeomorphisms $\phi_0, \phi_1 : S^2 \rightarrow S^2$ such that

- (1) the maps ϕ_0, ϕ_1 coincide on Q_f , send Q_f to Q_g and are isotopic rel Q_f ;
- (2) the diagram

$$\begin{array}{ccc}
 S^2 & \xrightarrow{\phi_1} & S^2 \\
 \downarrow f & & \downarrow g \\
 S^2 & \xrightarrow{\phi_0} & S^2
 \end{array}$$

commutes.

We will use this notion of equivalence for Thurston maps throughout the article.

Orbifold of a Thurston Map

Given a Thurston map $f : S^2 \rightarrow S^2$, we define a function $N_f : S^2 \rightarrow \mathbb{N} \cup \infty$ as follows:

$$N_f(x) = \begin{cases} 1 & \text{if } x \notin P_f, \\ \infty & \text{if } x \text{ is in a cycle containing a critical point,} \\ \text{lcm}_{f^k(y)=x} \deg_y(f^{ok}) & \text{otherwise.} \end{cases}$$

The pair $O_f = (S^2, N_f)$ is called the orbifold of f . The signature of the orbifold (S^2, N_f) is the set $\{N_f(x) \text{ for } x \text{ such that } 1 < N_f(x) < \infty\}$. The Euler characteristic of the orbifold is given by

$$\chi(S^2, N_f) := 2 - \sum_{x \in P_f} \left(1 - \frac{1}{N_f(x)} \right). \tag{1}$$

One can prove that $\chi(S^2, N_f) \leq 0$. In the case where $\chi(S^2, N_f) < 0$, we say that the orbifold is hyperbolic. Observe that most orbifolds are hyperbolic: indeed, as soon as the cardinality $|P_f| > 4$, the orbifold is hyperbolic.

Thurston Maps with Parabolic Orbifolds

A complete classification of postcritically finite branched covers with parabolic orbifolds has been given in [Douady and Hubbard \(1993\)](#). All postcritically finite rational functions with parabolic orbifolds have been extensively described in [Milnor \(2006\)](#). In this section, we remind the reader of basic results on Thurston maps with parabolic orbifolds.

Recall that a map $f : (S_1, v_1) \rightarrow (S_2, v_2)$ is a covering map of orbifolds if

$$v_1(x) \deg_x f = v_2(f(x))$$

for any $x \in S_1$. The following proposition is found in [Douady and Hubbard \(1993\)](#):

- Proposition 3.1** i. *If $f : S^2 \rightarrow S^2$ is a postcritically finite branched cover, then $\chi(O_f) \leq 0$.*
 ii. *If $\chi(O_f) = 0$, then $f : O_f \rightarrow O_f$ is a covering map of orbifolds.*

Equation (1) gives six possibilities for $\chi(O_f) = 0$. If we record all the values of v_f that are bigger than 1, we get one of the following orbifold signatures.

- (1) (∞, ∞) ,
- (2) $(2, 2, \infty)$,
- (3) $(2, 4, 4)$,
- (4) $(2, 3, 6)$,
- (5) $(3, 3, 3)$,
- (6) $(2, 2, 2, 2)$.

In cases (1)–(5) the orbifolds have a unique complex structure, and can be realized as a quotient \mathbb{C}/G of the complex plane by a discrete group of automorphisms G as follows (cf. [Douady and Hubbard 1993](#)):

- (1) $G = \langle z \mapsto z + 1 \rangle$,
- (2) $G = \langle z \mapsto z + 1, z \mapsto -z \rangle$,
- (3) $G = \langle z \mapsto z + a, z \mapsto iz \rangle$, where $a \in \mathbb{Z}[i]$,
- (4) $G = \langle z \mapsto z + a, z \mapsto wz \rangle$, where $w = e^{i\pi/3}$, $a \in \mathbb{Z}[w]$,
- (5) $G = \langle z \mapsto z + a, z \mapsto w^2z \rangle$, where $w = e^{i\pi/3}$, $a \in \mathbb{Z}[w]$.

We are mostly interested in the last case. We will refer to a Thurston map that has orbifold with signature $(2, 2, 2, 2)$ simply as a $(2, 2, 2, 2)$ -map. An orbifold with signature $(2, 2, 2, 2)$ is a quotient of a torus T by an involution i ; the four fixed points of the involution i correspond to the points with ramification weight 2 on the orbifold. The corresponding branched cover $p : T \rightarrow S^2$ has exactly four simple critical points which are the fixed points of i . It follows that a $(2, 2, 2, 2)$ -map f can be lifted to a covering self-map \hat{f} of T .

An orbifold with signature $(2, 2, 2, 2)$ has a unique affine structure of the quotient \mathbb{R}^2/G where

$$G = \langle z \mapsto z + 1, z \mapsto z + i, z \mapsto -z \rangle .$$

We will denote this quotient by the symbol \mathbb{H} , which graphically represents a “pillowcase”—a sphere with four corner points.

An important example of a $(2, 2, 2, 2)$ -map is a *flexible Lattés rational map* constructed as follows. Let

$$\mathbb{T} \simeq \mathbb{T}_\Lambda = \mathbb{C}/\Lambda$$

where the lattice $\Lambda = \langle 1, \tau \rangle$, with $\tau \in \mathbb{H}$. Set $i(z) = -z$. Then $\mathbb{T}_\lambda/i \simeq \hat{\mathbb{C}}$ and the branched cover

$$\wp : \mathbb{C} \rightarrow \mathbb{C}/\Lambda \rightarrow \hat{\mathbb{C}}$$

is the Weierstrass elliptic function \wp with periods $1, \tau$. Consider the parallelogram P with vertices $0, 1, \tau$, and $1 + \tau$ which is the fundamental domain of Λ . The four simple fixed points of the involution i are the \wp -images of $0, 1/2, \tau/2$ and $(1 + \tau)/2$. They are the critical points of the branched cover $\mathbb{T}_\Lambda \rightarrow \hat{\mathbb{C}}$ (Fig. 1).

Set

$$A(z) \equiv az + b, \text{ where } a \in \mathbb{Z} \text{ with } |a| > 1, \text{ and } b = (m + n\tau)/2 \in \Lambda/2.$$

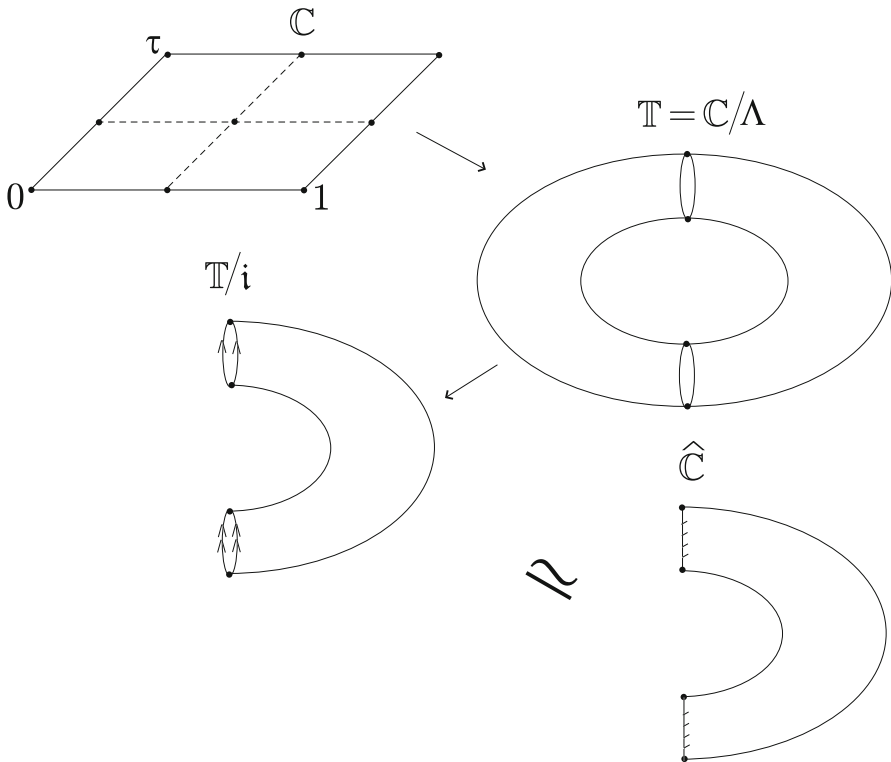


Fig. 1 Illustration of the branched cover \wp . The critical points of \wp are marked in a fundamental parallelogram of the lattice Λ , as well as their images

The complex-affine map A projects to a well-defined rational map

$$L : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$$

of degree a^2 . Trivially, all of the postcritical set of L lies in the projection of $\Lambda/2$ in $\hat{\mathbb{C}}$ and hence is finite. Note that as long as the values of $a, m,$ and n are the same, two different maps L are topologically conjugate for all values of τ . In particular, they cannot be distinguished by Thurston equivalence, which shows that the uniqueness part of Thurston’s Theorem does not generally hold in the parabolic orbifold case.

As before, let f be a $(2, 2, 2, 2)$ -map, and $p : T \rightarrow S^2$. Take any simple closed curve γ on $S^2 \setminus Q_f$. Then $p^{-1}(\gamma)$ has either one or two components that are simple closed curves.

The following propositions are straightforward (see, for example, [Selinger 2013](#)):

Proposition 3.2 *If there are exactly two postcritical points of f in each complementary component of γ , then the p -preimage of γ consists of two components that are homotopic in T and non-trivial in $H_1(T, \mathbb{Z})$. Otherwise, all preimages of γ are trivial.*

Every homotopy class of simple closed curves γ on T defines, up to sign, an element $\langle \gamma \rangle$ of $H_1(T, \mathbb{Z})$. If a simple closed curve γ on $S^2 \setminus Q_f$ has two p -preimages, then they are homotopic by the previous proposition. Therefore, every homotopy class of simple closed curves γ on $S^2 \setminus Q_f$ also defines, up to sign, an element $\langle \gamma \rangle$ of $H_1(T, \mathbb{Z})$. It is clear that for any $h \in H_1(T, \mathbb{Z})$ there exists a homotopy class of simple closed curves γ such that $h = n\langle \gamma \rangle$ for some $n \in \mathbb{Z}$.

Since $H_1(T, \mathbb{Z}) \cong \mathbb{Z}^2$, the push-forward operator \hat{f}_* is a linear operator. It is easy to see that the determinant of \hat{f}_* is equal to the degree of \hat{f} , which is in turn equal to the degree of f . Existence of invariant multicurves for f is related to the action of \hat{f}_* on $H_1(T, \mathbb{Z})$.

Proposition 3.3 *Suppose that a component γ' of the f -preimage of a simple closed curve γ on $S^2 \setminus Q_f$ is homotopic γ . Take a p -preimage α of γ . Then $\hat{f}_*(\langle \alpha \rangle) = \pm d\langle \alpha \rangle$, where d is the degree of f restricted to γ' .*

More generally, we obtain the following.

Proposition 3.4 *Let γ be a simple closed curve on $S^2 \setminus Q_f$ such that there are two points of the postcritical set P_f in each complementary component of γ . If all components of the f -preimage of γ have zero intersection number with γ in $S^2 \setminus Q_f$, then $\hat{f}_*(\langle \gamma \rangle) = \pm d\langle \gamma \rangle$, where d is the degree of f restricted to any preimage of γ .*

Geometrization of a Thurston Map with Parabolic Orbifold

As seen above, every parabolic orbifold, which is a topological 2-sphere, can be obtained by considering a quotient of \mathbb{R}^2 by the action of a discrete group G of Euclidean isometries that depends only on the signature of the orbifold. We will call G the *orbifold group*. Up to equivalence, we may thus assume that a Thurston map f with parabolic orbifold is a self-map of the $O_f = \mathbb{R}^2/G$.

Theorem 3.5 *Let f be a Thurston map with postcritical set $P = P_f$ and no extra marked points ($Q_f = P_f$) with parabolic orbifold. Then f is equivalent to a quotient of a real affine map by the action of the orbifold group. This affine map is unique up to affine conjugation.*

Proof Since O_f is parabolic there are three cases: $\#P$ is either 2, 3 or 4. In the first two cases, the orbifold has a unique complex structure and f is equivalent to a quotient of a complex affine map (see Douady and Hubbard 1993). In the third case, the orbifold $O_f = \square$, so it is the quotient of \mathbb{R}^2 by the action of

$$G = \langle z \mapsto z + 1, z \mapsto z + i, z \mapsto -z \rangle.$$

Note that the elements of G are either translations by an integer vector or symmetries around a preimage of a marked point. We will denote

$$S_w \cdot z = 2w - z$$

the symmetry around a point $w \in \mathbb{R}^2$. Consider a lift $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of f and denote

$$\tilde{P} = \{1/2(\mathbb{Z} + i\mathbb{Z})\}$$

the full preimage of P by the projection map.

Lemma 3.6 *A lift F of a continuous map $f : \square \rightarrow \square$ is affine on \tilde{P} .*

Proof Since F is a lift of f , it defines a push-forward map $F_* : G \rightarrow G$ such that

$$F(g \cdot z) = F_*g \cdot F(z) \text{ for any } z \in \mathbb{R}^2 \text{ and } g \in G.$$

It is clear that F_* is a homeomorphism and it sends translations to translations and symmetries to symmetries: $F_*S_z = S_{F(z)}$. We immediately see that

$$F(z + w) = Aw + F(z)$$

for some integer matrix A and any $w \in \mathbb{Z} + i\mathbb{Z}$. Since

$$\begin{aligned} F(0) &= F(S_{1/2} \cdot 1) = S_{F(1/2)} \cdot F(1) = S_{F(1/2)} \cdot (A(1, 0)^T + F(0)) \\ &= 2F(1/2) - (A(1, 0)^T + F(0)), \end{aligned}$$

we see that

$$F(1/2) = F(0) + A(1/2, 0)^T.$$

Similar computations for $F(1/2i)$ and $F(1/2 + 1/2i)$ conclude the proof of the lemma. □

Thus $F(z)$ agrees with an affine map $L(z) = Az + b$ on \tilde{P} , where A is an integer matrix and $b \in 1/2(\mathbb{Z} + i\mathbb{Z})$ and $F_*g = L_*g$ for all $g \in G$. Therefore the map $\tilde{\phi} = L^{-1} \circ F$ is G -equivariant and projects to a self-homeomorphism ϕ of O_f which fixes P .

Lemma 3.7 *Let $l(z)$ be a quotient of an affine map $L(z) = Az + b$ where A is an integer matrix and $b \in 1/2(\mathbb{Z} + i\mathbb{Z})$ by the action of G , and ϕ be an element of $PMCG(\square)$. If $l(z) \circ \phi$ has a lift L' to \mathbb{R}^2 such that $L'(z) = Az + b$ for all points in \tilde{P} , then ϕ is trivial.*

Proof If $l(z) \circ \phi$ and $l(z)$ have lifts that agree on \tilde{P} , then ϕ must have a lift that is identical on \tilde{P} .

The pure mapping class group $PMCG(\square)$ is a free group generated by Dehn twists T_α and T_β around simple closed curves α and β that lift to horizontal and vertical straight lines in \mathbb{R}^2 respectively. As a representative of T_α and T_β we can take unique homeomorphisms on \square that are quotients of

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

on \mathbb{R}^2 by the action of G . This representation of $PMCG(\square)$ is faithful, and therefore only the trivial element can have a lift which is identical on \tilde{P} . □

By the previous lemma, the homeomorphism ϕ represents the trivial element of $PMCG(\square)$ and, hence, is homotopic to the identity relative to P . Define l to be the quotient of L by the action of G . Then the commutative diagram

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\tilde{\phi}} & \mathbb{R}^2 \\ F \downarrow & & \downarrow L \\ \mathbb{R}^2 & \xrightarrow{\text{id}} & \mathbb{R}^2 \end{array}$$

projects to the commutative diagram

$$\begin{array}{ccc} \square & \xrightarrow{\phi} & \square \\ f \downarrow & & \downarrow l \\ \square & \xrightarrow{\text{id}} & \square \end{array}$$

which realizes Thurston equivalence between f and l .

On the other hand, suppose that l_1 and l_2 are quotients of two affine maps, which are Thurston equivalent. Then l_1 and l_2 are conjugate on P , hence lifts thereof are conjugate on \tilde{P} by an affine map (in the case when $O_f = \square$ this follows from Lemma 3.6; the other cases are similar) and the uniqueness part of the statement follows.

Thurston Linear Transformation

Let Q be a finite collection of points in S^2 . We recall that a simple closed curve $\gamma \subset S^2 - Q$ is *essential* if it does not bound a disk, is *non-peripheral* if it does not bound a punctured disk.

Definition 3.8 A multicurve Γ on (S^2, Q) is a set of disjoint, nonhomotopic, essential, non-peripheral simple closed curves on $S^2 \setminus Q$. Let (f, Q_f) be a Thurston map, and set $Q = Q_f$. A multicurve Γ on $S \setminus Q$ is *f-stable* if for every curve $\gamma \in \Gamma$, each component α of $f^{-1}(\gamma)$ is either trivial (meaning inessential or peripheral) or homotopic rel Q to an element of Γ .

To any multicurve is associated its *Thurston linear transformation* $f_\Gamma : \mathbb{R}^\Gamma \rightarrow \mathbb{R}^\Gamma$, best described by the following transition matrix

$$M_{\gamma\delta} = \sum_{\alpha} \frac{1}{\deg(f : \alpha \rightarrow \delta)}$$

where the sum is taken over all the components α of $f^{-1}(\delta)$ which are isotopic rel Q to γ . Since this matrix has nonnegative entries, it has a leading eigenvalue $\lambda(\Gamma)$ that is real and nonnegative (by the Perron–Frobenius theorem).

We can now state Thurston’s theorem:

Thurston’s Theorem *Let $f : S^2 \rightarrow S^2$ be a marked Thurston map with a hyperbolic orbifold. Then f is Thurston equivalent to rational function g with a finite set of marked pre-periodic orbits if and only if $\lambda(\Gamma) < 1$ for every f -stable multicurve Γ . The rational function g is unique up to conjugation by an automorphism of \mathbb{Q}^1 .*

The proof of Thurston’s Theorem for Thurston maps without additional marked points is given in [Douady and Hubbard \(1993\)](#), for the proof for marked maps see e.g. [Buff et al. \(2014\)](#).

When a multicurve Γ has a leading eigenvalue $\lambda(\Gamma) \geq 1$, we call it a *Thurston obstruction* for f . A Thurston obstruction Γ is *minimal* if no proper subset of Γ is itself an obstruction. We call Γ a *simple* obstruction if no permutation of the curves in Γ puts M_Γ in the block form

$$M_\Gamma = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix},$$

where the leading eigenvalue of M_{11} is less than 1. If such a permutation exists, it follows that M_{22} is a Thurston matrix of the corresponding sub-multicurve with the same leading eigenvalue as M_Γ . It is thus evident that every obstruction contains a simple one.

In the original formulation in [Douady and Hubbard \(1993\)](#), a Thurston obstruction was required to be invariant. Omitting this requirement makes the statement of the theorem weaker in one direction and stronger in the other direction. However, in [Selinger \(2011\)](#) is shown that if there exists a Thurston obstruction for f , then there also exists a simple f -stable obstruction.

The following is an exercise in linear algebra (c.f. [Selinger 2012](#)):

Proposition 3.9 *A multicurve Γ is a simple obstruction if and only if there exists a positive vector v such that $M_\Gamma v \geq v$.*

For what follows, let us make a note of a particular kind of Thurston obstructions:

Definition 3.10 *A Levy cycle is a multicurve*

$$\Gamma = \{\gamma_0, \gamma_1, \dots, \gamma_{n-1}\}$$

such that each γ_i has a nontrivial preimage γ'_i , where the topological degree of f restricted to γ'_i is 1 and γ'_i is homotopic to $\gamma_{(i-1) \bmod n}$ rel Q . A Levy cycle is *degenerate* if each γ_i has a pre-image γ'_i as above such that γ'_i bounds a disk D_i and the restriction of f to D_i is a homeomorphism and $f(D_i)$ is homotopic to $D_{(i+1) \bmod n}$ rel Q .

Proposition 3.11 *A Thurston map f admits a Levy cycle if and only if so does its iterate f^n .*

Proof If a simple closed curve γ generates a Levy cycle for f , then it obviously does so for f^n .

Suppose γ generates a Levy cycle for f^n of length m , i.e. there is a collection of simple closed curves $\{\gamma_0, \dots, \gamma_{nm-1}\}$ such that $f(\gamma_i) = \gamma_{i+1}$ for all $0 \leq i \leq nm - 1$ and $f(\gamma_{nm-1}) = \gamma$ is isotopic to γ_0 , and f restricted to any γ_i is a homeomorphism. Moreover, all γ_{in} represent pairwise disjoint isotopy classes of simple closed curves. If all γ_i were to represent pairwise disjoint isotopy classes of simple closed curves, then γ_0 would generate a Levy cycle for f , by definition. However this does not have to be the case. Changing f by isotopy if needed, we may assume that all intersections between γ_i are transversal.

Consider a connected component A of $\bigcup \gamma_i$ containing γ_0 . Let $k = \min\{i > 0 \mid \gamma_i \in A\}$. Then f^k send A homeomorphically to $B = A \cup \gamma \setminus \gamma_0$ and B is isotopic to A . Note that $\bigcup \gamma_i$ is a union of disjoint nm/k homeomorphic copies of A . The graphs A and B separate the sphere into a collection of discs C_i and D_i with boundaries α_i and β_i respectively, where α_i is isotopic to β_i . If β_i is peripheral, then f has at most one singular value inside D_i , which implies that there exists C_j such that $f^k(C_j) = D_i$ and $\deg f^k|_{C_j} = \deg f^k|_{\alpha_j} = 1$. Since $\deg f > 1$, we see that there exists β_0 which is non-peripheral (we allow ourselves to slightly abuse the notation here).

Define inductively $\beta_{i+1} = f^k(\alpha_i)$. Then all β_i are also non-peripheral and since they are chosen from a finite set, $\beta_s = \beta_0$ for some $s \geq 0$. Therefore β_0 has a degree 1 f^{ks} -preimage which is homotopic to β_0 . Note that all the β_i represent pairwise disjoint isotopy classes of simple closed curves. We conclude that β_0 generates a Levy cycle for f . □

A Thurston map f is called a *topological polynomial* if there exists a point w such that $f^{-1}(w) = \{w\}$. The following was proved by [Levy \(1985\)](#):

Theorem 3.12 *If f is a topological polynomial and Γ is a Thurston obstruction for f , then Γ contains a degenerate Levy cycle.*

Example Let us give a simple yet instructive example. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a postcritically finite polynomial of degree $\deg f = d \geq 2$ and let p be a fixed point of f which does not lie in P_f . Perform a topological surgery on $\hat{\mathbb{C}} \simeq S^2$ inserting a topological disk D_x at each point $x \in \cup_{j \geq 0} f^{-j}(p)$. Modify the map f accordingly to send D_x to $D_{f(x)}$. Finally, select a new dynamics on $D = D_p$ so that there are at least two fixed points $a, b \in D$. The resulting topological polynomial g has the same degree as f . Select the marked set

$$Q_g \equiv P_g \cup \{a, b\}.$$

With this choice of the marked, the Thurston map (g, Q_g) is clearly obstructed—a simple closed curve $\gamma \subset D$ which separates a and b from P_g is a degenerate Levy cycle.

Thurston Iteration on the Teichmüller Space

For the basics of the Teichmüller Theory see e.g. [Imayoshi and Taniguchi \(1992\)](#). Let S_n^2 denote the two-sphere with n marked points. The moduli space $\mathcal{M}(S_n^2)$ parametrizes distinct complex structures on S_n^2 . For $n \leq 3$ it consists of a single point. For $n > 3$, it is naturally identified with the $n - 3$ dimensional complex manifold consisting of all n -tuples (z_1, \dots, z_n) of points in $\hat{\mathbb{C}}$ defined up to a Möbius transformation. The Teichmüller space $\mathcal{T}(S_n^2)$ is the universal covering space of $\mathcal{M}(S_n^2)$. We will use the notation $\|\cdot\|_T$ for the Teichmüller norm on $\mathcal{T}(S_n^2)$.

The Teichmüller space $\mathcal{T}(S_n^2)$ can be naturally constructed as the space of equivalence classes of almost complex structures on S_n^2 with $\mu_1 \equiv \mu_2$ if $\mu_1 = h^*\mu_2$ where h is a quasiconformal mapping of $S_n^2 = \hat{\mathbb{C}}$ isotopic to the identity relative the marked points. Another interpretation of $\mathcal{T}(S_n^2)$ is as the space of equivalence classes of quasiconformal mappings $\phi : S_n^2 \rightarrow \hat{\mathbb{C}}$ with $\phi_1 \equiv \phi_2$ if and only if there exists a Möbius map $h : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ such that $h \circ \phi_1$ is isotopic to ϕ_2 relative the marked points. The correspondence between the two viewpoints is standard: an almost complex structure μ on S_n^2 is obtained as the pullback of the standard structure σ_0 on $\hat{\mathbb{C}}$ by ϕ :

$$\mu = \phi^*(\sigma_0).$$

Let $f : S^2 \rightarrow S^2$ be a Thurston map of the 2-sphere with marked set Q_f . We will further assume that f is quasi-regular (even piecewise linear, see Sect. 3.2). We denote \mathcal{M}_f and \mathcal{T}_f the moduli space and the Teichmüller space respectively of the sphere S^2 with marked points Q_f . It is straightforward to verify that the operation defined on almost complex structures by $[\mu] \mapsto [f^*\mu]$ yields a well-defined analytic mapping

$$\sigma_f : \mathcal{T}_f \rightarrow \mathcal{T}_f$$

which we call the *Thurston pullback mapping*. It is equally easy to see that if f and g are two equivalent Thurston maps then σ_f coincides with σ_g up to isomorphism of Teichmüller spaces \mathcal{T}_f and \mathcal{T}_g .

In terms of the description of \mathcal{T}_f by equivalence classes of homeomorphisms $\phi : S_n^2 \rightarrow \hat{\mathbb{C}}$, the mapping σ_f is defined as follows. We can pull back the almost complex structure $\mu = \phi^* \sigma_0$ by f to

$$\mu' \equiv f^* \mu = f^* \phi^* \sigma_0.$$

Using Measurable Riemann Mapping Theorem to integrate μ' , we get a mapping $\phi' : S_n^2 \rightarrow \hat{\mathbb{C}}$ satisfying

$$\phi'^* \sigma_0 = \mu'.$$

We now set

$$\sigma_f[\phi] = [\phi'].$$

The key starting point of the proof of Thurston Theorem is the following:

Proposition 3.13 (cf. Douady and Hubbard (1993), Proposition 3.2.2) *A Thurston map f is equivalent to a rational function if and only if σ_f has a fixed point.*

Proof Since the standard almost complex structure σ_0 on $\hat{\mathbb{C}}$ is invariant under the pullback by a rational function, the “if” direction is obvious. For the “only if” direction, consider a pair of homeomorphisms ϕ and ϕ' which describe the same point in the Teichmüller space and such that $\phi' = \sigma_f(\phi)$. The mapping

$$f_\tau \equiv \phi \circ f \circ (\phi')^{-1} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$$

preserves the almost complex structure σ_0 by construction, and therefore is analytic, and hence rational. Let h be a Möbius map such that ϕ' is isotopic to $h \circ \phi$ relative Q_f . Then the rational mapping $f_\tau \circ h$ is Thurston equivalent by h : the diagram

$$\begin{CD} (S^2, Q_f) @>{h^{-1} \circ \phi'}>> \hat{\mathbb{C}} \\ @V{f}VV @VV{f_\tau \circ h}V \\ (S^2, Q_f) @>{\phi}>> \hat{\mathbb{C}} \end{CD}$$

commutes up to isotopy relative Q_f . □

It is straightforward that

$$\|d\sigma_f\|_T \leq 1.$$

Moreover, when f has a hyperbolic orbifold, there exists $k \in \mathbb{N}$ such that

$$\|d(\sigma_f^k)\|_T < 1$$

(see e.g. Buff et al. 2014). It follows that:

Proposition 3.14 *Suppose that f has a hyperbolic orbifold, and σ_f has a fixed point in \mathcal{T}_f . Then the fixed point is unique, and every σ_f -orbit converges to it.*

Weak contraction properties of σ_f imply that non-existence of a fixed point means that for every compact subset $K \subseteq \mathcal{T}_f$ and every starting point $[\tau_0] \in \mathcal{T}_f$ there is a moment $j \in \mathbb{N}$ when $[\sigma_f^j \tau_0] \notin K$. The next section gives a more precise explanation, due to Pilgrim (2001).

Canonical Obstructions

For a general hyperbolic Riemann surface W we denote ρ_W , d_W , and length_W the hyperbolic metric, distance, and length on W . When we want to emphasize the dependence of the hyperbolic metric on the choice of the complex structure τ on a surface S , we will write ρ_τ for the hyperbolic metric on $S_\tau \equiv (S, \tau)$, length_τ for the hyperbolic length, and d_τ for the hyperbolic distance. For a non-trivial homotopy class of closed curves $[\gamma]$ on S we let $\ell_\tau([\gamma])$ denote the length of the unique geodesic representative of $[\gamma]$ in S_τ .

The following is straightforward (see e.g. Pilgrim 2001):

Proposition 3.15 *Suppose there exists $\tau \in \mathcal{T}_f$ such that for a non-trivial homotopy class of simple closed curves $[\gamma]$ the lengths*

$$\ell_{\sigma_f^n \tau}([\gamma]) \xrightarrow{n \rightarrow \infty} 0.$$

Then the same property holds for any other starting point $\tau' \in \mathcal{T}_f$.

Definition 3.16 The canonical obstruction Γ_f of f is the collection of all non-trivial homotopy classes γ such that

$$\ell_{\sigma_f^n \tau}([\gamma]) \xrightarrow{n \rightarrow \infty} 0$$

for some (equivalently, for all) $\tau \in \mathcal{T}_f$.

Pilgrim proved the following:

Theorem 3.17 (Pilgrim 2001) *Suppose f is a Thurston map with a hyperbolic orbifold. If the canonical obstruction is empty, then f is Thurston equivalent to a rational function. If the canonical obstruction is non-empty, then it is a Thurston obstruction.*

Pilgrim further showed:

Theorem 3.18 (Pilgrim 2001) *Let $\tau_0 \in \mathcal{T}_f$. There exists a constant $E = E(\tau_0)$ such that for every non-trivial simple closed curve $\gamma \notin \Gamma_f$ we have*

$$\inf \ell_{\sigma_f^n \tau_0}([\gamma]) > E.$$

Pilgrim’s Decompositions of Thurston Maps

What follows is a very brief review; the reader is referred to K. Pilgrim’s book *Pilgrim (1827)* for details. We adhere to the notation of *Pilgrim (1827)*, for ease of reference.

As a motivation, consider that for the canonical Thurston obstruction $\Gamma_c \ni \gamma$, there is a choice of complex structure τ for which $\ell_\tau([\gamma])$ is arbitrarily small, and remains small under pullbacks by f . It is thus natural to think of the punctured sphere $S^2 \setminus P_f$ as pinching along the homotopy classes $[\gamma] \in \Gamma_c$; instead of a single sphere we then obtain a collection of spheres interchanged by f .

More specifically, let f be a Thurston map, and $\Gamma = \cup \gamma_j$ an f -stable multicurve. Consider also a finite collection of disjoint closed annuli $A_{0,j}$ which are homotopic to the respective γ_j . For each $A_{0,j}$ consider only non-trivial preimages; these form a collection of annuli $A_{1,k}$, each of which is homotopic to one of the curves in Γ . Pilgrim says that the pair (f, Γ) is in a *standard form* (see Fig. 2) if there exists a collection of annuli $A_{0,j}$, which we call *decomposition annuli*, as above such that the following properties hold:

- (a) for each curve γ_j the annuli $A_{1,k}$ in the same homotopy class are contained inside $A_{0,j}$;
- (b) moreover, the two outermost annuli $A_{1,k}$ as above share their outer boundary curves with $A_{0,j}$.

We call the components of the complement of the decomposition annuli the *thick parts*.

A Thurston map with a multicurve in a standard form can be decomposed as follows. First, all annuli $A_{0,j}$ are removed, leaving a collection of spheres with holes, denoted $S_0(j)$. For each j , there exists a unique connected component $S_1(j)$ of $f^{-1}(\cup S_0(j))$ which has the property $\partial S_0(j) \subset \partial S_1(j)$. Any such $S_1(j)$ is a sphere with holes, with boundary curves being of two types: boundaries of the removed annuli, or boundaries of trivial preimages of the removed annuli.

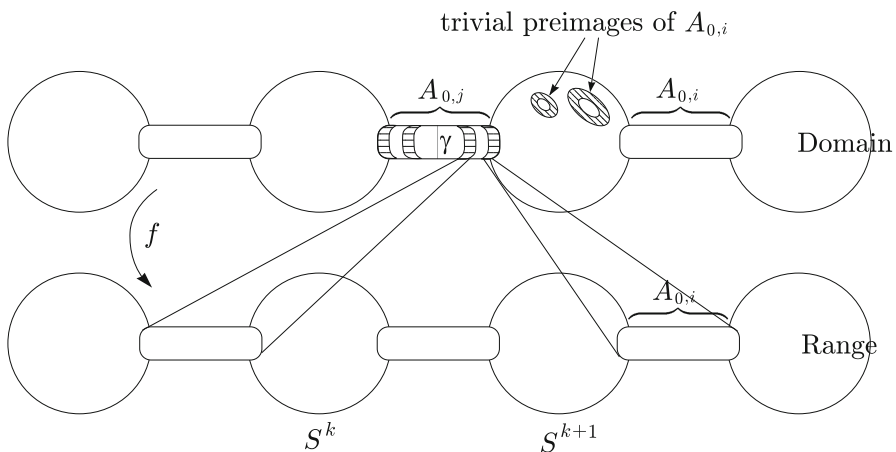


Fig. 2 Pilgrim’s decomposition of a Thurston map

The holes in $S_0(j) \subset S^2$ can be filled as follows. Let χ be a boundary curve of a component D of $S^2 \setminus S_0(j)$. Let $k \in \mathbb{N}$ be the first iterate $f^k : \chi \rightarrow \chi$, if it exists. For each $0 \leq i \leq k - 1$ the curve $\chi_i \equiv f^i(\chi)$ bounds a component D_i of $S^2 \setminus S_0(m_i)$ for some m_i . Denote d_i the degree of $f : \chi_i \rightarrow \chi_{i+1}$. Select homeomorphisms

$$h_i : \tilde{D}_i \rightarrow \tilde{\mathbb{D}} \text{ so that } h_{i+1} \circ f \circ h_i^{-1}(z) = z^{d_i}.$$

Set $\tilde{f} \equiv f$ on $\cup S_0(j)$. Define new punctured spheres $\tilde{S}(j)$ by adjoining cups $h_i^{-1}(\tilde{\mathbb{D}} \setminus \{0\})$ to $S_0(j)$. Extend the map \tilde{f} to each D_i by setting

$$\tilde{f}(z) = h_{i+1}^{-1} \circ (h_i(z))^{d_i}.$$

We have thus replaced every hole with a cap with a single puncture. We call such a procedure *patching* a thick component.

By construction, the map

$$\tilde{f} : \cup \tilde{S}(j) \rightarrow \cup \tilde{S}(j)$$

contains a finite number of periodic cycles of punctured spheres. For every periodic sphere $\tilde{S}(j)$ denote by \mathcal{F} the first return map $f^{kj} : \tilde{S}(j) \rightarrow \tilde{S}(j)$. This is again a Thurston map. The collection of maps \mathcal{F} and the combinatorial information required to glue the spheres $S_0(j)$ back together is what Pilgrim calls a *decomposition* of f .

Pilgrim shows:

Theorem 3.19 *For every obstructed marked Thurston map f with an obstruction Γ there exists an equivalent map g such that (g, Γ) is in a standard form, and thus can be decomposed.*

Topological Characterization of Canonical Obstructions

The first author showed in [Selinger \(2013\)](#):

Theorem 3.20 (Characterization of Canonical Obstructions) *The canonical obstruction Γ is a unique minimal obstruction with the following properties.*

- *If the first-return map F of a cycle of components in \S_Γ is a $(2, 2, 2, 2)$ -map, then every curve of every simple Thurston obstruction for F has two postcritical points of f in each complementary component and the two eigenvalues of \hat{F}_* are equal or non-integer.*
- *If the first-return map F of a cycle of components in \S_Γ is not a $(2, 2, 2, 2)$ -map or a homeomorphism, then there exists no Thurston obstruction of F .*

3.2 Algorithmic Preliminaries

A Piecewise-Linear Thurston Map

For the purposes of algorithmic analysis, we will require a finite description of a branched covering $f : S^2 \rightarrow S^2$.

Since we will work mainly in the piecewise linear category, it is convenient to recall here some definitions.

Simplicial complexes Following [Thurston \(1997\)](#) (chapter 3.2 and 3.9) we call a *simplicial complex* any locally finite collection Σ of simplices satisfying the following two conditions:

- a face of a simplex in Σ is also in Σ , and
- the intersection of any two simplices in Σ is either empty or a face of both.

The union of all simplices in Σ is the *polyhedron* of Σ (written $|\Sigma|$).

Piecewise linear maps A map $f : M \rightarrow N$ from a subset of an affine space into another affine space is *piecewise linear* (PL) if it is the restriction of a simplicial map defined on the polyhedron of some simplicial complex.

We also define *piecewise linear* (PL) *manifolds* as manifolds having an atlas where the transition maps between overlapping charts are piecewise linear homeomorphisms between open subsets of \mathbb{R}^n . It is well known that any piecewise linear manifold has a triangulation: there is a simplicial complex Σ together with a homeomorphism $|\Sigma| \rightarrow X$ which is assumed to be a PL map (see [Thurston \(1997\)](#), proof of theorem 3.10.2).

One example of such a manifold is the standard piecewise linear (PL) 2-sphere, which is nicely described in [Thurston \(1997\)](#) as follows: pick any convex 3-dimensional polyhedron $K \subset \mathbb{R}^3$, and consider the charts corresponding to all the possible orthogonal projections of the boundary (topological) sphere ∂K onto hyperplanes in \mathbb{R}^3 . The manifold thus obtained is the *standard piecewise linear 2-sphere*. One can prove that another choice of polyhedron would lead to an isomorphic object [see exercise 3.9.5 in [Thurston \(1997\)](#)].

It is known that in dimension three or lower, every topological manifold has a PL structure, and any two such structures are PL equivalent [in dimension 2, see [Radó \(1925\)](#), for the dimension 3 consult [Bing \(1959\)](#)].

Piecewise linear branched covers We begin by formulating the following proposition which describes how to lift a triangulation by a PL branched cover [see [Douady and Douady \(2005\)](#), section 6.5.4]:

Proposition 3.21 (Lifting a triangulation) *Let B be a compact topological surface, $\pi : X \rightarrow B$ a finite ramified cover of B . Let Δ be the set of branch points of π , and let \mathcal{T} be a triangulation of B such that Δ is a subset of vertices of \mathcal{T} ($\Delta \subset K_0(\mathcal{T})$ in the established notation). Then there exists a triangulation \mathcal{T}' of X , unique up to bijective change of indices, so that the branched covering map $\pi : X \rightarrow B$ sends vertices to vertices, edges to edges and faces to faces. Moreover, if $X = B$ is a standard PL 2-sphere and π is PL, then \mathcal{T}' can be produced constructively given a description of \mathcal{T} .*

We consider PL maps f of the standard PL 2-sphere which are postcritically finite topological branched coverings with a finite forward-invariant set Q_f of marked points. We call such a map a *piecewise linear Thurston map*.

Remark 3.22 Note that any such covering may be realized as a piecewise-linear branched covering map of a triangulation of $\hat{\mathbb{C}}$ with rational vertices. An algorithmic description of a PL branched covering could thus either be given by the combinatorial data describing the simplicial map, or as a collection of affine maps of triangles in $\hat{\mathbb{C}}$ with rational vertices. We will alternate between these descriptions as convenient.

We note:

Proposition 3.23 (Bonnot et al. 2012) *Every marked Thurston map f is Thurston equivalent to a PL Thurston map.*

For ease of reference we state:

Theorem 3.24 *There exists an algorithm \mathcal{A}_1 which, given a finite description of a marked Thurston map f with hyperbolic orbifold, outputs 1 if there exists a Thurston obstruction for f and 0 otherwise. In the latter case, \mathcal{A}_1 also outputs a finite description which uniquely identifies the rational mapping R which is Thurston equivalent to f , and the pre-periodic orbits of R that correspond to points in Q_f .*

The paper (Bonnot et al. 2012) contains a proof of the above theorem for the case of an unmarked Thurston map ($Q_f = P_f$), the proof extends to the general case *mutatis mutandis*.

Verifying Homotopy

Let us quote several useful results from Bonnot et al. (2012):

Proposition 3.25 *There exists an algorithm \mathcal{A}_2 to check whether two simple closed polygonal curves on a triangulated surface S are homotopic.*

Proposition 3.26 *There exists an algorithm \mathcal{A}_3 which does the following. Given a triangulated sphere with a finite number of punctures $S = S^2 - Z$ and a triangulated homeomorphism $h : S \rightarrow S$, the algorithm identifies whether h is isotopic to the identity.*

Enumeration of the Multicurves and Elements of the Mapping Class Group

We again quote Bonnot et al. (2012):

Proposition 3.27 *Given a finite set of punctures W , there exist algorithms $\mathcal{A}_5, \mathcal{A}_6$ which enumerate the elements of $MCG(S \setminus W)$ and $PMCG(S \setminus W)$ respectively.*

Proposition 3.28 *Given a finite set of punctures W , there exists an algorithm \mathcal{A}_7 which enumerates all non-peripheral multicurves on $S^2 \setminus W$.*

We combine Propositions 3.28 and 3.25 to formulate:

Proposition 3.29 *Given a marked PL Thurston map f , there exists an algorithm \mathcal{A}_8 which enumerates all f -stable multicurves.*

In Bonnot et al. (2012), Propositions 3.26 and 3.27 are combined in a straightforward fashion to prove:

Proposition 3.30 *There exists an algorithm \mathcal{A}_9 which, given two equivalent marked PL Thurston maps f and g verifies the equivalence, by presenting an element of $MCG(S^2 \setminus Q)$ which realizes it.*

We also need to state a constructive version of Theorem 3.19:

Proposition 3.31 *There exists an algorithm \mathcal{A}_{10} which, given an obstructed marked PL Thurston map f and an obstruction Γ , finds an equivalent PL Thurston map which is in a standard form, and such that the boundary curves of the thick parts are polygons.*

Sketch of proof We use a brute force search combined with algorithm \mathcal{A}_3 (Proposition 3.26) to find a PL approximation of the map g from Theorem 3.19. We then modify the triangulation near the boundary curves of the thick parts to obtain the desired map. We leave it to the reader to fill in the straightforward details.

Algorithmic Complexity of the Mapping Class Group

Let us recall that a group G is *finitely generated* if it is isomorphic to a quotient of the free group F_S on a finite set S by a normal subgroup $N \triangleleft F_S$. The elements of S are *generators* of G . A finitely generated group is *finitely presented* if there exists a finite set of words $R \subset F_S$ such that N is the normal closure of R (the smallest normal subgroup of F_S which contains R). The words in R are called *relators*; thus a finitely presented group can be described using a finite set of generators and relators.

The *Word Problem* for a finitely presented group G can be stated as follows:

Let S and R be given. For a word w in F_S decide whether or not w represents the identity in G . Equivalently, for two words $w_1, w_2 \in F_S$ decide whether w_1 and w_2 represent the same element of G .

The *Conjugacy Problem* is stated similarly:

Let S and R be given. For two words w_1, w_2 decide whether w_1 and w_2 are conjugate elements of G , that is, whether there exists $x \in G$ such that $w_1 = x w_2 x^{-1}$.

The Word Problem is a particular case of the conjugacy problem, since being conjugate to the identity element $e \in G$ is the same as being equal to it.

Both problems were explicitly formulated by Dehn (1911), who subsequently produced an algorithm deciding the Conjugacy Problem for a fundamental group of a closed orientable surface Dehn (1992). An example of a finitely presented group with an algorithmically unsolvable word problem was first given in 1955 by Novikov (1955), a different construction was obtained by Boone (1958).

We begin by noting the following (cf. Lickorish 1964; Farb and Margalit 2011):

Theorem 3.32 *Let S be an orientable surface of finite topological type. Then there exists an explicit finite presentation of $MCG(S)$ and of $PMCG(S)$. This presentation can be computed from a PL presentation of S .*

As was shown by [Hemion \(1979\)](#):

Theorem 3.33 *Let S be an orientable surface of finite topological type. Then the Conjugacy Problem in $MCG(S)$ is algorithmically solvable.*

It is known that the Conjugacy Problem in $MCG(S)$ is solvable in exponential time ([Tao 2013](#); [Hamenstädt 2009](#)).

Hurwitz Classification of Branched Covers

Let X and Y be two finite type Riemann surfaces. We say that two finite degree branched covers ϕ and ψ of Y by X are *equivalent in the sense of Hurwitz* if there exist homeomorphisms $h_0, h_1 : X \rightarrow X$ such that

$$h_0 \circ \phi = \psi \circ h_1.$$

An equivalence class of branched covers is known as a *Hurwitz class*. Enumerating all Hurwitz classes with a given ramification data is a version of the *Hurwitz Problem*. The classical paper of [Hurwitz \(1891\)](#) gives an elegant and explicit solution of the problem for the case $X = \hat{\mathbb{C}}$.

We will need the following narrow consequence of Hurwitz's work (for a modern treatment, see [Bartholdi et al. \(2013\)](#)):

Main Theorem III *There exists an algorithm \mathcal{A} which, given PL branched covers ϕ and ψ of PL spheres and a PL homeomorphism h_0 mapping the critical values of ϕ to those of ψ , does the following:*

- (1) *decides whether ϕ and ψ belong to the same Hurwitz class or not;*
- (2) *if the answer to (1) is affirmative, decides whether there exists a homeomorphism h_1 such that $h_0 \circ \phi = \psi \circ h_1$.*

4 Classification of Marked Thurston Maps with Parabolic Orbifolds

Let f be a Thurston map with postcritical set P_f and marked set $Q_f \supset P_f$. In what follows, we will drop the subscript f and will denote these sets simply P and Q . The goal of this section is to prove the following theorem:

Theorem 4.1 *Let f be a Thurston map with postcritical set P and marked set $Q \supset P$ such that the associated orbifold is parabolic and the associated matrix is hyperbolic. Then either f is equivalent to a quotient of an affine map or f admits a degenerate Levy cycle.*

Furthermore, in the former case the affine map is defined uniquely up to affine conjugacy.

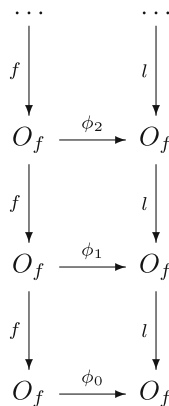
Remark 4.2 We note that in the case when the associated matrix has eigenvalue ± 1 , the two options are not mutually exclusive. We do not consider this case since it is not needed to prove Theorem 6.1. Indeed, the curve γ corresponding to the eigenvalue ± 1 will have d preimages of degree 1, which are all homotopic to γ relative P_f , where d is the degree of the Thurston map f . Thus the Thurston matrix corresponding to $\{\gamma\}$ is equal to $d > 1$ and the canonical obstruction of f necessarily contains a curve isotopic to γ relative P_f ; in particular, it is not empty.

4.1 The Case When the Associated Matrix is Expanding

We will first derive Theorem 4.1 in the case when the matrix of the corresponding affine map is expanding.

Theorem 4.3 *Let (f, Q) be a Thurston map with postcritical set P and marked set $Q \supset P$ with parabolic orbifold, such that (f, P) is equivalent to a quotient l of a real affine map $L(z) = Az + b$ by the orbifold group where both eigenvalues of A have absolute value greater than 1. Then (f, Q) is equivalent to a quotient of a real affine map by the action of the orbifold group if and only if f admits no degenerate Levy cycle.*

Proof Let ϕ_0 and ϕ_1 realize Thurston equivalence between (f, P) and l , i.e. $\phi_0 \circ f = l \circ \phi_1$ and ϕ_0 is isotopic to ϕ_1 relative P . The following argument is fairly standard (compare Shishikura 2000; Bonk and Meyer 2010). We can lift the isotopy between ϕ_0 and ϕ_1 by l to obtain an isotopy between ϕ_1 and the lift ϕ_2 of ϕ_1 (see the commutative diagram below).



Since both eigenvalues of A have absolute value greater than 1, the map l is expanding with respect to the Euclidean metric d_E on O_f . This implies that the distance between $\phi_n(z)$ and $\phi_{n+1}(z)$ decreases at a uniform geometric rate for all $z \in O_f$. We see that the sequence $\{\phi_n\}$ converges uniformly to a semi-conjugacy ϕ_∞ between f and l . One of the following is true then.

Case I Suppose ϕ_∞ is injective on Q . Let n be such that $d_E(\phi_\infty(z), \phi_n(z)) < \varepsilon$ and $d_E(\phi_{n+1}(z), \phi_n(z)) < \varepsilon$ for all $z \in O_f$, where ε is small. Consider an isotopy,

which is nontrivial only in the ε -neighborhood of $Q \setminus P$ that transforms ϕ_n to ϕ'_n such that ϕ'_n agrees with ϕ_∞ on Q . We lift the isotopy between ϕ_n and ϕ'_n starting at ϕ_{n+1} to get the lift ϕ'_{n+1} of ϕ'_n which is 2ε -close to ϕ_∞ and, hence, agrees with ϕ_∞ on Q if ε was chosen small enough. It is also clear that for ε small ϕ'_n and ϕ'_{n+1} are homotopic relative Q , realizing Thurston equivalence between (f, Q) and $(l, \phi_\infty(Q))$.

Case II Suppose ϕ_∞ is not injective on Q . Consider a point z_0 which is the image of at least two different points q_1 and q_2 in Q (i.e. $\phi_\infty(q_1) = \phi_\infty(q_2) = z_0$); obviously z_0 is either periodic or pre-periodic.

If $f(q_1) = f(q_2)$, then the distance between $\phi_n(q_1)$ and $\phi_n(q_2)$ is tending to 0, while $l(\phi_n(q_1)) = l(\phi_n(q_2))$ so the map l is not injective in any neighborhood of $z_0 = \phi_\infty(q_1)$, i.e. z_0 has to be a critical point of l . We see that $l(z_0)$ is one of the critical values of l , but $f(q_1)$ is not a critical value of f . Hence, $l(z_0)$ will be the image of at least two distinct points in Q : one of the critical values of f and $f(q_1)$. Of course, if $f(q_1)$ and $f(q_2)$ are distinct, then $l(z_0)$ is also the image of at least two different points in Q . Thus, we can assume that z_0 is periodic with period, say, m .

Consider a small simple closed curve γ around z_0 (for example, we can take a circle around z_0 of radius ε). Since z_0 is periodic it is not a critical point of l ; the m -th iterate of l sends γ to another simple closed curve γ' around z_0 , which is evidently homotopic to γ relative $\phi_\infty(Q)$, in one-to-one fashion, moreover the disk bounded by γ that contains z_0 is mapped homeomorphically to the disk bounded by γ' . This yields that, for n large enough, $\alpha' = \phi_n^{-1}(\gamma')$ and $\alpha = \phi_{n+m}^{-1}(\gamma)$ are homotopic relative Q and f^m homeomorphically maps a disk bounded by α to a disk bounded by α' . We see that $\alpha, f(\alpha), \dots, f^{m-1}(\alpha)$ form a degenerate Levy cycle. □

Remark 4.4 Note that if P has only three points, the matrix A is a multiplication by a complex number and both eigenvalues of A have the same absolute value, which is greater than 1.

4.2 When the Associated Matrix is Hyperbolic

We now want to prove Theorem 4.1 for any $(2, 2, 2, 2)$ -map such that the corresponding linear transformation is hyperbolic but not expanding. Throughout this section we assume that (f, Q) is a Thurston $(2, 2, 2, 2)$ -map with postcritical set P and marked set $Q \supset P$, such that (f, P) is equivalent to a quotient l of a real affine map $L(z) = Az + b$ by the orbifold group where both eigenvalues of A are not equal to ± 1 .

Definition 4.5 Let f be a $(2, 2, 2, 2)$ -map and let z be an f -periodic point with period n . Fix a lift F of f to the universal cover and take a point \tilde{z} in the fiber of z . If $z \notin P$, we define the *Nielsen index* $\text{ind}_{F,n}(\tilde{z})$ to be the unique element g of the orbifold group G such that $F^n(\tilde{z}) = g \cdot \tilde{z}$. If $z \in P$ then the Nielsen index of z is defined up to pre-composition with the symmetry around z .

Below, when we say that a point z has a period n , we do not imply that n is the minimal period of z .

Definition 4.6 Let f be a $(2, 2, 2, 2)$ -map and let z_1, z_2 be f -periodic points with period n . We say that z_1 and z_2 are in the same *Nielsen class of period n* if there

exists a lift F_n of f^n to the universal cover and points \tilde{z}_1, \tilde{z}_2 in the fibers of z_1, z_2 respectively, such that both \tilde{z}_1 and \tilde{z}_2 are fixed by F_n . We say that z_1 and z_2 are in the same *Nielsen class* if there exists an integer n such that they are in the same class of period n .

Note that if two points are in the same Nielsen class of period n , then they are in the same Nielsen class of period mn for any $m \geq 1$. Clearly, being in the same Nielsen class (without specifying a period) is an equivalence relation, which is preserved under Thurston equivalence for points in Q .

Lemma 4.7 *Periodic points z_1 and z_2 of period n are in the same Nielsen class if and only if, for any lift F of f to the universal cover, there exist points \tilde{z}_1, \tilde{z}_2 in the fibers of z_1, z_2 respectively such that $\text{ind}_{F,n}(\tilde{z}_1) = \text{ind}_{F,n}(\tilde{z}_2)$.*

Proof If $g = \text{ind}_{F,n}(\tilde{z}_1) = \text{ind}_{F,n}(\tilde{z}_2) \in G$ for some lift F of f to the universal cover and points \tilde{z}_1, \tilde{z}_2 , then $g^{-1} \cdot F^n(\tilde{z}_i) = \tilde{z}_i$ for $i = 1, 2$ and hence z_1 and z_2 are in the same Nielsen class.

In the other direction, suppose $F_n(\tilde{z}_i) = \tilde{z}_i$ for $i = 1, 2$ and some cover F_n of f^n . For any cover F of f , its iterate F^n can be written in the form $F^n = g \cdot F_n$ where $g \in G$. Therefore $\text{ind}_{F,n}(\tilde{z}_1) = \text{ind}_{F,n}(\tilde{z}_2) = g$. □

We will need the following statement.

Lemma 4.8 *Let A be a 2×2 integer matrix with determinant greater than 1 and both eigenvalues not equal to ± 1 . If v is a non-zero integer vector, then $A^{-n} \cdot v$ is non-integer for some $n > 0$.*

Proof Suppose, on contrary, that $A^{-n} \cdot v = (p_n, q_n)^T$ where $p_n, q_n \in \mathbb{Z}$ for all $n > 0$. If both eigenvalues of A have absolute values greater than 1, then evidently both p_n and q_n tend to 0. Thus for some n , $p_n = q_n = 0$ and, multiplying $(p_n, q_n)^T$ by A^n , we see that v is also a zero vector. Since by assumption, eigenvalues are not equal to ± 1 , the only case we need to consider is when A has two distinct real irrational eigenvalues $|\lambda_1| > 1$ and $|\lambda_2| < 1$.

In this case, A is diagonalizable; write v as a linear combination $v = v_1 + v_2$ of two eigenvectors $v_1 = (x_1, y_1)^T$ and $v_2 = (x_2, y_2)^T$. Then $A^{-n} \cdot v = \lambda_1^{-n} v_1 + \lambda_2^{-n} v_2$ so $p_n = x_1 \lambda_1^{-n} + x_2 \lambda_2^{-n}$ and $q_n = y_1 \lambda_1^{-n} + y_2 \lambda_2^{-n}$. Note that $q_n \lambda_1^{-2n} = y_1 \lambda_1^{-2n} + y_2 (\lambda_1 \lambda_2)^{-n} \rightarrow 0$ because $|\lambda_1| > 1$ and $\lambda_1 \lambda_2 = \det A > 1$; thus $|\lambda_1|^{-n} = o(1/|q_n|)$. Then

$$\left| \frac{p_n}{q_n} - \frac{x_2}{y_2} \right| = \left| \frac{(x_1 y_2 - x_2 y_1) \lambda_1^{-n}}{y_2 q_n} \right| = O\left(\left| \frac{\lambda_1^{-n}}{q_n} \right|\right) = o\left(\frac{1}{q_n^2}\right).$$

Since λ_2 is a quadratic algebraic number, the ratio x_2/y_2 must also be quadratic algebraic. No quadratic algebraic number, however, can be approximated by rationals this way and we arrive at a contradiction. □

Corollary 4.9 *Let $L(z) = Az + b$ be a real affine map such that A is an integer matrix with $|\det A| > 1$ and b is a vector with entries in $\frac{1}{q}\mathbb{Z}$ for some $q \in \mathbb{N}$, and assume*

that A has eigenvalues not equal to ± 1 . If $L^{-n}(v) \in \frac{1}{q}\mathbb{Z}$ for all $n \geq 0$, then v is equal to the fixed point of L .

Proof The case when $b = 0$ follows immediately from the previous lemma. If $b \neq 0$, we conjugate $L(z)$ by $t(z) = z - x$, where x is the unique fixed point of $L(z)$ (note that x is rational), to obtain a real linear map $L'(z)$. Then $L'(z)$ and $t(v)$ also satisfy the assumption of this corollary (possibly with a different q) and we conclude that $t(v) = v - x = 0$. □

Definition 4.10 Suppose that one of the complementary components to a simple closed curve γ in (\mathbb{H}, Q) contains at most one point of P (so that γ is trivial in (\mathbb{H}, P)). We call that component $\text{int}(\gamma)$ the *interior* of γ .

Proposition 4.11 Let $\{\gamma_n\}$ be a sequence of simple closed curves in (\mathbb{H}, Q) that are inessential in (\mathbb{H}, P) such that a $(2, 2, 2, 2)$ -map f sends γ_{n+1} to γ_n and $Q' = \text{int}(\gamma_n) \cap Q$ is the same for all n . Then there exists m such that all points in Q' are periodic with period m and lie in the same Nielsen class.

Proof Since all γ_n are inessential in (\mathbb{H}, P) , the map f sends $\text{int}(\gamma_{n+1})$ homeomorphically onto $\text{int}(\gamma_n)$. Indeed, $\text{int}(\gamma_n)$ contains at most one critical value of f , and if it does contain a critical value p , then the unique preimage of p in $\text{int}(\gamma_{n+1})$ must be p itself, which is not a critical point. Therefore f is a bijection on Q' and every point in Q' is periodic; denote m the least common multiple of the periods of points in Q' . It is enough to prove that for f^m , the subset Q' of the set of fixed points lies in a single Nielsen class.

Let F be a lift of f^m to the universal cover such that a point \tilde{r} in the fiber of $r \in Q'$ is fixed by F . Let s be any other point in Q' . Since $Q' \cap P$ contains at most one point, we may assume that $s \notin P$. Connect r and s by a curve α_1 in $\text{int}(\gamma_1) \setminus P$. The lift $\tilde{\alpha}_1$ of α_1 that starts at \tilde{r} will end at some point \tilde{s}_1 in the fiber of s ; in the case $r \in P$ take either of the two lifts. Denote $g_1 = \text{ind}_{F,1}(\tilde{s}_1)$; in other words, g_1 is a unique transformation in G such that $F(\tilde{s}_1) = g_1 \cdot \tilde{s}_1$. Consider the lift α_2 of α_1 by f^m that starts at r . Since $r \in \text{int}(\gamma_{m+1})$ the whole curve α_2 lies in $\text{int}(\gamma_{m+1})$ and, thus, ends in the unique preimage of s within $\text{int}(\gamma_{m+1})$, which is s itself. Therefore the lift $\tilde{\alpha}_2 = F^{-1}(\tilde{\alpha}_1)$ of α_2 starts at \tilde{r} and ends at some point \tilde{s}_2 in the fiber of s . We conclude by induction that

$$F^{-n}(\tilde{s}_1) = \tilde{s}_{n+1}$$

with \tilde{s}_{n+1} in the fiber of s for all n . Denote

$$g_n = \text{ind}_{F,1}(\tilde{s}_n),$$

which is a unique element of G such that

$$\tilde{s}_{n-1} = F(\tilde{s}_n) = g_n \cdot \tilde{s}_n \quad \text{for all } n \geq 2.$$

Then

$$g_n \cdot \tilde{s}_n = \tilde{s}_{n-1} = F(\tilde{s}_n) = F(g_{n+1} \cdot \tilde{s}_{n+1}) = F_*(g_{n+1}) \cdot F(\tilde{s}_{n+1}) = F_*(g_{n+1}) \cdot \tilde{s}_n.$$

Since $\tilde{s}_n \notin \tilde{P}$, this yields $F_*(g_{n+1}) = g_n$ for all n .

By Theorem 3.5 (f^m, P) is Thurston equivalent to a quotient of an affine map $L(z) = Az + b$; the push-forward map F_* is easily computed: for a translation

$$T_v \cdot z = z + v$$

we get

$$F_*(T_v) = T_{Av}$$

and for a symmetry

$$S_v \cdot z = 2v - z$$

we get

$$F_*(S_v) = S_{Av+b}.$$

It follows that if g_1 is equal to a translation T_v , then

$$g_n = T_{A^{-n+1}v} \text{ for all } n \geq 2.$$

In particular, all $A^{-n+1}v$ are integer vectors and Corollary 4.9 yields $v = 0$. We see that

$$\text{ind}_{F,1}(\tilde{s}_1) = g_1 = \text{id}$$

and \tilde{s}_1 is fixed by F so r and s are in the same Nielsen class.

Similarly, if $g_1 = S_v$, then $g_n = S_{L^{-n+1}(v)}$ and Corollary 4.9 also applies, implying that $L(v) = v$ and $g_1 = g_n$ for all n . But then

$$\tilde{s}_1 = g_2 \cdot g_3 \cdot \tilde{s}_3 = S_v^2 \tilde{s}_3 = \tilde{s}_3$$

and F^2 fixes both \tilde{r} and \tilde{s}_1 . □

Lemma 4.12 *A map f admits a degenerate Levy cycle if and only if so does its iterate f^n . Two points z_1, z_2 are in the same Nielsen class for f if and only if they are in the same Nielsen class for f^n .*

Proof The first statement follows from Proposition 3.11. In our case every Levy cycle is degenerate. Indeed, Proposition 3.4 implies that if a Levy cycle consists of curves that are non-peripheral in (\mathfrak{H}, P) , then the associated matrix must have an eigenvalue with absolute value 1, which contradicts the assumptions we made in the beginning of this section. If all curves in the Levy cycle are peripheral in (\mathfrak{H}, P) then the interiors thereof are mapped homeomorphically.

The second statement follows immediately from the definitions. □

Proposition 4.13 *A map f admits a degenerate Levy cycle if and only if there exist two distinct periodic points in Q in the same Nielsen class.*

Proof In view of Lemma 4.12, we can freely replace f by any iterate of f and assume that all periodic points in Q are fixed. Suppose an essential simple closed curve γ forms a Levy cycle of length 1, i.e. $f(\gamma)$ is homotopic to γ and the degree of f restricted to γ is 1. Recall that (f, P) is equivalent to a quotient of $z \mapsto Az + b$. Essential simple closed curves on (\mathfrak{H}, P) are in one-to-one correspondence with non-zero integer vectors $(p, q)^T$ such that $\gcd(p, q) = 1$. The action of f on the first homology group of (\mathfrak{H}, P) is (in the appropriate basis) the multiplication by A so that if γ_1 and γ_2 are simple closed essential curves labeled by (p_1, q_1) and (p_2, q_2) such that $f(\gamma_2) = \gamma_1$, then

$$A(p_2, q_2)^T = \pm d(p_1, q_1)^T$$

where d is the degree of f restricted to γ_2 . This yields that γ is inessential in (\mathfrak{H}, P) because otherwise A must have an eigenvalue ± 1 , which contradicts the assumptions. We are now in the setting of Proposition 4.11 and we see that all points in $\text{int}(\gamma)$ are in the same Nielsen class with some period m .

Suppose now that there are at least two fixed points of f in Q in the same Nielsen class C . Consider all points of Q in this class. Replacing f by an iterate, we may assume that all of them are in the same Nielsen class of period 1, i.e. there exists a lift F of f to the universal cover such that for each point $q \in C$, some lift \tilde{q} is fixed by F . Note that these will be the only fixed points of F . Note that C contains at most 1 point of P . □

Lemma 4.14 *There exists a simple closed curve γ on $(\mathfrak{H}, \mathbb{Q})$, which is inessential in (\mathfrak{H}, P) , such that $\text{int}(\gamma) \cap Q = C$ and some lift $\tilde{\gamma}$ of γ separates F -fixed lifts of points in C from the rest of the lifts of points in Q .*

Proof Let \tilde{s} be a fixed lift of a point $s \in C$ and assume $s \notin P$. Let \tilde{s}' be any other point in $\mathbb{R}^2 \setminus \tilde{Q}$. Take a path connecting \tilde{s} and \tilde{s}' in $\mathbb{R}^2 \setminus \tilde{Q}$ and construct a G -equivariant isotopy $H_t(z)$ that moves \tilde{s} along this path. Since H is G -equivariant and the chosen path is disjoint from \tilde{P} , the former projects to an isotopy h on (\mathfrak{H}, P) . Thus we find a map $f' = h_1 \circ f \circ h_1^{-1}$, which is conjugate to f , such that its lift

$$F' = H_1 \circ F \circ H_1^{-1}$$

fixes \tilde{s}' instead of \tilde{s} . Continuing in this manner, we can move all fixed points of F into a small round disk D , which contains no other lifts of points in Q . If $C \cap P = \emptyset$, then D projects homeomorphically to \mathfrak{H} ; if $p \in C \cap P$, then we can take the lift of p to be in the center of D . In both cases, the boundary of this disk and the projection thereof satisfy the conclusion of the proposition for the modified f ; the images of these curves by the conjugating maps will do the same for the map f itself. □

Consider a curve γ_0 as in the lemma above. Since $\tilde{\gamma}_0$ surrounds all fixed points of F , so does its preimage $\tilde{\gamma}_n = F^{-n}(\tilde{\gamma}_0)$, which projects to a simple closed curve γ_n on (\mathfrak{H}, Q) . Let a_n be the intersection number of γ_n and γ_{n+1} (we may always assume that γ_0 and γ_1 have only finitely many intersections, all of which are transversal). Clearly a_n is non-increasing. If $a_n = 0$ for some n , then γ_n and γ_{n+1} are disjoint and have

the same marked points in their interiors, hence they are homotopic and γ_n forms a Levy cycle of length 1. Otherwise, by truncating the sequence, we may assume that $a_n = a > 0$ for all $n \geq 0$. In this case, $\gamma_n \cup \gamma_{n+1}$ is mapped homeomorphically to $\gamma_{n-1} \cup \gamma_n$. Let $\beta_0 \subset \gamma_0 \cup \gamma_1$ be a simple closed curve and denote β_n to be a unique one-to-one f^n -preimage of β_0 that is a subset of $\gamma_n \cup \gamma_{n+1}$.

Lemma 4.15 *Let $\{\beta_n\}$ be a sequence of simple closed curves in (\mathbb{H}, Q) such that f sends β_{n+1} to β_n with degree 1. Then all β_n are inessential in (\mathbb{H}, P) .*

Proof Since the degree of f restricted to any β_n is 1, the following holds (see the proof of Proposition 4.13) :

$$A^n(p_n, q_n)^T = \pm(p_0, q_0)^T,$$

where β_n corresponds to $\pm(p_n, q_n)^T$ in the first homology group of (\mathbb{H}, P) . By Lemma 4.8, we see that $p_0 = q_0 = 0$. □

Thus, all β_n are inessential in (\mathbb{H}, P) . As β_0 was any simple closed curve in $\gamma_0 \cup \gamma_1$, we infer that there exists a connected component M of $\mathbb{H} \setminus (\gamma_0 \cup \gamma_1)$ that contains at least 3 points of P . Indeed, if there exists a component with exactly 2 points of P , then the boundary β thereof is essential in (\mathbb{H}, P) , which is a contradiction. If there are exactly 4 components, each containing a single point of P , one can find a simple closed curve in $\gamma_0 \cup \gamma_1$ that has exactly 2 points in each complementary component by induction on the number of components. Indeed, it is easy to see that there always exists a pair of adjacent components such that their closures intersect at exactly 1 boundary arc; removing that arc reduces the number of components by 1. From now on we assume that $\beta_0 = \partial M$.

Denote

$$Q_n = \text{int}(\beta_n) \cap Q.$$

Since $\#Q_n$ is non-increasing, we may assume, by further truncating the sequence if necessary, that $\#Q_n$ is constant. Recall that we assumed that all points in Q are either fixed or strictly pre-periodic. This implies $Q_n = Q_0$ for all n . We are now in the setting of Proposition 4.11, which yields all points in Q_0 are in the same Nielsen class. Recall that

$$\text{int}(\gamma_0) \cap Q = \text{int}(\gamma_1) \cap Q = C$$

contains at most 1 point of P so M is in the complement of $\text{int}(\gamma_0) \cup \text{int}(\gamma_1)$. We see that all marked points in the complement of M are in C . This readily implies that all γ_n are homotopic to β and γ_0 forms a Levy cycle of length 1, which concludes our proof of Proposition 4.13.

Proposition 4.16 *Let $\{\gamma_n\}$ be a sequence of essential simple closed curves in (\mathbb{H}, Q) such that f sends γ_{n+1} to γ_n with degree 1. Then f admits a degenerate Levy cycle.*

Proof By Lemma 4.15 all γ_n are inessential in (\mathfrak{H}, P) . Replacing $\{\gamma_n\}$ by a subsequence $\{\gamma_{nk+l}\}$, for some integers k, l , we can always assume that $Q' = \text{int}(\gamma_n) \cap Q$ is the same for all n (see the previous proof). Since γ_n are essential in (\mathfrak{H}, Q) , the set Q' contains at least two points. By Proposition 4.11, these two points are in the same Nielsen class and Proposition 4.13 implies existence of a Levy cycle. \square

Corollary 4.17 *If f admits no Levy cycle, then for every simple closed curve γ in (\mathfrak{H}, Q) , which is inessential in (\mathfrak{H}, P) , there exists an integer d such that all connected components of $f^{-d}(\gamma)$ are inessential in (\mathfrak{H}, Q) .*

Proof Define the *depth* of γ to be the largest integer $d(\gamma)$ such that $f^{-d(\gamma)}(\gamma)$ has an essential component. The goal is to prove that d_γ is finite for all inessential in (\mathfrak{H}, P) curves. Clearly,

$$d(\alpha) = 1 + \max d(\alpha_i)$$

where α_i are the connected components of the preimage of a simple closed curve α . Therefore, if γ has infinite depth, so does at least one of its preimages γ_1 . We construct thus an infinite sequence of essential in (\mathfrak{H}, Q) curves γ_n such that f maps γ_{n+1} to γ_n . Since a preimage of a trivial in (\mathfrak{H}, P) curve is also trivial in (\mathfrak{H}, P) , truncating the sequence if necessary, we may assume that all γ_n are either all trivial or all non-trivial in (\mathfrak{H}, P) . In both cases, the degree of f restricted to γ_n is 1 for all n and the previous proposition yields existence of a Levy cycle. \square

The above result immediately implies:

Corollary 4.18 *If f admits no Levy cycle, then every curve of every simple Thurston obstruction for f is essential in (\mathfrak{H}, P) .*

For future reference, let us summarize:

- Corollary 4.19** • *Let f be marked $(2, 2, 2, 2)$ -map such that the corresponding matrix does not have eigenvalues ± 1 . Then f is equivalent to a quotient of an affine map with marked pre-periodic orbits if and only if every curve of every simple Thurston obstruction for f has two postcritical points of f in each complimentary component.*
- *A marked Thurston map f with a parabolic orbifold that is not $(2, 2, 2, 2)$ is equivalent to a quotient of an affine map if and only if it admits no Thurston obstruction.*

Proof The first statement follows immediately from the previous corollary and Theorem 4.26. The second statement follows from Theorem 4.3. Indeed, suppose that a Thurston map f with a parabolic orbifold with signature other than $(2, 2, 2, 2)$ admits a simple obstruction Γ . If the signature is (∞, ∞) or $(\infty, 2, 2)$, then f (or at least its second iterate in the former case) is an obstructed topological polynomial and therefore admits a Levy cycle (Theorem 3.12) which is necessarily degenerate. In other cases all points in the postcritical set of f are not critical. As before, we pass to an iterate of f such that all marked points are either fixed or pre-fixed (in particular, all postcritical points are fixed in this case) and set the interior $\text{int}(\gamma)$ to be the unique

component of the complement to γ which contains at most 1 postcritical point. Up to passing to yet another iterate of f , we may assume that some $\gamma \in \Gamma$ has a preimage γ' homotopic to γ . If $\text{int}(\gamma)$ contains no postcritical points, then $\text{int}(\gamma')$ contains no critical points. If $\text{int}(\gamma)$ contains a postcritical point p , then p is the unique preimage of itself within $\text{int}(\gamma')$, and again $\text{int}(\gamma')$ contains no critical points. Therefore in both cases $\{\gamma\}$ is a degenerate Levy cycle. \square

Definition 4.20 Denote by $\text{RMCG}(\sqsupset, Q)$ the *relative mapping class group* of (\sqsupset, Q) , which is the group of all mapping classes ϕ for which there exists a lift $\tilde{\phi}$ to the universal cover that is identical on \tilde{Q} .

We now need the following generalization of Lemma 3.7.

Theorem 4.21 *The group $\text{RMCG}(\sqsupset, Q)$ is generated by Dehn twists around trivial curves in (\sqsupset, P) and by second powers of Dehn twists around non-trivial inessential curves in (\sqsupset, P) .*

Proof The proof of this theorem is similar to the proof of classical results on generators of MCG (cf. [Farb and Margalit 2011](#)). We proceed by induction on the number of points in Q . When $Q = P$, the group $\text{RMCG}(\sqsupset, Q)$ is trivial by Lemma 3.7.

Suppose that the statement is true for the marked set $Q \subset \sqsupset$ and let us prove it for $Q' = Q \cup \{q\}$ where $q \notin Q$. There exists an obvious projection map

$$\text{Forget} : \text{PMCG}(\sqsupset, Q') \rightarrow \text{PMCG}(\sqsupset, Q),$$

which simply regards a self-homeomorphism of (\sqsupset, Q') as a self-homeomorphism of (\sqsupset, Q) , forgetting about the existence of q . Take any $\phi \in \text{RMCG}(\sqsupset, Q')$; by inductive assumption, $\text{Forget}(\phi) \in \text{RMCG}(\sqsupset, Q)$ can be represented in $\text{PMCG}(\sqsupset, Q)$ as a product $\prod T_{\gamma_i}^{n_i}$ of Dehn twists around trivial curves in (\sqsupset, P) and second powers of Dehn twists around non-trivial inessential curves in (\sqsupset, P) . We may assume that every γ_i does not pass through the point q ; otherwise we replace γ_i by a curve γ'_i , which is homotopic to γ_i relative Q , that does not pass through q (note that in this case the homotopy class of γ'_i in (\sqsupset, Q) is not uniquely defined). Then

$$\text{Forget}(T_{\gamma_i}) = T_{\gamma_i}$$

where T_{γ_i} is viewed as an element of both $\text{PMCG}(\sqsupset, Q')$ and $\text{PMCG}(\sqsupset, Q)$. Thus

$$\psi = \phi \circ \left(\prod T_{\gamma_i}^{n_i} \right)^{-1}$$

is a well defined element of $\text{PMCG}(\sqsupset, Q')$ such that $\text{Forget}(\psi) = \text{id}$. It is, hence, sufficient to show that every $\psi \in \text{RMCG}(\sqsupset, Q')$ such that $\text{Forget}(\psi) = \text{id}$ is generated by (squares of) Dehn twists.

Recall the Birman exact sequence (cf. [Farb and Margalit 2011](#)):

$$1 \longrightarrow \pi_1(\sqsupset \setminus Q, q) \xrightarrow{\text{Push}} \text{PMCG}(\sqsupset, Q') \xrightarrow{\text{Forget}} \text{PMCG}(\sqsupset, Q) \longrightarrow 1,$$

where Push is the map that sends a loop based at q to a homeomorphism, which can be obtained at the end of a isotopy relative Q that pushes the point q along this loop. Since ψ lies in the kernel of Forget, we infer $\psi = \text{Push}(\gamma)$ for some loop $\gamma \in \pi_1(\sqcup \setminus Q, q)$. Since ψ is also an element of $\text{RMCG}(\sqcup, Q')$, it has a lift $\tilde{\psi}$ to the universal cover which is identical on the fiber of q . Pick a point \tilde{q} in this fiber; in particular, \tilde{q} is fixed by $\tilde{\psi}$. It is clear that the lift $\tilde{\gamma}$ of γ starting at \tilde{q} ends at $\tilde{\psi}(\tilde{q}) = \tilde{q}$, i.e. $\tilde{\gamma}$ is a loop based at \tilde{q} . On the other hand, each loop $\tilde{\gamma}$ in $\mathbb{R}^2 \setminus \tilde{Q}$ based at \tilde{q} produces a unique homeomorphism $\text{Push}'(\tilde{\gamma}) := \text{Push}(\gamma)$ where γ is the projection of $\tilde{\gamma}$. We see that Push' is an isomorphism between $\text{RMCG}(\sqcup, Q') \cap \ker(\text{Forget})$ and $\pi_1(\mathbb{R}^2 \setminus \tilde{Q}, \tilde{q})$, where the latter is generated by simple loops around a single point in \tilde{Q} .

Applying the same approach as in the proof of Lemma 4.14, one proves that for every point a in \tilde{P} there exists a simple loop $\tilde{\alpha}$ based at \tilde{q} , such that the bounded component of the complement of the loop contains a and no other points from \tilde{Q} , which projects two-to-one to a simple loop α based at q in \sqcup . Then α is inessential in (\sqcup, P) and

$$\text{Push}'(\tilde{\alpha}) = \text{Push}(\alpha^2) = T_\alpha^2.$$

Similarly, for every point b in $\tilde{Q} \setminus \tilde{P}$ there exists a simple loop $\tilde{\beta}$ based at \tilde{q} , such that the bounded component of the complement of the loop contains b and no other points from \tilde{Q} , which projects one-to-one to a simple loop β based at q in \sqcup . Then β is trivial in (\sqcup, P) and

$$\text{Push}'(\tilde{\beta}) = \text{Push}(\beta) = T_\beta.$$

As $\pi_1(\mathbb{R}^2 \setminus \tilde{Q}, \tilde{q})$ is generated by all possible curves α and β , the statement of the theorem follows. □

Definition 4.22 Denote by $\text{Lift}(\phi)$ the virtual endomorphism of $\text{PMCG}(\sqcup, Q)$ that acts by lifting by f , i.e. we write $\text{Lift}(\phi) = \psi$ whenever there exists $\psi \in \text{PMCG}(\sqcup, Q)$ such that $\phi \circ f = f \circ \psi$ as mapping classes.

Proposition 4.23 $\text{Lift}(\phi): \text{RMCG}(\sqcup, Q) \rightarrow \text{RMCG}(\sqcup, Q)$ is a well-defined endomorphism. If f admits no Levy cycles, then for every $\phi \in \text{RMCG}(\sqcup, Q)$, there exist an n such that $\text{Lift}^n(\phi) = \text{id}$.

Proof It is enough to prove the statement for a generating set of $\text{RMCG}(\sqcup, Q)$. By Theorem 4.21 we only need to consider two cases.

Case I Suppose $\phi = T_\alpha$ where α is a simple closed curve in (\sqcup, Q) , which is trivial in (\sqcup, P) . All connected components α_i of $f^{-1}(\alpha)$ are pairwise disjoint simple closed curves that are trivial in (\sqcup, P) and are mapped by f to α with degree 1. It is straightforward to see that

$$T_\alpha \circ f = f \circ \prod T_{\alpha_i}.$$

Thus

$$\text{Lift}(T_\alpha) = \prod T_{\alpha_i} \in \text{RMCG}(\sqcup, Q)$$

is well-defined. Similarly, denote by α_i^n all connected components of $f^{-n}(\alpha)$; then

$$T_\alpha \circ f^n = f^n \circ \prod T_{\alpha_i^n} \text{ and } \text{Lift}^n(T_\alpha) = \prod T_{\alpha_i^n}.$$

By Corollary 4.17, there exists an integer n such that all α_i^n are inessential in (\sqcup, Q) , implying

$$\text{Lift}^n(T_\alpha) = \prod T_{\alpha_i^n} = \text{id}.$$

Case II Suppose $\phi = T_\beta^2$ where β is a simple closed curve in (\sqcup, Q) , which is non-trivial and inessential in (\sqcup, P) . The interior of β contains a unique critical value p of f . All connected components β_i of $f^{-1}(\beta)$ are pairwise disjoint simple closed curves that are inessential in (\sqcup, P) . Each $\text{int}(\beta_i)$ contains a unique f -preimage p_i of p . If $p_i \in P$, then it is not a critical point of f and β_i is mapped by f to β with degree 1. If $p_i \notin P$, then it is a critical point and β_i is trivial in (\sqcup, P) and is mapped by f to β with degree 2. As in the Case I, we see that

$$T_\beta^2 \circ f = f \circ \left(\prod_{p_i \in P} T_{\beta_i}^2 \circ \prod_{p_i \notin P} T_{\beta_i} \right)$$

and

$$\text{Lift}(T_\beta^2) = \prod_{p_i \in P} T_{\beta_i}^2 \circ \prod_{p_i \notin P} T_{\beta_i} \in \text{RMCG}(\sqcup, Q)$$

is well-defined. We also see that

$$\text{Lift}^n(T_\beta^2) = \prod_{p_i^n \in P} T_{\beta_i^n}^2 \circ \prod_{p_i^n \notin P} T_{\beta_i^n}$$

where β_i^n are the connected components of $f^{-n}(\beta)$ and p_i^n denote the corresponding f^n -preimages of p . By Corollary 4.17, there exists an integer n such that all β_i^n are inessential in (\sqcup, Q) , implying

$$\text{Lift}^n(T_\beta^2) = \prod_{p_i^n \in P} T_{\beta_i^n}^2 \circ \prod_{p_i^n \notin P} T_{\beta_i^n} = \text{id}.$$

□

Lemma 4.24 *If $\psi = \text{Lift}(\phi)$ for some $\phi \in \text{PMCG}(\sqcup, Q)$, then $f \circ \phi$ is Thurston equivalent to $f \circ \psi$.*

Proof $f \circ \psi = \phi \circ f = \phi \circ (f \circ \phi) \circ \phi^{-1}$.

□

We arrive at the following statement.

Theorem 4.25 *If f admits no Levy cycle and $\phi \in RMCG(\square, Q)$ then $f \circ \phi$ is Thurston equivalent to f .*

Proof By proposition 4.23 and the previous lemma, there exists n such that $f \circ \phi$ is equivalent to $f \circ \text{Lift}^n(\phi) = f \circ \text{id} = f$. □

We can now prove the first part of the statement of Theorem 4.1.

Theorem 4.26 *Let (f, Q) be a Thurston $(2, 2, 2, 2)$ -map with postcritical set P and marked set $Q \supset P$, such that (f, P) is equivalent to a quotient l of a real affine map $L(z) = Az + b$ by the orbifold group where both eigenvalues of A are not equal to ± 1 . Then (f, Q) is equivalent to a quotient of a real affine map by the action of the orbifold group if and only if f admits no degenerate Levy cycle.*

Proof Necessity Suppose a quotient (l, Q) of a real affine map $L(z) = Az + b$ by the orbifold group G admits a degenerate Levy cycle. By Proposition 4.13, there exist two distinct periodic points $q_1, q_2 \in Q$ in the same Nielsen class and Lemma 4.7 implies that there exist points \tilde{q}_1, \tilde{q}_2 in the fibers of q_1, q_2 respectively such that

$$\text{ind}_{L,n}(\tilde{q}_1) = \text{ind}_{L,n}(\tilde{q}_2) = g \in G, \text{ i.e. } L^n(\tilde{q}_i) = g(\tilde{q}_i) \text{ for } i = 1, 2.$$

Since

$$L^n(z) = A^n z + b' \text{ and } g(z) = c \pm z$$

for some integer vectors b' and c , the equation

$$L^n(\tilde{q}_i) = g(\tilde{q}_i)$$

is equivalent to

$$(A^n \pm I)z = c - b'$$

where I denotes the identity matrix. By assumption, the eigenvalues of A are not equal to ± 1 , hence the matrix $(A^n \pm I)$ is non-degenerate. This yields $\tilde{q}_1 = \tilde{q}_2$, which is a contradiction.

Sufficiency Suppose f admits no Levy cycles and, hence, no two distinct points of Q are in the same Nielsen class by Proposition 4.13. Consider a lift F of f to the universal cover; by Lemma 3.6

$$F(z) = L(z) = A'z + b' \text{ for all } z \in \tilde{P}.$$

Conjugating the original f by a quotient of an affine transformation, if necessary, we may assume that $A' = A$ and $b' = b$. Pick a point \tilde{q} in the fiber of a periodic point $q \in Q$ of period n . Let s be a unique solution of the equation

$$L^n(z) = \text{ind}_{F,n}(\tilde{q}) \cdot z.$$

We can push the point \tilde{q} by a G -equivariant isotopy $\Phi_t(z): \mathbb{R}^2 \rightarrow \mathbb{R}^2$ along some path α in $\mathbb{R}^2 \setminus \tilde{Q}$ (except for the starting point \tilde{q} and, possibly, the end point) that ends at s . Since Φ is G -equivariant, it pushes the point

$$F^n(\tilde{q}) = \text{ind}_{F,n}(\tilde{q}) \cdot \tilde{q}$$

along the path $\text{ind}_{F,n}(\tilde{q}) \cdot \alpha$ to the point

$$\text{ind}_{F,n}(\tilde{q}) \cdot s = L^n(s).$$

Therefore, for $F_1 = \Phi_1 \circ F \circ \Phi_1^{-1}$, we have $F_1^n(s) = L^n(s)$. Let $s' = g \cdot s$, where $g \in G$, be any other point in the same fiber as s . Then G -equivariance of Φ implies

$$\begin{aligned} F_1^n(s') &= \Phi_1 \circ F^n \circ \Phi_1^{-1}(g \cdot s) = \Phi_1 \circ F^n(g \cdot \Phi_1^{-1}(s)) = \\ \Phi_1(F_*^n g \cdot F^n \circ \Phi_1^{-1}(s)) &= F_*^n g \cdot \Phi_1 \circ F^n \circ \Phi_1^{-1}(s) = F_*^n g \cdot F_1^n(s) = F_*^n g \cdot L^n(s). \end{aligned}$$

Since $F = L$ on \tilde{P} , their actions on the orbifold group are the same: $F_* = L_*$. Thus,

$$F_1^n(s') = F_*^n g \cdot L^n(s) = L_*^n g \cdot L^n(s) = L^n(g \cdot s) = L^n(s').$$

We repeat this procedure for each of the periodic points in Q to obtain a G -equivariant isotopy $\Psi_t(z): \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and set $F_2 = \Psi_1 \circ F \circ \Psi_1^{-1}$, such that for any point $s = \Psi_1(\tilde{q})$, where \tilde{q} is in the fiber of a periodic point of any period n from Q , we have $F_2^n(s) = L^n(s)$. The only possible obstacle can occur when we need to push some point \tilde{q} from the fiber of q into the fiber of some other point q' , which has already been adjusted. This would immediately imply that q and q' are in the same Nielsen class, which contradicts our assumptions.

Note that our construction automatically implies $F_2(s) = L(s)$ for all $s = \Psi_1(\tilde{q})$, where \tilde{q} is in the fiber of a periodic point q of any period n . Indeed, if $F^n(z) = g \cdot z$, then

$$F^n(F(z)) = F(F^n(z)) = F(g \cdot z) = F_*g \cdot F(z),$$

hence

$$\text{ind}_{F,n}(F(z)) = F_*\text{ind}_{F,n}(z) = L_*\text{ind}_{F,n}(z).$$

Therefore, if $s = \Psi_1(\tilde{q})$ is a unique solution of the equation

$$L^n(z) = \text{ind}_{F,n}(\tilde{q}) \cdot z,$$

then $L(s)$ is a unique solution of the equation

$$L^n(z) = \text{ind}_{F,n}(F(\tilde{q})) \cdot z = L_*\text{ind}_{F,n}(\tilde{q}) \cdot z,$$

because

$$L^n(L(z)) = L(L^n(z)) = L(\text{ind}_{F,n}(\tilde{q}) \cdot z) = L_* \text{ind}_{F,n}(\tilde{q}) \cdot L(z).$$

This yields $\Psi_1(F(\tilde{q})) = L(s)$ and

$$F_2(s) = F_2 \circ \Psi_1(\tilde{q}) = \Psi_1 \circ F(\tilde{q}) = L(s).$$

Now we perform an analogous procedure on all strictly pre-periodic points. Let $q \in Q$ be a strictly pre-periodic point and \tilde{q} be some point in its fiber. Denote by n the pre-period of q , i.e. the smallest integer such that $f^n(q)$ is periodic. We find a G -equivariant isotopy that pushes \tilde{q} to $L^{-n}(F_2^n(\tilde{q}))$ and leaves all fibers of other points of Q in place. After repeating this process for all pre-periodic points of f , we construct a G -equivariant isotopy $\Xi_t(z) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $F_3 = \Xi_1 \circ F \circ \Xi_1^{-1}$ agrees with $L(z)$ on $\Xi_1(\tilde{Q})$, in particular $F_{3*} = L_*$.

Denote by f_3 and ξ the quotients of F_3 and Ξ_1 respectively by the action of G . Then $f_3 = \xi_1 \circ f \circ \xi_1^{-1}$, and (f, Q) is conjugate (and, hence, Thurston equivalent) to $(f_3, \xi(Q))$. Set $\Theta(z) = L^{-1} \circ F_3(z)$; we see that

$$\Theta(g \cdot z) = L^{-1} \circ F_3(g \cdot z) = L^{-1}(F_{3*}g \cdot F_3(z)) = L^{-1}(L_*g \cdot F_3(z)) = g \cdot L^{-1} \circ F_3(z),$$

i.e. Θ is G -equivariant. Therefore $f_3 = l \circ \theta$ where θ is the quotient of Θ by the action of G . Since $F_3 = L$ on \tilde{Q} , the lift Θ of θ to the universal cover is identical on \tilde{Q} so $\theta \in \text{RMCG}(\sqcup, Q)$. By Theorem 4.25 $(f_3, \xi(Q))$ and $(l, \xi(Q))$ are Thurston equivalent, which concludes our proof. □

4.3 Uniqueness

We now prove the uniqueness part of the statement of Theorem 4.1:

Theorem 4.27 *Let (l_i, Q_i) be a Thurston map that is a quotient of an affine map $L_i(z) = A_i z + b$ ($A_i \in M_2(\mathbb{Z})$) by the action of an orbifold group G for $i = 1, 2$. Suppose that eigenvalues of A_i are not equal to ± 1 for $i = 1, 2$. If (l_1, Q_1) and (l_2, Q_2) are Thurston equivalent, then (l_1, Q_1) and (l_2, Q_2) are conjugate by a quotient of an affine map. In other words, there exist $g \in G$ and a real affine map S with linear part in $SL_2(\mathbb{Z})$ such that $L_2 = g \cdot S \circ L_1 \circ S^{-1}$ and S sends Q_1 to Q_2 .*

Proof Let ϕ, ψ realize the Thurston equivalence of (l_1, Q_1) and (l_2, Q_2) :

$$\begin{array}{ccc} (\sqcup, Q_1) & \xrightarrow{\psi} & (\sqcup, Q_2) \\ \downarrow l_1 & & \downarrow l_2 \\ (\sqcup, Q_1) & \xrightarrow{\phi} & (\sqcup, Q_2) \end{array}$$

where ϕ and ψ are homotopic relative Q_1 . Then there exist lifts $\tilde{\phi}$ and $\tilde{\psi}$ of ϕ and ψ to the universal cover such that the following diagram commutes:

$$\begin{CD} (R^2, \tilde{Q}_1) @>\tilde{\psi}>> (R^2, \tilde{Q}_2) \\ @V L_1 VV @V L_2 VV \\ (R^2, \tilde{Q}_1) @>\tilde{\phi}>> (R^2, \tilde{Q}_2) \end{CD}$$

By Lemma 3.6, both $\tilde{\phi}$ and $\tilde{\psi}$ are affine on \tilde{P} . Since ϕ and ψ are homotopic relative $P \subseteq Q_1$, there exists $g \in G$ such that $\tilde{\psi} = S$ and $\tilde{\phi} = g \cdot \tilde{\psi} = g \cdot S$ for all points in \tilde{P} , where S is a real affine map. Note that the linear part of S has determinant ± 1 because ϕ and ψ are one-to-one maps. Therefore

$$L_2 = \tilde{\phi} \circ L_1 \circ \tilde{\psi}^{-1} = g \cdot \tilde{\psi} \circ L_1 \circ \tilde{\psi}^{-1} = g \cdot S \circ L_1 \circ S^{-1} \text{ on } \tilde{P}.$$

As both sides of the last equation are real affine, the equation holds for all points in \mathbb{R}^2 .

Replace L_2 by another lift to the universal cover given by $L'_2 = g^{-1} \cdot L_2$ of l_2 so that $L'_2 = S \circ L_1 \circ S^{-1}$; set $\tilde{\phi}' = g^{-1} \cdot \tilde{\phi}$. Then both $\tilde{\phi}'$ and $\tilde{\psi}$ agree with S on \tilde{P} and it follows that $\tilde{\phi}'$ and $\tilde{\psi}$ agree on \tilde{Q}_1 and act by $\tilde{\phi}'_* = \tilde{\psi}_* = S_*$ on the first homology group of \sqsupset . Consider a lift \tilde{q} of a periodic point $q \in Q_1$ of some period n . Recall that \tilde{q} is a unique solution of $L_1^n(z) = g_1 \cdot z$, where $g_1 = \text{ind}_{L_1, n}(\tilde{q})$. Then

$$\tilde{\psi}(L_1^n(\tilde{q})) = \tilde{\psi}(g_1 \cdot \tilde{q}) = S_*(g_1) \cdot \tilde{\psi}(\tilde{q}).$$

This yields

$$\text{ind}_{L'_2, n}(\tilde{\psi}(\tilde{q})) = S_*(g_1)$$

and $\tilde{\psi}(\tilde{q})$ is a unique solution of

$$L'_2{}^n(z) = S_*(g_1) \cdot z,$$

which is equivalent to

$$S \circ L_1 \circ S^{-1}(z) = S(g_1 \cdot S^{-1}(z))$$

or

$$L_1 \circ S^{-1}(z) = g_1 \cdot S^{-1}(z).$$

We conclude that $\tilde{\psi}(\tilde{q}) = S(\tilde{q})$ for all lifts of periodic points in Q_1 .

For a lift \tilde{p} of a pre-periodic point $p \in Q_1$, consider some k such that $L_1^k(\tilde{p}) = \tilde{q}$, where $q = l_1^k(p)$ is periodic. Then

$$\tilde{\psi}(\tilde{p}) = L_2'^{-k} \circ \tilde{\psi} \circ L_1^k(\tilde{p}) = L_2'^{-k} \circ \tilde{\psi}(\tilde{q}) = L_2'^{-k} \circ S(\tilde{q}) = S \circ L_1^k(\tilde{q}) = S(\tilde{p}).$$

We have shown that S sends \tilde{Q}_1 to \tilde{Q}_2 , therefore the quotient of S to \square not only conjugates l_1 and l_2 , but sends Q_1 to Q_2 . □

5 Constructive Geometrization of Thurston Maps with Parabolic Orbifolds

Theorem 5.1 *There exists an algorithm \mathcal{A}_9 which for any marked Thurston map f with a parabolic orbifold whose matrix does not have eigenvalues ± 1 finds either a degenerate Levy cycle or an equivalence to a quotient of an affine map with marked pre-periodic orbits.*

Proof The proof is completely analogous to the argument given in Bonnot et al. (2012). We begin by identifying the orbifold group G and finding an affine map $L(x) = Ax + b$ such that f without marked points is equivalent to the quotient l of L by G (Theorem 3.5).

We now execute two sub-programs in parallel:

(I) we use algorithm \mathcal{A}_8 (Proposition 3.29) to enumerate all f -stable multicurves Γ_n . Using algorithm \mathcal{A}_2 (Proposition 3.25) we check whether Γ_n is a degenerate Levy cycle. If yes, we output *degenerate Levy cycle found* and halt;

(II) we identify all forward invariant sets S_k of pre-periodic orbits of l of the same cardinality as the set of marked points of f . We use algorithm \mathcal{A}_6 (Proposition 3.27) to enumerate the sequence ψ_n of all elements of $\text{PMCG}(S^2, Q)$. For every ψ_n and each of the finitely many sets S_k we use algorithm \mathcal{A}_3 (Proposition 3.26) to check whether $h_k \circ \psi_n$ realizes Thurston equivalence between f and l with marked points S_k , where $h_k: (S^2, Q) \rightarrow (S^2, S_k)$ is an arbitrary chosen homeomorphism. If yes, we output **Thurston equivalence found**, list the maps $l, h_k \circ \psi_n$ and the set S_k and halt.

By Theorem 4.1 either the first or the second sub-program, but not both, will halt and deliver the desired result.

6 Constructive Canonical Geometrization of a Thurston Map

Theorem 6.1 *There exists an algorithm which for any Thurston map f finds its canonical obstruction Γ_f .*

Furthermore, let \mathcal{F} denote the collection of the first return maps of the canonical decomposition of f along Γ_f . Then the algorithm outputs the following information:

- for every first return map with a hyperbolic orbifold, the unique (up to Möbius conjugacy) marked rational map equivalent to it;
- for every first return map of type $(2, 2, 2, 2)$ the unique (up to affine conjugacy) affine map of the form $z \mapsto Az + b$ where $A \in \text{SL}_2(\mathbb{Z})$ and $b \in \frac{1}{2}\mathbb{Z}^2$ with marked points which is equivalent to f after quotient by the orbifold group G ;

- for every first return map which has a parabolic orbifold not of type $(2, 2, 2, 2)$ the unique (up to Möbius conjugacy) marked rational map map equivalent to it, which is a quotient of a complex affine map by the orbifold group.

Proof The result of Bonnot et al. (2012) together with Theorem 5.1 implies the existence of the subprogram \mathcal{P} which given a marked Thurston map f does the following:

- (1) if f has a hyperbolic orbifold and is obstructed, it outputs a Thurston obstruction for f ;
- (2) if f has a parabolic orbifold not of type $(2, 2, 2, 2)$ and a degenerate Levy cycle it outputs such a Levy cycle;
- (3) if f is a $(2, 2, 2, 2)$ map such that the corresponding matrix has two distinct integer eigenvalues outputs a Thurston obstruction for f ;
- (4) if f is a $(2, 2, 2, 2)$ map with a degenerate Levy cycle outputs such a Levy cycle;
- (5) in the remaining cases output a geometrization of f as described in the statement of the theorem.

We apply the subprogram \mathcal{P} recursively to decompositions of f along the found obstructions until no new obstructions are generated (this will eventually occur by Theorem 3.20 and Corollary 4.19).

Denote by Γ the union of all obstructions thus generated. Use algorithm \mathcal{A}_2 and sub-program \mathcal{P} to find the set S of all subsets $\Gamma' \subset \Gamma$ such that:

- Γ' is a Thurston obstruction for f ;
- denote \mathcal{F}' the union of first return maps obtained by decomposing along Γ' . Then no $h \in \mathcal{F}'$ is a $(2, 2, 2, 2)$ map whose matrix has distinct integer eigenvalues, and every $h \in \mathcal{F}'$ which is not a homeomorphism is geometrizable.

Set

$$\Gamma_c \equiv \bigcap_{\Gamma' \in S} \Gamma'.$$

By Theorem 3.20 and Corollary 4.19, Γ_c is the canonical obstruction of f . □

7 Partial Resolution of the Problem of Decidability of Thurston Equivalence

Denote by \mathcal{H} the class of Thurston maps f such that every first return map in the canonical decomposition of f has hyperbolic orbifold. In this section we prove the following theorem:

Theorem 7.1 *There exists an algorithm which given a PL Thurston map $f \in \mathcal{H}$ and any PL Thurston map g decides whether f and g are equivalent or not.*

We will need several preliminary statements.

Proposition 7.2 *If (f, Q_f) and (g, Q_g) are Thurston equivalent marked rational maps with hyperbolic orbifolds, then the pair ϕ, ψ realizing the equivalence*

$$\phi \circ f = g \circ \psi$$

is unique up to isotopy relative Q_f .

Proof The statement is equivalent to saying that there are no non-trivial self equivalences of f . If $\phi \circ f = f \circ \psi$, where ϕ and ψ represent the same mapping class h , then $\sigma_f \circ h = h \circ \sigma_f$. If τ is the unique fixed point of σ_f , then $h(\tau)$ is also fixed, yielding a contradiction. \square

For the following see [Pilgrim \(1827\)](#):

Theorem 7.3 *Let f and g be two Thurston maps, and $\Gamma_f = \{\alpha_1, \dots, \alpha_n\}$ and $\Gamma_g = \{\beta_1, \dots, \beta_n\}$ be the corresponding canonical obstructions. Let A_i, B_i be decomposition annuli isotopic to α_i, β_i respectively. If f and g are equivalent then there exists an equivalence pair h_1, h_2 such that $h_1(A_i) = B_i$ (up to a permutation of indexes) and h_1 on ∂A_i is any given orientation-preserving homeomorphism of the boundary curves.*

Recall that the components of the complement of all A_i (resp B_i) are called *thick parts*.

Corollary 7.4 *If h_1, h_2 are as above then each thick component C is mapped by h_1 to a thick component C' . The components C and C' must have the same period and pre-period. When both are periodic, consider the patched components \tilde{C} and \tilde{C}' and consider the corresponding first-return maps $\mathcal{F}_{\tilde{C}}$ and $\mathcal{F}_{\tilde{C}'}$. Then these maps are Thurston equivalent.*

Proposition 7.5 *If f and g are equivalent Thurston maps in standard form then there exists an equivalence pair (h_1, h_2) such that h_1 and h_2 restrict to the identity map on all ∂A_i and h_1 is homotopic to h_2 on each thick component relative ∂A_i and Q_f .*

Proof As was shown above, there exists an equivalence pair (h_1, h_2) of f and g that descends to an equivalence of respective canonical decompositions. More precisely, there is a correspondence between thick components of f and thick components of g which conjugates the component-wise action of f to the action of g such that the first return maps of corresponding periodic components are Thurston equivalent. We fix coordinates on A_i and B_i and can choose h_1 and h_2 such that both restrict to the identity map on all ∂A_i . Since h_1 and h_2 are homotopic relative Q_f and coincide on ∂A_i , restricted to each thick component of f the two homeomorphisms can differ (up to isotopy relative the boundary of the component and the marked set) only by a composition of some powers of Dehn twists around the boundary components. Pushing this Dehn twists inside the annuli A_i , we can further normalize the pair (h_1, h_2) so that h_1 is homotopic to h_2 on each thick component.

The following is standard (see e.g. [Farb and Margalit 2011](#)):

Proposition 7.6 *For every Thurston obstruction $\Gamma = \{\alpha_1, \dots, \alpha_n\}$, the Dehn twists $T_{\alpha_j}, j = 1 \dots n$ generate a free Abelian subgroup of $PMCG(S \setminus Q_f)$.*

We write $\mathbb{Z}^\Gamma \simeq \mathbb{Z}^n$ to denote the subgroup generated by T_{α_j} .

Let us try to glue together a Thurston equivalence (h_1, h_2) of two maps $f, g \in \mathcal{H}$ using canonical decompositions thereof. By Proposition 7.5, we may look for h_1 and h_2 in standard form. For every periodic thick component of f with first return map

that has hyperbolic orbifold, the restriction of h_1 to that component (after patching) will represent the unique (by Proposition 7.2) mapping class that realizes Thurston equivalence to the corresponding periodic thick component of g . This in turn defines h_2 , and thus h_1 , uniquely up to isotopy relative ∂A_i and Q_f on every thick component that is a preimage of a periodic thick component by pulling back h_1 by f . Repeating the pullback procedure we can recover h_1 on all thick components in the decomposition of f . Therefore using the decomposition data we can define a mapping class h_0 which is homotopic to h_1 on all thick components and defined arbitrarily on A_i . We conclude the following.

Proposition 7.7 *If $f, g \in \mathcal{H}$ are equivalent Thurston maps, then there exists an equivalence pair (h_1, h_2) such that $h_1 = h_0 \circ m$ where h_0 is constructed as above and $m \in \mathbb{Z}^\Gamma$.*

Proof The restriction of $m = h_0^{-1} \circ h_1$ to every thick component is homotopic to the identity and the restriction of m to every annulus A_i is some power of the corresponding Dehn twist T_{α_i} , i.e. $m \in \mathbb{Z}^\Gamma$.

Notice that by construction if $h_1 \circ f = g \circ h_2$ where $h_1 = h_0 \circ m$ for some $m_1 \in \mathbb{Z}^\Gamma$, then h_2 is homotopic to $h_0 \circ m_2$ for some other $m_2 \in \mathbb{Z}^\Gamma$. Since we cannot check all elements of \mathbb{Z}^Γ we will require the following proposition.

Proposition 7.8 *There exists explicitly computable $N \in \mathbb{N}$ such that if $n \in \mathbb{Z}^\Gamma$ where all coordinates of n are divisible by N , then $(h_0 \circ (m_1 + n)) \circ f = g \circ h_1$, with h_1 homotopic to $h_0 \circ (m_2 + M_\Gamma n)$ rel Q_f , whenever $(h_0 \circ m_1) \circ f = g \circ h_2$, with h_2 homotopic to $h_0 \circ m_2$ rel Q_f .*

Proof We can take N to be the least common multiple of all degrees of f restricted to preimages of the annuli A_i . Then n lifts through f to $M_\Gamma n$ and we have the following commutative diagram:

$$\begin{array}{ccccc}
 S^2 & \xrightarrow{M_\Gamma n} & S^2 & \xrightarrow{h_1} & S^2 \\
 \downarrow f & & \downarrow f & & \downarrow g \\
 S^2 & \xrightarrow{n} & S^2 & \xrightarrow{h_0 \circ m_1} & S^2
 \end{array}$$

We can now present the proof of Theorem 7.1:

Proof The following algorithm solves the problem.

- (1) Find the canonical obstructions $\Gamma_f = \{\alpha_1, \dots, \alpha_n\}$ and $\Gamma_g = \{\beta_1, \dots, \beta_n\}$ (Theorem 6.1).
- (2) Check whether the cardinality of the canonical obstructions $\Gamma_f = \{\alpha_1, \dots, \alpha_n\}$ and $\Gamma_g = \{\beta_1, \dots, \beta_n\}$ is the same, and whether Thurston matrices coincide. If not, output **maps are not equivalent** and halt.
- (3) Construct decomposition annuli A_i and B_i as above. Geometrize the first return maps of patched thick parts (Theorem 6.1).
- (4) **for all** $\sigma \in S_n$ **do**

- (5) Is there a homeomorphism h_σ of S^2 sending $A_i \rightarrow B_{\sigma(i)}$? If not, **continue**. Check that the component-wise dynamics of f and g are conjugated by h_σ . If not, **continue**.
- (6) Construct equivalences between first return maps \mathcal{F}_i and \mathcal{G}_i of periodic thick components corresponding by h_σ . If the maps of some pair are not equivalent, **continue**.
- (7) For all thick components C_j^f check whether the Hurwitz classes of the patched coverings

$$\tilde{f} : \widetilde{C_j^f} \rightarrow \widetilde{f(C_j^f)} \text{ and } \tilde{g} : \widetilde{h_\sigma(C_j^f)} \rightarrow \widetilde{g(h_\sigma(C_j^f))}$$

are the same (Theorem 3.2). If not, **continue**.

- (8) Can the equivalences between first return maps \mathcal{F}_i and \mathcal{G}_i constructed at step (6) be lifted via branched covers \tilde{f} and \tilde{g} to every thick component (Theorem 3.2)? If not, **continue**.
- (9) Check if the lifted equivalences preserve the set of marked points. If not, **continue**.
- (10) Lift the equivalences, to obtain a homeomorphism h_0 defined on all thick parts.
- (11) Pick some initial homomorphisms $a_i : A_i \rightarrow B_i$ so that the boundary values agree with already defined boundary values of h_0 . This defines h_0 on the whole sphere.
- (12) **for all** $n \in \mathbb{Z}^\Gamma$ with coordinates between 0 and $N - 1$, where N is as in Proposition 7.8 **do**
 - (a) Try to lift $h_0 \circ n$ through f and g so that $(h_0 \circ n) \circ f = g \circ h_2$. If this does not work, **continue**.
 - (b) By the discussion above $h_2 = h_0 \circ m$ with $m \in \mathbb{Z}^\Gamma$. Compute m .
 - (c) Find a solution in \mathbb{Z}^Γ of the equation $Nx + n = M_\Gamma Nx + m$. If there is no integer solution, **continue**.
 - (d) Output **maps are equivalent** and $h_0 \circ (Nx + n)$ halt.
- (13) **end do**
- (14) **end do**
- (15) output **maps are not equivalent** and halt.

If the algorithm outputs $h_0 \circ (Nx + n)$ at step 12(d), then by Proposition 7.8 $h_0 \circ (Nx + n)$ lifts through f and g to a map which is isotopic to $h_0 \circ (M_\Gamma Nx + m) = h_0 \circ (Nx + n)$ producing an equivalence between f and g . If the algorithm fails to find an equivalence pair in this way, then Proposition 7.7 implies that f and g are not equivalent. □

8 Concluding Remarks

In this paper the problem of algorithmic decidability of Thurston equivalence of two Thurston maps f and g is resolved partially, when the decomposition of f (or g) does not contain any parabolic elements or homeomorphisms. Note that if the first return map \mathcal{F} of a periodic component \tilde{S} of the canonical decomposition of f is a

homeomorphism, then the problem of equivalence restricted to \tilde{S} is the conjugacy problem in $\text{MCG}(\tilde{S})$. By Theorem 3.33, it can be resolved algorithmically.

By Theorem 4.27, in the case when \mathcal{F} is parabolic, Thurston equivalence problem restricted to \tilde{S} reduces to a classical conjugacy problem of integer matrices:

Are two matrices in $\text{M}_2(\mathbb{Z})$ conjugate by an element of $\text{SL}_2(\mathbb{Z})$?

This problem is solvable algorithmically as well (see e.g. Grunewald and Segal 1979).

Thus in both exceptional cases, we can constructively determine whether the first return maps of the thick parts in the decompositions of f and g are Thurston equivalent or not. However, in contrast with Proposition 7.2, in this case the homeomorphism realizing equivalence is not unique. This poses an obvious difficulty with checking whether f is equivalent to g , as we have to check not one, but all possible equivalences of parabolic and homeomorphic components of the decomposition. In other words, the homeomorphism h_0 constructed in the proof of Theorem 7.1 is no longer unique; instead we get a certain subgroup of the Mapping Class Group of possible candidates. Extending our proof of decidability of Thurston equivalence to this case is an interesting problem, which may require, in particular, an algorithm for computing this subgroup.

Acknowledgments The authors gratefully acknowledge the support and hospitality of the IMS at Stony Brook where a part of this work has taken place.

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