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RESEARCH CONTRIBUTION

Galois Correspondence Theorem for Picard-Vessiot Extensions

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Abstract For a homogeneous linear differential equation defined over a differential field K, a Picard-Vessiot extension is a differential field extension of K differentially generated by a fundamental system of solutions of the equation and not adding constants. When K has characteristic 0 and the field of constants of K is algebraically closed, it is well known that a Picard-Vessiot extension exists and is unique up to K-differential isomorphism. In this case the differential Galois group is defined as the group of K-differential automorphisms of the Picard-Vessiot extension and a Galois correspondence theorem is settled. Recently, Crespo, Hajto and van der Put have proved the existence and unicity of the Picard-Vessiot extension for formally real (resp. formally p-adic) differential fields with a real closed (resp. p-adically closed) field of constants. This result widens the scope of application of Picard-Vessiot theory beyond the complex field. It is then necessary to give an accessible presentation of

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Picard-Vessiot theory for arbitrary differential fields of characteristic zero which eases its use in physical or arithmetic problems. In this paper, we give such a presentation avoiding both the notions of differential universal extension and specializations used by Kolchin and the theories of schemes and Hopf algebras used by other authors. More precisely, we give an adequate definition of the differential Galois group as a linear algebraic group and a new proof of the Galois correspondence theorem for a Picard-Vessiot extension of a differential field with non algebraically closed field of constants, which is more elementary than the existing ones.

Keywords Differential field · Picard-Vessiot extension · Linear algebraic group · Galois correspondence

Mathematics Subject Classification 12H05 · 12F10

1 Introduction

For a homogeneous linear differential equation $\mathcal{L}(Y) = 0$ defined over a differential field K with field of constants C, a Picard-Vessiot extension is a differential field L, differentially generated over K by a fundamental system of solutions of $\mathcal{L}(Y) = 0$ and with constant field equal to C. A classical result states that the Picard-Vessiot extension exists and is unique up to K-differential isomorphism in the case C algebraically closed (see Kolchin 1948, Crespo and Hajto 2011 or Put and Singer 2003). Recently, an existence and uniqueness result for Picard-Vessiot extensions has been established in the case when the differential field K is a formally real field (resp. a formally p-adic field) with real closed (resp. p-adically closed of the same rank than K) field of constants C (see Crespo et al. 2015). In Crespo et al. (2013) we presented a Galois correspondence theorem for Picard-Vessiot extensions of formally real differential fields with real closed field of constants. In this paper we establish a Galois correspondence theorem for general Picard-Vessiot extensions, i.e. without assuming the field of constants of the base field to be algebraically closed, valid in particular for Picard-Vessiot extensions of formally p-adic differential fields with p-adically closed field of constants.

Kolchin introduced the concept of strongly normal differential field extension and obtained a satisfactory Galois correspondence theorem for this class of extensions without assuming the field of constants of the differential base field to be algebraically closed (see Kolchin 1973, Chapter VI). Note that, for a strongly normal extension $L|K$, in the case when the constant field of K is not algebraically closed, the differential Galois group is no longer the group $\mathrm{DAut}_K L$ of K-differential automorphisms of L, rather one has to consider as well K-differential morphisms of L in larger differential fields. It is worth noting that a Picard-Vessiot extension is always strongly normal. The following simple example illustrates the necessity to change the definition of the differential Galois group when the field of constants is not algebraically closed. Let q be an odd prime number and C a field not containing the q-th roots of unity (e.g. $C = \mathbb{Q}_p$, with $q \nmid p-1$). Let $k = C(t)$ be the field of rational functions over C in the variable t endowed with the derivation determined by $t' = 1$. Let $L = k(y)$ be the field of rational functions over k in the variable y and extend the derivation to L by $y' = y$. Let $K = k(y^q)$. Then $L|K$ is clearly a Picard-Vessiot extension for the differential equation $Y' = Y$ with $[L : K] = q$ and $\mathrm{DAut}_K L = \mathrm{Aut}_K L = \{Id\}$. In the case of Picard-Vessiot extensions, we can adopt a definition of the differential Galois group inspired by Kolchin's but simpler than his one. We obtain then a Galois correspondence theorem which classifies intermediate differential fields of a Picard-Vessiot extension in terms of its differential Galois group. The fact that Kolchin used the notion of differential universal extension and Weil's algebraic geometry, which has been later replaced by Grothendieck's one, has led several authors to redo Kolchin's theory using the more modern language of schemes or Hopf algebras. However, as the knowledge of scheme and Hopf algebra theories is not extended to non-algebraists, we have chosen a simpler approach which will make the theory accessible to a wider range of mathematicians.

In this paper, we shall deal with fields of characteristic 0. For the sake of simplicity in the exposition we consider ordinary differential fields.

2 Main Result

We recall now the precise definition of Picard-Vessiot extension.

Definition 1 Given a homogeneous linear differential equation

$$\mathcal{L}(Y) := Y^{(n)} + a_{n-1}Y^{(n-1)} + \ldots + a_1 Y' + a_0 Y = 0$$

of order n over a differential field K, with field of constants C, a differential extension $L|K$ is a *Picard-Vessiot extension* for $\mathcal{L}$ if

1. $L = K\langle \eta_1, \ldots, \eta_n \rangle$, where $\eta_1, \ldots, \eta_n$ is a fundamental set of solutions of $\mathcal{L}(Y) = 0$ in L and $K\langle \eta_1, \ldots, \eta_n \rangle$ denotes the differential field generated by $\eta_1, \ldots, \eta_n$ over K.
2. Every constant of L lies in K, i.e. the field of constants of L is C.

As mentioned in the introduction, a Picard-Vessiot extension is strongly normal. Hence, the fundamental theorem established by Kolchin (1973, chapter VI) applies to Picard-Vessiot extensions. However, for a strongly normal extension $L|K$, Kolchin defines the differential Galois group $\mathrm{DGal}(L|K)$ by means of differential K-isomorphisms of L in the differential universal extension U of L. He obtains then that $\mathrm{DGal}(L|K)$ has the structure of an algebraic group defined over the field of constants C_U of U. Afterwards, by using the notion of specialization in Weil's algebraic geometry, he proves that there exists an algebraic group G defined over the field of constants C of K such that $G(C_U) = \mathrm{DGal}(L|K)$. It is worth noting that Kovacic in Kovacic (2003, 2006) established a theory of strongly normal extensions which avoids the differential universal extension by using tensor products. He defines the differential Galois group using a differential scheme. Umemura (1996) introduced the notion of automorphic differential field extensions, which allows a finite extension of the constant field, and includes both strongly normal extensions and almost classically Galois extensions (see Umemura 1996 Corollary (2.26)), a special class of Hopf-Galois finite extensions introduced in Greither and Pareigis (1987). Here a finite extension of fields is considered as a differential field extension using the trivial derivation. The differential Galois group of an automorphic differential field extension is defined as a group functor. Using Kovacic's theory of strongly normal extensions, Wibmer (2014) gave an intrinsic characterization of a Picard-Vessiot extension $L|K$ in terms of K-differential embeddings of L in differential L-algebras.

In this section, we give a more natural definition of the differential Galois group of a Picard-Vessiot extension, which avoids the use both of the differential universal extension and of scheme theory. We endow the differential Galois group with a linear algebraic group structure over the field of constants C and establish a Galois correspondence theorem in our setting.

2.1 Differential Galois Group

For K a differential field with field of constants C, we fix an algebraic closure K^c of K and let $\overline{C}$ denote an algebraic closure of C contained in K^c. We shall denote by $\overline{K}$ the composition field of K and $\overline{C}$ inside K^c. Whenever F is a differential field extension of K, we fix F^c containing K^c, so that $\overline{K} \subset \overline{F}$. Let $\{\alpha_i\}_{i\in I}$ be a C-basis of $\overline{C}$. If the field of constants of the differential field extension F of K is equal to C, the extensions $\overline{C}|C$ and $F|C$ are linearly disjoint over C and then $\{\alpha_i\}_{i\in I}$ is an F-basis of $\overline{F}$.

Remark 1 If K is a differential field with field of constants C, then $K \otimes_C \overline{C}$ is a field by Jacobson (1976, Chapter IV, Theorem 21 (2)). Since K and $\overline{C}$ are linearly disjoint over C, we have $\overline{K} \simeq K \otimes_C \overline{C}$. Therefore we can also define $\overline{K}$ as $K \otimes_C \overline{C}$.

For a Picard-Vessiot extension $L|K$, we shall consider the set $\mathrm{DHom}_K(L, \overline{L})$ of K-differential morphisms from L into $\overline{L}$. We shall see that we can define a group structure on this set and we shall take it as the differential Galois group $\mathrm{DGal}(L|K)$ of the Picard-Vessiot extension $L|K$. We shall prove that it is a C-defined (Zariski) closed subgroup of some $\overline{C}$-linear algebraic group.

We observe that we can define mutually inverse bijections

$$\begin{array}{ccc} \mathrm{DHom}_K(L,\overline{L}) & \rightarrow & \mathrm{DAut}_{\overline{K}}\overline{L} \\ \sigma & \mapsto & \widehat{\sigma} \end{array}, \quad \begin{array}{ccc} \mathrm{DAut}_{\overline{K}}\overline{L} & \rightarrow & \mathrm{DHom}_K(L,\overline{L}) \\ \tau & \mapsto & \tau_{|L} \end{array},$$

where $\widehat{\sigma}$ is the extension of σ to $\overline{L}$. For an element $\sum \lambda_i \alpha_i$ in $\overline{L}$, where $\lambda_i \in L$, we define $\widehat{\sigma}(\sum \lambda_i \alpha_i) = \sum \sigma(\lambda_i)\alpha_i$. We may then transfer the group structure from $\mathrm{DAut}_{\overline{K}}\overline{L}$ to $\mathrm{DHom}_K(L, \overline{L})$. Let us note that $\mathrm{DAut}_{\overline{K}}\overline{L}$ is the differential Galois group of the Picard-Vessiot extension $\overline{L}|\overline{K}$.

Let now $\eta_1, \ldots, \eta_n$ be C-linearly independent elements in L such that $L = K\langle \eta_1, \ldots, \eta_n\rangle$ and $\sigma \in \mathrm{DHom}_K(L, \overline{L})$. We have then $\sigma(\eta_j) = \sum_{i=1}^n c_{ij}\eta_i$, $1 \leq j \leq n$, with $c_{ij} \in \overline{C}$. We may then associate to σ the matrix (c_{ij}) in $\mathrm{GL}(n, \overline{C})$. The proofs of Propositions 16 and 17 and Corollary 18 in Crespo et al. (2013) remain valid in our present setting. We obtain then the following results.

Proposition 1 *Let K be a differential field with field of constants C, $L = K\langle \eta_1, \dots, \eta_n \rangle$ a Picard-Vessiot extension of K, where $\eta_1, \dots, \eta_n$ are C-linearly independent. There exists a set S of polynomials $P(\{X_{ij}\}_{1 \leq i,j \leq n})$, with coefficients in C such that*

1) If $\sigma \in \mathrm{DHom}_K(L, \overline{L})$ and $\sigma(\eta_j) = \sum_{i=1}^n c_{ij}\eta_i$, then $P(c_{ij}) = 0, \forall P \in S$.
2) Given a matrix $(c_{ij}) \in \mathrm{GL}(n, \overline{C})$ with $P(c_{ij}) = 0, \forall P \in S$, there exists a differential K-morphism σ from L to $\overline{L}$ such that $\sigma(\eta_j) = \sum_{i=1}^n c_{ij}\eta_i$.

The preceding proposition gives that $\mathrm{DGal}(L|K)$ is a C-defined closed subgroup of $\mathrm{GL}(n, \overline{C})$.

Proposition 2 *Let K be a differential field with field of constants C, $L|K$ a Picard-Vessiot extension. For $a \in L \setminus K$, there exists a K-differential morphism $\sigma : L \to \overline{L}$ such that $\sigma(a) \neq a$.*

For a subset S of $\mathrm{DGal}(L|K)$, we set $L^S := \{a \in L : \sigma(a) = a, \forall \sigma \in S\}$.

Corollary 1 *Let K be a differential field with field of constants C, $L|K$ a Picard-Vessiot extension. We have $L^{\mathrm{DGal}(L|K)} = K$.*

As mentioned above, some authors have also considered Picard-Vessiot extensions over differential fields with non algebraically closed field of constants with different approaches than ours. Takeuchi (1989) defines a Picard-Vessiot extension of C-ferential fields (a notion which generalizes differential, partial differential and iterative differential fields) and the Hopf algebra of such an extension. If $L|K$ is a Picard-Vessiot extension of differential fields with field of constants C and Hopf algebra H, then the group of $\overline{C}$-points of the affine group scheme $\mathrm{Spec} H$ is isomorphic to the group of $\overline{C}$-points of $\mathrm{DGal}(L|K)$. Dyckerhoff (2008) and Maurischat (2010) define the differential Galois group as a group functor which is representable by an affine group scheme coinciding with Takeuchi's one.

2.2 Fundamental Theorem

Let K be a differential field with field of constants C and $L|K$ a Picard-Vessiot extension. For a closed subgroup H of $\mathrm{DGal}(L|K)$, L^H is a differential subfield of L containing K. If E is an intermediate differential field, i.e. $K \subset E \subset L$, then $L|E$ is a Picard-Vessiot extension and $\mathrm{DGal}(L|E)$ is a C-defined closed subgroup of $\mathrm{DGal}(L|K)$.

Theorem 1 *Let $L|K$ be a Picard-Vessiot extension,* $\mathrm{DGal}(L|K)$ *its differential Galois group.*

1. The correspondences

$$H \mapsto L^H \quad , \quad E \mapsto \mathrm{DGal}(L|E)$$

define inclusion inverting mutually inverse bijective maps between the set of C-defined closed subgroups H of $\mathrm{DGal}(L|K)$ *and the set of differential fields E with $K \subset E \subset L$.*

2. *The intermediate differential field E is a Picard-Vessiot extension of K if and only if the subgroup* $\mathrm{DGal}(L|E)$ *is normal in* $\mathrm{DGal}(L|K)$. *In this case, the restriction morphism*

$$\begin{array}{ccc} \mathrm{DGal}(L|K) & \to & \mathrm{DGal}(E|K) \\ \sigma & \mapsto & \sigma_{|E} \end{array}$$

induces an isomorphism

$$\mathrm{DGal}(L|K)/\mathrm{DGal}(L|E) \simeq \mathrm{DGal}(E|K).$$

Proof 1. It is clear that both maps invert inclusion. If E is an intermediate differential field of $L|K$, we have $L^{\mathrm{DGal}(L|E)} = E$, taking into account that $L|E$ is Picard-Vessiot and corollary 1. For H a C-defined closed subgroup of $\mathrm{DGal}(L|K)$, let $\widehat{H} = \{\widehat{\sigma} : \sigma \in H\}$, for $\widehat{\sigma}$ the extension of σ to $\overline{L}$, as defined in section 2.1. Then $\widehat{H}$ is a closed subgroup of $\mathrm{DGal}(\overline{L}|\overline{K})$ and from Picard-Vessiot theory for differential fields with algebraically closed field of constants (see e.g. Crespo and Hajto 2011, Theorem 6.3.8), we obtain the equality $\widehat{H} = \mathrm{DGal}(\overline{L}|\overline{L}^{\widehat{H}})$. Now, from the definition of $\widehat{\sigma}$, the equality $\overline{L}^{\widehat{H}} = \overline{L^H}$ follows and hence $H = \mathrm{DGal}(L|L^H)$.
2. If E is a Picard-Vessiot extension of K, then $\overline{E}$ is a Picard-Vessiot extension of $\overline{K}$ and so $\mathrm{DGal}(L|E)$ is normal in $\mathrm{DGal}(L|K)$. Reciprocally, if $\mathrm{DGal}(L|E)$ is normal in $\mathrm{DGal}(L|K)$, then the subfield of $\overline{L}$ fixed by $\mathrm{DGal}(L|E)$ is a Picard-Vessiot extension of $\overline{K}$. Now, this field is $\overline{E}$. So, $\overline{E}$ is differentially generated over $\overline{K}$ by a $\overline{C}$-vector space V of finite dimension. Let $\{v_1, \dots, v_n\}$ be a $\overline{C}$-basis of V and let $\{\alpha_i\}_{i\in I}$ be a C-basis of $\overline{C}$, as above. We may write each $v_j, 1 \leq j \leq n$ as a linear combination of the elements α_i with coefficients in E. Now, there is a finite number of α_i's appearing effectively in these linear combinations. Let $\widetilde{C}$ be a finite Galois extension of C containing all these α_i's. We have then $v_i \in \widetilde{E} := \widetilde{C} \cdot E$ and $\widetilde{E}$ is differentially generated over $\widetilde{K} := \widetilde{C} \cdot K$ by V. We may extend the action of $\mathrm{Gal}(\widetilde{C}|C)$ to $\widetilde{E}$ and consider the transform $c(V)$ of V by $c \in \mathrm{Gal}(\widetilde{C}|C)$. Let $\widetilde{V} = \oplus_{c\in \mathrm{Gal}(\widetilde{C}|C)} c(V)$. We have that $\widetilde{E}$ is differentially generated over $\widetilde{K}$ by $\widetilde{V}$ and $\widetilde{V}$ is $\mathrm{Gal}(\widetilde{C}|C)$-stable, hence E is differentially generated over K by the C-vector space $\widetilde{V}^{\mathrm{Gal}(\widetilde{C}|C)} = \{y \in V : c(y) = y, \forall c \in \mathrm{Gal}(\widetilde{C}|C)\}$. We may then conclude that $E|K$ is a Picard-Vessiot extension. Finally, from the fundamental theorem of Picard-Vessiot theory in the case of algebraically closed fields of constants (Crespo and Hajto 2011 theorem 6.3.8), we obtain an isomorphism $\mathrm{DAut}_{\overline{K}}\overline{L}/\mathrm{DAut}_{\overline{E}}\overline{L} \simeq \mathrm{DAut}_{\overline{K}}\overline{E}$, induced by restriction, which implies $\mathrm{DGal}(L|K)/\mathrm{DGal}(L|E) \simeq \mathrm{DGal}(E|K)$. □

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