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RESEARCH CONTRIBUTION

The Coadjoint Operator, Conjugate Points, and the Stability of Ideal Fluids

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Abstract We give a new description of the coadjoint operator $\mathrm{Ad}^*_{\eta^{-1}(t)}$ along a geodesic $\eta(t)$ of the L^2 metric in the group of volume-preserving diffeomorphisms, important in hydrodynamics. When the underlying manifold is two dimensional the coadjoint operator is given by the solution operator to the linearized Euler equations modulo a compact operator; when the manifold is three dimensional the coadjoint operator is given by the solution operator to the linearized Euler equations plus a bounded operator. We give two applications of this result when the underlying manifold is two dimensional: conjugate points along geodesics of the L^2 metric are characterized in terms of the coadjoint operator and thus determining the conjugate locus is a purely algebraic question. We also prove that Eulerian and Lagrangian stability of the 2D Euler equations are equivalent and that instabilities in the 2D Euler equations are contained and small.

Keywords Diffeomorphism group · Hydrodynamics · Euler equations · Geodesic · Conjugate point · Stability

Mathematics Subject Classification 35Q05 · 35Q35 · 47H99 · 58C99 · 58B25 · 76E99 · 37K65 · 37K45

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1 Introduction

For a closed Riemannian manifold M with metric g and volume form μ, let $\mathcal{D}^s_\mu(M)$ denote those diffeomorphisms of Sobolev class H^s preserving μ. When $s > \frac{\dim M}{2} + 1$, $\mathcal{D}^s_\mu(M)$ becomes an infinite dimensional manifold whose tangent space at the identity $T_e\mathcal{D}^s_\mu(M)$ is given by H^s divergence free vector fields on M. Using right translations, the L^2 inner-product

$$(u, v)_{L^2} = \int_M g(u, v)\, d\mu \qquad u, v \in T_e\mathcal{D}^s_\mu(M) \tag{1}$$

defines a weak, right-invariant Riemannian metric with smooth exponential map $\exp^\mu_e$, right-invariant Levi-Civita connection ∇^μ and right invariant curvature tensor R^μ, cf. Ebin and Marsden (1970).

A curve $\eta(t)$ in the group $\mathcal{D}^s_\mu$ is a geodesic of the L^2 metric (1) if and only if the time dependent vector field $v \in T_e\mathcal{D}^s_\mu$, defined by $v = \dot{\eta} \circ \eta^{-1}$, solves the Euler equations of hydrodynamics

$$\begin{aligned} \partial_t v + P^\mu\left(\nabla_v v\right) &= 0 \\ \mathrm{div} v &= 0 \\ v(0) &= v_o, \end{aligned} \tag{2}$$

which describe the motion of a perfect incompressible fluid filling a compact, boundary-less domain M with Levi-Civita connection ∇ and P^μ the orthogonal projection onto the space of divergence free vector fields on M. We refer the reader to Arnold (1966), who first made this observation for the group of smooth volume-preserving diffeomorphisms, and Ebin and Marsden (1970) where this was made precise in the context of Sobolev H^s diffeomorphisms. When M is two dimensional it is well known that solutions of the Euler equations (2) exist for all time, see Wolibner (Wolibner 1933). Consequently the manifold $\mathcal{D}^s_\mu$ is L^2 geodesically complete. However, when M is three dimensional solutions of (2) are defined only on short time intervals, cf. (Ebin and Marsden 1970), and the global existence problem is wide open. In this case, L^2 geodesics in $\mathcal{D}^s_\mu$ exist only locally in time. Geodesics in the group $\mathcal{D}^s_\mu$ describe the paths of fluid particles and this known as the Lagrangian formulation of hydrodynamics. The Euler equations (2) describe particle velocities and this is known as the Eulerian formulation of hydrodynamics. The battleground for the Euler equations (2) has not been the space of vector fields but rather its corresponding dual space; not the adjoint representation, but the coadjoint representation. The coadjoint representation is the representation that associates to each element η of $\mathcal{D}^s_\mu(M)$ the linear operator Ad^*_η. The coadjoint orbit of a point ξ in the dual is the set of all points $\mathrm{Ad}^*_\eta\xi$ $(\eta \in \mathcal{D}^s_\mu(M))$ in the dual. When M is a closed surface the natural dual to the space of divergence free vector fields is the space of functions on M with zero mean. The coadjoint representation is given by the natural action of diffeomorphisms, preserving the area form of M, on functions on M. When M is three dimensional, the dual to the space of divergence free vector fields is the space of 1-forms on M. The adjoint and coadjoint representations are the standard actions of diffeomorphisms on the corresponding vector fields and 1-forms. We refer the reader to Arnold and Khesin (Arnold and Khesin 1998) for a discussion and proof of these statements.

In this paper we give a new description of the coadjoint operator $\mathrm{Ad}^*_{\eta^{-1}(t)}$ along an L^2 geodesic $\eta(t)$ in $\mathcal{D}^s_\mu$. We first identify the space of divergence free vector fields with its weak L^2 dual through the metric (1). The adjoint representation is given by the usual pushforward of vector fields and the formal coadjoint representation is defined as the L^2 operator adjoint of the pushforward operation. In particular, we study the coadjoint operator defined on the space of divergence free vector fields; one can always transform the coadjoint representation in this space of vector fields back to the coadjoint representation defined on the dual. When M is two dimensional the coadjoint operator $\mathrm{Ad}^*_{\eta^{-1}(t)}$ is given by the solution operator to the linearized Euler equations modulo a compact operator. When M is three dimensional the coadjoint operator $\mathrm{Ad}^*_{\eta^{-1}(t)}$ is given by the solution operator to the linearized Euler equations plus a bounded linear operator. This is the content of Theorem 5 and sheds some light on the subtle difference between coadjoint orbits in two and three dimensional hydrodynamics.

We give two applications of this result when M is a closed surface. The first solves the Jacobi equation explicitly and gives a new characterization of conjugate points. These results are contained in Proposition 6 and Theorem 7. Conjugate points are characterized completely in terms of the coadjoint operator and finding the conjugate locus is a purely algebraic question. The proofs make use of a result by Ebin, Misiołek, and Preston which states that the exponential map of the L^2 metric on $\mathcal{D}^s_\mu(M)$ is a non-linear Fredholm map of index zero, cf. (Ebin et al. 2006). The second application concerns the stability of a stationary solution v_o to the Euler equations (2) with corresponding stationary geodesic $\eta(t)$ in $\mathcal{D}^s_\mu$. Preston (2004) demonstrated that if the L^2 norm of a solution $z(t)$, $z(0) = z_o$, to the linearized Euler equations (at v_o) remains bounded in time then the L^2 norm of the Jacobi field $J(t)$ along $\eta(t)$, with $J(0) = 0$ and $J'(0) = z_o$, grows at most quadratically in time. Also, if v_o is analytic with isolated non-degenerate fixed points then v_o is at most Lagrangian polynomial unstable if it is at most Eulerian polynomial unstable. Here we prove that the notions of Eulerian and Lagrangian stabilities of the Euler equations (2) are equivalent in that the L^2 norm of a Jacobi field remains bounded in time if and only if the L^2 norm of the corresponding linearized Euler solution remains bounded in time. We make no assumptions on the topology of M^2, nor on the qualitative behavior of the stationary solutions to (2). This completes the results of (Preston 2004). Furthermore, we find that the instabilities present in the 2D Euler equations are contained and small whereas the instabilities in the 3D Euler equations seem to be much more wide-spread, see Remark 14.

In the next section we define the objects relevant to our discussion. Section 3 contains the main results, Proposition 4 and Theorem 5, while Sects. 4 and 5 contain the applications. We close with some concluding remarks in Sect. 6.

2 Preliminaries

In what follows M^n denotes a closed Riemannian manifold of dimension $n = 2, 3$. Define a map $g^\flat : TM^n \to T^*M^n$ which assigns to each vector field v the one form $i_v g$, where i_v denotes the interior product with v. This is an isomorphism from the tangent bundle to the cotangent bundle of M^n with inverse defined as $g^\sharp : T^*M^n \to TM^n$ and corresponds to contracting with the inverse components of the metric tensor.

The space of all Sobolev H^s vector fields on M^n decomposes, according to the Hodge decomposition, as

$$T_e\mathcal{D}^s = g^\sharp\left(d\delta H^{s+2}(T^*M^n) \oplus \delta d H^{s+2}(T^*M^n) \oplus \mathcal{H}\right), \tag{3}$$

where d denotes the usual differential operator, δ its formal L^2 adjoint and $\mathcal{H}$ denotes the finite dimensional space of harmonic vector fields, cf. Morrey (1966). The set of divergence free vector fields is given by

$$T_e\mathcal{D}^s_\mu = g^\sharp\left(\delta d H^{s+2}(T^*M^n) \oplus \mathcal{H}\right),$$

and since the action of $\mathcal{D}^s_\mu$ on $\mathcal{D}^s$ by composition on the right is an isometry of (1) we obtain an L^2-orthogonal splitting in each tangent space

$$T_\eta\mathcal{D}^s = T_\eta\mathcal{D}^s_\mu \oplus \nabla H^{s+1}(M^n) \circ \eta. \tag{4}$$

Denote by P^μ_η the orthogonal projection onto the first summand and the corresponding projection $Q^\mu_\eta = I - P^\mu_\eta$ onto the second. We refer the reader to [E-M] where it is shown that the projections depend smoothly on the base-point η.

For $\sigma \geq 0$, let $T_e\mathcal{D}^\sigma_\mu(M^n)$ denote the closure of the space of exact divergence free vector fields in the H^σ norm. By the Hodge decomposition this is a closed subspace in the space of all H^σ vector fields, cf. (Morrey 1966). For $\sigma > \frac{n}{2} + 1$ this coincides with the actual tangent space to $\mathcal{D}^\sigma_\mu$, however, for smaller σ the group $\mathcal{D}^\sigma_\mu$ is not necessarily a smooth manifold. In what follows, $s > \frac{n}{2} + 1$ and $\sigma \leq s - 1$.

In standard Lie group notation, the group adjoint representation is given by $\mathrm{Ad}_\eta = dL_\eta dR_{\eta^{-1}}$ for any $\eta \in \mathcal{D}^s_\mu$, where $dR_{\eta^{-1}}v = v \circ \eta^{-1}$ and $dL_\eta v = D\eta \cdot v$. Thus the adjoint representation is

$$\mathrm{Ad}_\eta v = \eta_*(v) = D\eta \cdot v \circ \eta^{-1}, \tag{5}$$

which is the usual pushfoward operation on vector fields. Observe that right multiplication dR_η is smooth whilst left multiplication dL_η is only continuous. Therefore, if $\eta \in \mathcal{D}^s_\mu$ the adjoint representation is only defined on $T_e\mathcal{D}^\sigma_\mu$.

For a curve $\eta(s) \in \mathcal{D}^s_\mu$, with $\eta(0) = e$, $\dot\eta(0) = v \in T_e\mathcal{D}^s$, the differential of $\mathrm{Ad}_{\eta(s)}$ at the group identity defines the algebra adjoint representation ad_v. The vector field v is of class C^1, by the Sobolev embedding theorem, and ad_v is the unbounded operator from $T_e\mathcal{D}^\sigma_\mu$ to itself, which is given by the negative of the usual Lie bracket of vector fields:

$$\mathrm{ad}_v u = -[v, u], \tag{6}$$

cf. (Arnold and Khesin 1998).

The formal L^2 adjoint of the operator (5) in $T_e\mathcal{D}^0_\mu$ is called the group coadjoint operator and is defined by

$$\left(\mathrm{Ad}^*_\eta u, v\right)_{L^2} = \left(u, \mathrm{Ad}_\eta v\right)_{L^2} \ \forall v \in T_e\mathcal{D}^0_\mu. \tag{7}$$

The algebra coadjoint operator in $T_e\mathcal{D}^0_\mu$ is defined by

$$\left(\mathrm{ad}^*_u v, w\right)_{L^2} = (v, \mathrm{ad}_u w)_{L^2} \ \forall w \in T_e\mathcal{D}^s_\mu. \tag{8}$$

Explicitly, we can compute that the group coadjoint operator is given by

$$\mathrm{Ad}^*_\eta v = P^\mu_e(g^\sharp \eta^* g^\flat v) = P^\mu_e(D\eta^T(v \circ \eta)), \tag{9}$$

where η^* is the pullback operator on differential forms. Although Ad_η maps $T_e\mathcal{D}^0_\mu$ to itself, the L^2 adjoint does not, which is why we must compose with the projection P^μ_e. Using the metric (1), the definition of ad_v and the definition of ad^*_v we compute that

$$\mathrm{ad}^*_v z + \mathrm{ad}^*_z v = P^\mu_e(\nabla_v z + \nabla_z v). \tag{10}$$

Recall that the L^2 metric admits a smooth exponential mapping $\exp^\mu_e$ which is a local diffeomorphorphism in a nighbourhood of the identity, cf. (Ebin and Marsden 1970). Consider a geodesic $\eta(t) = \exp^\mu_e(tv_o)$ starting from the identity in the direction $v_o \in T_e\mathcal{D}^s_\mu$. The Jacobi field $J(t)$ along $\eta(t)$, with initial conditions $J(0) = 0$ and $\dot{J}(0) = z_o \in T_e\mathcal{D}^s_\mu$, is defined by

$$J(t) \equiv D\exp^\mu_e(tv_o)tz_o,$$

and satisfies the Jacobi equation

$$\nabla^\mu_{\dot\eta}\nabla^\mu_{\dot\eta} J + R^\mu(J, \dot\eta)\dot\eta = 0. \tag{11}$$

The fact that R^μ is a bounded multi-linear operator implies that Jacobi fields exist, are unique and are defined on $T_e\mathcal{D}^s_\mu$ for as long as η is, cf. (Misiołek 1993). Denote the solution operator to the Jacobi equation (11) by $\Phi(t)$,

$$\Phi(t)z_o := J(t), \tag{12}$$

which is a bounded operator from $T_e\mathcal{D}^s_\mu$ to $T_{\eta(t)}\mathcal{D}^s_\mu$.

Proposition 1 *Let $\eta(t)$ be a geodesic of (1) in $\mathcal{D}^s_\mu$ with $\eta(0) = e$ and $\dot\eta(0) = v_o$. Then every Jacobi field $J(t)$ along $\eta(t)$ satisfies the following system of equations:*

$$\partial_t y - \mathrm{ad}_v y = z, \tag{13}$$

$$\partial_t z + \mathrm{ad}^*_v z + \mathrm{ad}^*_z v = 0, \tag{14}$$

where $J(t) = y(t) \circ \eta(t)$ and $\dot\eta(t) = v(t) \circ \eta(t)$.

Proof Recall that a curve $\eta(t)$ is a geodesic in $\mathcal{D}^s_\mu$ if and only if the vector field $v(t)$ defined by

$$\partial_t \eta(t) = v(t) \circ \eta(t) \tag{15}$$

solves the Euler equations (2). Let $\eta(t,s) = \exp_e t(v_o + s z_o)$ be a smooth variation of $\eta(t)$ through geodesics. Set $y(t) = dR_{\eta(t)^{-1}} \partial_s|_{s=0} \eta(t,s)$ and $z(t) = \partial_s|_{s=0} dR_{\eta(t)^{-1}} \dot\eta(t,s)$. Then, differentiating (15) with respect to s and evaluating at $s = 0$ yields (13). Differentiating (2) with respect to s and using (10) yields (14). $\square$

Although $v(t)$ is defined on $T_e\mathcal{D}^s_\mu$, Eqs. (13) and (14) are only defined on $T_e\mathcal{D}^\sigma_\mu$ due to the loss of derivatives in v. Equation (14) are the linearized Euler equations and we denote the solution operator to (14) by

$$S(t) z_o := z(t), \tag{16}$$

which is a bounded operator from $T_e\mathcal{D}^\sigma_\mu$ to $T_e\mathcal{D}^\sigma_\mu$, for each t, and is defined for as long as $v(t)$ is defined.

For any $v \in T_e\mathcal{D}^s_\mu$ define an operator

$$K_v : T_e\mathcal{D}^\sigma_\mu \to T_e\mathcal{D}^\sigma_\mu$$

$$u \mapsto K_v(u) := \mathrm{ad}^*_u v, \tag{17}$$

which, using the metric (1), may be written explicitly as

$$K_v u = P^\mu_e \left(g^\sharp i_u d g^\flat v \right). \tag{18}$$

The operator K_v is bounded on $T_e\mathcal{D}^\sigma_\mu$ since $dg^\flat v$ is of class H^{s-1} and P^μ_e is continuous. It is skew self adjoint since $dg^\flat v$ is antisymmetric.

Proposition 2 *Let $\eta(t)$ be a geodesic of (1) in $\mathcal{D}^s_\mu$ with $\dot\eta(t) = v(t) \circ \eta(t)$, $\eta(0) = e$ and $\dot\eta(0) = v_o$. Then*

$$K_{v(t)} = \mathrm{Ad}^*_{\eta^{-1}(t)} K_{v_o} \mathrm{Ad}_{\eta^{-1}(t)}. \tag{19}$$

Proof For the Euler equations of ideal fluids, we know the vorticity 2-form $dg^\flat v(t)$ is transported by the flow; i.e., that $dg^\flat v(t) = \left(\eta^{-1}\right)^* dg^\flat v_o$. So $K_{v(t)} u =$

$P_e^\mu\left(g^\sharp\iota_u\left(\eta^{-1}\right)^* dg^\flat v_o\right)$. Let $u, w \in T_e\mathcal{D}_\mu^\sigma$, we have

$$\begin{aligned}\iota_u\left(\eta^{-1}\right)^* dg^\flat v_o(w) &= \left(\eta^{-1}\right)^* dg^\flat v_o(u, w)\\ &= dg^\flat v_o(\eta_*^{-1}u, \eta_*^{-1}w)\\ &= \iota_{\eta_*^{-1}u} dg^\flat v_o(\eta_*^{-1}w)\\ &= \left(\eta^{-1}\right)^*\left(\iota_{\eta_*^{-1}u} dg^\flat v_o\right)(w).\end{aligned}$$

Thus by (5) and (9) we obtain (19). □

The next Proposition holds only for two dimensional manifolds M^2.

Proposition 3 *Let M^2 be a closed surface. If $s > 2$ and $s \geq \sigma + 1$ then for any vector field $v \in T_e\mathcal{D}_\mu^s(M^2)$ the operator*

$$K_v : T_e\mathcal{D}_\mu^\sigma(M^2) \to T_e\mathcal{D}_\mu^\sigma(M^2),$$

defined by formula (18)*, is compact.*

Proof See Ebin et al. (2006). □

3 The Coadjoint Operator

The adjoint representation is given by the usual pushforward of vector fields and the formal coadjoint representation on $T_e\mathcal{D}_\mu^0(M)$ is defined as the L^2 operator adjoint of the pushforward operation, as in Sect. 2 above. One can always transform the coadjoint representation on $T_e\mathcal{D}_\mu^0(M)$ to the coadjoint representation on the dual spaces mentioned in the introduction. A volume form is enough to define the space of divergence free vector fields and its dual so that the coadjoint operator is an algebraic object and does not rely on the metric; here we are identifying the space of divergence free vector fields with its weak L^2 dual and considering the coadjoint operator defined on the space of divergence free vector fields. In the next Proposition we relate the coadjoint operator in the space of vector fields, which is algebraic, to solutions of the linearized Euler equations and Jacobi fields, which are Riemannian.

Proposition 4 *Let $\eta(t)$ be any geodesic of the L^2 metric in $\mathcal{D}_\mu^s(M^n)$, $n = 2, 3$, with corresponding Euler solution $v(t) = \dot\eta(t) \circ \eta^{-1}(t) \in T_e\mathcal{D}_\mu^s$, and initial velocity $v_o \in T_e\mathcal{D}_\mu^s$. Then, the coadjoint operator along $\eta^{-1}(t)$ is given by*

$$\mathrm{Ad}^*_{\eta^{-1}(t)} = S(t) + K_{v(t)} \circ dR_{\eta^{-1}(t)} \circ \Phi(t), \tag{20}$$

on $T_e\mathcal{D}_\mu^\sigma$.

Proof Recall that

$$\partial_t \mathrm{Ad}_{\eta^{-1}(t)} = -\mathrm{Ad}_{\eta^{-1}(t)} \mathrm{ad}_{v(t)}$$

and, by general properties of adjoints,

$$\partial_t \mathrm{Ad}^*_{\eta^{-1}(t)} = -\mathrm{ad}^*_{v(t)} \mathrm{Ad}^*_{\eta^{-1}(t)}.$$

Using these we can rewrite Eq. (13) as

$$\partial_t \left(\mathrm{Ad}_{\eta^{-1}(t)} y(t) \right) = \mathrm{Ad}_{\eta^{-1}(t)} z(t).$$

Applying $\mathrm{Ad}^*_{\eta^{-1}(t)} K_{v_o}$ to both sides of this equation, then using the product rule and Proposition 2 yields

$$\partial_t \left(K_{v(t)} y(t) \right) + \mathrm{ad}^*_{v(t)} K_{v(t)} y(t) = K_{v(t)} z(t).$$

In view of (14), this equation becomes

$$\partial_t \left(K_{v(t)} y(t) + z(t) \right) + \mathrm{ad}^*_{v(t)} \left(K_{v(t)} y(t) + z(t) \right) = 0,$$

whose solution is given by

$$\mathrm{Ad}^*_{\eta^{-1}(t)} z_o = z(t) + K_{v(t)} y(t).$$

Writing this identity in terms of the solution operators (12) and (16) yields (20). □

Theorem 5 *Let M^n be a closed Riemmanian manifold of dimension n, $s > \frac{n}{2} + 1$, $s \geq \sigma + 1$, and let $\eta(t)$ be a geodesic of the metric (1) in $\mathcal{D}^s_\mu$ with velocity field $v(t) = \dot{\eta}(t) \circ \eta^{-1}(t) \in T_e\mathcal{D}^s_\mu$. Then,*

1. *For $n = 2$, the coadjoint representation $\mathrm{Ad}^*_{\eta^{-1}(t)}$ on $T_e\mathcal{D}^\sigma_\mu$ is given by the solution operator to the linearized Euler equations (14) modulo a compact operator.*
2. *For $n = 3$, the coadjoint representation $\mathrm{Ad}^*_{\eta^{-1}(t)}$ on $T_e\mathcal{D}^\sigma_\mu$ is given by the solution operator to the linearized Euler equations (14) plus a bounded operator.*

Proof Let $\eta(t)$ and $v(t)$ be as in the statement of the Theorem.

1. When $n = 2$ we have, by Propositions 2 and 3, that the operator $K_{v(t)}$ is a compact operator on $T_e\mathcal{D}^\sigma_\mu$. Since $dR_{\eta^{-1}(t)}$ and $\Phi(t)$ are bounded operators the composition $K_{v(t)} \circ dR_{\eta^{-1}(t)} \circ \Phi(t)$ is compact for each t on $T_e\mathcal{D}^\sigma_\mu$. The statement now follows from Proposition 4.
2. When $n = 3$ the composition $K_{v(t)} \circ dR_{\eta^{-1}(t)} \circ \Phi(t)$ is only bounded on $T_e\mathcal{D}^\sigma_\mu$ for each t, since $K_{v(t)}$ is only bounded on $T_e\mathcal{D}^\sigma_\mu$. The statement follows from Proposition 4.

□

4 Conjugate Points

Here M^2 is a closed surface. Using Propositions 3 and 4 we will derive an explicit expression for Jacobi fields along L^2 geodesics in $\mathcal{D}^s_\mu$. A new characterization of conjugate points is given in terms of the coadjoint operator; thus, finding the conjugate locus is a purely algebraic question.

Let $\eta(t)$ be a geodesic of the L^2 metric (1) in $\mathcal{D}^s_\mu(M^2)$. The point $\eta(t^*)$ is said to be conjugate to $\eta(0)$, $t^* \in (0, t]$, if the linear operator

$$D\exp^\mu_e(t^* v_o) : T_e\mathcal{D}^s_\mu \to T_{\eta(t^*)}\mathcal{D}^s_\mu$$

fails to be an isomorphism. If $\dim\ker D\exp^\mu_e(t^* v_o) = k$, k is called the multiplicity of the conjugate point.

A linear operator between Hilbert spaces with empty kernel need not be an isomorphism. Therefore, $\eta(t^*)$ may be a conjugate point even if $D\exp^\mu_e(t^* v_o)$ has empty kernel. A point $\eta(t^*)$ is monoconjugate to e if $D\exp^\mu_e(t^* v_o)$ fails to be injective and a point $\eta(t^*)$ is epiconjugate to e if $D\exp^\mu_e(t^* v_o)$ fails to be surjective. If M^2 has positive curvature and enough symmetry then conjugate points can be located along those geodesics which are contained in the isometry subgroup of $\mathcal{D}^s_\mu(M^2)$. It is also known that conjugate points exist in $\mathcal{D}^s_\mu(M^2)$ even if M^2 is flat; see Misiołek (1996) for the case where M^2 is the flat torus and Shnirelman (1994) for the ball in $\mathbb{R}^3$. It was shown in (Ebin et al. 2006) that the exponential mapping $\exp^\mu_e$ is a non-linear Fredholm map of index zero. In particular, conjugate points in $\mathcal{D}^s_\mu(M^2)$ are isolated, of finite multiplicity, and the two types of conjugacies coincide.

Let $\eta(t)$ be a geodesic with velocity field $v(t) \in T_e\mathcal{D}^s_\mu$, $v(0) = v_o$. For each t, the operator $K_{v(t)}$ is compact and skew self-adjoint on $T_e\mathcal{D}^0_\mu$, by Proposition 3. According to the spectral theorem for compact (skew) self-adjoint operators there exists an orthonormal basis of $T_e\mathcal{D}^0_\mu$ consisting of eigenvectors $\{\psi_i(t)\}$ of $K_{v(t)}$, and

$$T_e\mathcal{D}^0_\mu = \ker K_{v(t)} \oplus \left(\ker K_{v(t)}\right)^\perp .$$

Proposition 6 *Suppose $\eta(t)$ is a geodesic of the L^2 metric in $\mathcal{D}^s_\mu(M^2)$, with initial velocity $v_o \in T_e\mathcal{D}^s_\mu$, and let $J(t) = y(t) \circ \eta(t)$ be the Jacobi field along $\eta(t)$ with initial conditions $J(0) = 0$, $J'(0) = z_o \in T_e\mathcal{D}^s_\mu$. Furthermore, let $z(t)$ be the solution to the linearized Euler equations (14) with initial condition $z_o \in T_e\mathcal{D}^s_\mu$. Then,*

1. *If $z_o \in \left(T_e\mathcal{D}^s_\mu\right) \cap \ker K_{v_o}$ then $y(t) \in \left(T_e\mathcal{D}^s_\mu\right) \cap \ker K_{v_o}$ for all t and is never zero for $t > 0$. Moreover,*

$$z(t) = \mathrm{Ad}^*_{\eta^{-1}(t)} z_o, \tag{21}$$

$$y(t) = \mathrm{Ad}_{\eta(t)} \int_0^t \mathrm{Ad}_{\eta^{-1}(s)} \mathrm{Ad}^*_{\eta^{-1}(s)} z_o \, ds. \tag{22}$$

2. *If* $z_o \in \left(T_e\mathcal{D}^s_\mu\right) \cap \left(\ker K_{v_o}\right)^\perp$ *then* $y(t) \in \left(T_e\mathcal{D}^s_\mu\right) \cap \left(\ker K_{v_o}\right)^\perp$ *for all t and*

$$y(t) = \sum_i \frac{1}{\lambda_i(t)}\left(a_i(t) - g_i(t)\right)\psi_i(t),$$

where $\mathrm{Ad}^*_{\eta^{-1}(t)} z_o = \sum_i a_i(t)\psi_i(t)$ *and* $z(t) = \sum_i g_i(t)\psi_i(t)$.

Before proving Proposition 6 we state and prove the main result of this section.

Theorem 7 *Let* $\eta(t)$, $z(t)$ *and* z_o *be as in Proposition 6. Then* $\eta(t^*)$, $t^* > 0$ *is conjugate to the identity if and only if*

$$z(t^*) = \mathrm{Ad}^*_{\eta^{-1}(t^*)} z_o.$$

Proof The L^2 exponential map is a non-linear Fredholm map of index zero so that $\eta(t^*)$, $t^* > 0$, is conjugate to the identity if and only if there exists a Jacobi field along $\eta(t)$ vanishing at $t = 0$ and at $t = t^*$. This fact, in combination with Propositions 6–2. yields the statement of Theorem 7. □

The proof of Proposition 6 will be broken up into a series of Lemmas, all of which amount to keeping track of how the operators $\mathrm{Ad}_{\eta(t)}$ and $\mathrm{Ad}^*_{\eta(t)}$ act on the subspace $\ker K_{v_o}$. Let $\{\psi_i(t)\}$ be the time-dependent orthonormal basis of eigenvectors corresponding to $K_{v(t)}$ and spanning $T_e\mathcal{D}^0_{\mu,ex}$. Given a curve $x(t)$ in $T_e\mathcal{D}^0_{\mu,ex}$ we shall write $x^{\ker(t)}(t)$ for the part of $x(t)$ which lies in $\ker K_{v(t)}$, and $x^{\perp(t)} = x(t) - x^{\ker(t)}(t)$.

Lemma 8 *For each t we have*

$$\ker K_{v(t)} = \mathrm{Ad}_{\eta(t)} \ker K_{v_o}, \quad \left(\ker K_{v(t)}\right)^\perp = \mathrm{Ad}_{\eta(t)}\left(\ker K_{v_o}\right)^\perp \tag{23}$$

$$\ker K_{v(t)} = \mathrm{Ad}^*_{\eta^{-1}(t)} \ker K_{v_o} \quad \left(\ker K_{v(t)}\right)^\perp = \mathrm{Ad}^*_{\eta^{-1}(t)}\left(\ker K_{v_o}\right)^\perp. \tag{24}$$

Proof The first relation (23) follows directly from Proposition 2. To see the second, let w be any vector in $\ker K_{v_o}$ and u be any vector in $\left(\ker K_{v(t)}\right)^\perp$. Then, according to (23)

$$0 = \left(u, \mathrm{Ad}_{\eta(t)} w\right)_{L^2} = \left(\mathrm{Ad}^*_{\eta(t)} u, w\right)_{L^2}$$

so that $\mathrm{Ad}^*_{\eta(t)} u$ is orthogonal to every vector in $\ker K_{v_o}$. □

Lemma 9 *Let* $z(t)$ *be the solution of* (14) *with* $z(0) = z_o$. *Then*

$$z^{\ker(t)}(t) = \left(\mathrm{Ad}^*_{\eta^{-1}(t)} z_o\right)^{\ker(t)}. \tag{25}$$

Proof Expand $z(t)$ and $\mathrm{Ad}^*_{\eta^{-1}(t)} z_o$ in the orthonormal basis of eigenvectors provided by $K_{v(t)}$:

$$z(t) = \sum_i g_i(t)\psi_i(t), \quad \mathrm{Ad}^*_{\eta^{-1}(t)} z_o = \sum_{i=1} a_i(t)\psi_i(t).$$

Let $J(t) = y(t) \circ \eta(t)$ be the Jacobi field along $\eta(t)$, with initial conditions $J(0) = 0$, $\dot{J}(0) = z_o$, and expand $y(t)$ as

$$y(t) = \sum_{i=1} h_i(t)\psi_i(t).$$

By Proposition 4

$$\sum_{i=1} a_i(t)\psi_i(t) = \sum_{i=1} g_i(t)\psi_i(t) + \sum_{i=1} \lambda_i(t) h_i(t)\psi_i(t).$$

For those $\lambda_j(t) = 0$, at any time t, we have

$$a_j(t)\psi_j(t) = g_j(t)\psi_j(t)$$

and the relation (25) follows. □

The solution operator $\Phi(t)$ to the Jacobi equation (11) is a non-linear Fredholm operator of index zero on $T_e\mathcal{D}^s_{\mu,ex}$, cf. (Ebin et al. 2006). Any Fredholm operator decomposes as the sum of an invertible operator and a compact operator. The operator

$$\Omega(t) : T_e\mathcal{D}^s_\mu \to T_e\mathcal{D}^s_\mu$$
$$z_o \mapsto \int_0^t \mathrm{Ad}_{\eta^{-1}(s)} \mathrm{Ad}^*_{\eta^{-1}(s)} z_o \, ds,$$

is the invertible part of $\Phi(t)$ on $T_e\mathcal{D}^s_\mu$ and we will make use of this in the following.

Lemma 10 *For each t we have*

$$\Omega(t) \ker K_{v_o} = \ker K_{v_o} \tag{26}$$
$$\Omega(t) \left(\ker K_{v_o}\right)^\perp = \left(\ker K_{v_o}\right)^\perp . \tag{27}$$

Proof Choose any $w \in \ker K_{v_o}$ and write it as a linear combination of the eigenvectors spanning $\ker K_{v_o}$. Then, by Lemma 8, $\mathrm{Ad}_{\eta^{-1}(s)} \mathrm{Ad}^*_{\eta^{-1}(s)} w$ lies in $\ker K_{v_o}$. Writing $\mathrm{Ad}_{\eta^{-1}(s)} \mathrm{Ad}^*_{\eta^{-1}(s)} w$ as a linear combination of the eigenvectors $\psi_i(0)$ spanning $\ker K_{v_o}$, with coefficients $x_j(t)$, we see that $\Omega(t)w \in \ker K_{v_o}$. Since $\Omega(t)$ is invertible on $T_e\mathcal{D}^s_\mu$ we obtain (26) and (27). □

Proof of Proposition 6: Recall that

$$\partial_t \mathrm{Ad}_{\eta^{-1}(t)} = -\mathrm{Ad}_{\eta^{-1}(t)} \mathrm{ad}_{v(t)}$$

so we can rewrite Eq. (13) as

$$\partial_t \left(\mathrm{Ad}_{\eta^{-1}(t)} y(t) \right) = \mathrm{Ad}_{\eta^{-1}(t)} z(t),$$

whose solution is given by

$$y(t) = \mathrm{Ad}_{\eta(t)} \int_0^t \mathrm{Ad}_{\eta^{-1}(s)} z(s)\, ds.$$

Writing $z(s) = z^{\ker(s)}(s) + z^{\perp(s)}(s)$, we have

$$y(t) = \mathrm{Ad}_{\eta(t)} \int_0^t \mathrm{Ad}_{\eta^{-1}(s)} z^{\ker(s)}(s)\, ds + \mathrm{Ad}_{\eta(t)} \int_0^t \mathrm{Ad}_{\eta^{-1}(s)} z^{\perp(s)}(s)\, ds. \tag{28}$$

By Lemma 8

$$y^{\ker(t)}(t) = \mathrm{Ad}_{\eta(t)} \int_0^t \mathrm{Ad}_{\eta^{-1}(s)} z^{\ker(s)}(s)\, ds,$$
$$y^{\perp(t)}(t) = \mathrm{Ad}_{\eta(t)} \int_0^t \mathrm{Ad}_{\eta^{-1}(s)} z^{\perp(s)}(s)\, ds.$$

In view of Lemma 9 we have

$$y^{\ker(t)}(t) = \mathrm{Ad}_{\eta(t)} \int_0^t \mathrm{Ad}_{\eta^{-1}(s)} \left(\mathrm{Ad}^*_{\eta^{-1}(t)} z_o \right)^{\ker(s)} ds,$$

which, using Lemma 8 once more, becomes

$$y^{\ker(t)}(t) = \mathrm{Ad}_{\eta(t)} \left(\int_0^t \mathrm{Ad}_{\eta^{-1}(s)} \mathrm{Ad}^*_{\eta^{-1}(t)} z_o\, ds \right)^{\ker(0)}.$$

Finally, using the definition of $\Omega(t)$ and Lemma 10 we obtain

$$\begin{aligned} y(t)^{\ker(t)} &= \mathrm{Ad}_{\eta(t)} \left(\Omega(t) z_o \right)^{\ker(0)} \\ &= \mathrm{Ad}_{\eta(t)} \left(\Omega(t) z^{\ker(0)} + \Omega(t) z^{\perp(0)} \right)^{\ker(0)} \\ &= \mathrm{Ad}_{\eta(t)} \Omega(t) z^{\ker(0)}. \end{aligned}$$

Therefore, if $z_o \in \ker K_{v_o}$ then

$$y(t) = \mathrm{Ad}_{\eta(t)} \Omega(t) z_o$$

is never zero for $t > 0$ and $y(t) \in \ker K_{v_o}$ for all t. Since the Jacobi equation is well posed on $T_e\mathcal{D}^s_\mu$ it follows that $y(t) \in T_e\mathcal{D}^s_\mu \cap \ker K_{v_o}$, proving 1.

If $z_o \in \left(\ker K_{v_o}\right)^\perp$ then $y(t)^{\ker(t)} = 0$ for all t and $y(t) \in T_e\mathcal{D}^s_\mu \cap \left(\ker K_{v(t)}\right)^\perp$. Expanding as a sum of eigenvectors of $K_{v(t)}$: $z(t) = \sum_i g_i(t)\psi_i(t)$, $\mathrm{Ad}^*_{\eta^{-1}(t)} z_o = \sum a_i(t)\psi_i(t)$, $y(t) = \sum_i h_i(t)\psi_i(t)$ and applying Proposition 4 we have for each i,

$$g_i(t) = a_i(t) - \lambda_i(t)h_i(t).$$

Since each $\lambda_i(t)$ is non-zero,

$$y(t) = \sum_i \frac{1}{\lambda_i(t)} \left(a_i(t) - g_i(t)\right) \psi_i(t),$$

giving (22).

5 Eulerian and Lagrangian Stability

As in Sect. 4 we will consider a closed surface M^2. We turn our attention to the stability of a stationary solution v (and its corresponding stationary geodesic $\eta(t)$ in $\mathcal{D}^s_\mu(M^2)$) to the Euler equations (2).

Let $k \geq 0$. A geodesic $\eta \in \mathcal{D}^s_\mu(M^2)$ is said to be Lagrangian H^k (linearly) stable if every solution of the Jacobi equation (11) along η, remains bounded in the Sobolev H^k norm. Intuitively, a geodesic is Lagrangian stable if configurations of particles which are initially nearby remain nearby.

We also have the notion of Eulerian stability, in which the initial velocity of a solution to the Euler equations (2) is perturbed. A stationary solution v to the Euler equations is said to be Eulerian H^k (linearly) stable if every solution of the linearized Euler equations (14) remains bounded in the Sobolev H^k norm. Intuitively, a solution is Eulerian stable if nearby velocity fields remain nearby.

In Preston (2004) it was shown that if a solution to the linearized Euler equations (14) (at v) remains bounded in time then the corresponding Jacobi field along $\eta(t)$ grows at most quadratically in time. Also, if v is analytic with isolated non-degenerate fixed points, then v is at most Lagrangian polynomial unstable if it is at most Eulerian polynomial unstable. We show more; that is, a solution to the linearized Euler equation remains bounded in time if and only if the corresponding Jacobi field remains bounded in time, and if a Jacobi field is unbounded in time then the corresponding solution to the linearized Euler equation is also unbounded and grows at the same rate. This holds for any stationary H^s solution to the Euler equations and proves that the notions of Eulerian and Lagrangian stabilities of the Euler equations (2) are equivalent, which completes the results of Preston (2004).

Recall that the set of divergence free vector fields is given by

$$T_e\mathcal{D}^s_\mu = g^\sharp\left(d\delta H^{s+2}(T^*M^2)\right) \oplus \mathcal{H},$$

where the second summand is the finite dimensional space of harmonic vector fields and the first the set of rotated gradients; that is, vector fields of the form $v_F = \nabla^{\perp} F$ for some H^{s+1} function F with zero mean on M^2. $T_e\mathcal{D}^s_{\mu,ex} := g^{\sharp}\left(d\delta H^{s+2}(T^*M^2)\right)$ is known as the set of exact divergence free vector fields and is a closed subspace of finite codimension in $T_e\mathcal{D}^s_{\mu}$. The Lie bracket of any two exact divergence free vector fields is again an exact divergence free vector field and therefore we can restrict the adjoint and coadjoint operators to this space.

Following Smolentsev (1986), define another inner product on $T_e\mathcal{D}^s_{\mu,ex}$ by

$$\langle v_F, v_H \rangle_S = \int_M F H d\mu. \tag{29}$$

The weak L^2 metric (1) on vector fields is related to the inner product (29) by

$$(v_F, v_H)_{L^2} = \left\langle v_{-\triangle F}, v_H \right\rangle_S, \tag{30}$$

where $-\triangle$ is the Hodge Laplacian.

Lemma 11 *For a closed surface M^2 the group adjoint and coadjoint operators on $T_e\mathcal{D}^{\sigma}_{\mu,ex}(M^2)$ are given by*

$$\mathrm{Ad}_{\eta} v_F = v_{F\circ\eta^{-1}} \tag{31}$$

$$\mathrm{Ad}^*_{\eta} v_F = v_{\triangle^{-1}(\triangle F\circ\eta)}, \tag{32}$$

where $\eta \in \mathcal{D}^s_{\mu,ex}$ and $v_F \in T_e\mathcal{D}^{\sigma}_{\mu,ex}$.

Proof Formula (31) is well known (see Arnold and Khesin 1998) so we proceed to compute formula (32). Let v_F and v_G be in $T_e\mathcal{D}^{\sigma}_{\mu,ex}$ and compute

$$\begin{aligned}\left(\mathrm{Ad}^*_{\eta} v_F, v_G\right)_{L^2} &= \left(v_F, \mathrm{Ad}_{\eta} v_G\right)_{L^2} = \left(v_F, v_{G\circ\eta^{-1}}\right)_{L^2} \\ &= -\int_M (\triangle F)\left(G\circ\eta^{-1}\right) d\mu = -\int_M (\triangle F\circ\eta)\, G\, d\mu \\ &= \left(v_{\triangle^{-1}(\triangle F\circ\eta)}, v_G\right)_{L^2}.\end{aligned}$$

As this holds for any v_G in $T_e\mathcal{D}^{\sigma}_{\mu,ex}$ formula (32) follows. □

Lemma 12 *Let M^2 be a closed surface and $\eta(t)$ a geodesic in $\mathcal{D}^s_{\mu}$ with velocity v_o. For any vector u in $T_e\mathcal{D}^{\sigma}_{\mu}(M^2)$, $\sigma \geq 1$, we have*

$$\left\| \mathrm{Ad}^*_{\eta^{-1}(t)} u \right\|_{L^2} \leq C, \tag{33}$$

for some finite constant $C > 0$.

Proof The vector u decomposes as $u = v_G + h$ where $v_G \in T_e\mathcal{D}^s_{\mu,ex}$ and $h \in \mathcal{H}$. Using formula (32) for the coadjoint operator and the relation (30), we estimate

$$\begin{aligned}\left\|\mathrm{Ad}^*_{\eta^{-1}(t)} v_G\right\|^2_{L^2} &= \left\|v_{\triangle^{-1}((\triangle G)\circ\eta^{-1}(t))}\right\|^2_{L^2}\\ &= \left\langle(\triangle G)\circ\eta^{-1}(t), \triangle^{-1}\left((\triangle G)\circ\eta^{-1}(t)\right)\right\rangle_S\\ &\lesssim \|\triangle G\|^2_S .\end{aligned}$$

Since v_G is in H^σ, $\sigma \geq 1$, the function G is at least H^2 and therefore

$$\left\|\mathrm{Ad}^*_{\eta^{-1}(t)} v_G\right\|^2_{L^2} \leq c, \tag{34}$$

for some finite non-zero constant c. Also, $\mathrm{ad}^*_{v_o} h = 0$ so that $\mathrm{Ad}^*_{\eta^{-1}(t)} h = h$ for all t. The desired estimate now follows. □

Theorem 13 *Let M^2 be a closed surface and $\eta(t)$ a stationary geodesic of the L^2 metric in $\mathcal{D}^s_\mu(M^2)$, $s > 2$, with velocity field $v_o = \dot\eta(t)\circ\eta^{-1}(t) \in T_e\mathcal{D}^s_\mu$ solving the Euler equations (2) on M^2. Let $z(t)$ be the solution to the linearized Euler equations (14) at v_o and $J(t) = y(t)\circ\eta(t)$ the Jacobi field along $\eta(t)$ satisfying $J(0) = 0$, $\dot J(0) = z_o$.*

1. *If $z(0) = z_o \in \left(T_e\mathcal{D}^s_\mu\right) \cap \left(\ker K_{v_{F_o}}\right)^\perp$, then*

$$\|z(t)\|_{L^2} = \mathcal{O}(\|J(t)\|_{L^2}) \tag{35}$$
$$\|J(t)\|_{L^2} = \mathcal{O}(\|z(t)\|_{L^2}), \tag{36}$$

   as $t \to \infty$.
2. *If $z_o \in \left(T_e\mathcal{D}^s_\mu\right) \cap \ker K_{v_{F_o}}$ then*

$$z(t) = \mathrm{Ad}^*_{\eta^{-1}(t)} z_o$$

   which remains bounded in the L^2 norm and

$$J(t) = D\eta(t)\cdot\int_0^t \mathrm{Ad}_{\eta^{-1}(s)}\mathrm{Ad}^*_{\eta^{-1}(s)} z_o\, ds.$$

Proof We first prove (35). Let $z(t)$ and $J(t)$ be as in the Theorem. Using Proposition 4 we estimate

$$\begin{aligned}\|J(t)\|_{L^2} &\geq C_1\left\|K_{v_o} y(t)\right\|_{L^2}\\ &\geq C_1\|z(t)\|_{L^2} - \left\|\mathrm{Ad}^*_{\eta^{-1}(t)} z_o\right\|_{L^2} .\end{aligned}$$

Applying estimate (33) gives

$$\|z(t)\|_{L^2} \leq C\left(\|J(t)\|_{L^2}+1\right) \tag{37}$$

for some positive non-zero constant C. Thus, as $t \to \infty$, $\|z(t)\|_{L^2} = \mathcal{O}(\|J(t)\|_{L^2})$.
To prove (36) we estimate using Proposition 4 and Lemma 12 once more,

$$\begin{aligned}\left\|K_{v_o} y(t)\right\|_{L^2} &\leq \|z(t)\|_{L^2} + \left\|\mathrm{Ad}^*_{\eta^{-1}(t)} z_o\right\|_{L^2}\\ &\leq \|z(t)\|_{L^2} + C.\end{aligned}$$

Thus, as $t \to \infty$, $\left\|K_{v_o} y(t)\right\|_{L^2} = \mathcal{O}(\|z(t)\|_{L^2})$. Observe that

$$\begin{aligned}\left\|K_{v_o} y(t)\right\|_{L^2} = \left\|\sum_i \lambda_i h_i(t)\psi_i\right\|_{L^2} &= \sum_i |\lambda_i|^2\,|h_i(t)|^2\\ \|J(t)\|_{L^2} &= \sum_i |h_i(t)|^2,\end{aligned}$$

where $y(t) = \sum_i h_i(t)\psi_i$ and $\{\psi_i\}$ is the orthonormal basis of eigenvectors of K_{v_o} with each eigenvalue λ_i non-zero. The former norm is a re-weighting of the latter by positive non-zero constants. That is, for large t $\|J(t)\|_{L^2} \sim \left\|\sum_i \lambda_i h_i(t) v_{\psi_i}\right\|_{L^2}$, and hence $\|J(t)\|_{L^2} = \mathcal{O}(\|z(t)\|_{L^2})$ as $t \to \infty$.
The second statement of the Theorem follows from Propositions 6–1. and the estimate (33). □

We now present two simple examples illustrating Theorem 13.

Example 1 Let M be the flat 2-Torus and define $F(x, y) = \sin(x)$ on M. The function F is an eigenfunction of the Laplacian and hence generates a stationary solution to the Euler equations. The corresponding geodesic in $\mathcal{D}^s_\mu$ is given by

$$\eta(t)(x, y) = (x, y + t\cos(x)).$$

Let $G(x, y) = \cos(y)$ with corresponding divergence free vector field $v_G \in \left(\ker K_{v_F}\right)^\perp$. Then, v_G defines a time-independent solution to the linearized Euler equations and the corresponding Jacobi field is given by

$$J(t) = (-\sec(x)\cos(y + t\cos(x)), \sec(x)\tan(x)\sin(y + t\cos(x))).$$

Both v_G and $J(t)$ remain bounded in the L^2 norm for all time.

Example 2 Let M, F, and $\eta(t)$ be as in the above example. Consider the one-parameter variation of η given by

$$\zeta(t, s)(x, y) = \gamma(s) \circ \eta(t) \circ \gamma^{-1}(s)(x, y) = (x, y + t\cos(x - s)),$$

where

$$\gamma(s)(x, y) = (x + s, y).$$

Then $\zeta(t, 0) = \eta(t)$, $\zeta(0, s)(x, y) = (x, y)$, and we will show that $\zeta(t, s)$ is a family of geodesics in $\mathcal{D}^s_\mu$ for each s. The vector field

$$w(t, s)(x, y) = \partial_t \zeta(t, s) \circ \zeta^{-1}(s, t)(x, y) = (0, \cos(x - s))$$

is independent of t and since $\nabla_w w = 0$ it defines a stationary solution to the Euler equations for each s. The Jacobi field and solution to the linearized Euler equation corresponding to this variation are given by

$$J(t) = \partial_s|_{s=0}\zeta(t, s) = (0, t\sin(x)) \in \left(\ker K_{v_F}\right)^\perp,$$

$$z(t) = \partial_s|_{s=0} w(t, s) = (0, \sin(x)) \in \left(\ker K_{v_F}\right)^\perp,$$

which differ in growth by a linear factor of t. To see that this agrees with the formula given in 2. of the Theorem, observe that $z(t) = v_G$ with $G(x, y) = -\cos(x)$, and G is an eigenfunction of the laplacian with the property $G \circ \eta(t) = G$. Therefore v_G is invariant under the adjoint and coadjoint action and

$$J(t) = \int_0^t v_G \, dt = t v_G.$$

Remark 14 Combining Theorem 13 with Theorem 5 we observe an interesting distinction between the stability of the 2D Euler equations and 3D Euler equations. Let $\eta(t)$ be a stationary geodesic in $\mathcal{D}^s_{\mu,ex}(M^2)$, $s > 2$, with velocity field $v_o = \dot{\eta} \circ \eta^{-1} \in T_e\mathcal{D}^s_{\mu,ex}$ solving the Euler equations (2), and denote by U the unit sphere in $T_e\mathcal{D}^2_{\mu,ex} \cap \left(\ker K_{v_o}\right)^\perp$. Consider the solution set $S(t)U$ to the linearized Euler equations (14). By Theorem 5 this solution set can be described by the set $\mathrm{Ad}^*_{\eta^{-1}(t)} U$ plus $\kappa_t = K_{v_o} dR_{\eta^{-1}(t)} \Phi(t) U$ which is a compact set for each t since $K_{v_o} dR_{\eta^{-1}(t)} \Phi(t)$ is compact for each t. The set $\mathrm{Ad}^*_{\eta^{-1}(t)} U$ is bounded in the L^2 norm for all t by estimate (33). Since Eulerian and Lagrangian stabilities are equivalent, the L^2 growth of the solution set $S(t)U$ is determined by the compact set κ_t and the instabilities present in the 2D Euler equations are small in this sense.

6 Concluding Remarks

Let M be a closed symplectic manifold of dimension $2n$. It has been shown by Ebin in (2012) that the group of Sobolev symplectic diffeomorphisms $\mathcal{D}^s_\omega(M)$, equipped with the L^2 metric (1), is geodesically complete and the corresponding symplectic Euler equations on M are globally well-posed for $s > \frac{\dim M}{2} + 1$. The subgroup of Hamiltonian diffeomorphisms plays a role in plasma dynamics analogous to the role played by the volume preserving diffeomorphism group in incompressible hydrodynamics, cf. Arnold and Khesin (1998). It has also been shown in Benn (2016) that 3 remains true for any closed Symplectic manifold of dimension $2n$ and the exponential map of the L^2 metric on $\mathcal{D}^s_\omega(M)$ is a non-linear Fredholm map of index zero. Consequenty, all results derived in this paper hold true for the symplectomorphism group $\mathcal{D}^s_\omega(M)$. Theorem 7 also provides an interesting contrast to the Hofer metric which is a bi-invariant metric on the group of symplectomorphisms whose conjugate points do not have such a nice description, see Ustilovsky (1996).

References

Arnold, V. I.: Sur la geometrie differentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluids parfait. Ann. Inst. Grenoble 16 (1966)

Arnold, V.I., Khesin, B.A.: Topological methods in hydrodynamics, Springer-Verlag (1998)

Benn, J.: Fredholm Properties of the L^2 Exponential Map on the Symplectomorphism Group. J. Geom. Mech. **8**(1) (2016)

Ebin, D., Marsden, J.: Groups of diffeomorphisms and the motion of an incompressible fluid. Ann. Math. 92 (1970)

Ebin, D., Misiolek, G., Preston, S.: Singularities of the exponential map on the volume-preserving diffeomorphism group. Geom. Funct. Anal. **16** (2006)

Ebin, D.G.: Geodesics on the symplectomorphism group. Geom. Funct. Anal. **22**, 202–212 (2012)

Khesin, B., Wendt, R.: The geometry of infinite dimensional groups. Springer-Verlag (2009)

Misiołek, G.: Stability of flows of ideal fluids and the geometry of the groups of diffeomorphisms. Ind. Uni. Math. J. **42**(1), 215–235 (1993)

Misiołek, G.: Conjugate points in $\mathcal{D}_\mu(T^2)$. Proc. Amer. Math. Soc. **124**, 977–982 (1996)

Morrey, C. B.: Multiple integrals in the calculus of variations. Springer-Verlag (1966)

Preston, S.: For ideal fluids, Eulerian and Lagrangian instabilities are equivalent. Geom. Funct. Anal. **14** (2004)

Smolentsev, N.K.: A biinvariant metric on the group of symplectic diffeomorphisms and the equation $\partial_t \triangle F = \{\triangle F, F\}$. Trans. Sibirskii Matematicheskii Shurnal **27**(1), 150–156 (1986)

Shnirelman, A.: Generalized fluid flows, their approximation and applications, GAFA. Geom. Funct. Anal. **4**, 586–620 (1994)

Ustilovsky, I.: Conjugate points on geodesics of Hofer's metric. Diff. Geometry Appl. **6**, 327–342 (1996)

Wolibner, W.: Un Theoreme sur l'existence du mouvement plan d'un fluide parfait, homogene, incompressible, pendant un temps infiniment longue. Math. Z. **37**, 698–726 (1933)