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RESEARCH EXPOSITION

Proof of the Broué–Malle–Rouquier Conjecture in Characteristic Zero (After I. Losev and I. Marin—G. Pfeiffer)

Pavel Etingof[1]

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Abstract We explain a proof of the Broué–Malle–Rouquier conjecture on Hecke algebras of complex reflection groups, stating that the Hecke algebra of a finite complex reflection group W is free of rank $|W|$ over the algebra of parameters, over a field of characteristic zero. This is based on previous work of Losev, Marin– Pfeiffer, and Rains and the author.

Keywords Hecke algebra · Complex reflection group · Broué–Malle–Rouquier conjecture

The goal of this note is to explain a proof of the Broué–Malle–Rouquier conjecture (Broué et al. 1998, p. 178), stating that the Hecke algebra of a finite complex reflection group W is free of rank $|W|$ over the algebra of parameters, over a field of characteristic zero. This result is not original—it follows immediately from the results of Losev (2015), Marin and Pfeiffer (2017), and Etingof and Rains (2006), but it does not seem to have been stated explicitly in the literature, so we state and prove it for future reference.We note that there have been a lot of results on this conjecture for particular complex reflection groups, reviewed in Marin (2015), e.g. Ariki (1995), Ariki and Koike (1994), Marin (2012, 2014); we are not giving the full list of references here.

1 The Main Result

Let V be a finite dimensional complex vector space, and $W \subset GL(V)$ a finite complex reflection group, i.e., W is generated by complex reflections (elements s such that $\mathrm{rank}(1-s) = 1$). Let $S \subset W$ be the set of reflections, and $V_{\mathrm{reg}} := V \setminus \cup_{s\in S} V^s$. Then by Steinberg's theorem, W acts freely on V_{reg}. Let $x \in V_{\mathrm{reg}}/W$ be a base point. The *braid group* of W is the group $B_W := \pi_1(V_{\mathrm{reg}}/W, x)$. We have a surjective homomorphism

✉ Pavel Etingof
etingof@math.mit.edu

[1] Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

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$\pi : B_W \to W$ (corresponding to gluing back the reflection hyperplanes V^s), and $\mathrm{Ker}\pi$ is called the *pure braid group* of W, denoted by PB_W. For each $s \in S$, let $T_s \in B_W$ be a path homotopic to a small circle around V^s (it is defined uniquely up to conjugation). Also let n_s be the order of s. Then $T_s^{n_s} \in PB_W$, and by the Seifert–van Kampen theorem, PB_W is the normal closure of the subgroup of B_W generated by $T_s^{n_s}$, $s \in S$. In other words, W is the quotient of B_W by the relations $T_s^{n_s} = 1, s \in S$.

Let $u_{s,i}$, $i = 1, \ldots, n_s$, be variables such that $u_{s,i} = u_{t,i}$ if s is conjugate to t in W. Let $R := \mathbb{Z}[u_{s,i}^{\pm 1}, s \in S, i \in [1, n_s]]$.

Definition 1.1 (Broué et al. 1998) The Hecke algebra $H(W)$ is the quotient of the group algebra RB_W by the relations

$$\prod_{i=1}^{n_s}(T_s - u_{s,i}) = 0, \quad s \in S.$$

Conjecture 1.2 (Broué et al. 1998, p. 178) *$H(W)$ is a free R-module of rank $|W|$.*

This conjecture is currently known for all irreducible complex reflection groups except $G_{17}, \ldots, G_{21}$ (according to the Shephard–Todd classification), and there is a hope that these cases can be proved as well using a sufficiently powerful computer (see Chavli 2016a, b; Marin 2015 for more details). Also, it is shown in Broué et al. (1998) that to prove the conjecture, it suffices to show that $H(W)$ is spanned by $|W|$ elements.

Our main result is

Theorem 1.3 *If K is a field of characteristic zero then $K \otimes_{\mathbb{Z}} H(W)$ is a free module over $K \otimes_{\mathbb{Z}} R$ of rank $|W|$. In particular, if $q : R \to K$ is a homomorphism, then the specialization $H_q(W) := K \otimes_R H(W)$ is a $|W|$-dimensional K-algebra.*

Remark 1.4 Theorem 1.3 is useful in many situations, for instance in the representation theory of rational Cherednik algebras, where a number of results were conditional on its validity for W; see e.g. Ginzburg et al. (2003), 5.4, or Shan (2011), Section 2. Also, Theorem 1.3 implies a positive answer to a question by Deligne and Mostow (1993, (17.20), Question 3), which served as one of the motivations in Broué et al. (1998) (see Broué et al. 1998, p. 127).

2 Proof of Theorem 1.3

First assume that $K = \mathbb{C}$. It also suffices to assume that W is irreducible. In this case, possible groups W are classified by Shephard and Todd (1954). Namely, W belongs to an infinite series, or W is one of the exceptional groups G_n, $4 \leq n \leq 37$. Among these, G_n with $4 \leq n \leq 22$ are rank 2 groups, while G_n for $n \geq 23$ are of rank ≥ 3.

The case of the infinite series of groups is well known, see Ariki (1995), Ariki and Koike (1994), Broué et al. (1998). So it suffices to focus on the exceptional groups. Among these, the result is well known for Coxeter groups, which are $G_{23} = H_3$, $G_{28} = F_4$, $G_{30} = H_4$, $G_{35} = E_6$, $G_{36} = E_7$, $G_{37} = E_8$.

For the groups G_n for $n = 24, 25, 26, 27, 29, 31, 32, 33, 34$, the result was established in Marin and Pfeiffer (2017) and references therein, see Marin (2015), Subsection 4.1. Thus, Theorem 1.3 is known (in fact, over any coefficient ring) for all W except those of rank 2.

In the rank 2 case, the following weak version of Theorem 1.3 was established.

Theorem 2.1 (Etingof and Rains 2006, Theorem 6.1) *If $W = G_n$, $4 \leq n \leq 22$, then $\mathbb{C} \otimes_{\mathbb{Z}} H(W)$ is a finitely generated $\mathbb{C} \otimes_{\mathbb{Z}} R$-module. In particular, every specialization $H_q(W)$ is finite dimensional.*

Theorem 1.3 now follows from Theorem 2.1 and the following theorem due to I. Losev.

Theorem 2.2 (Losev 2015, Theorem 1.1) *For any W and any $q : R \to \mathbb{C}$, there is a minimal two-sided ideal $I \subset H_q(W)$ such that $H_q(W)/I$ is finite dimensional. Moreover, we have $\dim H_q(W)/I = |W|$.*

Namely, Theorems 2.1 and 2.2 imply that for any character $q : R \to \mathbb{C}$, the specialization $H_q(W)$ has dimension $|W|$. This implies that for $K = \mathbb{C}$ the algebra $K \otimes_{\mathbb{Z}} H(W)$ is a projective $K \otimes_{\mathbb{Z}} R$-module of rank $|W|$ (Hartshorne 1977, Exercise 2.5.8(c)). Hence the same is true for any field K of characteristic zero. But by Swan's theorem (Lam 2006, Corollary 4.10), any finitely generated projective module over the algebra of Laurent polynomials over a field is free. Hence, the algebra $K \otimes_{\mathbb{Z}} H(W)$ is a free $K \otimes_{\mathbb{Z}} R$-module of rank $|W|$ (cf. also Marin 2014, Proposition 2.5). This proves Theorem 1.3.

Corollary 1 *Let $K = \mathbb{Z}[1/N]$ for $N \gg 0$. Then $K \otimes_{\mathbb{Z}} H(W)$ is a free $K \otimes_{\mathbb{Z}} R$-module of rank $|W|$. Hence the same holds when K is a field of sufficiently large positive characteristic.*

Proof Theorem 2.1 is valid (with the same proof) over any coefficient ring (see e.g. Marin 2014, Theorem 2.14), i.e., for any W, the algebra $H(W)$ is module-finite over R. Hence by Grothendieck's Generic Freeness Lemma (Eisenbud 1994, Theorem 14.4), there exists an integer $L > 0$ such that $H(W)[1/L]$ is a free $\mathbb{Z}[1/L]$-module.

Now let $v_1, \ldots, v_r$ be generators of $H(W)$ over R, and $e_i, \ldots, e_{|W|} \in H(R)$ be elements defining a basis of $\mathbb{Q} \otimes_{\mathbb{Z}} H(W)$ over $\mathbb{Q} \otimes_{\mathbb{Z}} R$ (they exist by Theorem 1.3). Then $v_i = \sum_j a_{ij} e_j$ for some $a_{ij} \in \mathbb{Q} \otimes_{\mathbb{Z}} R$. So for some integer $D > 0$ we have $Dv_i = \sum_j b_{ij} e_j$, with $b_{ij} \in R$. Since $H(W)[1/L]$ is a free $\mathbb{Z}[1/L]$-module, the same relation holds in $H(W)[1/L]$. Thus, for $N = LD$, $H(W)[1/N]$ is a free $R[1/N]$-module with basis $e_1, \ldots, e_{|W|}$. □

Remark 2.3 1. The proof of Theorem 1.3 does not extend to positive characteristic, since the proof of Theorem 2.2 uses complex analysis (the Riemann–Hilbert correspondence).

2. The last step of the proof of Theorem 1.3 (Swan's theorem) is really needed for purely aesthetic purposes, to establish the original formulation of the conjecture on the nose. As usual, for practical purposes it is normally sufficient to know only that the algebra $K \otimes_{\mathbb{Z}} H(W)$ is a *projective* $K \otimes_{\mathbb{Z}} R$-module. In fact, for

most applications, including the ones mentioned in Remark 1.4, already Losev's Theorem 2.2 is sufficient.

3. One would like to have a stronger version of Theorem 1.3, giving a set-theoretical splitting $W \to B_W$ of the homomorphism π whose image is a basis of the Hecke algebra. For instance, when W is a Coxeter group, then such a splitting is well known and is obtained by taking reduced expressions in the braid group. Such a version is currently available (over any base ring) for all irreducible complex reflection groups except $G_{17}, \dots, G_{21}$, see Marin (2015), Chavli (2016a, b).
4. Here is an outline of the proof of Theorem 2.2 given in Losev (2015). Let $q = e^{2\pi i c}$, and let $\boldsymbol{H}_c(W)$ be the rational Cherednik algebra of W with parameter c, Ginzburg et al. (2003). Let $M \in \mathcal{O}_c(W)$ be a module from the category $\mathcal{O}$ for this algebra. It is shown in Ginzburg et al. (2003) that the localization of M to the set $\mathfrak{h}^{\text{reg}}$ of regular points of the reflection representation $\mathfrak{h}$ of W is a vector bundle on $\mathfrak{h}^{\text{reg}}$ with a flat connection. So for every $x \in \mathfrak{h}^{\text{reg}}$ we get a monodromy representation of the braid group $\pi_1(\mathfrak{h}^{\text{reg}}/W)$ on the fiber M_x, which is shown in Ginzburg et al. (2003) to factor through $H_q(W)$. This representation is denoted by $KZ(M)$, and the functor $M \mapsto KZ(M)$ is called the Knizhnik–Zamolodchikov (KZ) functor. It is shown in Ginzburg et al. (2003) that the representation $KZ(M)$ of $H_q(W)$ factors through a certain quotient $H'_q(W)$ of $H_q(W)$ of dimension $|W|$. Thus, Theorem 2.2 is equivalent to the statement that every finite dimensional representation of $H_q(W)$ is of the form $KZ(M)$ for some M.
   To show this, let $\mathfrak{h}^{\text{sr}}$ be the complement of the intersections of pairs of distinct reflection hyperplanes in $\mathfrak{h}$. Take a finite dimensional representation V of $H_q(W)$, and let $N = N_V$ be the vector bundle with a flat connection with regular singularities on $\mathfrak{h}^{\text{reg}}$ corresponding to V under Deligne's multidimensional Riemann–Hilbert correspondence. One then extends N to a vector bundle $\widetilde{N}$ on $\mathfrak{h}^{\text{sr}}$ compatibly with the $\boldsymbol{H}_c(W)$-action. One then defines $M := \Gamma(\mathfrak{h}^{\text{sr}}, \widetilde{N})$ and shows that $M \in \mathcal{O}_c(W)$ and $KZ(M) = V$, as desired.
5. Here is an outline of the proof of Theorem 2.1 given in Etingof and Rains (2006). For the infinite series of complex reflection groups the result was proved in Broué et al. (1998). Thus, let $W \subset GL_2(\mathbb{C})$ be an exceptional complex reflection group of rank 2, of type $G_4, \dots, G_{22}$. Then the intersection of W with the scalars is a finite cyclic group generated by an element Z. This element defines a central element of $H_q(W)$, which we will also call Z. Let $W/\langle Z\rangle = G \subset PGL_2(\mathbb{C}) = SO_3(\mathbb{C})$. Then G is the group of even elements in a Coxeter group of type A_3, B_3, or H_3. Using the theory of length in these Coxeter groups, it is shown that $\mathbb{C} \otimes_{\mathbb{Z}} H(W)$ is generated by $|G|$ elements as a module over $\mathbb{C} \otimes_{\mathbb{Z}} R[Z, Z^{-1}]$. Moreover, taking the determinant of the braid relation of this algebra in its finite dimensional representations, we find that Z^d is an element of $\mathbb{C} \otimes_{\mathbb{Z}} R$ for some d. This implies that $\mathbb{C} \otimes_{\mathbb{Z}} H(W)$ is a finite rank module over $\mathbb{C} \otimes_{\mathbb{Z}} R$, as desired.
   We note that this argument works over an arbitrary base ring. A much more detailed description of this argument is given in Chavli (2016b).

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