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The Number π and a Summation by $SL(2, \mathbb{Z})$

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Abstract The sum (resp. the sum of squares) of the defects in the triangle inequalities for the area one lattice parallelograms in the first quadrant has a surprisingly simple expression.

Namely, let $f(a, b, c, d) = \sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} - \sqrt{(a + c)^2 + (b + d)^2}$. Then,

$$\sum f(a, b, c, d)^2 = 2 - \pi/2, \tag{Ж}$$

$$\sum f(a, b, c, d) = 2, \tag{ж}$$

where the sum runs by all $a, b, c, d \in \mathbb{Z}_{\geq 0}$ such that $ad - bc = 1$. We present a proof of these formulae and list several directions for the future studies.

Keywords pi · Special linear group · Summation · Tropical geometry · Lattice geometry

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1 History: Geometric Approach to π

> What good your beautiful proof on the
> transcendence of π: why investigate such
> problems, given that irrational numbers do not
> even exists?
>
> Apocryphally attributed to Leopold Kronecker by
> Ferdinand Lindemann

The digit computation of π probably is one of the oldest research directions in mathematics. Due to Archimedes, we may consider the inscribed and circumscribed equilateral polygons for the unit circle. Let q_n (resp., Q_n) be the perimeter of such an inscribed (resp., circumscribed) $3 \cdot 2^n$-gon. The sequences $\{q_n\}, \{Q_n\}$ obey the recurrence

$$Q_{n+1} = \frac{2q_n Q_n}{q_n + Q_n}, q_{n+1} = \sqrt{q_n Q_{n+1}}$$

and both converge to 2π. However this gives no closed formula.

A major breakthrough was made by Euler, Swiss-born (Basel) German–Russian mathematician. In his Saint-Petersburg Academy of Science talk (December 5, 1735) and, then, paper (Euler 1740), he calculated (*literally*) that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{1}$$

Euler's idea was to use the identity

$$1 - \frac{z^2}{6} + \cdots = \frac{\sin(z)}{z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right),$$

where the first equality is the Taylor series and the second equality is due to the fact that these two functions have the same set of zeroes. Equating the coefficients in z^2, we get (1). This reasoning was not justified until Weierstrass, but there appeared many other proofs. A nice exercise to get (1) is by considering the residues of $\frac{\cot(\pi z)}{z^2}$.

We would like to mention here a rather elementary geometric proof of (1) which is contained in Cauchy's notes, (Cauchy 1821).

Let $\alpha = \frac{\pi}{2m+1}$. Then $\sin(n\alpha) < n\alpha < \tan(n\alpha)$ for $n = 1, \ldots, m$, see Fig. 1. Therefore, $\cot^2(n\alpha) \leq \frac{1}{n^2\alpha^2} \leq \csc^2(n\alpha)$. Writing $\frac{\sin((2m+1)x)}{(\sin x)^{2m+1}}$ as a polynomial in $\cot x$ and using the fact that $\frac{\pi n}{2m+1}$ are the roots of this polynomial by Vieta's Theorem, we can find the sum of $\cot^2 \alpha$ and $\csc^2 \alpha$ for $\alpha = \frac{\pi n}{2m+1}, n = 1, \ldots, m$.

So, the above geometric consideration gives a two-sided estimate for $\frac{1}{\pi^2} \sum_{n=1}^{m} \frac{1}{n^2}$ whose both sides converge to $\frac{1}{6}$ as $m \to \infty$.
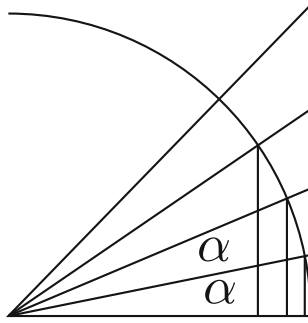
Fig. 1 An illustration for Cauchy's proof about the sum of reciprocals of the squares of natural numbers
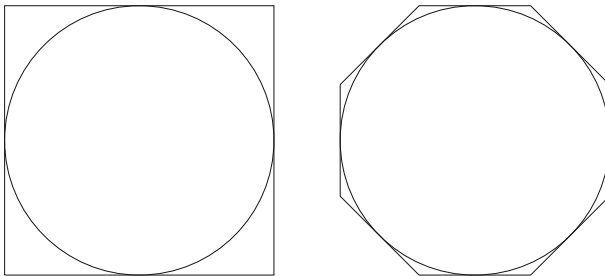

Fig. 2 The disc is inscribed into the square P_0. Then, P_1 is the only unimodular octagon circumscribing D^2, which can be obtained by corner cuts of P_0

2 $SL(2, \mathbb{Z})$-Way to Cut Corners

Recall that $SL(2, \mathbb{Z})$ is the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. With respect to the matrix multiplication, $SL(2, \mathbb{Z})$ is a group. We may identify such a matrix with the pair $(a, b), (c, d) \in \mathbb{Z}^2$ of lattice vectors. These vectors span the parallelogram of area one and, consequently, give a basis of $\mathbb{Z}^2$.

Definition 1 A vector $v \in \mathbb{Z}^2$ is *primitive* if its coordinates are coprime. A polygon $P \subset \mathbb{R}^2$ is called *unimodular* if

- the sides of P have rational slopes;
- two primitive vectors in the directions of every pair of adjacent sides of P give a basis of $\mathbb{Z}^2$.

Example 1 The polygons P_0 and P_1 in Fig. 2 are unimodular.

Let $P_0 = [-1, 1]^2$, and D^2 be the unit disk inscribed into P_0, Fig. 2, left. Cutting all corners of P_0 by tangent lines to D^2 in the directions $(\pm 1, \pm 1)$ results in the octagon P_1, into which D^2 is inscribed, Fig. 2, right.

Remark 1 Note that if we cut a corner of P_0 by any other tangent line to D^2, then the resulting pentagon would not be unimodular. Then, a polygon's property of being

unimodular is $SL(2, \mathbb{Z})$-invariant. This implies that if two sides of a corner are orthogonal to (a, b), $(c, d) \in \mathbb{Z}^2$ with $ad - bc = 1$, then the only direction to cut this corner in order to have two unimodular corners after this procedure is to use a line orthogonal to $(a + c, b + d)$.

Definition 2 For $n \geq 0$, the unimodular polygon P_{n+1} circumscribing D^2 is defined to be the result of cutting all 2^{n+2} corners of P_n by tangent lines to D^2 in such a way that P_{n+1} is a unimodular polygon.

Note that passing to P_{n+1} is unambiguous due to Remark 1.

Remark 2 $\lim_{n \to \infty} \text{Area}(P_n) = \pi$, $\lim_{n \to \infty} \text{Perimeter}(P_n) = 2\pi$.

Example 2 The primitive vector $(1, 1)$ is orthogonal to a side S of P_1, belongs to the positive quadrant, and goes outside P_1. Two vectors orthogonal to the neighboring to S sides of P_2 are $(2, 1)$ and $(1, 2)$.

Thus, we start with four vectors $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$—the outward directions for the sides of P_0. To pass from P_n to P_{n+1}, we order by angle all primitive vectors orthogonal to the sides of P_n. Then, for each two neighboring vectors, we cut the corresponding corner of P_n by the tangent line to D^2, orthogonal to their sum. In particular, every line with a rational slope tangent to D^2 contains a side of P_n for n large enough.

We can reformulate the above observation as follows:

Lemma 1 *For all $a, b, c, d \in \mathbb{Z}_{\geq 0}$ with $ad - bc = 1$, such that (a, b), (c, d) belong to the same quadrant, there is a corner of P_n for some $n \geq 0$ with sides orthogonal to (a, b) and (c, d). In P_{n+1}, this corner is cropped by the line orthogonal to $(a+c, b+d)$ and tangent to D^2.*

The following lemma can be proven by a direct computation.

Lemma 2 *In the above notation, the area of the cropped triangle is $\frac{1}{2} f(a, b, c, d)^2$.*

3 Proofs of (ЖК) and (ж)

The area of the intersection of $P_0 \backslash D^2$ with the first quadrant is $1 - \frac{\pi}{4}$. Therefore, Remark 2 and Lemma 2 prove (ЖК) in the abstract.

Definition 3 Let v be a primitive vector. We define the *lattice length* of a vector kv, $k \in \mathbb{R}_{\geq 0}$, to be k.

In other words, the length is normalized in each direction in such a way that all primitive vectors have length one. Note that the lattice length is $SL(2, \mathbb{Z})$-invariant.

The lattice perimeter of P_n is the sum of the lattice lengths of its sides. For example, the usual perimeter of the octagon P_1 is $16(\sqrt{2} - 1)$, and the lattice perimeter is $4\sqrt{2}$.

Lemma 3 *The lattice perimeter of P_n*
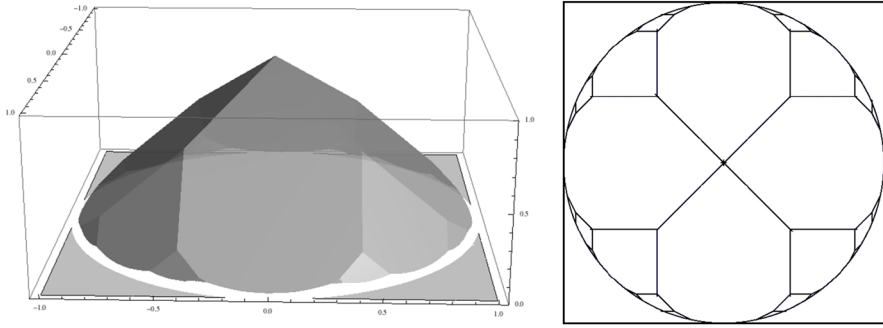
- *tends to zero as $n \to \infty$;*

Fig. 3 The plot of F and its corner locus (tropical analytic curve) C for a disc

- *is given by* $4\big(2 - \sum f(a, b, c, d)\big)$, *where the sum runs over* $a, b, c, d \in \mathbb{Z}_{\geq 0}$, $ad - bc = 1$, (a, b) *and* (c, d) *are orthogonal to a pair of neighboring sides of some* P_k *with* $k < n$.

Proof The second statement follows from the cropping procedure by a direct computation. To prove the first statement, we note that for each primitive direction w, the length of the side of P_n orthogonal to w tends to 0 as $n \to \infty$. Therefore, for each $N > 0$, the sum of the lattice lengths of the sides of P_n orthogonal to w with $|w| < N$ tends to zero, and the rest part of the lattice perimeter of P_n is less than $\frac{2\pi}{N}$ because of the definition of the lattice length. We conclude the proof by letting $N \to \infty$. $\square$

Finally, ($\divideontimes$) follows from Lemmata 1, 3.

4 Questions

One may ask what happens for other powers of $f(a, b, c, d)$. There is a partial answer in degree 3, which also reveals the source of our formulae.

For every primitive vector $w \in \mathbb{Z}^2$, a tangent line to D^2 orthogonal to w is given by $w \cdot p + |w| = 0$. Consider a piecewise linear function $F : D^2 \to \mathbb{R}$ defined as

$$F(p) = \inf_{w \in \mathbb{Z}^2 \setminus 0} (w \cdot p + |w|). \tag{2}$$

Performing *verbatim* the analysis of cropped tetrahedra applied to the graph of F, one can prove the following lemma.

Lemma 4 $4 - 2 \sum f(a, b, c, d)^3 = 3 \int_{D^2} F$.

Now we describe the general idea behind ($\divideontimes\!\!\divideontimes$), ($\divideontimes$). Denote by $C \subset D^2$ the locus of all points p where the function F is not smooth. The set C is a locally finite tree (see Fig. 3). In fact, it is naturally a tropical curve (see Kalinin and Shkolnikov 2016, 2017). The numbers $f(a, b, c, d)$ represent the values of F at the vertices of C and can be computed from the equations of tangent lines.

Below we list some directions, which we find interesting to explore.

Coordinates on the space of compact convex domains For every compact convex domain Ω, we can define F_Ω as the infimum of all support functions with integral

slopes, exactly as in (2). Consider the values of F_Ω at the vertices of C_Ω, the corner locus of F_Ω. These values are the complete coordinates on the set of convex domains; therefore, the characteristics of Ω (for example, the area) can be potentially expressed in terms of these values. How to relate these coordinates of Ω with those of the dual domain Ω^*?

Higher dimensions We failed to reproduce this line of arguments "by cropping" for three-dimensional bodies, but it seems that we need to sum up by all quadruples of vectors v_1, v_2, v_3, v_4 such that $\mathrm{ConvHull}(0, v_1, v_2, v_3, v_4)$ contains no lattice points.

Zeta function We may consider the sum $Z(s) = \sum f(a, b, c, d)^s$ as an analog of the Riemann zeta function. We can prove that $Z(s)$ converges as long as $s > 1/2$. Can we extend this function for complex values of s?

Other proofs It would be nice to give another proof of (К), (ж) with the methods which were used to prove (1). Note that the vectors (a, b), (c, d) can be uniquely reconstructed from the vector $(a+c, b+d)$, and our construction resembles the Farey sequence a lot. So we can think of $f(a, b, c, d)$ as a kind of a measure on $\mathbb{Q} \cap [0, 1]$, and can integrate a function along it. Can we interpret $f(a, b, c, d)$ as a residue of a certain function at $(a+b)+(c+d)i$? The Riemann zeta function is related to integers; could it be that f is related to the Gauss integers?

Modular forms We can extend f to the whole $SL(2, \mathbb{Z})$. If both vectors (a, b), (c, d) belong to the same quadrant, we use the same definition. For (a, b), (c, d) from different quadrants, we could define

$$f(a, b, c, d) = \sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} - \sqrt{(a - c)^2 + (b - d)^2}.$$

Then

$$\sum_{m \in SL(2,\mathbb{Z})} f(m) = \sum_{\substack{a,b,c,d \in \mathbb{Z} \\ ad - bc = 1}} f(a, b, c, d)$$

is well defined. Can we naturally extend this function to the $\mathbb{C}/SL(2, \mathbb{Z})$? Can we make similar series for other lattices or tessellations of the plane?

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