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RESEARCH CONTRIBUTION

On Centro-Affine Curves and Bäcklund Transformations of the KdV Equation

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Abstract
We continue the study of the Korteweg-de Vries equation in terms of cento-affine curves, initiated by Pinkall. A centro-affine curve is a closed parametric curve in the affine plane such that the determinant made by the position and the velocity vectors is identically one. The space of centro-affine curves is acted upon by the special linear group, and the quotient is identified with the space of Hill's equations with periodic solutions. It is known that the space of centro-affine curves carries two pre-symplectic structures, and the KdV flow is identified with is a bi-Hamiltonian dynamical system therein. We introduce a one-parameter family of transformations on centro-affine curves, prove that they preserve both presymplectic structures, commute with the KdV flow, and share the integrals with it. Furthermore, the transformation commute with each other (Bianchi permutability). We also describe integrals of the KdV equation as arising from the monodromy of Riccati equations associated with centro-affine curves. We are motivated by our work (joint with M. Arnold, D. Fuchs, and I. Izmenstiev), concerning the cross-ratio dynamics on ideal polygons in the hyperbolic plane and hyperbolic space, whose continuous version is studied in the present paper.

Keywords Korteweg-de Vries equation · Centro-affine curves · Bäcklund transformation · Riccati equation · Bianchi permutability · Bi-Hamiltonian structure

1 A Family of Transformations on the Space of Curves

Our motivation for this note is twofold. On the one hand, the results that we present here are continuous versions of some of the results obtained in our recent paper Arnold et al. (2018) (see below).

On the other hand, this is a contribution to the study of the Korteweg-de Vries equation viewed as evolution of curves in centro-affine geometry, that goes back to

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Pinkall (1995); see Sect. 2 for a brief overview. The Bäcklund transformations of the KdV equation take a simple and natural form in terms of centro-affine curves.

In Arnold et al. (2018) we study the integrable dynamics of a 1-parameter family of correspondences on ideal polygons in the hyperbolic plane and hyperbolic space: two n-gons $P = (p_1, p_2, \ldots)$ and $Q = (q_1, q_2, \ldots)$ in $\mathbb{RP}^1$ or in $\mathbb{CP}^1$ are in correspondence $P \overset{c}{\sim} Q$ if $[p_i, p_{i+1}, q_i, q_{i+1}] = c$ for all i; the constant c is a parameter.

In the limit $n \to \infty$, a polygon becomes a parameterized curve. The ground field can be either $\mathbb{R}$ of $\mathbb{C}$; to fix ideas, choose $\mathbb{R}$. Let us use the following definition of cross-ratio to define our correspondence (other five definitions result in the change of the constant c):

$$[p_i, p_{i+1}, q_i, q_{i+1}] = \frac{(q_{i+1} - q_i)(p_{i+1} - p_i)}{(q_i - p_i)(q_{i+1} - p_{i+1})} = c. \tag{1}$$

We replace polygons by non-degenerate closed curves $\gamma : \mathbb{R} \to \mathbb{RP}^1$ with $\gamma'(t) > 0$; to be concrete, let the period be π: $\gamma(t + \pi) = \gamma(t)$. Also let us assume that the rotation number of the curve γ is 1, that is, $\gamma : \mathbb{R}/\pi\mathbb{Z} \to \mathbb{RP}^1$ is a diffeomorphism. Denote the space of such curves by $\widetilde{\mathcal{C}}$ and let $\mathcal{C} = \widetilde{\mathcal{C}}/\operatorname{PSL}(2, \mathbb{R})$ be the moduli space.

Then a continuous version of (1) is

$$\frac{\gamma'(t)\delta'(t)}{(\delta(t) - \gamma(t))^2} = c. \tag{2}$$

Write $\gamma \overset{c}{\sim} \delta$ to denote this relation on $\widetilde{\mathcal{C}}$. Since cross-ratio is Möbius invariant, we also have a relation on $\mathcal{C}$ which we continue to denote by $\overset{c}{\sim}$. Note that $\overset{c}{\sim}$ is a symmetric relation.

Lemma 1.1 *For a generic curve γ, the relation $\gamma \overset{c}{\sim} \delta$ is a (partially defined) 2-2 map $T_c : \gamma \mapsto \delta$.*

Proof Given $\gamma(t)$, Eq. (2) is a Riccati equation on $\delta(t)$, its monodromy is a Möbius transformation (see, e.g., Ince 1944) which has either two or no fixed points, unless it is the identity. Over $\mathbb{C}$, there are always two fixed points (possibly, coinciding), and over $\mathbb{R}$, we need to assume that they exist. Then $\overset{c}{\sim}$ defines a 2–2 map. □

Thus, given γ, there are two choices of $\delta = T_c(\gamma)$. Once a choice is made, one similarly has two choices for $T_c(\delta)$, but one of them is γ, so we choose the other one, and so on. Hence the choice of δ determines the map T_c; the other choice gives the inverse map T_c^{-1}.

Following the standard procedure (see, e.g., Ovsienko and Tabachnikov 2005), lift a curve $\gamma(t)$ from $\mathbb{RP}^1$ to $\mathbb{R}^2$, normalizing the lift $\Gamma(t)$ so that $[\Gamma, \Gamma'] = 1$ (here and elsewhere [,] denotes the determinant made by two vectors).

Explicitly, $\Gamma = ((\gamma')^{-1/2}, (\gamma')^{-1/2}\gamma)$. Note the square root: the curve $-\Gamma$ will do as well, the lift is defined up to the sign, and the action of $\operatorname{PSL}(2, \mathbb{R})$ is replaced by that of $\operatorname{SL}(2, \mathbb{R})$. We obtain centro-affine realizations of the spaces $\widetilde{\mathcal{C}}$ and $\mathcal{C}$.

The curve Γ satisfies a Hill equation

$$\Gamma''(t) = p(t)\Gamma(t) \tag{3}$$

with a π-periodic potential $p(t)$, and $\Gamma(t+\pi) = -\Gamma(t)$ (the curve makes exactly half-rotation on $[0,\pi]$, but its projection to $\mathbb{RP}^1$ closes up). In geometric terms, the potential p is the (negative) centro-affine curvature of the curve Γ.

In these terms, the reciprocal of Eq. (1) becomes

$$\frac{[\Delta(t), \Gamma(t)][\Delta(t+\varepsilon), \Gamma(t+\varepsilon)]}{[\Delta(t), \Delta(t+\varepsilon)][\Gamma(t), \Gamma(t+\varepsilon)]} = \text{const},$$

and, in the limit $\varepsilon \to 0$, after multiplying by ε^2, we obtain an analog of Eq. (2):

$$[\Gamma(t), \Delta(t)]^2 = c^2.$$

Alternatively, one can check directly that $[\Gamma, \Delta] = (\gamma')^{-1/2}(\delta')^{-1/2}(\delta - \gamma)$, and then the above equation follows from (2) without recourse to a small ε expansion.

Break the symmetry between Γ and Δ by taking square root:

$$[\Gamma(t), \Delta(t)] = c. \tag{4}$$

This defines a map on the lifted curves: $T_c(\Gamma) = \Delta$. Note that $T_c(\Delta) = -\Gamma$.

Lemma 1.2 $T_c : \Gamma \mapsto \Delta$ *is a* (*partially defined*) $2-2$ *map.*

Proof Let us search for Δ in the form $\Delta = a\Gamma + b\Gamma'$, where a and b are π-periodic functions. Then equation $[\Gamma, \Delta] = c$ implies that $b(t) = c$, $\Delta = a\Gamma + c\Gamma'$, and then $\Delta' = (a' + cp)\Gamma + a\Gamma'$. The condition $[\Delta, \Delta'] = 1$ now implies

$$a' = \frac{a^2 - 1}{c} - cp. \tag{5}$$

This is a Riccati equation on function $a(t)$ with periodic coefficients. The monodromy of this equation is a Möbius transformation, hence it has two fixed points (always, if one works over $\mathbb{C}$, and over $\mathbb{R}$ one needs to assume that it does), corresponding to two periodic solutions of (5). Each solution defines a curve Δ with $T_c(\Gamma) = \Delta$. □

As before, once a choice of a fixed point of the monodromy is made, the map becomes 1–1: of the two choices available for the next curve Δ, one is extraneous because it takes one back to $-\Gamma$.

2 Two Pre-Symplectic Forms and a Bi-Hamiltonian Structure

Starting with Pinkall (1995), a number of recent papers were devoted to the study of the Korteweg-de Vries equation in terms of centro-affine curves Calini et al. (2009), Fujioka and Kurose (2010, 2014) and Terng and Wu (2013). Let us present the relevant results.

A tangent vector at a cento-affine curve Γ is a vector field along Γ that can be written as a linear combination $h\Gamma + f\Gamma'$ where h, f are π-periodic functions.

Lemma 2.1 *The function f is arbitrary, and $h = -\frac{1}{2}f'$.*

Proof If $\Gamma + \varepsilon v$ is a deformation of Γ, then $[\Gamma, v'] = [\Gamma', v]$ because $[\Gamma, \Gamma'] = 1$. For $v = h\Gamma + f\Gamma'$, this implies that $h = -\frac{1}{2}f'$. □

Denote the tangent vectors by U, V or U_f, V_f, in the format $-\frac{1}{2}f'\Gamma + f\Gamma'$.

The following pre-symplectic structure on space $\widetilde{\mathcal{C}}$ was introduced in Pinkall (1995). Let U, V be tangent vector fields along Γ; define

$$\omega(U, V) = \int_\Gamma [U, V]\, dt,$$

that is,

$$\omega(U_f, V_g) = \frac{1}{2}\int_0^\pi (fg' - f'g)dt.$$

The kernel of ω is spanned by the field Γ', that is, by the reparameterizations $t \mapsto t$+const.

Pinkall observed that the Hamiltonian vector field of the function $\int p\ dt$ is U_p, which induces the KdV evolution of the potential p

$$\dot{p} = -\frac{1}{2}p''' + 3p'p$$

(the signs differ from those of Pinkall because he used the opposite sign for the potential of Hill's equation).

The second pre-symplectic structure was introduced in Fujioka and Kurose (2014): for tangent vector fields U, V along Γ, let

$$\Omega(U, V) = \int_\Gamma ([U', V'] + p[U, V])\, dt,$$

that is,

$$\Omega(U_f, V_g) = \int_0^\pi \left[\frac{1}{4}(f'g'' - f''g') + p(fg' - f'g)\right] dt.$$

Concerning the kernel of Ω, one has

Lemma 2.2 (Fujioka and Kurose 2014) *The kernel of Ω is 3-dimensional, it is generated by the Killing vector fields $A(\Gamma)$ with $A \in$ SL(2).*

Proof One has

$$\Omega(U,V)=\int[pU-U'',V]\,dt.$$

Hence U is in the kernel if and only if $U''=pU$, that is, $U(t)$ is SL(2)-equivalent to $\Gamma(t)$. □

Thus the form Ω descends on the moduli space $\mathcal{C}$ as a symplectic form.

It is shown in Calini et al. (2009), Terng and Wu (2013) and Fujioka and Kurose (2014) that the forms ω and Ω provide a bi-Hamiltonian structure on $\widetilde{\mathcal{C}}$, corresponding to a pair of compatible Poisson brackets for the KdV equation.

Namely, let $X_0, X_1, \ldots$ and $H_1, H_2, \ldots$ be the vector fields and the Hamiltonians of the KdV hierarchy in terms of centro-affine curves:

$$X_0=U_1=\Gamma',\, X_1=U_p=-\frac{p'}{2}\Gamma+p\Gamma',\ldots,\, H_1=\int p\,dt,\, H_2=\frac{1}{2}\int p^2dt,\ldots$$

Then one has

$$\Omega(X_{j-1},\cdot)=dH_j=\omega(X_j,\cdot),\quad j=1,2,\ldots \tag{6}$$

see Fujioka and Kurose (2014).

The forms ω and Ω **on projective curves.** Let us calculate these forms in terms of the curves $\gamma:\mathbb{R}\to\mathbb{RP}^1$.

In Arnold et al. (2018), the following differential 2-form on the space of polygons $(p_1,\ldots,p_n)\subset\mathbb{RP}^1$ was considered

$$\omega'=\sum_i\frac{dp_i\wedge dp_{i+1}}{(p_{i+1}-p_i)^2},$$

and it was proved that this form was T_c-invariant. In the continuous limit, a polygon becomes a curve $\gamma(t)$. Let $u(t), v(t)$ be two vector fields along $\gamma(t)$, that is, two periodic functions. Then, in the continuous limit, we obtain the form

$$\omega'(u,v)=\int\frac{uv'-u'v}{(\gamma')^2}\,dt.$$

Lemma 2.3 *One has* $\omega=\frac{1}{2}\omega'$.

Proof Since

$$\Gamma_1=(\gamma')^{-1/2},\ \Gamma_2=(\gamma')^{-1/2}\gamma,$$

one calculates the respective vector field along Γ:

$$U=\left(-\frac{1}{2}u'\Gamma_1^3,-\frac{1}{2}u'\Gamma_1^2\Gamma_2+u\Gamma_1\right),$$

and likewise for V. Then

$$[U, V] = \frac{1}{2}\Gamma_1^4(uv' - u'v),$$

and the result follows. □

By Lemma 2.2, the 2-form Ω descends to the moduli space of projective curves, that is, to the space of Hill's equations. This space is a coadjoint orbit of the Virasoro algebra, and Ω coincides (up to a factor) with the celebrated Kirillov–Kostant–Souriau symplectic structure, see, e.g., Khesin and Wendt (2009) and Ovsienko and Tabachnikov (2005) for this material.

Namely, let γ be a curve in $\mathbb{RP}^1$, and let u and v be vector fields along γ. The Kirillov–Kostant–Souriau symplectic form is given by the formula

$$\Omega'(u, v) = \int \frac{u''(t)v'(t) - u'(t)v''(t)}{\gamma'(t)^2}\, dt,$$

see, e.g., Ovsienko and Tabachnikov (2015).

Lemma 2.4 *One has*

$$\Omega = -\frac{1}{4}\Omega'.$$

Proof As in the proof of Lemma 2.3,

$$U = \left(-\frac{1}{2}u'\Gamma_1^3, -\frac{1}{2}u'\Gamma_1^2\Gamma_2 + u\Gamma_1\right),$$

and then

$$U' = \left(-\frac{1}{2}u''\Gamma_1^3 - \frac{3}{2}u'\Gamma_1^2\Gamma_1', -\frac{1}{2}u''\Gamma_1^2\Gamma_2 - u'\Gamma_1\Gamma_2\Gamma_1' - \frac{1}{2}u'\Gamma_1^2\Gamma_2' + u'\Gamma_1 + u\Gamma_1'\right).$$

Similar formulas hold for V.

Now one computes, using the fact that $\Gamma_2'\Gamma_1 - \Gamma_1'\Gamma_2 = 1$,

$$[U', V'] = -\frac{1}{4}\Gamma_1^4(u''v' - u'v'') - \frac{1}{2}\Gamma_1^3\Gamma_1'(u''v - uv'') - \frac{3}{2}\Gamma_1^2(\Gamma_1')^2(u'v - uv'),$$

and

$$p[U, V] = -\frac{1}{2}p\Gamma_1^4(u'v - uv') = -\frac{1}{2}\Gamma_1^3\Gamma_1''(u'v - uv').$$

Integrating by parts,

$$-\int \Gamma_1^3\Gamma_1'(u''v - uv'')\, dt = \int (\Gamma_1^3\Gamma_1'' + 3\Gamma_1^2(\Gamma_1')^2)(u'v - uv')\, dt,$$

and collecting terms,

$$\Omega(U,V)=\int([U',V']+p[U,V])\,dt=-\frac{1}{4}\int\Gamma_1^4(u''v'-u'v'')\,dt=-\frac{1}{4}\Omega'(u,v),$$

as claimed. □

3 T_c-Invariance of the Bi-Hamiltonian Structure and Complete Integrability of the Transformations T_c

Let $T_c(\Gamma)=\Delta$ with $\Delta''=q\Delta$; one can write $\Delta(t)=a(t)\Gamma(t)+c\Gamma'(t)$, where $a(t)$ is a periodic function.

Lemma 3.1 *One has:*

$$\Gamma=a\Delta-c\Delta',\;\; p+q=\frac{2}{c^2}(a^2-1),\;\; q-p=\frac{2}{c}a'.$$

Proof Since $[\Delta,-\Gamma]=c$, we can write $-\Gamma=b\Delta+c\Delta'$ where $b(t)$ is a periodic function. Substitute $\Delta=a\Gamma+c\Gamma'$ in this equation to find that $b=-a$. We also have an analog of (5) for function $b(t)$: $cb'=b^2-1-c^2q$. This implies the relations between p and q stated in the lemma. □

Let $T_c(\Gamma)=\Delta$, and let U_f, V_g be two tangent vectors, at Γ and Δ, respectively, related by the differential of T_c.

Lemma 3.2 *One has*

$$\frac{c}{2}(f'+g')=a(g-f), \tag{7}$$

where the function $a(t)$ is as above.

Proof One has $[U,\Delta]+[\Gamma,V]=0$, or

$$\frac{c}{2}(f'+g')=f[\Gamma',\Delta]+g[\Gamma,\Delta']=a(g-f),$$

where the last equality makes use of $\Delta=a\Gamma+c\Gamma'$ and of $[\Gamma',\Delta]+[\Gamma,\Delta']=0$. □

Remark 3.3 In the above lemmas, one recognizes the usual Bäcklund transformation for the KdV equation

$$u'+v'=C-(u-v)^2,$$

where, in our case, $p=u'$, $q=v'$, and $C=-2/c^2$.

The following theorem is our main observation.

Theorem 1 *The forms ω and Ω are invariant under the maps T_c:*

$$T_c^*(\omega) = \omega,\ T_c^*(\Omega) = \Omega.$$

Proof Let $T_c(\Gamma) = \Delta$, and let $U_{f_i}, V_{g_i}, i = 1, 2$, be two pairs of tangent vectors, at Γ and Δ, respectively, related by the differential of T_c. One has

$$\begin{aligned}\int [(g_1' g_2 - g_1 g_2') - (f_1' f_2 - f_1 f_2')]dt &= \int [(g_1' g_2 - g_1 g_2') - (f_1' f_2 - f_1 f_2') \\ &\quad - (g_1' f_2 + g_1 f_2') + (g_2' f_1 + g_2 f_1')]dt \\ &= \int [(f_1' + g_1')(g_2 - f_2) - (f_2' + g_2')(g_1 - f_1)]dt = 0,\end{aligned}$$

where the first equality follows from the fact that $g_1' f_2 + g_1 f_2' = (g_1 f_2)'$ and $g_2' f_1 + g_2 f_1' = (g_2 f_1)'$, which integrates to zero, and the last equality follows from (7). Thus $T_c^*(\omega) = \omega$.

To prove that $T_c^*(\Omega) = \Omega$, we argue similarly, although the computation is more involved.

Differentiate (7) to obtain

$$\frac{c}{2}(f'' + g'') = a'(g - f) + a(g' - f'). \tag{8}$$

We want to show that the integral

$$\int \left(\frac{1}{4}(f_1' f_2'' - f_1'' f_2') + p(f_1 f_2' - f_1' f_2) - \frac{1}{4}(g_1' g_2'' - g_1'' g_2') - q(g_1 g_2' - g_1' g_2) \right) dt \tag{9}$$

vanishes. One has

$$\begin{aligned}f_1' f_2'' - f_1'' f_2' - g_1' g_2'' + g_1'' g_2' &= (f_1'' + g_1'')(g_2' - f_2') \\ &\quad -(f_2'' + g_2'')(g_1' - f_1') + (f_2' g_1' - f_1' g_2')',\end{aligned}$$

hence

$$\begin{aligned}&\frac{1}{4}\int (f_1' f_2'' - f_1'' f_2' - g_1' g_2'' + g_1'' g_2')dt \\ &= \frac{1}{2c}\int \{[a'(g_1 - f_1) + a(g_1' - f_1')](g_2' - f_2') \\ &\quad - [a'(g_2 - f_2) + a(g_2' - f_2')](g_1' - f_1')\}dt \\ &= \int \frac{a'}{2c}[(g_1 - f_1)(g_2' - f_2') - (g_2 - f_2)(g_1' - f_1')]dt,\end{aligned}$$

where the first equality follows from (8).

Next we evaluate the remaining part of the integral (9), using Lemma 3.1:

$$\int [p(f_1 f_2' - f_1' f_2) - q(g_1 g_2' - g_1' g_2)]dt$$
$$= \int \frac{a^2-1}{c^2}(f_1 f_2' - f_1' f_2 - g_1 g_2' + g_1' g_2)dt$$
$$- \int \frac{a'}{c}(f_1 f_2 - f_1' f_2 + g_1 g_2' - g_1' g_2)dt.$$

Collecting the integrals together, we obtain

$$\int \frac{a'}{2c}[(f_1' + g_1')(f_2 + g_2) - (f_2' + g_2')(f_1 + g_1)]dt$$
$$+ \int \frac{a^2-1}{c^2}(f_1 f_2' - f_1' f_2 - g_1 g_2' + g_1' g_2)dt$$
$$= \int \frac{2aa'}{c^2}(f_2 g_1 - f_1 g_2)dt + \int \frac{a^2-1}{c^2}(f_1 f_2' - f_1' f_2 - g_1 g_2' + g_1' g_2)dt,$$

where the equality is due to (7).

Finally, notice that $(a^2-1)' = 2aa'$, and integrate by parts to obtain

$$\int \frac{a^2-1}{c^2}[(f_1 f_2' - f_1' f_2 - g_1 g_2' + g_1' g_2) - (f_2' g_1 + f_2 g_1' - f_1' g_2 - f_1 g_2')]dt$$
$$= \int \frac{a^2-1}{c^2}[(f_2' + g_2')(f_1 - g_1) - (f_1' + g_1')(f_2 - g_2)]dt = 0,$$

since the last integrand vanishes due to (7). □

Corollary 2 *The maps T_c commute with the KdV flows and preserve the KdV integrals.*

Proof One argues inductively using formulas (6):

$$\Omega(X_{j-1}, \cdot) = dH_j = \omega(X_j, \cdot).$$

If T_c preserves X_{j-1} then, since it also preserves Ω, it preserves dH_j. If T_c preserves dH_j then, since it preserves ω, it also preserves X_j.

To start the induction, we check that $\int p\, dt$ is invariant:

$$\int (q(t) - p(t))\, dt = \frac{2}{c} \int a'(t)\, dt = 0$$

due to Lemma 3.1.

Since dH_j is preserved, it could be that T_c changes H_j by a constant. To see that this constant is zero, let Γ be the circle $(\cos t, \sin t)$. Then Δ differs from Γ by a parameter shift, and the values of the functions H_j on Γ and Δ are equal. □

Thus the transformations T_c are symmetries of the Korteweg-de Vries equation.

Remark 3.4 The argument above is similar to the one given in Tabachnikov (2017) which concerned with the filament equation and the bicycle transformations as its symmetries.

Additional integrals. Let $\Gamma = (\Gamma_1, \Gamma_2)$. Consider the functions

$$I = \int \Gamma_1^2 \, dt, \;\; J = \int \Gamma_1 \Gamma_2 \, dt, \;\; K = \int \Gamma_2^2 \, dt$$

on the space of centro-affine curves.

Proposition 3.5 *The functions I, J, K are the Hamiltonians of the generator of the action of* $\mathrm{sl}(2, \mathbb{R})$ *on* $\widetilde{\mathcal{C}}$ *with respect to the 2-form ω. The function $IK - J^2$ is* $\mathrm{SL}(2, \mathbb{R})$*-invariant.*

Proof The generators of $\mathrm{sl}(2, \mathbb{R})$ are the fields

$$(\Gamma_2, 0), \;\; (\Gamma_1, -\Gamma_2), \;\; (0, \Gamma_1).$$

Let us consider the first one; the other ones are dealt with similarly.

We claim that $(\Gamma_2, 0) = -V_{\Gamma_2^2}$. Indeed,

$$V_{\Gamma_2^2} = -\Gamma_2 \Gamma_2' \Gamma + \Gamma_2^2 \Gamma'.$$

The first component of this vector is $-\Gamma_2(\Gamma_2'\Gamma_1 - \Gamma_1'\Gamma_2) = -\Gamma_2$, and the second component is $-\Gamma_2'\Gamma_2^2 + \Gamma_2'\Gamma_2^2 = 0$.

Let U_f be a test vector field. Then

$$\begin{aligned} dK(U_f) &= \int \Gamma_2 \left(\Gamma_2' f - \frac{1}{2} \Gamma_2 f' \right) \, dt = \int \Gamma_2 \Gamma_2' f \, dt - \frac{1}{2} \int \Gamma_2^2 f' \, dt \\ &= 2 \int \Gamma_2 \Gamma_2' f \, dt, \end{aligned}$$

the last equality due to integration by parts. On the other hand,

$$\omega(U_f, U_{\Gamma_2^2}) = \int 2\Gamma_2 \Gamma_2' f \, dt,$$

as needed.

As to $\mathrm{sl}(2, \mathbb{R})$ invariance of $IK - J^2$, let us again check invariance under the field $(\Gamma_2, 0)$ (the rest is similar). Calculating mod ε^2, one has

$$\begin{aligned} &\left(\int (\Gamma_1 + \varepsilon \Gamma_2)^2 \, dt \right) \left(\int \Gamma_2^2 \, dt \right) - \left(\int (\Gamma_1 + \varepsilon \Gamma_2) \Gamma_2 \, dt \right)^2 = IK - J^2 \\ &\quad + 2\varepsilon \left[\left(\int \Gamma_1 \Gamma_2 \, dt \right) \left(\int \Gamma_2^2 \, dt \right) - \left(\int \Gamma_1 \Gamma_2 \, dt \right) \left(\int \Gamma_2^2 \, dt \right) \right] = IK - J^2, \end{aligned}$$

as needed. □

Next we show that I, J, K are integrals of the transformations T_c.

Theorem 3 *Let $T_c(\Gamma) = \Delta$, then*

$$I(\Gamma) = I(\Delta),\ J(\Gamma) = J(\Delta),\ K(\Gamma) = K(\Delta).$$

Proof Consider the case of I; the other two cases are similar.

We have $\Delta = a\Gamma + c\Gamma'$, and we want to show that $\int \Delta_1^2 = \int \Gamma_1^2$. Indeed,

$$\begin{aligned}\int (\Delta_1^2 - \Gamma_1^2)\, dt &= \int [(a^2-1)\Gamma_1^2 + 2ca\Gamma_1\Gamma_1' + c^2(\Gamma_1')^2]\, dt \\ &= \int [(a^2-1-ca')\Gamma_1^2 + c^2(\Gamma_1')^2]\, dt = c^2 \int [p\Gamma_1^2 + (\Gamma_1')^2]\, dt \\ &= c^2 \int [\Gamma_1''\Gamma_1 + (\Gamma_1')^2]\, dt = 0,\end{aligned}$$

where the second equality is integration by parts, the third is due to (5), the fourth is due to $\Gamma'' = p\Gamma$, and the last one is again integration by parts. □

4 Monodromy Integrals and Permutability

Now we describe an infinite collection of $\mathrm{SL}(2,\mathbb{R})$-invariant integrals of the maps T_c that arise from the monodromy of the Riccati equations.

Let x be an affine coordinate on $\mathbb{RP}^1$. The Lie algebra $\mathrm{sl}(2,\mathbb{R})$ is generated by the vector fields $\partial_x, x\partial_x, x^2\partial_x$. Introduce time-dependent vector fields, depending on $\gamma(t)$ or $\delta(t)$, respectively, taking values in $\mathrm{sl}(2,\mathbb{R})$ for each t:

$$\xi_\gamma = \left(\frac{\gamma^2}{\gamma'} - 2\frac{\gamma}{\gamma'}x + \frac{1}{\gamma'}x^2\right)\partial_x,\quad \xi_\delta = \left(\frac{\delta^2}{\delta'} - 2\frac{\delta}{\delta'}x + \frac{1}{\delta'}x^2\right)\partial_x.$$

Then Eq. (2) describes δ as evolving under the field $c\xi_\gamma$ and, equivalently, γ as evolving under $c\xi_\delta$.

Fix a (spectral) parameter λ, and consider the time-π flows of the fields $\lambda\xi_\gamma$ and $\lambda\xi_\delta$, where γ and δ are related by (2). Denote these projective transformations of $\mathbb{RP}^1$ by $\Phi_{\lambda,\gamma}$ and $\Phi_{\lambda,\delta}$.

Theorem 4 *For every λ, the maps $\Phi_{\lambda,\gamma}$ and $\Phi_{\lambda,\delta}$ are conjugate in* $\mathrm{PSL}(2,\mathbb{R})$.

It follows that the spectral invariants of $\Phi_{\lambda,\gamma}$, say $\mathrm{Tr}^2/\det$, as functions of λ, are integrals of the maps T_c for all values of c.

Proof Let γ and δ satisfy (2). Introduce a time-dependent matrix, also depending on parameter μ:

$$A_{\mu,\gamma,\delta}(t)=\frac{1}{\gamma(t)-\delta(t)}\begin{bmatrix}\gamma(t)-\mu\delta(t), & \gamma(t)\delta(t)(\mu-1)\\ 1-\mu, & \gamma(t)\mu-\delta(t)\end{bmatrix}.$$

We claim that if $\lambda = c(1-\mu)$, then $A_{\mu,\gamma,\delta}(t)$ conjugates the vector fields $\lambda\xi_\gamma$ and $\lambda\xi_\delta$.

Namely, let ε be an infinitesimal parameter, and set

$$V_\gamma(t,\varepsilon)=\begin{bmatrix}1-\frac{\varepsilon\lambda\gamma(t)}{\gamma(t)'}, & \frac{\varepsilon\lambda\gamma(t)^2}{\gamma(t)'}\\ -\frac{\varepsilon\lambda}{\gamma(t)'}, & 1+\frac{\varepsilon\lambda\gamma(t)}{\gamma(t)'}\end{bmatrix}$$

This time-dependent Möbius transformation is the time-ε flow of the vector field $\lambda\xi_\gamma$.

Then one has

$$V_\delta(t,-\varepsilon)A_{\mu,\gamma,\delta}(t+\varepsilon)V_\gamma(t,\varepsilon)=V_\delta(t,\varepsilon)A_{\mu,\gamma,\delta}(t-\varepsilon)V_\gamma(t,-\varepsilon) \mod \varepsilon^2,$$

which is verified by a direct calculation or, in the limit $\varepsilon \to 0$,

$$\begin{bmatrix}\frac{\delta(t)}{\delta'(t)}, & \frac{\delta(t)^2}{\delta'(t)}\\ -\frac{1}{\delta'(t)}, & \frac{\delta(t)}{\delta'(t)}\end{bmatrix}A_{\mu,\gamma,\delta}(t)-A_{\mu,\gamma,\delta}(t)\begin{bmatrix}\frac{\gamma(t)}{\gamma'(t)}, & \frac{\gamma(t)^2}{\gamma'(t)}\\ -\frac{1}{\gamma'(t)}, & \frac{\gamma(t)}{\gamma'(t)}\end{bmatrix}=\frac{1}{\lambda}A'_{\mu,\gamma,\delta}(t).$$

This equality implies that the vector fields $\lambda\xi_\gamma$ and $\lambda\xi_\delta$ are conjugate, and so are $\Phi_{\lambda,\gamma}$ and $\Phi_{\lambda,\delta}$:

$$\Phi_{\lambda,\delta}=A_{\mu,\gamma,\delta}(0)\Phi_{\lambda,\gamma}A^{-1}_{\mu,\gamma,\delta}(0), \tag{10}$$

as needed. □

Remark 4.1 The above theorem is also a continuous analog of a result for ideal polygons in Arnold et al. (2018).

Bianchi permutability. Let us show that the maps T_c commute; the argument is similar to that given in Arnold et al. (2018) for ideal polygons.

Theorem 5 *Let three closed curves satisfy* $\gamma \overset{c_1}{\sim} \gamma_1$ *and* $\gamma \overset{c_2}{\sim} \gamma_2$*. Then there exists a fourth curve* γ_{12} *such that* $\gamma_1 \overset{c_2}{\sim} \gamma_{12}$ *and* $\gamma_2 \overset{c_1}{\sim} \gamma_{12}$*.*

Proof We use (10), writing A instead of $A(0)$.

Since $\gamma \overset{c_1}{\sim} \gamma_1$ and $\gamma \overset{c_2}{\sim} \gamma_2$, we have

$$\Phi_{c_1,\gamma}(\gamma_1(0))=\gamma_1(0),\ \Phi_{c_2,\gamma}(\gamma_2(0))=\gamma_2(0).$$

By (10),

$$\Phi_{c_1,\gamma_2}=A_{\mu,\gamma,\gamma_2}\Phi_{c_1,\gamma}A^{-1}_{\mu,\gamma,\gamma_2},\ \Phi_{c_2,\gamma_1}=A_{\nu,\gamma,\gamma_1}\Phi_{c_2,\gamma}A^{-1}_{\nu,\gamma,\gamma_1}$$

with

$$c_1 = c_2(1-\mu), c_2 = c_1(1-\nu). \tag{11}$$

It follows that

$$\Phi_{c_1,\gamma_2}(A_{\mu,\gamma,\gamma_2}(\gamma_1(0))) = A_{\mu,\gamma,\gamma_2}(\gamma_1(0)),\ \Phi_{c_2,\gamma_1}(A_{\nu,\gamma,\gamma_1}(\gamma_2(0))) = A_{\nu,\gamma,\gamma_1}(\gamma_2(0)).$$

Thus we need to show that

$$A_{\mu,\gamma,\gamma_2}(\gamma_1(0)) = A_{\nu,\gamma,\gamma_1}(\gamma_2(0)). \tag{12}$$

This is indeed the case: (11) implies that $\frac{1}{\mu} + \frac{1}{\nu} = 1$, and then a calculation shows that

$$\frac{1}{\mu}\begin{bmatrix} \gamma - \mu\gamma_2, & \gamma\gamma_2(\mu-1) \\ 1-\mu, & \gamma\mu - \gamma_2 \end{bmatrix}\begin{bmatrix} \gamma_1 \\ 1 \end{bmatrix} = \frac{1}{\nu}\begin{bmatrix} \gamma - \nu\gamma_1, & \gamma\gamma_1(\nu-1) \\ 1-\nu, & \gamma\nu - \gamma_1 \end{bmatrix}\begin{bmatrix} \gamma_2 \\ 1 \end{bmatrix},$$

as needed. □

Remark 4.2 The above considerations can be extended to centro-affine *twisted* curves, that is, curves with monodromy, $\Gamma(t+\pi) = M(\Gamma(t))$, where the monodromy $M \in \mathrm{SL}(2,\mathbb{R})$ is not necessarily $-\mathrm{Id}$. One can define the maps T_c on twisted curves: given Γ, consider the respective π-periodic potential of the Hill equation $p(t)$, find a π-periodic solution $a(t)$ to Eq. (5), and define $\Delta = a\Gamma + c\Gamma'$. Then the monodromy of Δ coincides with that of Γ. At the level of Hill's equations, this is the map $p \mapsto q$. We do not dwell on this extension here.

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