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RESEARCH CONTRIBUTION

Simplicial Equations for the Moduli Space of Stable Rational Curves

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Abstract
In this, largely expository, note, we show how the simplicial structure of the moduli spaces of stable rational curves with marked points allows to produce explicit equations for these spaces. The key argument is an elementary combinatorial statement about the sets of trees with marked leaves.

Keywords Trees with marked leaves · Deligne-Mumford compactifications

1 Introduction: Δ-sets

There exists a convenient combinatorial notion which allows to encode the structure of a triangulated topological space; namely, that of a Δ-*set* (Rourke and Sanderson 1971). A Δ-set X is a sequence of sets $X_0, X_1, X_2, \ldots$ together with the maps

$$\partial_i : X_n \to X_{n-1},$$

which are defined for all $n > 0$ and $i = 0, 1, \ldots, n$, and satisfy

$$\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i \tag{1}$$

To Rafail Kalmanovich Gordin on his 70th birthday.

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whenever $i < j$. This definition is a simplification of the standard definition of a *simplicial set*, a fundamental notion in algebraic topology and homological algebra, see, for instance, (May 1967) or (Weibel 1994)

Given a simplicial complex $\mathcal{K}$ with a totally ordered set of vertices, let X_n be the set of all n-dimensional simplices of $\mathcal{K}$. For $x \in X_n$ define $\partial_i(x)$ to be the $(n-1)$-dimensional face of the simplex x which does not contain the ith vertex of x. The identities (1) are then satisfied and $\mathcal{K}$ gives rise to a Δ-set X. Not all Δ-sets come from simplicial complexes; the simplest example is the Δ-set O such that O_0 and O_1 are one-point sets and O_n is empty for $n > 1$.

Definition 1 We shall say that a Δ-set X is *uniquely fillable in dimension* n if for each sequence $(x_0, x_1, \ldots, x_n)$ of elements of X_{n-1} that satisfies

$$\partial_i(x_j) = \partial_{j-1}(x_i)$$

for all $0 \leq i < j \leq n$, there exists a unique element $y \in X_n$ with $\partial_i(y) = x_i$.

If X is uniquely fillable in dimension n, the set X_n can be given by the system of equations $\partial_i(x_j) = \partial_{j-1}(x_i)$ inside the product of $n+1$ copies of X_{n-1}. In this note we shall see that this observation can be used in order to produce the equations for various algebraic varieties such as the Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,n}$ of the moduli space of rational curves with n marked points. The main argument is, actually, a combinatorial statement about certain sets of trees.

2 The Δ-set of Trees with Marked Leaves

For $n \geq 0$, let T_n be the set of all trees without bivalent vertices whose leaves are labelled by the numbers from 0 to n. In particular, T_0, T_1 and T_2 are one-point sets, T_3 has 4 elements and T_4 consists of 26 elements; see Fig. 1.

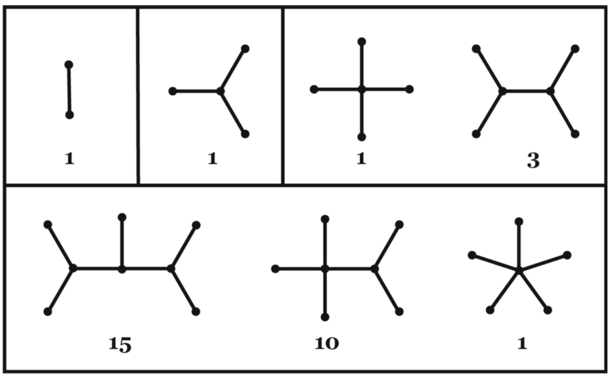


Fig. 1 Trees that form the sets T_1, T_2, T_3 and T_4

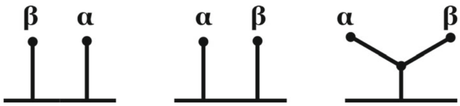


Fig. 2 The following three configurations of the adjacent leaves α and β cannot be distinguished after one of these leaves is erased

For each i between 0 to n define

$$\partial_i : T_n \to T_{n-1}$$

as the map that erases the ith leaf and, for $j > i$, replaces the label j by $j - 1$. If the resulting tree has a bivalent vertex, it is simply "smoothed out": the vertex is deleted and the incoming edges are joined together.

Theorem 2 *The sets T_n together with the maps ∂_i form a Δ-set which is uniquely fillable in dimensions 5 and greater.*

The fact that the T_n form a Δ-set is clear. For the purposes of the argument which establishes the unique fillability in dimension n, it will be more convenient to label the leaves of a tree with some fixed labels, rather than number them from 0 to n. Namely, consider a set A with $n + 1$ elements. We will assume that the leaves of the trees in T_n are marked by distinct elements of A; for $\mu \in A$ we write ∂_μ for the operation of deleting the leaf labelled by μ followed, if necessary, by smoothing a bivalent vertex.

Consider a tree $t \in T_n$. We will be interested in the following question: for which pairs of labels α, β can the tree t be uniquely reconstructed from $\partial_\alpha t$ and $\partial_\beta t$? The answer is expressed in terms of the *adjacency* of leaves in a tree.

Denote by $v(\alpha)$ the vertex to which the leaf α of t is connected. We shall call the leaves α and β of the tree t adjacent if either $v(\alpha) = v(\beta)$ or $v(\alpha)$ and $v(\beta)$ are both trivalent and connected by an edge. Then, the tree t can be uniquely reconstructed from $\partial_\alpha t$ and $\partial_\beta t$ if and only if the leaves α and β of t are not adjacent; see Fig. 2.

On the other hand, assume that α and β are not adjacent in t and consider the tree $\partial_\alpha \partial_\beta t$. The trees $\partial_\alpha t$ and $\partial_\beta t$ are obtained from it by adding one leaf. Each of these leaves is attached either at an internal vertex of $\partial_\alpha \partial_\beta t$ (that is, a vertex of valency greater than 1) or in the interior of an edge, say, at the midpoint. They cannot be attached at the same point since in this case the leaves α and β would be adjacent in t; this means that both of them can be added simultaneously and the result coincides with t.

Now, let us proceed to the proof of Theorem 2.

For $n \geq 5$, consider a collection of $n + 1$ trees (x_μ), $\mu \in A$, each with n marked leaves, such that the leaves of x_μ have labels in $A - \{\mu\}$. Assume that

$$\partial_\alpha x_\beta = \partial_\beta x_\alpha \tag{2}$$

for all pairs of distinct $\alpha, \beta \in A$. We must prove that there exists a unique tree y whose leaves are labelled by the elements of A, such that $x_\mu = \partial_\mu y$.

Consider first $n = 5$. Let us first assume that all the x_μ have only trivalent internal vertices. The equations (2) imply that there are two pairs of labels, say α_1, β_1, and

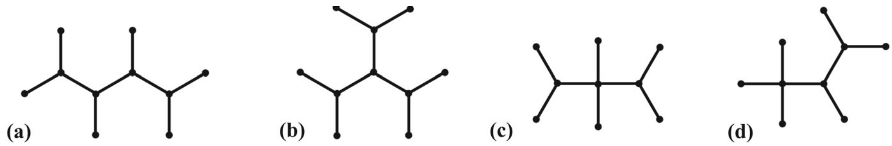


Fig. 3 Graphs y with six leaves such that some of the $\partial_\mu y$ have three internal vertices

α_2, β_2, such that whenever both labels α_i and β_i are present in x_μ, we have $v(\alpha_i) = v(\beta_i)$. It can be then verified that $x_\mu = \partial_\mu y$ where the tree y can have one of the two shapes, either (a) or (b) on Fig. 3. In the case when some, but not all, of the x_μ have only trivalent internal vertices, one verifies that all the solutions are of the type $(\partial_\mu y)$ where y is one of the graphs (c) and (d) on Fig. 3. When each of the x_μ has all its internal vertices of valency greater than three, the number of the internal vertices of x_μ is at most two; this situation is covered by the next two cases.

Let $n \geq 5$ be arbitrary and the maximal number of the internal vertices of the x_μ be two. Then, each of those x_μ that has two internal vertices, gives a decomposition of $A - \{\mu\}$ into two disjoint subsets; namely, the sets of leaves attached to each of the internal vertices.

Assume that the labels α and β belong to the same subset with respect to this decomposition of $A - \{\mu\}$ for some μ. Then, it follows from the condition (2) that this is true for *any* label $\mu \neq \alpha, \beta$ such that x_μ has two internal vertices. As a consequence, there is a well-defined decomposition of A into two subsets. If y is a graph with two internal vertices that corresponds to this decomposition of A, then we have $x_\mu = \partial_\mu y$ for all $\mu \in A$.

Let $n \geq 5$ be arbitrary and assume that for each $\mu \in A$ the graph x_μ has only one internal vertex. Then $x_\mu = \partial_\mu y$ where y also has only one internal vertex and the leaves of y are labelled by elements of A.

Finally, we are left with the case when $n > 5$ and at least one of the x_μ has more than two internal vertices. In this situation one can always find two labels α and β such that the corresponding leaves are not adjacent in each x_μ.

Indeed, if there exist two leaves in one of the x_μ which are separated by at least 4 internal vertices, their labels correspond to non-adjacent leaves for each μ. If any pair of leaves in each x_μ are separated by fewer than 4 internal vertices, it is sufficient to find in some x_{μ_0} two leaves α and β which are separated by precisely three internal vertices, say $v_1 = v(\alpha)$, v_2 and $v_3 = v(\beta)$ so that at least one of the v_i has valency 4. In this case, the labels α and β are not adjacent in any of the x_μ. Such x_{μ_0} can always be found. Indeed, suppose that in x_{μ_0} any pair of leaves which are separated by precisely three internal vertices, are separated by *trivalent* vertices. Then, $n = 6$ and the only possibility for x_{μ_0} is the graph (b) on the last figure. This, however, leads to a contradiction since in this case some other x_{μ_1} would either have a path of 4 internal vertices or a vertex of valency 4.

The existence of α and β which are not adjacent in any x_μ has the following consequence.

Take $z = \partial_\alpha x_\beta = \partial_\beta x_\alpha$. In order to obtain x_α and x_β from z one has to attach the leaves α and β, respectively, to z at two different points; hence, both of them can be added simultaneously so as to obtain an element $y \in T_n$ with $\partial_\alpha y = x_\alpha$ and $\partial_\beta y = x_\beta$. We have

$$\partial_\alpha x_\mu = \partial_\mu x_\alpha = \partial_\mu \partial_\alpha y = \partial_\alpha \partial_\mu y,$$

and similarly, that $\partial_\beta x_\mu = \partial_\beta \partial_\mu y$. Since the leaves α and β are not adjacent in x_μ, this implies that $\partial_\mu y = x_\mu$. Since $\mu \neq \alpha, \beta$ is arbitrary, the existence (and uniqueness) of y is established.

3 The Space of Stable Rational Curves $\overline{\mathcal{M}}_{0,n}$

The set T_{n-1} of trees with n marked leaves can be thought of as the combinatorial version of the Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,n}$ of the moduli spaces of rational curves with n marked points (Deligne and Mumford 1969).

Recall that the moduli space $\mathcal{M}_{0,n}$ is the space of all configurations of n distinct points on a complex projective line, considered modulo the action of the group of Möbius transformations. It has a compactification $\overline{\mathcal{M}}_{0,n}$ which consists of all *stable* rational curves with n marked points. Such a curve is a tree of projective lines with nodal singularities and n marked points, which has no automorphisms.The marked points are assumed to be distinct from the nodes and among themselves and carry n distinct labels; we may take these labels to be numbers from 0 to $n-1$. The absence of automorphisms means that each line contains at least three distinguished points; that is, either marked points or singularities. The complement to $\mathcal{M}_{0,n}$ in $\overline{\mathcal{M}}_{0,n}$ consists of curves with more than one irreducible component.

For curves with fewer than 5 marked points, the moduli spaces of stable curves are very simple. When $n < 4$ one defines $\overline{\mathcal{M}}_{0,n}$ to be a point. Assigning to a quadruple (z_1, z_2, z_3, z_4) of distinct points on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ its cross-ratio

$$\frac{(z_4 - z_1)(z_2 - z_3)}{(z_4 - z_3)(z_2 - z_1)} \tag{3}$$

we obtain the embedding of $\mathcal{M}_{0,4}$ into $\mathbb{P}^1$ which extends to an isomorphism between $\overline{\mathcal{M}}_{0,4}$ and $\mathbb{P}^1$.

The first non-trivial case $n = 5$ is already quite interesting. In particular, the real part of $\overline{\mathcal{M}}_{0,5}$ is a non-orientable surface with a natural decomposition into 12 pentagons; this led Devadoss (1999) to characterize it as "the evil twin of the dodecahedron" (in fact, it is a connected sum of 5 projective planes). The cohomology of $\overline{\mathcal{M}}_{0,n}$ for all n has been computed by Keel (1992), Etingof et al. (2010) described the cohomology of the real part. One can write down explicit equations for all the $\overline{\mathcal{M}}_{0,n}$, see the paper by Keel and Tevelev (2009). As we shall see here, one may think of the equations for arbitrary $\overline{\mathcal{M}}_{0,n}$ as "simplicial consequences" of the equations for $\overline{\mathcal{M}}_{0,5}$.

For each label i there is a forgetful morphism

$$\partial_i : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n-1}$$

which consists in:

1. erasing the point marked by i and, for each $j > i$, replacing the label j by $j - 1$;
2. collapsing the component with only two distinguished points if such a component appears after the previous step.

The forgetful morphisms satisfy the simplicial identities:

$$\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$$

for all pairs of labels $i < j$, and, therefore, the spaces $\overline{\mathcal{M}}_{0,n}$ form a Δ-set (with the space $\overline{\mathcal{M}}_{0,n}$ being the set of $(n-1)$-simplices).

Theorem 3 *The sets $\overline{\mathcal{M}}_{0,n}$, together with the maps ∂_i form a Δ-set which is uniquely fillable in dimensions 5 and greater.*

This, in particular, means that the simplicial identities can be thought of the equations for $\overline{\mathcal{M}}_{0,n}$ in a product of n copies of $\overline{\mathcal{M}}_{0,n-1}$ for $n > 5$.

Proof The space $\overline{\mathcal{M}}_{0,n}$ can be subdivided into strata indexed by the elements of T_{n-1}; see (Kock and Vainsencher 2007). Namely, a point in $\overline{\mathcal{M}}_{0,n}$ is uniquely specified by a tree in T_{n-1} each of whose k-valent internal vertices is labelled by a configuration in $\mathcal{M}_{0,k}$; the labels of the points of each configuration are the edges emanating from the corresponding vertex. Note that, since $\mathcal{M}_{0,k}$ is a one-point space for $k < 4$, the difference between $\overline{\mathcal{M}}_{0,n}$ and T_{n-1} consists in the labels at the vertices of valency 4 and more.

The effect of the map ∂_α on $\overline{\mathcal{M}}_{0,n}$ amounts to that of ∂_α on T_{n-1} together with forgetting the corresponding point in $\mathcal{M}_{0,k}$ for the vertex $v(\alpha)$ when $v(\alpha)$ is at least 4-valent. The question whether a point $x \in \overline{\mathcal{M}}_{0,n}$ can be uniquely reconstructed from $\partial_\alpha(x)$ and $\partial_\beta(x)$ has a somewhat simpler answer than in T_{n-1}: this can be always be done uniquely unless $v(\alpha)$ and $v(\beta)$ are both trivalent and connected by an edge. Other than this, no changes are necessary in the proof of Theorem 2 in order to adapt it for $\overline{\mathcal{M}}_{0,n}$. □

In fact, the embedding of $\overline{\mathcal{M}}_{0,5}$ into $(\mathbb{P}^1)^5$ defined as the product $\partial_0 \times \cdots \times \partial_4$ is also injective, although its image is not given by the simplicial identities alone (which are trivial in this case). The following is well-known:

Proposition 4 *The image of $\overline{\mathcal{M}}_{0,5}$ in $(\mathbb{P}^1)^5$ is the non-singular surface given by the equations*

$$\begin{aligned} a_1(a_4b_5 - a_5b_4) &= b_1b_5(a_4 - b_4) \\ a_2(a_4b_5 - a_5b_4) &= b_2a_4b_5 \\ a_3(a_4b_5 - a_5b_4) &= b_3a_4(b_5 - a_5), \end{aligned}$$

where $[a_k : b_k]$, for $1 \leq k \leq 5$, are the homogeneous coordinates in the kth copy of $\mathbb{P}^1$.

Proof For a point on $\mathcal{M}_{0,5}$, that is, an ordered quintuple $z = (z_1, z_2, z_3, z_4, z_5)$ of distinct points on $\mathbb{P}^1$, we have that $a_i/b_i \in \mathbb{C} \cup \{\infty\}$ is the cross-ratio of the quadruple obtained by omitting z_i from z; verifying the above equations is a straightforward matter. Since $\mathcal{M}_{0,5}$ is open in $\overline{\mathcal{M}}_{0,5}$, these equations are also satisfied on the image of $\overline{\mathcal{M}}_{0,5}$.

Conversely, if a point $c = (c_1, c_2, c_3, c_4, c_5)$ of $(\mathbb{P}^1)^5$ satisfies these equations, the corresponding curve $x \in \overline{\mathcal{M}}_{0,5}$ can be reconstructed as follows. The number of projective lines of x is: one if none of the c_i is 0, 1 or ∞, two if exactly three of the c_i are 0, 1 or ∞, and three if all of the c_i are equal to 0, 1 or ∞. The entries equal to 0, 1 or ∞ determine the combinatorics of the marked tree and the c_i different from 0,1 and ∞ gives in each case the cross-ratios of the marked points in each projective line.

The image of $\overline{\mathcal{M}}_{0,5}$ can be covered by explicit non-singular charts obtained by fixing three of the five points on $\mathbb{P}^1$ to be 0, 1, ∞. For instance, ordered quintuples of the form $(0, 1, \infty, x, y)$ with $x, y \in \mathbb{C}$ define the chart

$$(x, y) \mapsto \left(\frac{y-1}{y-x}, \frac{y}{y-x}, \frac{y(1-x)}{y-x}, y, x \right);$$

the other charts differ by the indices of the fixed points. □

The equations for $\overline{\mathcal{M}}_{0,5}$, together with the simplicial identities, produce the equations for all the $\overline{\mathcal{M}}_{0,n}$. For instance, consider the case $n = 6$. The moduli space $\overline{\mathcal{M}}_{0,6}$ is a subvariety of

$$(\overline{\mathcal{M}}_{0,5})^6 \subset ((\mathbb{P}^1)^5)^6.$$

Denote the $[a_{ij} : b_{ij}]$, where $1 \leq i \leq 5$ and $1 \leq j \leq 6$, the homogeneous coordinates in $(\mathbb{P}^1)^{30}$, with the index j being the number of the copy of $\overline{\mathcal{M}}_{0,5}$ and i the number of the coordinate in the corresponding copy of $(\mathbb{P}^1)^5$. The simplicial identities give rise to the equalities

$$[a_{ij} : b_{ij}] = [a_{(j-1)i} : b_{(j-1)i}]$$

whenever $i < j$. Therefore, the complete set of equations for $\overline{\mathcal{M}}_{0,6}$ in $(\mathbb{P}^1)^{30}$ is

$$\begin{aligned} a_{1j}(a_{4j}b_{5j} - a_{5j}b_{4j}) &= b_{1j}b_{5j}(a_{4j} - b_{4j}) \\ a_{2j}(a_{4j}b_{5j} - a_{5j}b_{4j}) &= b_{2j}a_{4j}b_{5j} \\ a_{3j}(a_{4j}b_{5j} - a_{5j}b_{4j}) &= b_{3j}a_{4j}(b_{5j} - a_{5j}) \\ a_{ij}b_{(j-1)i} &= a_{(j-1)i}b_{ij} \end{aligned}$$

where i, j vary over the set $1 \leq i < j \leq 6$.

4 Other Examples

There are other varieties similar to the moduli spaces of stable rational curves whose points can be thought of as trees with marked leaves and "decorations" at the internal vertices. The two principal examples are two compactifications of the configuration space $F_n(X)$ of n distinct points on an algebraic variety X: namely, the Fulton-MacPherson compactification $X[n]$ (Fulton and MacPherson 1994), and Ulyanov's (2002) polydiagonal compactification $X\langle n\rangle$.

The configuration space $F_n(X)$ is defined as the complement in X^n to the union of all the diagonals $z_i = z_j$. The spaces $F_n(X)$ form a Δ-set: the map ∂_i erases the ith point in the configuration. It is easy to see that this Δ-set is uniquely fillable in dimensions two and higher.

A point in $X[n]$ is a collection $(z_1, \ldots, z_n) \in X^n$ together with additional data: if two or more of the z_i coincide at a point $z \in X$, one specifies a *screen* at z. Denote by $I \subseteq \{1, \ldots, n\}$ the set of indices of the z_i which coincide with z. A screen at z is a configuration of points, labelled by the set I and not all equal to each other, in the tangent space $T_z X$; it is considered up to a translation and a multiplication by a nonzero scalar. If, in turn, some of the points in the screen coincide, one specifies another screen which corresponds to the set of coinciding points, and the procedure is iterated until in some screen all the points corresponding to different indices are distinct (Fulton and MacPherson 1994, page 191). The map ∂_i extends from $F_n(X)$ to $X[n]$: it erases z_i from $(z_1, \ldots, z_n)$ and deletes the corresponding points from all the screens; if the index i happens to occur in some screen with only two labels, this screen is also erased. It is clear that the ∂_i satisfy the simplicial identities.

Proposition 5 *The spaces $X[n]$ form a Δ-set which is uniquely fillable in dimensions three and greater.*

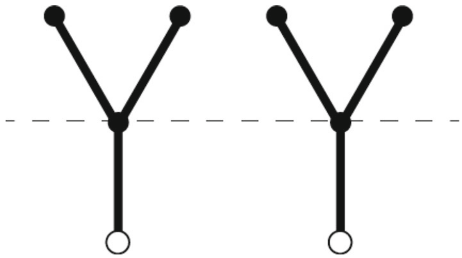
This result should not be surprising: for instance, Fulton and MacPherson (1994) explicitly point out that $X[n]$ form a Δ-set (without using this terminology) and that $X[n]$ is a subvariety in a product of several copies of $X[2]$ and $X[3]$.

The proof (whose details we omit) is very similar to the case of $\overline{\mathcal{M}}_{0,n}$. Indeed, a point of $X[n]$ can be represented by a forest of rooted trees with no bivalent vertices. The roots are univalent and marked by distinct points of X; the rest of the leaves are numbered from 1 to n; each internal vertex carries a label corresponding to a screen. The points on the screen at any internal vertex are labelled by the outgoing edges, assuming that every edge is oriented away from the root. Again, since a screen with two points in it is unique, it is sufficient to consider the labels only for the internal vertices of valency at least 4.

The points that are added to $F_n(X)$ in the construction of $X[n]$ carry the data that record the directions and the hierarchy of the collisions of several points. The polydiagonal compactification is a generalization of the Fulton-MacPherson compactification that allows to record, in addition, the velocities of collisions among several collisions. A point in $X\langle n\rangle$ is given by a forest of rooted trees as in the construction of $X[n]$, with the following differences:

1. there is a total order on the set of internal vertices which can be expressed by a *level* function which increases in the direction away from the root;

Fig. 4 Erasing one leaf in this forest destroys the scale factors at the internal vertices



2. for each screen, a non-zero real *scale factor* is given;
3. the screens, rather than being considered up to up to a translations and dilatations have a finer equivalence on them; namely, one is allowed to,
   (a) apply a translation to all the points in one screen;
   (b) apply a dilatation by a non-zero real λ of all the points in one screen and, at the same time, multiply its scale factor by λ^{-1};
   (c) multiply the scale factor of all the screens on the same level by a non-zero number.

Then, again, we have the forgetful maps $\partial_i : X\langle n\rangle \to X\langle n-1\rangle$ which satisfy the simplicial identities.

Proposition 6 *The $X\langle n\rangle$ form a Δ-set, uniquely fillable in dimensions four and greater.*

Here, the unique fillability dimension four, as opposed to three in the Fulton-MacPherson case, is due to the presence of the scale factors. For instance, consider two points of $X\langle 4\rangle$ which correspond to the forest shown on Fig. 4, with the same markings of roots and leaves but with different (even after any rescaling) scale factors. These points will map to the same elements in $X\langle 3\rangle$ under each ∂_i, since erasing any leaf destroys the scale factors. It can be seen that this problem does not arise when $n > 4$.

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