



Core Entropy of Quadratic Polynomials

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Abstract

We give a combinatorial definition of “core entropy” for quadratic polynomials as the growth exponent of the number of certain precritical points in the Julia set (those that separate the α fixed point from its negative). This notion extends known definitions that work in cases when the polynomial is postcritically finite or when the topology of the Julia set has good properties, and it applies to all quadratic polynomials in the Mandelbrot set. We prove that core entropy is continuous as a function of the complex parameter. In fact, we model the Julia set as an invariant quadratic lamination in the sense of Thurston: this depends on the external angle of a parameter in the boundary of the Mandelbrot set, and one can define core entropy directly from the angle in combinatorial terms. As such, core entropy is continuous as a function of the external angle. Moreover, we prove a conjecture of Giulio Tiozzo about local and global maxima of core entropy as a function of external angles: local maxima are exactly dyadic angles, and the unique global maximum within any wake occurs at the dyadic angle of lowest denominator. We also describe where local minima occur. An appendix by Wolf Jung relates different concepts of core entropy and biaccessibility dimension and thus shows that biaccessibility dimension is continuous as well.

Keywords Core entropy · Hubbard tree · Mandelbrot set

With an appendix by Wolf Jung.

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1 Introduction and Statement of Results

Topological entropy of a topological dynamical system $f: K \rightarrow K$ measures its complexity, roughly speaking as follows. If $K = \bigcup U_i$ is covered by some number of open sets, denote by $N(n)$ the number of allowed itineraries of length n : these are sequences $i(0), \dots, i(n-1)$ for which there exists some $x \in K$ with $f^{ok}(x) \in U_{i(k)}$ for $k = 0, 1, \dots, n-1$. Then the topological entropy of (K, f) is defined as the exponential growth rate of $N(n)$; more precisely, as $h = \limsup_n \frac{1}{n} \log N(n)$. For a precise definition and equivalent descriptions, see for instance Brin and Stuck (2002) or de Melo and van Strien (1993).

In their seminal paper Milnor and Thurston (1988) investigated topological entropy of continuous interval maps. Specifically for the quadratic family $f_\lambda: x \mapsto \lambda x(1-x)$ acting on $I = [0, 1]$ they proved that topological entropy as a function of λ is monotone and continuous.

Misiurewicz and Szlenk (1980) wrote one of the early papers that gave various equivalent interpretations of topological entropy for interval maps, for example as growth exponents of intervals of monotonicity of f^{on} or of the number of periodic points of period n . Since then, there has been a lot of activity on related questions; see for instance de Melo and van Strien (1993) for an overview.

In the last years of his life, William Thurston raised the issue of finding a good definition of topological entropy of a complex polynomial p of degree d . Viewing p as a map on \mathbb{C} or on the filled-in Julia set, the entropy is clearly $\log d$ and not very interesting. The same holds of course when p is a real polynomial (considered as a self-map of \mathbb{C}), but in that case there may be a dynamically interesting invariant interval in \mathbb{R} that contains the orbits of the critical points: the restriction to this interval is the dynamically interesting object, and the aforementioned studies were concerned with self-maps of such intervals. What is the analogous object for complex polynomials, or what is an interesting definition of topological entropy?

If the polynomial p is postcritically finite (that is, all critical points are periodic or preperiodic), then it has a natural invariant tree called its Hubbard tree H , and Thurston defined the *core entropy* of p as the topological entropy of the restriction to H ; see Baik et al. (2015), Li (2007). This entropy can also be defined in certain other cases: for instance, for certain maps one can define a finite tree in analogy to the Hubbard tree that is still invariant and contains the critical orbit, even if the latter is infinite (in analogy to the case of real polynomials). However, in the general case there are difficulties: the filled-in Julia set may not be path connected so that there may not be a tree, or the critical orbit may be dense in the filled-in Julia set so that any tree would have to be infinite, and topological entropy might seem to be equal to $\log d$ (this happens for a dense set of parameters on $\partial\mathcal{M}$, so if entropy with this definition was continuous then it would have to be constant). Thurston asked for a definition of core entropy that would work in all cases. Moreover, in a seminar in Cornell in the spring of 2012 he raised the question whether core entropy could be continuous as a function of the complex parameter. This resulted in a bet between him and John Hubbard, which was one original inspiration for starting our work.

One possible definition is in terms of *biaccessibility dimension*: that is the Hausdorff dimension of all angles $\vartheta \in \mathbb{S}^1$ for which there exists another angle $\vartheta' \neq \vartheta$ so that the

dynamic rays at angles ϑ and ϑ' land at a common point. Thurston had shown that in the cases for which his definition of core entropy applied it was equal to biaccessibility dimension (up to a factor of $\log d$). Earlier we had shown (Meerkamp and Schleicher 2013) that the biaccessibility dimension is always less than 1, except when the Julia set is an interval, strengthening earlier work by Smirnov (2001), Zakeri (2000), and Zdunik (2000). Quite recently, Tiozzo wrote an interesting thesis on core entropy and related questions (Tiozzo 2013), in particular with a relation to the thermodynamic formalism. Some additional history, as well as the relation between core entropy and biaccessibility dimension in the general case, are given in the appendix by Wolf Jung.

Bill Thurston inspired a number of people to investigate core entropy. In particular, there are a survey on current work and open problems by Lei (2014), a manuscript by Jung (2014), and two manuscripts by Tiozzo (2013, 2016).

We propose a general definition of core entropy that coincides with Thurston's in all cases he considered, and that does not require the Julia set to have particular properties (such as pathwise connectivity or the existence of an invariant compact tree that might generalize the Hubbard tree), nor does it require the concepts of thermodynamic formalism or biaccessibility dimension. Instead, our approach is purely combinatorial and thus applies in great generality; here we work only on the case of quadratic polynomials with connected Julia sets. Of course, in the cases where the usual definitions of topological entropy apply, our definition agrees with them.

More precisely, we define two entropy functions:

- $h: \mathbb{S}^1 \rightarrow [0, \log 2]$, assigning to every external angle $\vartheta \in \mathbb{S}^1$ the core entropy of the lamination associated to the angle ϑ .
- $\tilde{h}: \mathcal{M} \rightarrow [0, \log 2]$, assigning to every $c \in \mathcal{M}$ (the Mandelbrot set) the core entropy of the quadratic polynomial $z \mapsto z^2 + c$.

We describe these definitions in Sect. 2. Note that we define \tilde{h} for every $c \in \mathcal{M}$, postcritically finite or not (and could extend it to every parameter $c \in \mathbb{C}$), and similarly for every $\vartheta \in \mathbb{S}^1$. These definitions are such that if the parameter ray at angle ϑ lands (or accumulates) at $c \in \partial\mathcal{M}$, then $h(\vartheta) = \tilde{h}(c)$.

Our first result goes back to the bet between Bill Thurston and John Hubbard in the spring of 2012 and helped settle this bet soon after.

Theorem 1.1 (Continuity of core entropy) (Fig. 1).

Both entropy functions, $\tilde{h}: \mathcal{M} \rightarrow [0, \log 2]$ and $h: \mathbb{S}^1 \rightarrow [0, \log 2]$, are continuous.

We prove continuity of h in Theorem 5.9; the fact that this implies continuity of \tilde{h} is easy and is explained in Sect. 2. An independent proof of continuity of core entropy can be found in Tiozzo (2016).

Our second result settles a conjecture of Tiozzo (2013, Conjecture 1.6).

Theorem 1.2 (Local maxima of core entropy) *The entropy function $h: \mathbb{S}^1 \rightarrow [0, \log 2]$ has the following properties.*

- (a) *Every dyadic angle is an isolated local maximum of the entropy function.*
- (b) *Conversely, every local maximum of h is dyadic.*
- (c) *Within every wake, the entropy function has a unique global maximum, and it occurs at the unique dyadic angle of lowest denominator in the wake.*

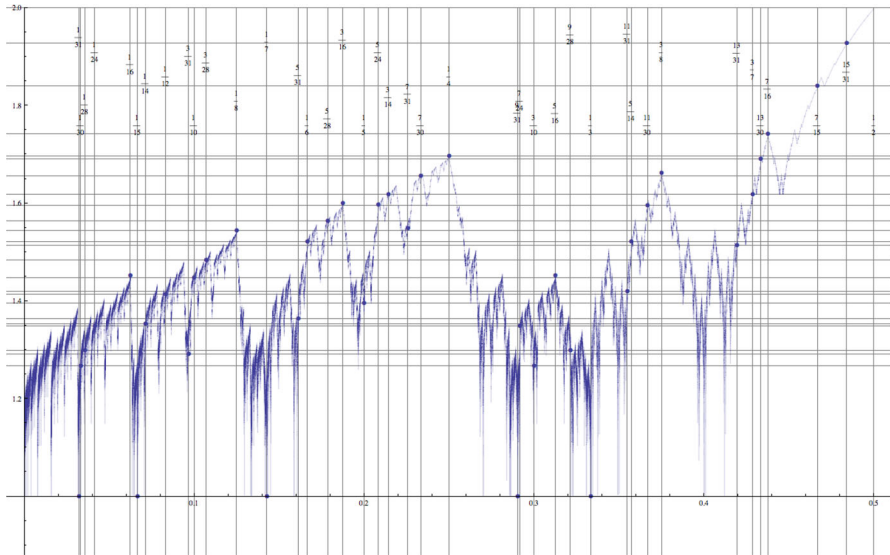


Fig. 1 The graph of the core entropy function $h: [0, 1/2] \rightarrow [0, \log 2]$ (by Bill Thurston)

We will prove this in Theorem 4.1.

In Sect. 2, we give all relevant definitions, in particular of the two entropy functions h and \tilde{h} , and of invariant laminations; we define a wake as a closed interval in \mathbb{S}^1 bounded by two rational angles so that the corresponding rays land together (or more generally as any closed interval for which the two boundary angles have the same angled internal address, so the corresponding parameter rays land together at the same point in \mathcal{M} , or at least in its combinatorial models).

The fundamental construction will be carried out in Sect. 3: we define a surgery construction of dyadic Hubbard trees (that is, Hubbard trees associated to external angles $a/2^k$) that lowers the dyadic exponent k and increases the core entropy. We also describe the vein structure of \mathcal{M} and introduce a relation \triangleleft between dyadic veins.

From this construction, it is no longer difficult to prove Tiozzo's conjecture on maxima of h ; this will be done in Sect. 4: here we have to extend the estimates on entropy to non-dyadic external angles. We also discuss local minima of h and \tilde{h} in this section.

In Sect. 5 we then discuss continuity of entropy. The main result is that for any parameter $c \in \mathcal{M}$, if entropy is continuous along the (combinatorial) path from 0 to that endpoint, then it is continuous in \mathcal{M} at each of these points on the path. The difference is that we *assume* continuity on a path, but continuity (the conclusion) makes a claim about an entire neighborhood in \mathbb{C} . It is known, by work of Tiozzo and Jung, that entropy along many paths in \mathcal{M} is continuous, in particular to all dyadic endpoints. It remains to prove that entropy is continuous along paths to irrational endpoints of \mathcal{M} , and that we prove in Sect. 6.

Finally, in a brief appendix, Wolf Jung discusses the relations of core entropy with biaccessibility dimension, and concludes that biaccessibility dimension is a continuous function as well.

2 Definitions

In this section we introduce our combinatorial definition of topological entropy that applies to all (quadratic) polynomials with connected Julia set, whether or not they are locally connected and whether or not they have (generalized) Hubbard trees, and if so whether the latter are compact. One advantage of our approach is that we work in an entirely combinatorial setting, so we never have to worry about topological issues. We work on invariant (quadratic) laminations as introduced by Thurston (2009).

For every angle $\vartheta \in \mathbb{S}^1 \setminus \{0\}$, there is a unique invariant quadratic lamination where the minor leaf either ends at ϑ or is the degenerate leaf at ϑ . This lamination will be called L_ϑ .

A *precritical leaf of generation n* in this lamination will be any leaf on the backward orbit of one of the two major leaves that takes n generations to map to the minor leaf. Arbitrarily choose one of these two major leaves as *preferred major leaf* (in the special case that the minor leaf is degenerate, there is only one major leaf, which is a diameter, and in this case there is no choice).

In an invariant quadratic lamination, the α gap is defined to be either the leaf connecting the two angles $1/3$ and $2/3$, or the unique gap that is fixed by the dynamics (a finite or infinite polygon).

Definition 2.1 (*Relevant precritical leaf and entropy associated to external angle*) We call a precritical leaf *relevant* if it separates the α gap from its negative and if it is on the backward orbit of the preferred major leaf, and define $N(n) = N_\vartheta(n)$ as the number of relevant precritical leaves of generation n . We define the *core entropy* of this lamination as $h = h(\vartheta) := \limsup_n \frac{1}{n} \log N_\vartheta(n)$.

In the special case $\vartheta = 0$, the lamination is trivial and we naturally define $N_\vartheta(n) := 0$.

If the α gap is an infinite polygon, then $N_\vartheta(n) = 0$ except for $n = 1$, so $h(\vartheta) = 0$.

Remark A natural question is whether the lim sup in the definition of entropy can be replaced by a simple lim. This is not always so: Wolf Jung kindly pointed out to us the example of $c = c(9/56)$ where the Hubbard tree has the shape of a Y so that the branch point is fixed and one endpoint maps to the second, which maps to the third, which in turn maps to the branch point. Here $N(n) = 0$ for infinitely many n while $h = (\log 2)/3 > 0$. There are thus counterexamples when the dynamics is renormalizable. We prove that renormalizability is the only obstruction (see Corollary 6.7). For now, observe that in the postcritically finite non-renormalizable case, there is a finite iterate for which every edge covers the entire tree, so the associated transition matrix for the edge dynamics has a power with all entries positive. Therefore the associated subshift of finite type is irreducible, which implies that the limit exists and equals the lim sup (compare for instance Jung 2014, Section 3.3).

In the definition of relevant precritical leaves, we only count those leaves that are preimages of the preferred major leaf. Without this restriction, the count of $N(n)$ would increase by a factor 2, which would yield the same definition of entropy, but we would lose monotonicity of $N(n)$ as in Lemma 2.5 for the stupid reason that the degenerate minor leaf has only half as many preimages (of course, an alternate way to remedy this problem would be to count preimages of a degenerate minor leaf with multiplicity two).

Thurston showed that the union of all minor leaves of all invariant quadratic laminations forms itself a lamination, called the *quadratic minor lamination* QML (Thurston 2009). It turns out that h is naturally defined on QML: since both ends of any leaf in QML define the same lamination, we can first extend the definition of h to each separate leaf on QML. Complementary components of leaves in QML are called *gaps*, and they come in two kinds: either they are finite polygons (corresponding to Misiurewicz–Thurston parameters) or have infinitely many boundary leaves (and describe hyperbolic components). In both cases, it is easy to see that h is constant on all boundary leaves of any gap, so h naturally extends to the disk on which QML is defined, as a constant function on the closure of each leaf and each gap.

The equivalence relation defining QML is closed, so the quotient of the supporting closed unit disk by collapsing all leaves to points yields a topological Hausdorff space called the “abstract Mandelbrot set” \mathcal{M}_{abs} (this construction is known as Douady’s “pinched disk model” of \mathcal{M}). Since h is constant on fibers of the quotient map $q: \mathbb{D} \rightarrow \mathcal{M}_{abs}$, it follows that h is naturally a function on \mathcal{M}_{abs} .

Finally, there is the natural projection $\pi: \mathcal{M} \rightarrow \mathcal{M}_{abs}$ from the Mandelbrot set \mathcal{M} to the abstract Mandelbrot set \mathcal{M}_{abs} . It is defined by mapping every landing point $c(\vartheta)$ of any rational parameter ray ϑ to the equivalence class of the angle ϑ ; then $\pi: \partial\mathcal{M} \rightarrow \partial\mathcal{M}_{abs}$ is the unique continuous extension, and from here it is easy to extend π to a continuous map $\mathcal{M} \rightarrow \mathcal{M}_{abs}$ that on each component of the interior is either injective or constant (for details, see Douady 1993).

We thus obtain a unique map $\tilde{h} = h \circ \pi: \mathcal{M} \rightarrow [0, \log 2]$, and continuity of \tilde{h} follows from continuity of h , with continuity of π being well known. More specifically, the map \tilde{h} can also be constructed explicitly as follows.

We define a *ray pair* $RP(\phi, \psi)$ to be the union of two rays with angles ϕ and ψ (in a dynamical plane or in parameter space) that land (or perhaps accumulate) at a common point, together with the union of their accumulation sets. Every ray pair divides \mathbb{C} into two open components.

Definition 2.2 (*Entropy associated to quadratic polynomial in \mathcal{M}*) Let $p_c(z) := z^2 + c$ be any quadratic polynomial with $c \in \mathbb{C}$ for which the critical value is in the Julia set, and let ϑ be the external angle of any parameter ray that lands the parameter c , or contains this parameter in its impression (or accumulates on the boundary of any non-hyperbolic component of \mathcal{M} that contains c). The *critical ray pair* will be the ray pair $RP(\vartheta/2, (1 + \vartheta)/2)$, and a precritical ray pair of generation n will be any ray pair on the backward orbit of the critical ray pair that takes n iterations to map to the ray $R(\vartheta)$.

If p_c is such that the critical value is in the Fatou set, then it has an attracting or parabolic orbit and there is a unique periodic characteristic ray pair that lands on the

boundary of the Fatou component containing the critical value; precritical ray pairs are then defined as ray pairs on its backward orbit, and their generation is the number of iterations it takes to map to the characteristic ray pair.

We call a precritical ray pair *relevant* if it separates the α fixed point from its negative, and define $\tilde{N}(n)$ as the number of relevant precritical ray pairs of generation n . We define the *core entropy* of p_c as $\tilde{h} = \tilde{h}(c) := \limsup_n \frac{1}{n} \log \tilde{N}(n)$.

Note that we use the term “separation” in a combinatorial sense: the two rays in a separating ray pairs either land together or accumulate at the same fiber.—We do not wish to (or need to) deal with topological subtleties such as whether, in the Cremer case, a dynamic ray indeed accumulates at the critical value: every $c \in \mathcal{M}$ has naturally associated one or several external angles ϑ that define its dynamics, and this is sufficient for our combinatorial approach.

We need to fix some notation on Hubbard trees of postcritically finite polynomials. By definition, the *Hubbard tree* is the unique minimal tree within the filled-in Julia set connecting the postcritical set (subject to a natural condition on how to traverse bounded Fatou components in case some critical points are periodic); see for instance Douady and Hubbard (1985a), Bielefeld (1992), Bruin and Schleicher (2008). The *marked points* or *vertices* of the Hubbard tree are the endpoints, branch points, and the postcritical points. (In fact, all endpoints are postcritical points; critical points are not included in their own right, but they might be postcritical, for instance when they are periodic, and when the degree is greater than 2 then they might also be branch points.) Since the set of vertices is forward invariant, every edge (a closed arc connecting two vertices) maps over one or several entire edges, so that the image contains every edge the interior of which intersects the image; in other words, the edges form a Markov partition on the Hubbard tree.—Here, we will only discuss quadratic polynomials and Hubbard trees.

The usual definition of core entropy is modeled after the postcritically finite case. The finite set of edges on the tree form a Markov chain with associated transition matrix, where the matrix element $M_{i,j}$ is 0, 1, or 2 if the edge e_i covers the edge e_j respectively 0, 1, or 2 times. Having only non-negative real entries, this matrix has a leading eigenvalue which is real, and its logarithm is defined as the core entropy of the given postcritically finite parameter. This definition coincides with the classical definition of topological entropy of general dynamical systems, and it also applies in the postcritically infinite case as long as the Hubbard tree is defined (i.e. the Julia set is path connected) and still finite (in this case the map is sometimes called *topologically finite*; Tiozzo 2013). However, the number of edges of the Hubbard trees is not locally bounded even among postcritically finite maps, which makes entropy estimates based on these transition matrices difficult.

It is well known—at least in the postcritically finite case, in which case the edges form a Markov partition—that if x is any point on the Hubbard tree and $N_x(n)$ is the number of preimages of x of generation n on the Hubbard tree, then $h = \limsup_n \log N_x(n)$ is the core entropy. Since each of the finitely many edges, except those within “renormalizable little Julia sets”, will cover the entire tree after finitely many iterations (Lemma 6.5), one can as well count only those preimages of x that are on an arbitrary subset of the edges of the Hubbard tree, as long as at least one of

these edges is not in a renormalizable little Julia set. For instance, instead of counting precritical leaves on $[\alpha, -\alpha]$ (as in our definition above) we may count preimages on $[\alpha, \beta]$ (as we will do in Sect. 5) or on $[\beta, -\beta]$.

Lemma 2.3 (Definitions of core entropy coincide) *For postcritically finite polynomials, the core entropy as in Definition 2.2 coincides with the usual definition (in terms of transition matrices on finite Hubbard trees).*

If for p_c the critical value is in the impression of the dynamic ray at angle ϑ , the core entropy of p_c equals the core entropy of the lamination L_ϑ .

Proof There is an $m \geq 1$ so that $\bigcup_{i=1}^m p_c^i[\alpha, -\alpha]$ is equal to the Hubbard tree of p_c . Therefore, the first statement follows from the remark above: the growth rate of precritical points on $[\alpha, -\alpha]$ equals the growth rate of precritical points on the Hubbard tree. Moreover, precritical point on $[\alpha, -\alpha]$ are in bijection with relevant precritical ray pairs; this implies the second statement.

It is well known that if several parameter rays accumulate at the same parameter in \mathcal{M} , then the laminations associated to their corresponding angles coincide, so these angles have the same entropy (if more than two rays accumulate at the same parameter, then the parameter is a Misiurewicz–Thurston parameter and the rays actually land there, see Schleicher (2004, Corollary 7.5); the reason is that parameter ray pairs at periodic angles alone subdivide \mathbb{C} and \mathcal{M} so that three parameter rays at non-periodic angles can fail to be separated only when they are either all preperiodic and land at the same Misiurewicz–Thurston-parameter, or they are all irrational and land at the same hyperbolic component).

If $\vartheta \in \mathbb{S}^1$ is so that the parameter ray of \mathcal{M} at angle ϑ lands (in particular, if ϑ is rational), we define $c(\vartheta)$ as the landing point in \mathcal{M} of the parameter ray at angle ϑ , and within any connected Julia set we define $z(\vartheta)$ as the landing point of the dynamic ray at angle ϑ (if the latter ray lands).

We define a partial order on \mathcal{M} as follows: if c, \tilde{c} are two parameters in \mathcal{M} , we say that $c < \tilde{c}$ if there is a parameter ray pair $RP(\phi^-, \phi^+)$ at periodic angles that separates \tilde{c} simultaneously from c and from the origin.

Following Milnor (2000b), a periodic or preperiodic ray pair $RP(\phi^-, \phi^+)$ (with $\phi^-, \phi^+ \in \mathbb{S}^1$) in the dynamical plane of p_c is called *characteristic* if the dynamic rays $R(\phi^-)$ and $R(\phi^+)$ land together at some point x in such a way that they separate the critical value c from the critical point 0 as well as from all other rays landing at $\bigcup_{k \geq 0} p_c^{ok}(x)$. It is well known that every periodic ray pair $RP(\phi^-, \phi^+)$ has a unique characteristic ray pair on its forward orbit.

Let us now state the Correspondence Theorem relating the combinatorics of dynamical and parameter ray pairs; see for instance Milnor (2000b) or Schleicher (2004).

Theorem 2.4 (Correspondence Theorem) *A ray pair $RP(\phi^-, \phi^+)$ in the dynamical plane of p_c with ϕ^\pm rational is characteristic if and only if the parameter rays with angles ϕ^- and ϕ^+ land together and separate the parameters c and 0 from each other.*

Lemma 2.5 (Monotonicity of lamination and of entropy) *If $c < \tilde{c}$, then $N_c(n) \leq N_{\tilde{c}}(n)$ for all n and thus $h(c) \leq h(\tilde{c})$. Moreover, any characteristic leaf in L_c also occurs in $L_{\tilde{c}}$.*

Proof It is routine to check that any precritical leaf of c also “occurs” in the dynamics (or the lamination) of \tilde{c} : the major leaf (or leaves) in $L_{\tilde{c}}$ separate the two major leaves in L_c . Precritical leaves in L_c are preimages of the pair of major leaves (the preimages of this pair are always “parallel”, i.e. not separated by the critical value, because otherwise the forward orbit of the critical value would have to intersect the domain bounded by the two major leaves). Each pair of preimages surrounds one preimage of the major leaf of \tilde{c} (or a pair of preimages of the major leaves), and when such a preimage separates the α gap from its negative in L_c , then it also does so for $L_{\tilde{c}}$. Therefore, $N_{\tilde{c}}(n) \geq N_c(n)$ and $\tilde{h}(\tilde{c}) \geq \tilde{h}(c)$.

The statement about characteristic leaves (or characteristic ray pairs) is well known and follows from the Correspondence Theorem 2.4.

For the record, we define a *dyadic parameter* as a parameter c that is the landing point of a parameter ray at a dyadic angle $a/2^m$; in this case, we call m the *generation* of c .

3 Topological Surgery on Dyadic Hubbard Trees

Combinatorial arcs and veins. Conjecturally, the Mandelbrot set is path connected: every $c \in \mathcal{M}$ has an arc $[0, c]$ that connects it to the origin. Such arcs are unique when requiring that they traverse hyperbolic components only along internal rays (radial curves with respect to the parameterization by the multiplier map). In fact, for many parameters $c \in \mathcal{M}$ one can prove that such an arc actually exists (using an observation of Jeremy Kahn using Yoccoz’ puzzle results, or work of Riedl (2000) using quasiconformal surgery). However, we only need a combinatorial version of such arcs. One way of defining them is as follows: the abstract Mandelbrot set \mathcal{M}_{abs} (as defined in Sect. 2) is well known to be locally connected and hence path connected, and it comes with a continuous projection $\pi : \mathcal{M} \rightarrow \mathcal{M}_{abs}$. For $c \in \mathcal{M}$, there exists a preferred path $\Gamma(c) \subset \mathcal{M}_{abs}$ connecting $\pi(0)$ to $\pi(c)$, and then we define $[0, c] := \pi^{-1}(\Gamma(c))$. It is not hard to describe this combinatorial arc $[0, c]$ in terms of internal rays of hyperbolic components and of fibers that separate 0 from c ; the details are somewhat tedious but not very enlightening.

Specifically if c is a dyadic parameter in \mathcal{M} , we call the combinatorial arc $[0, c]$ the *long vein* of c . The *vein* of c is the shortest closed sub-arc of the long vein connecting c to the union of the long veins of all dyadic parameters of lower generation than c .

If c and c' are two dyadic parameters, then it is well known that their long veins intersect in an arc $[0, x]$, where x is postcritically finite; this result is known as the “branch theorem” of \mathcal{M} (Douady and Hubbard 1985a; Schleicher 2004). Moreover, in the special case that c and c' are dyadic of equal generation, then x is on the vein of a dyadic parameter c'' of lower generation. This means that veins (minus endpoints) are disjoint.

Definition 3.1 (*Directly subordinate parameter* $c \triangleleft c'$) We say that c is *directly subordinate* to c' and write $c \triangleleft c'$ if the vein of c terminates at an interior point of the vein of c' ; in addition, any dyadic parameter other than $c = -2$ whose vein terminates at 0 is declared to be subordinate to $c(1/2) = -2$.

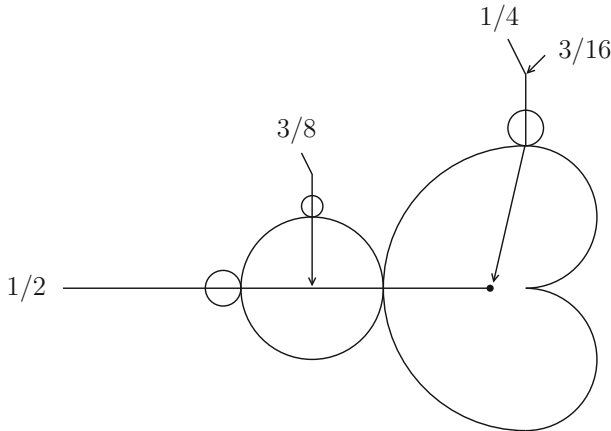


Fig. 2 Illustration of directly subordinate parameters: we have $c(3/16) \triangleleft c(1/4)$, $c(1/4) \triangleleft c(1/2)$, and $c(3/8) \triangleleft c(1/2)$. Arrows indicate where the vein to some dyadic parameter terminates at the vein of the dyadic parameter to which it is directly subordinate

If $c \triangleleft c'$, then necessarily the external angle of c' has lower denominator than that of c . Note that this is not a transitive relation and thus not a partial order. A few directly subordinate dyadic parameters are illustrated in Fig. 2.

Theorem 3.2 (Comparing entropy between dyadic Hubbard trees) *If $c \triangleleft c'$ are two dyadic parameters, then*

- (a) $N(n) \leq N'(n)$ for all n ,
- (b) $N(n) < N'(n)$ for all sufficiently large n , and
- (c) $\tilde{h}(c) < \tilde{h}(c')$.

Here $N(n)$ and $N'(n)$ denote the numbers of relevant precritical points of generation n for p_c and for $p_{c'}$. This immediately implies a weak version of the Tiozzo conjecture:

Corollary 3.3 (Dyadic version of Tiozzo conjecture) *Every dyadic angle ϑ has a neighborhood on which h , restricted to dyadic angles, assumes its unique maximum at ϑ .*

From here, we could use continuity of h to prove the unrestricted version of the Tiozzo conjecture (which indeed was our strategy in an early version of the proof). We will argue the other way around: it is much easier to deduce the conjecture directly and use this in the proof of continuity.

Let us state another corollary to the Correspondence Theorem 2.4: it is a more precise version than monotonicity of entropy (as stated in Lemma 2.5) because even the lamination is monotone.

Corollary 3.4 (Monotonicity of rational lamination on Hubbard tree). *Let $c, c' \in \mathcal{M}$ be two postcritically finite parameters such that $c < c'$. Suppose that in the dynamical plane of p_c two periodic or preperiodic rays $R(\phi^-)$ and $R(\phi^+)$ land together at some point in the Hubbard tree of p_c , but not on the backward orbit of the critical value. Then in the dynamical plane of c' the dynamic rays $R(\phi^-)$ and $R(\phi^+)$ also land together and the landing point is in the Hubbard tree of c' .*

Proof In the dynamical plane of p_c , let x be the landing point of the dynamic ray pair $RP_c(\phi^-, \phi^+)$; it is by hypothesis in the Hubbard tree of c and it must be a repelling periodic or preperiodic point because p_c is postcritically finite.

Let $W \subset \mathbb{C}$ be the set of parameters c'' for which the dynamic rays $R_{c''}(\phi^-)$ and $R_{c''}(\phi^+)$ land together at a repelling periodic or preperiodic point that is not precritical. By Theorem 2.4, the set W is open, and its boundary in \mathbb{C} consists of parameters $c'' \in \mathbb{C} \setminus \mathcal{M}$ where the critical value is on the forward orbit of one of the rays $R_{c''}(\phi^-)$ and $R_{c''}(\phi^+)$, as well as of parameters $c'' \in \mathcal{M}$ where at least one of the two rays lands at a parabolic orbit or at a precritical point. By hypothesis, the set W contains c and is thus non-empty, and any parameter $c'' \in \mathbb{C} \setminus \overline{W}$ must be separated from c by a parameter ray pair with angles on the forward orbit of ϕ^- or ϕ^+ .

However, in the dynamical plane of c , all dynamic rays at such angles land at the Hubbard tree, and their configuration shows that there is no parameter ray pair available that, for $c' \succ c$, could bound c' away from c . Thus $c' \in W$.

Definition 3.5 (*Dynamical counterpart to parameter*)

Let c be a postcritically finite parameter and suppose $c' \succ c$. Then a (pre)periodic point x in the dynamical plane of $p_{c'}$ is called the *dynamical counterpart to c* in the following cases:

- if c is preperiodic, then x is the landing point of the preperiodic dynamic rays at the same angles as c ;
- if c is periodic, then x is the landing point of the periodic dynamic rays bounding (in the parameter plane) the subwake of c containing c' .

In the periodic case, c is the center of a hyperbolic component, say H_c , and the subwake of c containing c' is bounded by a periodic parameter ray pair landing at ∂H_c . The angles of this ray pair are the angles of two rays landing at x . For example, the α fixed point is the dynamical counterpart of $c = 0$. In the preperiodic case, it is known that all dynamic rays with the same angles of the rays landing at c also land together in the dynamical plane of c' .

An equivalent definition is that x is the unique repelling periodic or preperiodic point in the dynamical plane of c' such that the itinerary of x (with respect to the critical point) equals the (upper) kneading sequence of c .

Lemma 3.6 (*Dynamical counterpart is characteristic*) *Every periodic or preperiodic point that is a dynamical counterpart is characteristic.*

Proof If, in the dynamical plane of $p_{c'}$, the point x is the dynamical counterpart of a parameter c , then $c' \succ c$, and x is the landing point of at least two dynamic rays. If p_c is critically strictly preperiodic, then the parameter rays landing at c bound the wake that c' is in, and this implies that x is a characteristic preperiodic point. If p_c is critically periodic, then the argument is similar, except that the parameter rays do not land at c but at the root of the hyperbolic component containing c .

Lemma 3.7 (*Directly subordinate dyadics*) *If c and c' are two dyadic parameters, then $c \triangleleft c'$ if and only if there is a postcritically finite parameter $c_* \in \mathcal{M}$ so that c' is the dyadic of least generation within any sublimb of c_* , and c is the dyadic of least generation within a different sublimb of c_* than c' .*

In this case, denoting the external angles of c and c' by ϑ and ϑ' , respectively, then in the dynamics of c (or any other parameter in the same sublimb of c_*) there is a repelling (pre)periodic point x_* that is the landing point of at least three dynamic rays that separate the dynamic rays at angles 0 , ϑ , and ϑ' . The point x_* is the dynamical counterpart to c_* .

Proof Any two dyadic parameters are endpoints of \mathcal{M} , so by the Branch Theorem of the Mandelbrot set there is a unique postcritically finite parameter c_* that contains c and c' in two different of its sublimbs. Let c_0 be the unique dyadic of least generation in any of the sublimbs of c_* ; then c_* is on the long vein of all three of c_0 , c , and c' , and it is on the vein of c_0 .

The assumption that $c \triangleleft c'$ means that the vein of c terminates at an interior point of the vein of c' , and hence it must terminate at the parameter c_* , so c_* is an interior point of the vein of c' . Since c_* is also an interior point of the vein of c_0 , it follows that $c' = c_0$ (two veins can never have more than one point in common).

Conversely, if c' is the dyadic of least generation in the sublimb of c_* and c is the dyadic of least generation within a different sublimb of c_* than c' , then c_* is in the interior of the vein of c' and the vein of c terminates at c_* . This proves the first claim.

For the second claim, we first consider the case that c_* is a Misiurewicz–Thurston parameter; it is then the landing point of $s \geq 3$ rational parameter rays, say at angles $\vartheta_1, \dots, \vartheta_s$, so that the parameter rays at angles 0 , ϑ , and ϑ' are in different sectors with respect to these parameter rays. Every parameter in any sublimb of c_* has the property that the dynamic rays at angles $\vartheta_1, \dots, \vartheta_s$ land together at a repelling preperiodic point, and the claim follows.

If c_* is the center of a hyperbolic component, then the parameter c is in a sublimb at internal angle $p/q \neq 1/2$, and in the dynamical plane of c (or any parameter within the same sublimb) there is a repelling periodic point that is the landing point of $q \geq 3$ dynamic rays that separates the angles 0 , ϑ and ϑ' so that ϑ is in the largest sector not containing the angle 0 .

Lemma 3.8 (No extra branch point) *Suppose $c' \in \mathcal{M}$ is a dyadic parameter and $c \in \mathcal{M}$ is the postcritically finite parameter where the vein of c' ends. Let x be the dynamical counterpart of c in H' . Then the arc $(x, c'] \subset H'$ does not contain a branch point.*

Proof Suppose the claim is false and there is a branch point on $(x, c']$. Then the sub-wake of x containing c' also contains a point, say c'' , in the forward orbit of c' (all endpoints of H' are on the orbit of c'). But then c'' is the landing point of a dyadic ray of lower generation than the dyadic ray landing at c' . This is impossible because in the parameter plane the dyadic ray landing at c' has the lowest possible generation among all dyadic rays in the sub-wake of c containing c' (by hypothesis that the vein of c' terminates at c).

The main step in proving Theorem 3.2 is a topological surgery on Hubbard trees, as follows:

Proposition 3.9 (Relation between subordinate dyadic Hubbard trees). *Let $c \triangleleft c'$ be two dyadic parameters with external angles ϑ and ϑ' , let H_c be the Hubbard tree*

of c , and let $H \supset H_c$ be the connected hull of the critical orbit and of the orbit of $z(\vartheta')$. Let x be the branch point of the arcs from 0 to c and to $z(\vartheta')$. If p_c is the natural map on H , define a map $f: H \rightarrow H$ as follows: choose a homeomorphism $\rho: [x, c] \rightarrow [x, z(\vartheta')]$ fixing x and let

$$f(z) := \begin{cases} \rho \circ p_c(z) & \text{if } p_c(z) \in [x, c] \\ p_c(z) & \text{otherwise.} \end{cases}$$

Let H' be the connected hull within H of the orbit of 0 under f . Then (H', f) is the Hubbard tree of $p_{c'}$ (up to isotopy rel branch points and endpoints).

Proof Since there are no branch points on $[x, c]$ (Lemma 3.8), the new map f is a branched covering where 0 is the unique critical point.

We have a connected tree H containing the critical point 0 , and with respect to f the orbit of 0 is still finite (it still terminates at the β fixed point). Therefore, (H', f) is a finite tree with a continuous self-map, and the dynamics is locally injective except at the critical point 0 . Since there are at most 2 branches at 0 , the map is globally at most $2:1$. Every endpoint is by definition on the critical orbit, and the tree comes with an embedding into \mathbb{C} that is compatible with the dynamics; i.e. the dynamics respects the cyclic order of the local arms at all branch points, see Bruin and Schleicher (2008). Therefore, up to isotopy rel branch points and endpoints, (H', f) is the Hubbard tree of a postcritically finite polynomial in which the critical orbit lands at the β fixed point at the desired number of iterations: this follows for instance from Bielefeld (1992) or Bruin et al. (2009), or by observing that for topological polynomials any obstruction must be a degenerate Levy cycle, but such cannot exist in this case where the periodic part of the postcritical set in \mathbb{C} consists of a single fixed point. Let c'' be the corresponding parameter and ϑ'' be the external angle; we have $\vartheta'' = a''/2^{k'}$. It remains to prove that $\vartheta'' = \vartheta'$ and thus $c'' = c'$.

Since $c \triangleleft c'$, there is a postcritically finite branch point in \mathcal{M} , say c_* , that separates c from c' , and the external angles of c_* are the external angles of x in the dynamical plane of c (Lemma 3.7). In the dynamical plane of c'' , the point x has the same external angles because it has the same period and preperiod and the dynamics of the subtree connecting the orbit of x is unaffected by the surgery (except the bit around the critical point that maps past x). Hence ϑ'' is the unique dyadic of least generation that is separated from the angle 0 by the angles of x , and the same is true for ϑ' .

Remark The fact that $c'' = c'$ can also be shown using Hubbard and Schleicher (1994) and Thurston's theorem. Let us topologically extend the map $f: H' \rightarrow H'$ to a continuous map on \mathbb{C} as follows. First, we set f to be p_c on the dynamic rays $R(2^t \vartheta')$ of p_c , where $t \in \{0, 1, \dots, k'\}$. There are $k' + 1$ topological discs in the complement of $H' \cup_{t \geq 0} R(2^t \vartheta')$ and the map f easily extends to each of them as a homeomorphism. The new map f is a topological polynomial for which $\bigcup_t R(2^t \vartheta')$ forms an invariant spider. Since this spider is equivalent to a standard invariant spider of c' , we get $c' = c''$ by Thurston rigidity (Douady and Hubbard 1993; Hubbard and Schleicher 1994).

Lemma 3.10 (Injective dynamics of last edge) *In any dyadic Julia set, consider any dyadic angle $\vartheta = a/2^k$ with $k \geq 1$ and let x be the point where the arc from $z(\vartheta)$ to α is attached to the minimal tree connecting all dyadic endpoints of generations $a'/2^{k'}$ with $k' < k$. Then $[z(\vartheta), x]$ maps injectively for k iterations to an interval $[\beta, y] \subset [\beta, \alpha]$.*

Proof For every integer $m \geq 0$, let T_m be the minimal tree connecting the α fixed point to all dyadic endpoints of generation at most m . Let f be the map on the Julia set.

The edge $[z(\vartheta), x] \subset T_k \setminus T_{k-1}$ certainly maps forward homeomorphically one generation to an arc $[z(2\vartheta), x']$, where $x' = f(x) \in f(T_{k-1})$. We claim that $[z(2\vartheta), x'] \subset T_{k-1} \setminus T_{k-2}$ so that the inductive step applies and completes the proof.

We first show that

$$f^{-1}(T_{k-2}) \subset T_{k-1}. \tag{3.1}$$

Indeed, T_{k-2} is the minimal tree connecting all dyadic endpoints of generation at most $k - 2$. Consider an endpoint y of the forest $f^{-1}(T_{k-2})$. If $f(y) \in T_{k-2}$ was not an endpoint, so it was connected to at least two edges in T_{k-2} , then y would have to be connected to at least two edges in $f^{-1}(T_{k-2})$, a contradiction. Thus every endpoint of the forest $f^{-1}(T_{k-2})$ is a dyadic endpoint of generation at least $k - 1$ and hence $f^{-1}(T_{k-2}) \subset T_{k-1}$.

Finally, if $[z(2\vartheta), x']$ intersects T_{k-2} , then $[z(\vartheta), x]$ intersects T_{k-1} , and by hypothesis this intersection is the single point x . Hence $[z(2\vartheta), x'] \subset (T_{k-1} \setminus T_{k-2}) \cup \{x'\}$.

Remark In this lemma, the hypothesis that the polynomial be dyadic was stated only for convenience. All we are using is that the Julia set is path connected (if there are bounded Fatou components, the notation needs minor adjustments).

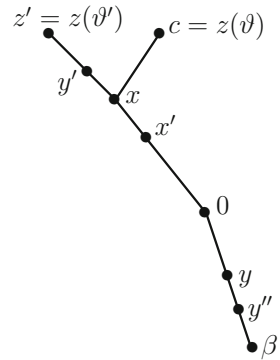
Lemma 3.11 (Homeomorphic preimage of arc) *Suppose that $c \triangleleft c'$ are two dyadic parameters and let ϑ, ϑ' be their external angles. In the Julia set of c , let x be the branch point between $0, z(\vartheta) = c$ and $z(\vartheta') =: z'$. If $\vartheta = a/2^k$ for odd a , then there is a point $y' \in (z', x)$ so that $p_c^{ok} : [z', y'] \rightarrow p_c^{ok}([z', y']) = p_c^{ok}([c, x])$ is a homeomorphism.*

Proof. We know from Lemma 3.10 that $[c, x]$ maps homeomorphically for k iterations to a subinterval of $[\beta, \alpha]$; define $y := p_c^{ok}(x) \in (\beta, \alpha]$, see Fig. 3. Similarly, there is a point $x' \in [z', 0]$ so that $[z', x']$ maps forward homeomorphically for k' iterations (if $\vartheta' = a'/2^{k'}$). We have $x' \in [x, 0]$ (or equivalently $x \in [z', x']$) because $c \triangleleft c'$, i.e. c is directly subordinate to c' .

More precisely, if c_* is the point where the vein of c terminates (Lemma 3.7), then the external angles of the dynamic rays landing at x are exactly the external angles of the parameter rays bounding the subwake of c_* containing c . Analogously, the same is true for the point x' and the vein of c' ; let c'_* be this branch point. But since $c \triangleleft c'$, it follows that c'_* separates c_* from the origin, and hence, by the Correspondence Theorem 2.4, x' separates x from the origin.

Let $y'' := p_c^{ok'}(x)$; then $p_c^{ok'} : [z', x] \rightarrow [\beta, y'']$ is a homeomorphism. Iterating this $k - k'$ further times, the image arc terminates at β and at y , but it can no longer be injective (the map $p_c^{ok} : [c, x] \rightarrow [\beta, y]$ is a homeomorphism, and p_c^{ok} is a local homeomorphism near x because x cannot be on the critical orbit).

Fig. 3 Illustration of the relative position of various points in the Hubbard tree of c in the proof of Lemma 3.11. Note that we do not know or need the relative position between $0, y,$ and y'' ; in particular, we do not claim $y'' \in [\beta, y]$



There is a branch $p_c^{-1} : [\beta, \alpha] \rightarrow [\beta, -\alpha]$; let $[\beta, y''']$ be the image of $[\beta, y]$ under the $k - k'$ -th iterate of this branch. Observe that $[y''', \beta] \subsetneq [y'', \beta]$ because otherwise $p_c^{\circ(k-k')}$ restricted to $[\beta, y'']$ would have degree 1. Pulling back k' times we obtain an interval $[z', y'] \subset [z', x]$ so that $p_c^{\circ k'} : [z', y'] \rightarrow [\beta, y''']$ and $p_c^{\circ k} : [z', y'] \rightarrow [\beta, y]$ are homeomorphisms, as claimed. \square

Proposition 3.12 (Injection between precritical points) *Given two dyadic parameters $c \triangleleft c'$, there exists a generation-preserving injection B from the set of precritical points in $[c, -\alpha]$ of p_c to the set of precritical points in $[c', -\alpha]$ of $p_{c'}$. Moreover, B can be taken to satisfy the following properties (A)–(C) for every precritical point $\zeta \in [c, -\alpha]$.*

- (A) *For $k \geq 0$ we have $p_c^{\circ k}(\zeta) \in [c, -\alpha]$ if and only if $p_{c'}^{\circ k}(B(\zeta)) \in [c', -\alpha]$. Moreover, if $p_c^{\circ k}(\zeta) \in [c, -\alpha]$, then*

$$B(p_c^{\circ k}(\zeta)) = p_{c'}^{\circ k}(B(\zeta)).$$

- (B) *Assume c_* is a postcritically finite parameter such that $c_* \prec c$ and $c_* \prec c'$. Let x_* and x'_* be the dynamical counterparts of c_* in the dynamical planes of c and c' . Then $\zeta \in [x_*, -\alpha]$ if and only if $B(\zeta) \in [x'_*, -\alpha]$.*
- (C) *There is a sub-interval $J \subset [c', \alpha]$ such that $p_{c'}^{\circ k}(J) = [-\alpha, \alpha]$ for some $k > 0$ and such that the image of B is in $[c', -\alpha] \setminus J$.*

Proof We continue to use Proposition 3.9 and its notation. In particular, in the Julia set of p_c , let x be the branch point between the arcs from 0 to c and to $z(\vartheta')$ (see Fig. 3); and we may identify the dynamics $p_{c'} : H_{c'} \rightarrow H_{c'}$ on the Hubbard tree of $p_{c'}$ with $f : H' \rightarrow H'$ in the dynamics of p_c .

The point $x_* \in H$ is periodic or preperiodic under iteration of p_c and never visits $[x, c]$ because the orbit of x_* never visits $[x_*, c]$ and x_* as the dynamic counterpart of c_* is before x ; i.e. $x_* \in [\alpha, x]$. Thus $x_* \in H'$ is identified with $x'_* \in H_{c'}$. We will now construct a bijection B between precritical points in $[c, -\alpha]$ of p_c and those precritical points in $[z', -\alpha] \setminus [y', x]$ of f for which the orbit never visits $[y', x]$, where y' is specified in Lemma 3.11 (see Fig. 3). In fact, our bijection will preserve

the itinerary with respect to $H \setminus \{0\}$, except for k iterations along the orbit from $[c, x]$ to $[\beta, y] = p_c^{ok}([c, x])$ (resp. from $[z', y']$ to $[\beta, y]$).

Our proof proceeds by induction on the number of times, say m , that an orbit of a precritical point ζ runs through $[c, x]$ (not counting ζ itself). We start by those precritical points on $[c, x]$ (for the map p_c that never run through (c, x) again, that is with the case $m = 0$). By Lemma 3.11, an appropriate branch of $f^{o(-k)} \circ p_c^{ok}$ sends $[c, x]$ homeomorphically to $[z', y']$; call this branch $\eta: [c, x] \rightarrow [z', y']$. Then for any $a \in [c, x]$, we have $p_c^{ok}(a) = f^{ok}(\eta(a))$ and the future orbits of these points under p_c respectively under f coincide as long as the orbits avoid $[c, x]$. Note that all precritical points on $[c, x]$ must have generation at least k . We thus obtain an injection, say B_0 , of precritical points with $m = 0$.

Every precritical point $\zeta \in [c, x]$ of generation n and with $m = 0$ is the common endpoint of two adjacent sub-intervals of $[c, x]$ that map homeomorphically onto $[c, x]$ after n iterations: we have $p_c^{on}(\zeta) = c$ and $p_c^{o(n-1)}(\zeta) = 0$, so we can pull the entire interval $[c, x]$ back in two ways (with a choice in the first step) until we end at ζ . The pull-back of the entire interval $[c, x]$ is possible because no critical value can interfere (the critical orbit visits only endpoints of the tree), and the resulting interval is in $[c, x]$ because the Hubbard tree is unbranched on $[c, x)$ and the orbit of x never enters $[c, x)$.

There is an analogous result about precritical points $\zeta' \in [z', y']$ of f with $m = 0$ and sub-intervals of $[z', y']$ that map to $[z', y']$. By construction of y' , the point ζ' has generation $n \geq k$, and there is a precritical point $\zeta = \eta^{-1}(\zeta') \in [c, x]$. The point ζ has a neighborhood, say $I_\zeta \subset [c, x]$, that maps 2 : 1 under p_c^{on} onto $[c, x]$ (the union of the two intervals constructed above), and then ζ' has a neighborhood $I_{\zeta'} \subset [z', y']$ with $p_c^{ok}(I_{\zeta'}) = p_c^{ok}(I_\zeta) \subset [\beta, y]$.

The bijection B_0 of precritical points with $m = 0$ thus extends to a bijection between intervals $I \subset [c, x]$ that map homeomorphically onto $[c, x]$ after some number of iterations without visiting (c, x) before, and intervals $I' \subset [z', y']$ that map homeomorphically onto $[z', y']$ after the same number of iterations and without ever visiting (z', y') ; this bijection respects the number of iterations as well as the order along the intervals within $[c, x]$ and $[z', y']$ (the intervals are obviously disjoint). Denote this bijection of intervals by B_0^* .

Now suppose the statement is shown for all precritical points on $[c, x]$ that visit (c, x) at most m times, for some $m \geq 0$; in particular, we have an injection, say B_m , from precritical points on (c, x) that map into (c, x) exactly m times, to precritical points on (z', y') that map into (z', y') exactly m times and never visit (y', x) . Consider any precritical point $\zeta \in (c, x)$ that visits (c, x) exactly $m + 1$ times, and let s be minimal such that $p_c^{os}(\zeta) \in (c, x)$. Then there is an interval $I \ni \zeta$ so that $p_c^{os}: I \rightarrow [c, x]$ is a homeomorphism (same reasoning as above). Let $I' := B_0^*(I)$. Then $p_c^{os}(\zeta) \in (c, x)$ is a precritical point that visits (c, x) only m times, and $B_{m+1}(\zeta) = \zeta' := f^{o(-s)} \circ B_m \circ p_c^{os}(\zeta)$, choosing the branch $f^{o(-s)}: [y', z'] \rightarrow I'$.

Since the map B_0^* is injective, different intervals I land in disjoint intervals I' , and since B_m is injective by induction, the restriction of B_{m+1} that runs through any particular I is injective too, so in total B_{m+1} is injective as claimed.

This takes care of all precritical points on $[c, x]$, and we still have to deal with those on $[x, -\alpha]$. But those with orbits that never run through $[c, x]$ are unaffected by the

changed dynamics, and the injection easily extends to those that map into $[c, x]$ under p_c .

It remains to show that the map B satisfies Properties (A)–(C). Let us extend B_0^* to all intervals in $[x, -\alpha]$ that are injective preimages of $[c, x]$ and never run through $[c, x]$ before mapping into $[c, x]$. It is easy to see that $B_0^*(I) \subset I$ for every such interval $I \not\subset [x, c]$ because $[z', y'] \subset [z', x]$ and $p_c^{\circ(-1)}([c, x]) = f^{-1}([z', x])$, and induction can be applied. Since x_* never visits $[x, c]$ (see the beginning of the proof), we see that $x_* \notin I$ for every maximal pre-image I of $[c, x]$. Also, by construction, $\zeta \in I$ if and only if $B(\zeta) \in I$ for every precritical ζ and interval $I \not\subset [x, c]$ as above. Therefore, ζ and $B(\zeta)$ are on the same side of x_* . It is clear that ζ and $B(\zeta)$ have the same return times to $[c, -\alpha]$ and $[z', -\alpha]$ because $p_c^{\circ k}([c, x]) = f^{\circ k}([z', y'])$. And the dynamical relation $B(p_c^{\circ k}(\zeta)) = p_{c'}^{\circ k}(B(\zeta))$ holds by construction. Thus Properties (A) and (B) hold. To prove (C), observe once again that every edge covers the entire tree after finitely many iterates; in particular, there is an interval $J \subset [x, y']$ for which the image contains $[-\alpha, \alpha]$ after some finite iterate, and by restricting appropriately if necessary we can make sure that the image equals $[-\alpha, \alpha]$.

Proof of Theorem 3.2 From Proposition 3.12 we have an injection of precritical points of p_c of any given generation on $[-\alpha, c]$ to precritical points of $p_{c'}$ of the same generation on $[-\alpha', c']$, and by claim (B) this injection restricts to the arcs $[-\alpha, \alpha]$ and $[-\alpha', \alpha']$. This immediately proves part (a), and part (b) follows from claim (C) of Proposition 3.12.

Part (c) of Theorem 3.2 also follows from claim (C) of Proposition 3.12 (compare also Jung 2014, Section 3.3): since $p_{c'}$ is dyadic, every edge of its Hubbard tree, say H' , maps over every other in a bounded number of iterations (the Markov partition associated to the edges is irreducible). Therefore, every typical orbit visits J with positive frequency, and the entropy of orbits in H' that are not allowed to visit J is strictly smaller than the full entropy in H' : this boils down to the statement that if two irreducible square matrices M and M' of real numbers have all non-negative entries, with those of M' at least as large as those of M and at least one strictly larger, then the leading eigenvalue of M' strictly exceeds that of M . Finally, precritical orbits in H inject to precritical orbits in H' avoiding J , and so H has smaller entropy than H' . □

4 Irrational Angles and the Tiozzo Conjecture

In this brief section, we will complete the proof of the Tiozzo Conjecture about local maxima of h , and we also describe local minima.

Theorem 4.1 (The Tiozzo conjectures) *The entropy function $h : \mathbb{S}^1 \rightarrow [0, \log 2]$ has the following properties.*

- (a) *Every dyadic angle is an isolated local maximum of the entropy function.*
- (b) *Conversely, every local maximum of h is dyadic.*
- (c) *Within every wake, the entropy function has a unique global maximum, and it occurs at the unique dyadic of lowest denominator in the wake.*

(d) *Within every wake, for each n the function $N_{\vartheta}(n)$ assumes its maximum at the dyadic of least generation (of course, this maximum is not unique).*

Proof Fix a dyadic angle ϑ_0 and let $I = I(\vartheta_0) \subset \mathbb{S}^1$ be the open interval of angles ϑ for which the combinatorial arcs to $c(\vartheta)$ intersect the interior of the vein of $c(\vartheta_0)$ (not the long vein). In other words, if c_* is the endpoint of the vein of $c(\vartheta_0)$, then I consists of the angles within the same subwake of c_* that $c(\vartheta)$ is in. Clearly $\vartheta_0 \in I$. Every dyadic angle in I is either directly or indirectly subordinate to ϑ_0 (where the latter means that there is a finite sequence of dyadic angles ending at ϑ_0 so that each is directly subordinate to the next).

By Theorem 3.2 part (c), we know that h restricted to dyadic angles in I has its unique maximum at ϑ_0 .

We claim that for every $\vartheta \in I$ and every $n \in \mathbb{N}$ we have $N_{\vartheta}(n) \leq N_{\vartheta_0}(n)$ and hence $h(\vartheta) \leq h(\vartheta_0)$. Indeed, if ϑ is a dyadic angle, then this is Theorem 3.2 part (a). And if not, then there is a dyadic angle ϑ' so that $c(\vartheta) < c(\vartheta')$ and we have $N_{\vartheta}(n) \leq N_{\vartheta'}(n) \leq N_{\vartheta_0}(n)$ for all n where the second inequality is again the dyadic argument and the first one is Lemma 2.5. Therefore ϑ_0 is a (weak) local maximum of $N_{\vartheta}(n)$ for all n and thus of h .

Part (a) will follow from the stronger claim that h has a unique global maximum on $I(\vartheta_0)$, and this occurs at ϑ_0 . We know that any dyadic $\vartheta \in I(\vartheta_0) \setminus \{\vartheta_0\}$ has $h(\vartheta) < h(\vartheta_0)$. If $\vartheta \in I(\vartheta_0)$ is such that $c(\vartheta) < c(\vartheta_0)$, then by Lemma 2.5 we have $h(\vartheta) \leq h(\vartheta_0)$; but in fact we have strict monotonicity because there is some dyadic ϑ' with $c(\vartheta) < c(\vartheta')$ and $h(\vartheta) \leq h(\vartheta') < h(\vartheta_0)$. And if not $c(\vartheta) < c(\vartheta_0)$, there is another dyadic ϑ' so that $\vartheta \in I(\vartheta')$ but $\vartheta_0 \notin I(\vartheta')$. We then have $h(\vartheta) \leq h(\vartheta') < h(\vartheta_0)$. Therefore, ϑ_0 is the unique global maximum within $I(\vartheta_0) \ni \vartheta_0$. This finishes the proof of the stronger version of claim (a).

For part (c), consider any hyperbolic component W and let I be the open interval of angles within its wake. Let ϑ_W be the unique dyadic of lowest generation within I . Then $I(\vartheta_W) \supset I$, and on this interval h has its unique global maximum at ϑ_W . Now suppose W is a wake that is not the wake of a hyperbolic component: then either it is one of the subwakes of a Misiurewicz–Thurston parameters, or an irrational wake (bounded by two irrational angles with equal angled internal address). But such wakes are exhausted by wakes of hyperbolic components, so the claim holds for them as well. Part d) also follows.

For claim (b), suppose ϑ is a local maximum of h , and let $I \subset \mathbb{S}^1$ be an interval on which ϑ is a global maximum. By monotonicity, we may assume that the ϑ -ray lands (combinatorially) at an endpoint of \mathcal{M} . If ϑ is not dyadic, then choose a dyadic angle $\vartheta' \in I$ with $\vartheta \in I(\vartheta') \subset I$. Then the unique global maximum of h within $I(\vartheta')$ is at ϑ' , so $\vartheta = \vartheta'$ is dyadic.

Remark For the record, we observe that along the way we proved that core entropy \tilde{h} is strictly monotone on arcs before dyadic endpoints: if c' is dyadic and $c < c'$, then $\tilde{h}(c) < \tilde{h}(c')$ (the general result in Lemma 2.5 would only give $\tilde{h}(c) \leq \tilde{h}(c')$). In fact, there are parameters $c < c'$ so that entropy is constant along $[c, c']$; this happens when c and c' are within the same little Mandelbrot sets, for instance within the “main molecule of \mathcal{M} ” (which is the locus of parameters with zero biaccessibility dimension; see Lemma 4.3 below).

This implies that core entropy is strictly monotone along all veins, even long veins, except within little Mandelbrot sets: if $c < c'$ are two postcritically finite parameters, then the Hubbard tree of c (more precisely, its marked points) can be recovered in the Hubbard tree of c' , so all precritical orbits of c are found for c' , while there is strictly more choice for c' . This choice strictly increases entropy except when c and c' are within the same little Mandelbrot set (or the main molecule with entropy 0): in the latter case, entropy may be dominated by the non-renormalizable dynamics while the extra choices are added only to the renormalizable dynamics (see Sect. 6.4).

The following is a rather obvious restatement of this result for $\tilde{h}: \mathcal{M} \rightarrow [0, \log 2]$.

Corollary 4.2 (Local maxima of \tilde{h} on \mathcal{M}) *The function $\tilde{h}: \mathcal{M} \rightarrow [0, \log 2]$ has local maxima exactly at parameters $c(\vartheta)$ with $\vartheta = k/2^q$, and these are all isolated.*

In order to provide an explicit reference, we restate a known result on where core entropy vanishes.

Lemma 4.3 (Zero core entropy) *We have $\tilde{h}(c) = 0$ if and only if c is in the main molecule of \mathcal{M} , and $h(\vartheta) = 0$ if and only if ϑ is associated to this main molecule.*

Proof In Bruin and Schleicher (2017, Proposition 2.11) it was shown that for the parameter angle ϑ the biaccessibility dimension is zero if and only if ϑ is associated to the main molecule, and up to a factor of $\log 2$ the biaccessibility dimension equals core entropy by Theorem A.3.

Remark Here is a proof in the context of our reasoning, and assuming our main result on continuity: a parameter c is in the main molecule of \mathcal{M} if and only if it is separated from 0 by at most finitely many parameter ray pairs at periodic angles, or if it is the limit of such parameters. In the first case, core entropy is zero, and by continuity it is zero also in the limit.

Conversely, if c is outside of the main molecule, then it is in the wake of a primitive hyperbolic component (its internal address must have an entry that is not a multiple of its predecessor), and for those again every edge eventually covers the entire Hubbard tree after finitely many iterations (one might prefer the argument that every primitive hyperbolic component is separated from the main molecule by non-renormalizable Misiurewicz–Thurston parameters, and these have positive core entropy as well).

The graph in Fig. 1 also shows local minima, and these can be classified relatively easily; we are grateful to Steffen Maass for his questions (see also the preprint by Jung (2014, Theorem 6.8) for similar results).

Theorem 4.4 (Local minima of entropy) *The function $h: \mathbb{S}^1 \rightarrow [0, \log 2]$ has an isolated local minimum at ϑ if and only if there are two angles ϑ_1, ϑ_2 with $0 < \vartheta_1 < \vartheta < \vartheta_2 < 1$ so that all three parameter rays at angles $\vartheta, \vartheta_1, \vartheta_2$ land at a common parameter; in this case all angles $\vartheta, \vartheta_1, \vartheta_2$ are rational with even denominators that are all equal, and the landing point of these rays is a Misiurewicz–Thurston parameter.*

The function $\tilde{h}: \mathcal{M} \rightarrow [0, \log 2]$ has no isolated local minimum.

Proof Since entropy is monotone along the combinatorial arc $[0, c]$ for every $c \in \mathcal{M}$, the entropy function \tilde{h} cannot have isolated local minima when $\tilde{h}(c) > 0$. Since $\tilde{h}^{-1}(0)$ is connected (the main molecule of \mathcal{M}), there is no isolated local minimum at all.

By strict monotonicity along veins, except within little copies of \mathcal{M} , it is obvious that h has an isolated minimum at every angle ϑ for which there are angles ϑ_1, ϑ_2 as claimed. Conversely, if h has an isolated local minimum at ϑ , then all angles associated to the combinatorial arc $[0, c(\vartheta)]$ must avoid a neighborhood of ϑ . For sufficiently small $\varepsilon_1 > 0, \varepsilon_2 > 0$ so that $\vartheta - \varepsilon_1 \in \mathbb{Q}$ and $\vartheta + \varepsilon_2 \in \mathbb{Q}$, the Branch Theorem applied to $c(\vartheta - \varepsilon_1)$ and $c(\vartheta + \varepsilon_2)$ shows that either $c(\vartheta)$ is within a little copy of \mathcal{M} , or it is a Misiurewicz–Thurston parameter and $\vartheta = p/q$ with odd p and even q . It remains to exclude the first case.

Indeed, if $c(\vartheta)$ is on the boundary of a hyperbolic component, then every neighborhood of ϑ contains angles for which the corresponding rays land at the boundary of the same hyperbolic component, and these have the same entropy (even the same number of relevant precritical leaves), so we have no isolated local minimum. If instead $c(\vartheta)$ is in a little copy of \mathcal{M} but not on the boundary of a hyperbolic component, then the combinatorial arc to $c(\vartheta)$ contains parameters with entropy at most that of $c(\vartheta)$, and hence there are angles in every neighborhood of ϑ with entropy at most that of ϑ , and again there is no isolated local minimum. The claim follows.

The same reasoning also allows us to describe non-isolated local minima of h . We need to use that statement, proved in Corollary 6.20 below, that core entropy is constant on all little copies of \mathcal{M} for which the center has positive entropy. (We avoid circular reasoning because the result in this corollary is not used elsewhere in this paper.)

Corollary 4.5 (Non-isolated local minima) *The function $\tilde{h}: \mathcal{M} \rightarrow [0, \log 2]$ assumes its global minimum 0 exactly at the main molecule of \mathcal{M} . It has further local minima exactly at all primitive little copies of \mathcal{M} (for primitive little copies that are maximal under inclusion, their root is not a local minimum).*

The function $h: \mathbb{S}^1 \rightarrow [0, \log 2]$ assumes its global and local minima exactly at angles that land or accumulate at the main molecule, resp. at primitive little copies minus their roots.

All these local minima are non-isolated.

Proof The claim about the global minimum is exactly Lemma 4.3. The fact that core entropy can have further local minima only at little copies of \mathcal{M} was shown in the proof of Theorem 4.4. Observe that if a local minimum c is not in any primitive copy of the Mandelbrot set, then c is infinitely many times renormalizable and every renormalization is satellite; otherwise c is not a local minimum: Corollary 6.20 reduces the statement to the non-renormalizable case. Therefore, c is in the main molecule.

Let \mathcal{M}' be a primitive little Mandelbrot set. It is outside of the main molecule, so its entropy is positive, and it has constant entropy by Corollary 6.20; denote this entropy by $h' > 0$. The set $\mathcal{M} \setminus \mathcal{M}'$ consists of one connected component containing 0 and countably further components within the wake of \mathcal{M}' . On every parameter in the first component entropy is strictly smaller than h' , and on all other components it is strictly greater than h' . This implies that \tilde{h} has local minima on all of \mathcal{M}' minus its root.

If \mathcal{M}' is non-primitive, then either it is the subset of a larger primitive component and the statement is contained in the one for the larger component, or its center is in

the main molecule and has zero entropy, while some parts of \mathcal{M}' are outside and have positive entropy.

5 Continuity of Entropy

Many of our estimates will involve “radial growth of entropy”: that is, comparing $\tilde{h}(c') - \tilde{h}(c)$ for parameters $c' \succ c$. A fundamental case will be when c' is a dyadic endpoint of \mathcal{M} and c is the parameter where the vein of c' terminates. We start by comparing the corresponding Hubbard trees. Since we count pre-critical points between α and $-\alpha$, we add the points α and $-\alpha$ to the set of vertices of the Hubbard trees (if they are not already there).

Lemma 5.1 (Marked points in related Hubbard trees) *Suppose $c' \in \mathcal{M}$ is a dyadic parameter and $c \in \mathcal{M}$ is the postcritically finite parameter where the vein of c' ends. We assume that $c \neq 0$. Denote by H and H' the Hubbard trees of p_c and $p_{c'}$. If $-\alpha \notin H$, then extend H to $-\alpha$ by adding an extra edge. Then H and H' are related as follows. Let V be the union of the postcritical points, $\{\alpha, -\alpha\}$, and branch points in H . Denote by x the dynamical counterpart of c in H' . Let V' be the union of the postcritical points, $\{\alpha, -\alpha\}$, branch points, and points on the orbit of x in H' , and let V'_* be the union of $\{\alpha, -\alpha\}$, the branch points, and the points on the orbit of x in H' . Then*

- (A) $V \setminus V'_*$ are exactly the endpoints of H' ; and
- (B) there is a bijection $J: V \rightarrow V'_*$ so that
 - (1) $J(p_c(v)) = p_{c'}(J(v))$ for all $v \in V$;
 - (2) if $v \in V$ is the landing point of a ray with angle ϕ , then $J(v) \in V'$ is also the landing point of the ray with angle ϕ ;
 - (3) $J(c) = x$; and
 - (4) if a ray pair $RP(\phi_1, \phi_2)$ lands at a non-precritical point of H so that $RP(\phi_1, \phi_2)$ separates two vertices $v_1, v_2 \in H$, then $RP(\phi_1, \phi_2)$ separates $J(v_1)$ and $J(v_2)$ in the dynamical plane of $p_{c'}$.

Remark Note that the tree H contains $-\alpha$ unless c is immediately satellite renormalizable (see Lemma 6.4); in all other cases $-\alpha$ is in the connected hull of the critical orbit.

Observe also that every $v \in V$ is either the landing point of a periodic or preperiodic dynamic ray, or on the orbit of c .

Proof The set of marked points of the Hubbard tree H are the postcritical points, which include the endpoints, the branch points, and $\{\alpha, -\alpha\}$. They form the set V , and we show that these points exist, with the same combinatorics, for all parameters $c'' \succ c$, including c' .

Let v be a branch point of H . Then v is the landing point of at least three dynamic rays, all periodic or preperiodic, and the rays at the same angles land at a common point for all parameters $c'' \succ c$, in particular for c' , by Corollary 3.4. It is not hard to see that the tree H' has (at least) as many branches at the corresponding point as H

does at v . This defines the map J on the set of branch points on H in a natural way. The case $v \in \{\alpha, -\alpha\}$ is treated similarly.

All further marked points of H are the critical value c and its finite forward orbit. In the dynamics of H' , there is the dynamical counterpart x of c , see Definition 3.5. Let us set $J(p_c^{on}(c)) := p_{c'}^{on}(x)$ and show that J is well defined and satisfies the requirements (1)–(4).

If c is a Misiurewicz–Thurston parameter, then $p_c^{ok}(c)$ is the landing point of at least two external rays (because $c < c'$ is not an endpoint). Moreover, $p^{ok}(c)$ is the landing point of a ray $R(\phi)$ if and only if $R(\phi)$ lands at $J(p^{ok}(c)) = p'^{ok}(x)$ in the dynamical plane of p' . Thus J is well defined and satisfies (1)–(4) in the Misiurewicz–Thurston case.

The other case is that c is the center of a hyperbolic component $W \subset \mathcal{M}$. In this case, no ray lands at $p^{ok}(c)$ for $k \geq 0$ because these points are in the Fatou set. By definition, $J(c) = x$ is the landing point of a periodic ray pair, say $RP(\phi_-, \phi_+)$, so that in parameter space the ray pair at the same angles bounds the subwake of c containing c' . Denote by $c_* \in \partial W$ the landing point of the parameter ray pair $RP(\phi_-, \phi_+)$. For a parameter $w \in \bar{W}$, let γ_w be a periodic point on the unique non-repelling orbit; there is a unique continuous choice so that $\gamma_c = c$. Then γ_{c_*} is the landing point of the dynamic ray pair $RP(\phi_-, \phi_+)$. Observe also that the cycle $\{p_w^{ok}(\gamma_w)\}_{k \geq 0}$ for $w \in W \cup \{c_*\}$ does not cross any ray pair $RP(\phi_1, \phi_2)$ as in the requirement (4) of J . This proves claim (4) because the rays landing at non-precritical points of H plus rays $R(\phi_-)$ and $R(\phi_+)$ are stable in the subwake of c containing c' . The requirements (1)–(3) are immediate.

We get a natural injection $J : V \rightarrow V'$. None of the image points are endpoints: the images of branch points are branch points, the images of $\alpha, -\alpha$ are $\alpha, -\alpha$, and the image of x is on $[0, c']$ and not an endpoint, so the forward iterates of x cannot be endpoints either. Hence $J(V) \subset V'_*$. Since there is no branch point on $(x, c']$ (Lemma 3.8), all branch points of H' are in the connected hull of the orbit of x . Therefore, $J : V' \rightarrow V'_*$ is a bijection. This completes the proof.

Lemma 5.2 (Corresponding dynamics on edges) *Let H and H' be the Hubbard trees of c and c' as in Lemma 5.1. Let V and V' be the vertex sets of H and H' (again as in Lemma 5.1). Then the bijection $J : V \rightarrow V'_*$ extends to an injection of edges in H to edges in H' . The image of H under this map on edges is the connected hull in H' containing the orbit of x .*

Let $e'_0 \subset H'$ be the edge that contains the critical point in its interior. The critical point of p_c is in the interior of $J^{-1}(e'_0) \subset H$ if c is pre-periodic, and it is on the boundary of $J^{-1}(e'_0) \subset H$ if c is periodic. Moreover, an edge $e_1 \subset H$ covers once (resp. twice) an edge $e_2 \subset H$ under p_c if and only if the edge $J(e_1) \subset H'$ covers once (resp. twice) an edge $J(e_2) \subset H'$ under $p_{c'}$.

If there are two edges $e'_1, e'_2 \subset H'$ with $e'_1 \subset J(H)$ and $e'_2 \not\subset J(H)$ so that $p_{c'}(e'_1) \supset e'_2$, then $e'_1 = e'_0$ and $e'_2 = [c', x]$.

Proof If an edge $e \subset H$ connects two vertices $v_1, v_2 \in V$, then by Lemma 5.1 the vertices $J(v_1), J(v_2) \in V'$ are adjacent; i.e. $J(v_1)$ and $J(v_2)$ are connected by an edge that we defined to be $J(e)$. This extends J to an injection of edges in H to edges in H' . Clearly, $J(H)$ is the connected hull of $V'_* = J(V)$.

Since $p_{c'}$ is a Misiurewicz–Thurston parameter, the critical point of $p_{c'}$ is in the interior of an edge; we denote this edge by $e'_0 = [a, b] \subset H'$. Then both pre-images of x must be in $[a, b]$ because (x, c') contains no branch point (Lemma 3.8) and thus no marked point other than c' .

If $c \in H$ is periodic, then so is $x \in H'$, and at least one of the two pre-images of x must be in V' , so this point must be in $\{a, b\}$. Thus one of $p_{c'}(a)$ and $p_{c'}(b)$ equals x . Since $x = J(c)$, one of $J^{-1}(a)$ or $J^{-1}(b)$ is the critical point in H .

If $c \in H$ is pre-periodic, then the critical point $0 \in H$ is not a marked point, so it is an interior point of some edge, say e_0 . All marked points in H are landing points of (pre)periodic dynamic rays, and the map J respects their external angles, and this implies that J must send e_0 to e'_0 (the critical point must be accessible by two rays with angles that differ by $1/2$).

It is now easy to see that J respects the dynamics of edges in H and in H' , except that e'_0 covers, in addition, twice $[c', x]$. Indeed, if an edge $[v, w] \in H$ is different from e_0 , then $[J(v), J(w)] \neq e'_0$; thus p_c maps $[v, w]$ homeomorphically onto $[p_c(v), p_c(w)]$, while $p_{c'}$ maps $[J(v), J(w)]$ homeomorphically onto $[p_{c'}(J(v)), p_{c'}(J(w))] = [J(p_c(v)), J(p_c(w))]$.

It remains to analyze the edges $e'_0 = [a, b]$ and $e_0 := [J^{-1}(a), J^{-1}(b)]$; this is done similarly: the arcs $[a, 0]$ and $[0, b]$ cover homeomorphically $[p_{c'}(a), c']$ and $[c', p_{c'}(b)]$ respectively, while $[J^{-1}(a), 0]$ and $[0, J^{-1}(b)]$ respectively cover $[p_c(J^{-1}(a)), c]$ and $[c, p_c(J^{-1}(b))]$.

Finally, if $e'_1 \subset J(H)$ but $p_{c'}(e'_1) \not\subset J(H)$, then $p_{c'}: e'_1 \rightarrow p_{c'}(e'_1)$ cannot be a homeomorphism (because $J(H)$ is connected), so e'_1 must be the unique edge containing the critical point, so $p_{c'}(e'_1) \ni c'$; since (x, c') contains no marked point of H' , we have $p_{c'}(e'_1) \setminus J(H) = (x, c')$.

Next we need a combinatorial estimate. A “combinatorial pattern of length n with gap size s ” is a finite sequence of integers $(j_1, j_2, \dots, j_m, n)$ with $1 \leq j_1 < j_2 < \dots < j_m < n$ and $j_{i+1} - j_i \geq s$ and $n - j_m \geq s$.

Lemma 5.3 (Number of combinatorial patterns) *The number of combinatorial patterns of length n with gap size s is at most $e^{(n/s) \log(s+1)}$.*

Proof The number of combinatorial patterns equals the number of binary sequences of length n where two consecutive digits 1 have distance at least s , and so that the final digit is a 1. Write $n = ks + r$ with $r < s$. Then each block of s consecutive entries has $s + 1$ possibilities because it has at most a single 1, and the last block has $r - 1$ digits 0 followed by a 1, so it has 1 possibility. The number of combinatorial patterns is thus at most $(s + 1)^k = e^{k \log(s+1)} \leq e^{(n/s) \log(s+1)}$.

In the proof of Proposition 5.5, it will be convenient to define “relevant precritical points” as precritical points on $[\alpha, \beta]$, i.e. precritical leaves separating the two fixed points α and β or their corresponding leaves in the lamination (rather than separating α from $-\alpha$ as before). We thus start by showing that this will not affect the value of the entropy.

Lemma 5.4 (Different counts yield identical entropy) *In any invariant quadratic lamination, let $N_1(n)$ be the number of precritical leaves that separate α from $-\alpha$, and*

let $N_2(n)$ be the number of precritical leaves of generation n that separate α from β . Then

$$\limsup_n \frac{1}{n} \log N_1(n) = \limsup_n \frac{1}{n} \log N_2(n);$$

in other words, both counting functions define the same entropy.

Proof Since $-\alpha \in [\alpha, \beta]$, we clearly have $N_1(n) \leq N_2(n)$. To show the converse, we claim that $N_2(n) \leq N_1(n) + N_1(n - 1) + N_1(n - 2) + \dots$.

To see this, denote $\alpha_0 := \alpha$ and, recursively, α_{k+1} to be the unique preimage of α_k on $[\alpha, \beta]$, so $\alpha_1 = -\alpha$. Then $[\alpha_{k+1}, \alpha_k]$ maps homeomorphically onto $[\alpha_k, \alpha_{k-1}]$ and $[\alpha, \beta] = \bigcup_{k \geq 0} [\alpha_k, \alpha_{k+1}]$. If $N_{[\alpha_k, \alpha_{k+1}]}(n)$ denotes the number of precritical points of generation n on $[\alpha_k, \alpha_{k+1}]$, then we have $N_{[\alpha_{k+1}, \alpha_{k+2}]}(n) = N_{[\alpha_k, \alpha_{k+1}]}(n - 1)$ and $N_{[\alpha_0, \alpha_1]}(n) = N_1(n)$ and indeed $N_2(n) = N_1(n) + N_1(n - 1) + N_1(n - 2) + \dots$. If $N_1(n) \leq Ce^{(h+\varepsilon)n}$ for all n , then $N_2(n) \leq Cne^{(h+\varepsilon)n}$. Therefore, N_1 and N_2 define the same entropy.

Proposition 5.5 (Bound on entropy increase) *Suppose $c_1 < c_2$ and $[c_1, c_2]$ is a (combinatorial) arc in \mathcal{M} such that $0 \leq \tilde{h}(c_2) - \tilde{h}(c_1) \leq \varepsilon$. Then there is an $s > 0$ with the following property: if $[c, c']$ is a dyadic vein of generation at least s that terminates at $c \in [c_1, c_2]$, then $\tilde{h}(c') - \tilde{h}(c_2) \leq \varepsilon$.*

Proof Consider first the case $c \neq 0$. Set $h := \tilde{h}(c_2)$. There is a $C > 0$ so that all $c \in [c_1, c_2]$ satisfy $N_c(n) \leq N_{c_2}(n) \leq Ce^{(h+\varepsilon/2)n}$ for all n (monotonicity, Lemma 2.5). We may suppose that s is large enough so that $2C \leq e^{(h+\varepsilon/2)s}$. Let $s' \geq s$ be the generation of c' .

As before (see Lemma 5.1), let H' be the Hubbard tree of $p_{c'}$, which is dyadic, and let H be the Hubbard tree of p_c , which is postcritically finite (as endpoint of a dyadic vein in \mathcal{M}), so both Hubbard trees exist and are finite. Since in this proof we count pre-critical points in $[\alpha, \beta]$ we add to H the arc, say $[\beta', \beta]$, connecting β to H ; denote by $H_\beta := H \cup [\beta', \beta]$ the extended Hubbard tree. Clearly, H_β is p_c -invariant.

Let $x \in H'$ be the dynamical counterpart of c . Recall that x is a characteristic periodic or preperiodic point in the sense that the entire orbit of x is contained in the closure of the component of $H' \setminus \{x\}$ that contains 0 (Lemma 3.7). In particular, the orbit of x is disjoint from (x, c') . It follows that any connected component I of $p_{c'}^{\circ(-n)}([x, c'])$ within H' is either contained in $[x, c']$ or intersects it at most in $\{x\}$. (Otherwise, x would be in the interior of I and after n iterates x would be mapped into (x, c') , but H' does not have a branch point on (x, c') by Lemma 3.8).

We know from Lemma 3.10 that

$$p_{c'}^{\circ s'}([x, c']) \subset [\alpha, \beta] \tag{5.1}$$

and, moreover, the orbit of $p_{c'}^{\circ i}([x, c'])$ for $i \in \{0, 1, \dots, s' - 1\}$ does not contain 0.

Before we finish the proof of Proposition 5.5, we need to define a few terms, and we need a lemma.

By a *maximal preimage* of $[x, c'] \subset H'$ of generation $n > 0$ we mean a connected component I of $p_c^{\circ(-n)}([x, c'])$ so that $I \not\subset p_c^{\circ(-i)}[x, c']$ for every $i \in \{1, 2, \dots, n\}$; equivalently, $p_c^{\circ i}(I) \not\subset [x, c']$ for all $i < n$ (we just proved that this implies that $p_c^{\circ i}(I) \cap [x, c'] \subset \{x\}$). We denote by M the set of all maximal preimages of $[x, c']$ that are in $[\alpha, \beta]$.

An *itinerary* of I will be a sequence $s_0s_1 \dots s_{n-2} \in \{0, 1\}^{n-1}$ where each s_i describes the connected component of $H' \setminus \{0\}$ containing $p_c^{\circ i}(I)$ (labeled for instance so that the critical value is in the component with label 1). Our construction assures that every itinerary is well-defined (the immediate preimage of I contains 0, but further preimages do not because $[x, c']$ does not contain postcritical points, and c' is not periodic).

Lemma 5.6 *There is an itinerary preserving injection from*

- *the set M of maximal preimages of $[x, c'] \subset H'$ of generation n to*
- *the set of precritical points of $f : H_\beta \rightarrow H_\beta$ of generation n .*

Moreover, a maximal preimage of $[x, c'] \subset H'$ belongs to the interval $[\alpha, \beta] \subset H'$ if and only if the corresponding precritical point belongs to $[\alpha, \beta] \subset H_\beta$.

Proof The map $J : H \rightarrow H'$ from Lemma 5.2 extends naturally into $J : H_\beta \rightarrow H'$ such that the new map embeds the sets of vertices and edges of H_β into the sets of vertices and edges of H' .

Every interval $I \in M$ is uniquely characterized by a sequence $\bar{\tau}' = (\tau'_0, \varepsilon_0, \tau'_1, \varepsilon_1, \dots, \varepsilon_{n-2}, \tau'_{n-1})$ so that τ'_i is the edge of H' containing $p_c^{\circ i}(I)$ and $\varepsilon_i \in \{0, 1\}$ describes the connected component of $H' \setminus \{0\}$ containing $p_c^{\circ i}(I)$, labeled for instance so that the critical value is in the component with label 1. Of course, ε_i is determined by τ_i unless $\tau_i = e'_0$; similarly, ε_i and τ_{i+1} uniquely determine τ_i .

The sequence $\bar{\tau}'$ is subject to the following conditions:

- $p_{c'}(\tau'_i) \supset \tau'_{i+1}$;
- all $\tau_i \neq [c', x]$ (by the condition of “maximal preimage”);
- $\tau_0 \subset [\alpha, -\alpha]$; and
- $\tau'_{n-1} = e'_0$.

Since all $\tau_i \neq [c', x]$ and $\tau_0 \subset [\alpha, -\alpha]$, all τ_i are in the image of J . Define the sequence

$$\bar{\tau} = J^{-1}(\bar{\tau}') := (J^{-1}(\tau'_0), \varepsilon_0, J^{-1}(\tau'_1), \varepsilon_1, \dots, J^{-1}(\tau'_{n-1})).$$

By Lemma 5.2, $J^{-1}(\tau'_{n-1}) = e_0$ while $J^{-1}(\tau'_0) \subset [\alpha, -\alpha]$.

Since e_0 contains the critical point (in its interior or in its boundary) the sequence $\bar{\tau}$ determines a unique relevant pre-critical point ℓ such that $p_c^i(\ell)$ is contained in the intersection of τ_i with component of $H_\beta \setminus \{0\}$ labeled by ε_i . It is straightforward that a different choice of I leads to a different choice of $\bar{\tau}'$, which leads to a different choice of ℓ . And the itinerary of I , which is $\varepsilon_0\varepsilon_1 \dots \varepsilon_{n-2}$, is preserved.

Now we continue the proof of Proposition 5.5. By Lemma 5.6 the number of intervals in M of generation n that are in $[\alpha, \beta]$ is bounded above by $Ce^{(h+\varepsilon/2)n}$.

In order to bound the number of relevant precritical points of any generation n in the dynamics of $p_{c'} : H' \rightarrow H'$ (these are, by definition, the precritical points in $[\alpha, \beta]$), consider any relevant precritical point ℓ of generation n . Let $j_1 < j_2 < \dots < j_m = n$ be the set of all iterates so that $p_{c'}^{\circ j_i}(\ell) \in [x, c']$. Then $\ell, p_{c'}^{\circ(j_1+s')}(\ell), p_{c'}^{\circ(j_2+s')}(\ell), \dots, p_{c'}^{\circ(j_{m-1}+s')}(\ell)$ are within $[\alpha, \beta]$; compare (5.1). This also implies that $j_{i+1} - j_i \geq s'$.

Let $I_0, I_1, \dots, I_{m-1} \in M$ be the unique intervals in M containing, respectively, $\ell, p_{c'}^{\circ(j_1+s')}(\ell), p_{c'}^{\circ(j_2+s')}(\ell), \dots, p_{c'}^{\circ(j_{m-1}+s')}(\ell)$; their respective generations are $j_1, j_2 - j_1 - s', \dots, j_m - j_{m-1} - s'$. We claim that ℓ has itinerary $\underline{s} = s_0s_1s_2 \dots s_{n-2}$ of ℓ as follows:

- $s_0s_1 \dots s_{j_1-2}$ is the itinerary of I_0 ;
- $s_{j_1}s_{j_1+1} \dots s_{j_1+s'-1}$ is the kneading sequence of c' ;
- $s_{j_1+s'}s_{j_1+s'+1} \dots s_{j_{i+1}-2}$ is the itinerary of I_i ; and
- all s_{j_i-1} are arbitrary in $\{0, 1\}$.

We justify this as follows: $p_{c'}^{\circ(j_1-1)}$ maps I_0 homeomorphically, while $p_{c'}^{\circ j_1} : I_0 \rightarrow [x, \tilde{c}]$ is a $2 : 1$ -map, so the first $j_1 - 2$ iterates do not contain 0 and all points in I_0 have the same entries in their itineraries up to entry number $j_1 - 2$.

Since $p_{c'}^{\circ j_i}(\ell) \in [x, c']$, the next iterates are the same as for $[x, c']$ and, in particular, for c' , hence equal to the kneading sequence of c' , at least before c' lands at the β fixed point, that is for $s' - 1$ iterations.

The iterate $p_{c'}^{\circ(j_i+s')}(\ell)$ is by definition in I_i , and this interval travels forward homeomorphically until it covers 0, which is the iteration before it reaches $[x, c']$ the next time; since the latter is at iterate j_{i+1} , the itinerary of ℓ coincides with that of I_i until position $j_{i+1} - 2$ (analogous to the beginning). In the subsequent iterate, the image interval $p_{c'}^{\circ(j_{i+1}-j_i-1)}(I_i)$ contains 0, so both entries in the itinerary are possible.

Now consider the set of all precritical points in $[\alpha, \beta]$ of generation n corresponding to a particular combinatorial pattern $(j_1, j_2, \dots, j_m, n)$. We just showed that in order to determine the itinerary of ℓ we only need to specify $s_{j_1-1}, s_{j_2-1}, \dots, s_{j_m-1} \in \{0, 1\}$ as well as the intervals I_0, I_1, \dots, I_{m-1} as above; their numbers we estimated in Lemma 5.6. Therefore, the total number of precritical points with pattern $(j_1, j_2, \dots, j_m, n)$ is at most

$$2C e^{(h+\varepsilon/2)(j_1-1)} \left(\prod_{i=1}^{m-1} 2C e^{(h+\varepsilon/2)(j_{i+1}-j_i-s'-1)} \right) \leq 2C e^{(h+\varepsilon/2)n}$$

because $2C e^{-(h+\varepsilon/2)s'} \leq 1$ by hypothesis.

Since the number of combinatorial patterns is at most $e^{(n/s') \log(s'+1)}$ (Lemma 5.3), it follows that $N_{\tilde{c}}(n) \leq 2C e^{n(h+\varepsilon/2+\log(s'+1)/s')}$. Therefore

$$\begin{aligned} \tilde{h}(\tilde{c}) &\leq \limsup_n \frac{1}{n} \left(\log 2 + \log C + n(h + \varepsilon/2) + \frac{n}{s'} \log(s' + 1) \right) \\ &\leq h + \varepsilon/2 + \frac{\log(s' + 1)}{s'} \leq h + \varepsilon \end{aligned}$$

if $s' \geq s$ is sufficiently large.

Consider now the case $c = 0$. We claim that $h(c') < \varepsilon$ if s is sufficiently big. Denote by K' the filled in Julia set of $p_{c'}$. Suppose that $K' \setminus \{\alpha\}$ consists of q connected components (this is equivalent to c' being in the primary p/q limb of the Mandelbrot set for some p coprime with q), we enumerate them as K_1, K_2, \dots, K_q such that K_1 contains the critical value, K_q contains the critical point, and $p_{c'}$ maps K_i homeomorphically onto K_{i+1} for all $i < q$. Observe that $\beta \in K_q$ and there is a unique $c_i \in p_{c'}^{q-i}(\beta)$ such that $c_i \in K_i$. Since c' is the dyadic endpoint of smallest generation in a limb of $c = 0$ we see that c_1 is the critical value of $p_{c'}$, in particular $q \geq s$. Thus $H' = \cup_{i \leq q} [\alpha, c_i]$ is a star-like tree. The associated transition matrix of $p_{c'}: H' \rightarrow H'$ is

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 2 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

It is easy to see that if $q \geq s$ is big enough, then the leading eigenvalue of M is close to 1, thus its logarithm is close to 0.

Theorem 5.7 (Continuity of entropy at hyperbolic components) *The core entropy function h is continuous at all angles that are associated to hyperbolic components of \mathcal{M} .*

Proof Let W be a hyperbolic component of \mathcal{M} of some period n and suppose an angle ϑ is associated to W , in the sense that the parameter ray at angle ϑ lands at ∂W (in fact, all we are using is that the ray accumulates at ∂W ; we are not assuming the known fact that all such rays actually land).

Let c_1 be the root of W and c_2 the bifurcation point in ∂W of the period $2n$ component. The combinatorial arc $[c_1, c_2]$ consists of the two internal rays of W connecting the center, say c_0 , to c_1 and to c_2 .

Every sublimb L of W has a leading dyadic, say c_L , at which the entropy within L is maximal (Theorem 4.1). Unless L is the $1/2$ -limb, the vein of c_L terminates at c_0 , so for given $\varepsilon > 0$, by Proposition 5.5 there are only finitely many limbs of W in which the entropy exceeds $\tilde{h}(c_1) + \varepsilon$. This immediately implies continuity of h at all irrational angles ϑ associated to W .

If ϑ is a rational angle associated to W , then either $c(\vartheta)$ is the root of W , or $c(\vartheta)$ is the parameter where W bifurcates to a component W' with period a proper multiple of n . In the latter case, Proposition 5.5 implies continuity of h at ϑ among all rays that are not in the wake of W' . The same argument, applied to W' , implies continuity at ϑ among all rays in the wake of W' .

We finally have to discuss the case that ϑ lands at the root of W . Continuity among rays in the wake of W is handled once again as before.

The only case left is when W is a primitive hyperbolic component of period $n \geq 2$ (in the period $n = 1$ case every ray is in the wake of W). Here we use the fact that entropy is continuous along the combinatorial arc from 0 to W (Jung 2014, Theorem 4.9), so there is a postcritically finite parameter $c_3 < c_1$ with $\tilde{h}(c_3) \geq \tilde{h}(c_1) - \varepsilon$. By Proposition 5.5 there are at most finitely many veins ending at $[c_3, c_1]$ with entropy variation greater than ε . Therefore, ϑ is continuous among all rays outside of the wake of W .

Theorem 5.8 (Continuity of entropy near veins) *Suppose $\vartheta \in \mathbb{S}^1$ is such that topological entropy is continuous along the (combinatorial) vein $[0, c(\vartheta)]$ connecting the parameters 0 to $c(\vartheta)$ in \mathcal{M} . Then h is continuous for all parameters on $[0, c(\vartheta)]$, and h is continuous at all angles ϕ that correspond to parameters on $[0, c(\vartheta)]$.*

Remark It may be helpful to explain the statement. Let $\gamma : [0, 1] \rightarrow \mathbb{C}$ be a parametrization of the (combinatorial) arc $[0, c(\vartheta)]$. Then the hypothesis says that $\tilde{h}(\gamma(t))$ is continuous for $t \in [0, 1]$ (only considering parameters along the arc). The conclusion is that then $\tilde{h} : \mathcal{M} \rightarrow [0, \log 2]$ is continuous at $\gamma(t)$ for all t (where $\gamma(t)$ is now viewed as an element of \mathbb{C} , not just of the arc). Note that this hypothesis is known to be true for all angles $\vartheta \in \mathbb{S}^1$, except when $c(\vartheta)$ is an endpoint of \mathcal{M} at an irrational angle (Jung 2014, Theorem 4.9); we will treat the missing case in Sect. 6.

Proof We start the proof with an auxiliary consideration that does not involve the angle ϑ . Suppose there are two parameters $c_a < c_b \in \mathcal{M}$ with $0 \leq \tilde{h}(c_b) - \tilde{h}(c_a) \leq \varepsilon$; we allow c_b to be a (combinatorial) endpoint of \mathcal{M} .

Denote by $\text{wake}(c_a)$ the open wake of c_a : this is the set of all parameters in \mathcal{M} that are separated from 0 by two parameter rays landing at (or accumulating at) c_a . If c_a is a Misiurewicz–Thurston-parameter, then we set $\text{wake}(c_a)$ to be the subwake of c_a containing c_b . Similarly $\text{wake}(c_b)$ is defined; if c_b is a combinatorial endpoint of the Mandelbrot set, then $\text{wake}(c_b) = \emptyset$. Set $W := \overline{\text{wake}(c_a)} \setminus \text{wake}(c_b)$. By Proposition 5.5 there are at most finitely many dyadic veins $[c_i, c'_i]$ with $c_i \in [c_a, c_b]$ such that the entropy variation along $[c_i, c'_i]$ exceeds ε . We may suppose that $(c_i, c'_i) \cap [c_a, c_b] = \emptyset$, possibly by replacing $[c_i, c'_i]$ with the closure of $[c_i, c'_i] \setminus [c_a, c_b]$.

We will construct a “reduced wake” $W' \subset W$ in which the entropy variation is at most 2ε .

By the Branch Theorem (Douady and Hubbard 1985a; Schleicher 2004, Theorem 3.1), the points c_i are either Misiurewicz–Thurston parameters or centers of hyperbolic components. In both cases, we will exclude a subwake at c_i from W where the entropy variation is large.

If c_i is a Misiurewicz–Thurston-parameter, let W_i be the subwake of c_i containing c'_i and thus $(c_i, c'_i]$. If c_i is the center of a hyperbolic component, say H_i , then let W_i be the subwake of H_i that contains c'_i (the root of this wake is a bifurcation parameter on ∂H_i). In both cases, W_i does not contain c_b .

Set $W' := W \setminus \bigcup_i \overline{W_i}$ (recall that the union is finite). The external angles corresponding to rays in W' occupy finitely many intervals, and the maximal entropy of these angles occurs either at an interior point or at an endpoint. In the first case, the

maximum is at a dyadic angle by Theorem 4.1 (the Tiozzo Conjecture), and our construction is such that this maximum is at most $\tilde{h}(c_a) + 2\varepsilon$. In the second case, it occurs at an angle corresponding to some $c_i \in [c_a, c_b]$ with entropy $\tilde{h}(c_i) \leq \tilde{h}(c_b) \leq \tilde{h}(c_a) + \varepsilon$ (if c_i is the center of a hyperbolic component, then the ray at the angle with entropy maximum does not land at c_i , but at a parabolic boundary point of the same hyperbolic component, with equal entropy). Therefore, for all parameter rays $R(\phi) \subset W'$ we have $\tilde{h}(c_a) \leq h(\phi) \leq \tilde{h}(c_a) + 2\varepsilon$.

Now we start the actual proof: consider a parameter $c \in [0, c(\vartheta)]$ with $c = c(\phi)$ for some $\phi \in \mathbb{S}^1$. We claim that \tilde{h} is continuous at c and that h is continuous at ϕ . Fix $\varepsilon > 0$.

We first consider the case that $c(\vartheta)$ is a combinatorial endpoint and $c = c(\vartheta)$. We prove continuity of \tilde{h} at $c(\vartheta)$ and of h at ϑ . By hypothesis, entropy is continuous along the combinatorial vein $[0, c(\vartheta)]$, so there is a $c_a \in \mathcal{M}$ with $0 < c_a < c(\vartheta)$ and $0 \leq \tilde{h}(c(\vartheta)) - \tilde{h}(c_a) \leq \varepsilon$. Now using the argument from above we construct a reduced wake W' so that all angles $\vartheta' \in W'$ satisfy $h(\vartheta') \in [\tilde{h}(c_a), \tilde{h}(c_a) + \varepsilon]$. These angles form a neighborhood of ϑ , while W' is a neighborhood of c . This completes the proof when $c = c(\vartheta)$ is a combinatorial endpoint.

If $c = c(\vartheta)$ is not a combinatorial endpoint, then we can extend the combinatorial arc $[0, c(\vartheta)]$, so from now on it suffices to assume that $c \in (0, c(\vartheta))$, possibly by replacing ϑ by a different angle.

The second case is that c is neither a Misiurewicz–Thurston-parameter nor on the boundary of a hyperbolic component. In this case, we may choose an arc $(c_a, c_b) \ni c$ (i.e., $c_a < c < c_b$) with $0 \leq \tilde{h}(c_b) - \tilde{h}(c_a) \leq 2\varepsilon$ and proceed as above. The assumptions on c mean that $c \notin \partial W_i$, so c is still an interior point of W' , and we conclude that \tilde{h} is continuous at c and h is continuous at ϕ .

If $c(\phi)$ is a Misiurewicz–Thurston-parameter, then there are finitely many branches, and the previous argument works separately for all the individual branches.

The final case is that $c(\phi)$ is on the boundary of a hyperbolic component—and that case was handled in Theorem 5.7.

Theorem 5.9 (Continuity of entropy) *The core entropy function $h: \mathbb{S}^1 \rightarrow [0, \log 2]$ is continuous.*

Proof If $\phi \in \mathbb{S}^1$ is such that $c(\phi)$ is on some combinatorial arc $[0, c(\vartheta)]$ along which entropy is continuous, then h is continuous at ϕ : this is the content of Theorem 5.8. By work of Tiozzo (2013) and Jung (2014, Theorem 4.9), this is true for all dyadic angles ϑ .

This proves continuity of h at all angles ϕ except when ϕ corresponds to a combinatorial endpoint of \mathcal{M} at irrational angle, or when ϕ corresponds to a boundary point of a hyperbolic component at irrational angle.

The second case, irrational boundary points of hyperbolic components, has been treated in Theorem 5.7.

The first case will be taken care of in Corollary 6.21: there we will prove continuity of entropy along all combinatorial arcs $[0, c(\phi)]$ for all irrational endpoints of \mathcal{M} , and then the claim follows as above from Theorem 5.8.

Remark Recall from Sect. 2 that continuity of $h: \mathbb{S}^1 \rightarrow [0, \log 2]$ implies continuity of $\tilde{h}: \mathcal{M} \rightarrow [0, \log 2]$ (there is a natural continuous projection from \mathcal{M} to the “abstract

Mandelbrot set”, and h is naturally defined on the latter, so \tilde{h} is the composition of two continuous maps).

6 Irrational Endpoints

In this section we prove that for every combinatorial endpoint c of \mathcal{M} , entropy is continuous along the combinatorial arc $[0, c]$. This is known when c is postcritically finite (Jung 2014, Theorem 4.9), but we need it in all cases. This proof provides the missing step in the continuity proof in Theorem 5.9. In fact, we prove the result somewhat more generally for endpoints that are non-dyadic, whether or not they are irrational.

We will approximate the non-dyadic endpoints by dyadic ones, and of course we need uniform estimates for the latter (Proposition 6.13).

6.1 Hubbard Trees, Automata, and Renormalization

Consider a dyadic endpoint c' of \mathcal{M} with external angle $q/2^m$ and Hubbard tree H' . Then the critical value and all further postcritical points are endpoints of H' , so vertices of H' are either endpoints or branch points. As in Lemma 5.1 we add α and $-\alpha$ to the vertex set of H' (if necessary). Easy calculations show that H' has

- $m + 1$ endpoints;
- at most $m - 1$ branch points;
- at most $2m + 2$ vertices (including α and $-\alpha$); and
- at most $2m + 1$ edges.

Let c be the postcritically finite parameter where the vein of c' terminates. We will assume that $c \neq 0$ so that we are in the setting of Lemma 5.1. We denote by H the Hubbard tree of c ; note that $\beta \notin H$. Denote by $x \in H'$ the dynamic counterpart of c as in Definition 3.5.

Lemma 6.1 *The set $\{p_{c'}^{ok}(x) : k \geq 0\} \setminus \{\text{vertices of } H'\}$ contains at most m points; equivalently, $p_{c'}^{om}(x)$ is a branch point of H' .*

Proof. Let \tilde{c} be the dyadic parameter with $c' \triangleleft \tilde{c}$, so c' is directly subordinate to \tilde{c} . Then $c < c'$ and $c < \tilde{c}$. Denote by $\tilde{q}/2^n$ the external angle of \tilde{c} ; note that $n < m$.

We will work in the dynamical plane of $p_{c'}$. The rays landing at x separate $R(0)$, $R(\tilde{q}/2^n)$, and $R(q/2^m)$ (Lemma 3.7; the point x is denoted x_* there), so x has at least three branches in $H' \cup [x(\tilde{q}/2^n), x]$, where $x(\tilde{q}/2^n)$ denotes the landing point of $R(\tilde{q}/2^n)$.

We claim that $p_{c'}^{on}[x(\tilde{q}/2^n), x] \subset H'$. Indeed, for $k \geq 0$ let T_k be the minimal tree connecting the α fixed point to all dyadic endpoints of generation at most k . Since endpoints of $p_{c'}(T_{k-1})$ are endpoints of T_{k-2} as well as the critical value, we have $p_{c'}(T_{k-1}) \subset T_{k-2} \cup H'$ and, by induction, indeed $p_{c'}^{on}(T_n) \subset H'$.

Therefore, $p_{c'}^{on}(x)$ has at least 3 branches in H' because $p_{c'}^{on} : H' \cup [x(\tilde{q}/2^n), x] \rightarrow H'$ is locally injective near x . Since $m > n$, also $p_{c'}^{om}(x)$ is a branch point (and also $p_{c'}^{o(m-1)}(x)$). □

Let us now refine H' by adding the finite set $\{p_c^{ok}(x) : k \geq 0\}$ to its vertex set. The new tree, still called H' , has

- at most $3m + 2$ vertices; and
- at most $3m + 1$ edges.

Corollary 6.2 *The tree H has*

- at most $2m + 1$ vertices; and
- at most $2m$ edges.

Proof. We have a natural injection of marked points (vertices) in H to marked points in H' (Lemma 5.1) that is compatible with the dynamics and with edges connecting marked points (Lemma 5.2), and the $m + 1$ endpoints of H' are not in the range of this injection (Lemma 5.1). □

Overview on the argument The key idea of our proof consists of identifying the dynamics of p on H as an embedded subset of the dynamics of p' on H' . Since entropy measures the growth rate of choice of orbits of length n , the entropy of p' on H' is no less than the entropy of p on H , and we need to give an upper bound on the difference. An orbit in H' that realizes the additional choice is one that leaves the embedded image of H in H' , and we show that it starts on a single edge $[x, c']$ at the critical value. We show that this edge maps forward homeomorphically a large number of iterations: so if some orbit uses the additional choice, then it will not have any choice for a long time, and this will give an upper bound on the entropy increase.

Automata. Here and elsewhere, we find it convenient to express some combinatorial properties in terms of *automata*. We would like to reassure the reader that we only use the basic notion without results from automata theory and hope it will not be distracting. The concept is simple: given a postcritically finite polynomial $p = p_c$ with Hubbard tree H , we associate to it an automaton A in a natural way, as follows. The states of A correspond to the edges of H . There is an arrow in A from edge e_1 to edge e_2 whenever $p(e_1) \supset e_2$; the number of arrows from e_1 to e_2 equals the number of times $p(e_1)$ covers e_2 under p (this is well defined because we have a Markov partition). This number of arrows equals 0 or 1, except for the unique edge (if any) that contains the critical point in its interior. Similarly, denote by A' the natural automaton associated with $p' = p_{c'} : H' \rightarrow H'$.

In Lemma 5.2, we had identified the dynamics on edges of H as a subset of the dynamics on edges of H' . We will find it convenient to express this fact by saying that the automaton A can be considered as a sub-automaton of A' ; this is done in the following lemma.

Lemma 6.3 (Automata and edges in Hubbard trees) *Let $J : H \rightarrow H'$ be the embedding of vertices and edges of the Hubbard tree of H into vertices and edges of the Hubbard tree of H' as in Lemma 5.2. Then J induces an inclusion of automata $A \hookrightarrow A'$ by mapping a state e of A into the state $J(e)$ of A' , and this inclusion is compatible with the number of arrows between states.*

There are exactly two arrows in A' going from $J(A)$ to $A' \setminus J(A)$, and they connect the same states (they start at the state corresponding to the edge containing the critical point, and they end at the state corresponding to the edge $[c', x]$).

Proof By Lemma 5.2, the inclusion J injects the set of states of A , which are edges of H , into the set of states of A' , which are edges of H' . Moreover, the number of arrows from a_1 to a_2 (i.e., the degree of the corresponding map on the edge) is equal to the number of arrows from $J(a_1)$ to $J(a_2)$.

The claim concerning the edges from $J(A)$ into $A' \setminus J(A)$ is a straightforward reformulation of the last claim of Lemma 5.2.

In view of this lemma, we may simply write $A \subset A'$; this will help us compare the two automata.

We need to discuss whether the dynamics on our trees is *renormalizable*; see e.g. Douady and Hubbard (1985b), McMullen (1994), Milnor (1989, 2000a), Riedl and Schleicher (1998) for details. A quadratic polynomial p is n -renormalizable if $p^{\circ n}$ admits a quadratic-like restriction $p^{\circ n} : U \rightarrow V$ with connected non-escaping set K_0 , called the *small filled-in Julia set*. The map $p^{\circ n} : U \rightarrow V$ is hybrid equivalent to a unique polynomial in \mathcal{M} ; its filled-in Julia set is homeomorphic to K_0 . For $i \in \{0, 1, \dots, n-1\}$ set $K_i := p^i(K_0)$. We obtain a periodic cycle $(K_i)_i$ of small filled-in Julia sets.

A renormalization is called *primitive* if all K_i are disjoint. It is called *of satellite type* if some pair(s) of the K_i meet at their β -fixed points. The third and last case of renormalization is called *crossed* (or of α -type): in this case some pair of filled-in Julia sets meet, and indeed “cross”, at their α -fixed points. A renormalization is called *simple* if it is either primitive or of satellite type. Finally, a renormalization that is of satellite type or crossed is called *immediate* if the periodic point at which some of the K_i meet is actually a fixed point of p ; in this case all K_i meet at this fixed point.

It is well known where in the Mandelbrot set renormalizable parameters occur. The renormalization is simple if and only if p belongs to a small embedded copy of the Mandelbrot set, and every map in the copy (except perhaps at the root) admits a simple renormalization of the same combinatorics. The difference between the primitive and satellite cases consists in the fact that exactly in the latter case the root of the main hyperbolic component of the little Mandelbrot set is a bifurcation from a component of lower period. Similarly, p is crossed renormalizable if and only if p belongs to an embedded copy of a primary limb of the Mandelbrot set with certain combinatorics, and every map in this embedded limb is crossed renormalizable of the same combinatorics (Riedl and Schleicher 1998) (except at the root).

The following result is standard and its proof is omitted; compare McMullen (1994, Sec. 7).

Lemma 6.4 (Immediate renormalization) *If p_c is n -renormalizable and n is minimal with this property, and the small Julia sets of the n -renormalization are denoted K_i , then either*

- all K_i contain the α fixed point of p_c ; in this case $H \subset \bigcup_{i=0}^{n-1} K_i$; or
- the K_i are pairwise disjoint.

Remark There are renormalizable Hubbard trees in which some K_i intersect and others do not, but this does not happen when the renormalization period n is chosen minimal.

In parameter space, Lemma 6.4 may be expressed as follows. Suppose that the minimal renormalization (corresponding to the least period) of p_c is simple. Let \mathcal{M}' be the

largest renormalization copy of \mathcal{M} containing p_c ; the two cases in the lemma depend on whether or not \mathcal{M}' touches the main cardioid of \mathcal{M} . If not, the main component of \mathcal{M}' is primitive. (There are many renormalization copies of \mathcal{M} within \mathcal{M} that are non-primitive and that do not touch the main cardioid of \mathcal{M} ; these are contained in larger renormalization copies of \mathcal{M} , and they describe n -renormalizable parameters for which n is not minimal.) If the minimal renormalization of p_c is of crossed type, then p_c belongs to a primary limb $\mathcal{L}_{a/b}$ with rotation number a/b and the minimal renormalization period n divides b (Riedl and Schleicher 1998). The set of maps in $\mathcal{L}_{a/b}$ renormalizable with period n is an embedded copy of the primary $\mathcal{L}_{an/b} = \mathcal{L}_{a/(b/n)}$ limb. In this case all K_i contain the α -fixed point of p_c .

Observe that a parameter is *immediately renormalizable* if the first case in Lemma 6.4 (the second case described in the remark) is realized.

Now we return to the dynamics of the Hubbard trees H and H' from Lemma 6.3. Recall that H' is dyadic; this implies that its dynamics is never renormalizable, but H may be. If the critical point of H is periodic of some period n , then the dynamics is trivially n -renormalizable, and the Hubbard tree resulting from n -renormalization is trivial (a single point). This particular case of renormalizability is not related easily to the dynamics of edges and thus to the automaton A . However, we have the following.

Lemma 6.5 (The Hubbard tree and renormalization) *For every edge e of H , at least one of the following is true:*

- *there is a $k \geq 0$ such that $p_c^{ok}(e) = H$; or*
- *p_c is renormalizable and there is a cycle K_0, K_1, \dots, K_{n-1} of small filled in Julia sets of p_c such that $e \subset \bigcup_{i=0}^{n-1} K_i$.*

Proof We start by assuming that p_c is n -renormalizable; let n be minimal with this property. We may assume that all K_i are pairwise disjoint: if not, then by Lemma 6.4 all K_i meet at the α fixed point of p_c , and thus $H \subset \bigcup_{i=0}^{n-1} K_i$, so the result is clear.

We need the following properties.

- (A) If T is a periodic connected subset of H such that $T \neq H$ but T contains at least two points, then $T \subset K_i$ for some i .

Indeed, since T is periodic but not a singleton, the forward orbit of T contains a critical point. Hence around T there is a small filled in Julia set of p_c .

- (B) If K is one of the K_i , or an (iterated) preimage thereof, then $H \setminus K$ has at most two connected components.

Indeed, there must be some K_i so that $H \setminus K_i$ is connected (start with an arbitrary K_0 , and if $H \setminus K_0$ is not connected, choose some $K_1 \neq K_0$, and then some K_2 in a different component of $H \setminus K_1$ than K_0 ; this process must terminate at some K_i for which $H \setminus K_i$ is connected). Further, if K_j does not contain the critical point, then $H \setminus K_j$ has no more connected components as $H \setminus p_c(K_j)$; and if K_j contains the critical point, then $H \setminus K_j$ can have at most twice as many components as $H \setminus p_c(K_j)$. Therefore, $H \setminus K$ has at most two connected components for any iterated pre-image K of K_0 .

- (C) Every vertex $v \in H$ either has $v \in K_i$ for some i or the forward orbit of v is disjoint from $\bigcup_{i=1}^{n-1} K_i$.

To see this, consider a non-periodic iterated pre-image K' of K_i . Then $H \setminus K'$ has at most two connected components by (B). Since the critical point is contained in some periodic K_0 , it follows that K' contains no postcritical point. A vertex in K' must thus be a branch point, and this is possible only if $H \setminus K'$ has at least three components, which is not the case.

Suppose now that $e \not\subset \bigcup_{i=1}^{n-1} K_i$. By Property (C) the edge e is not within a strictly preperiodic preimage of K_0 . Therefore, e contains a point z whose forward orbit is outside of $\bigcup_{i=1}^{n-1} K_i$. For $i \geq 0$ set $E_i := p_c^i(e)$; this is a connected subset of H . Since every E_i is a finite union of edges, we obtain $E_n = E_m$ for some $n > m \geq 0$. Then E_m is a periodic subset of H ; thus $E_m = H$ by (A) (because $E_m \ni p_c^i(z)$).

Finally, if p_c is not renormalizable, then $E_m := p_c^m(e)$ is periodic for a sufficiently big $m \geq 0$, and by (A) it follows that $E_m = H$.

We want to relate renormalization of H to properties of the associated automaton A . We write $A = A_n \cup A_r$ so that

- A_n (non-renormalizable edges) contains all states that, for some fixed finite iterate, reach all states of A simultaneously; and
- A_r (renormalizable edges) contains all states from which not all of A can be reached simultaneously: these correspond to edges within the Hubbard trees of “small Julia sets” corresponding to renormalization domains).

We should remark that similar notions can be found in Jung (2014, Sec 3.3).

Let us recall that the α fixed point is a vertex of H by our convention.

Lemma 6.6 (Renormalization and automata) *Suppose that p_c is postcritically finite. We have $A_r = \emptyset$ if and only if*

- either p_c is non-renormalizable
- or c is periodic of some period n and p_c is n' -renormalizable only for $n' = n$ and so that all little Julia sets are disjoint.

If p_c is renormalizable, let n be minimal so that p_c is n -renormalizable. Let K_0, K_1, \dots, K_{n-1} be the cycle of small filled-in Julia sets of p_c . Then A_r is the set of states of A so that the associated edges are within $\bigcup_{i=0}^{n-1} K_i$.

Proof Let again H denote the Hubbard tree of p_c . If p_c is non-renormalizable, then it follows from Lemma 6.5 that every edge $e \subset H$ has a k so that $p_c^{ok}(e) = H$ for all large k , so $A_r = \emptyset$.

Suppose p_c is renormalizable and let n be minimal so that p_c is n -renormalizable. If e is an edge of H such that $e \not\subset \bigcup_{i=1}^{n-1} K_i$, then every sufficiently high iterate of e will cover all of H (Lemma 6.5), and the associated state of A is in A_n . The other case is that $e \subset \bigcup_{i=0}^{n-1} K_i$; then by Lemma 6.4 either the K_i are disjoint or they all touch at the α fixed point (which is a vertex of H by our convention), and in both cases we have $e \subset K_i$ for a unique i . In this case, the orbit of e will follow the orbit of K_i and every edge is in only one K_i , so it follows that the associated state is in A_r .

We conclude that if p_c is renormalizable, then $A_r = \emptyset$ if and only if $\bigcup_{i=1}^{n-1} K_i$ contains no edge of H , and that is the case if and only if every “little Hubbard tree” $H_i \subset K_i$ (corresponding to the Hubbard tree after renormalization) is trivial and the

K_i are all disjoint. Finally, the little Hubbard trees are trivial if and only if c is periodic of period n .

An automaton is called *irreducible* if every state of A can be reached from every other. This is certainly the case when A is non-renormalizable, but may also happen in the renormalizable case: for instance, the Hubbard tree of the rabbit polynomial with a superattracting 3-cycle has its Hubbard tree in the form of a topological Y where the three edges are permuted cyclically: the automaton has the form $e_0 \rightarrow e_1 \rightarrow e_2 \rightarrow e_0$ and is irreducible, but the dynamics is renormalizable. The difference is that no edge covers all of A after the same number of iterations (see Lemma 6.6).

If $A_r \neq \emptyset$, so that the dynamics is renormalizable, we may have $A_n = \emptyset$ or $A_n \neq \emptyset$. The next lemma shows that the former case happens only in the case of immediate renormalization.

Corollary 6.7 (Existence of limit) *In the dynamics of p_c , the limit $\lim_{n \rightarrow \infty} (1/n) \log N(n)$ exists whenever p_c is not immediately renormalizable.*

Proof Consider a parameter $c \in \mathcal{M}$ such that c is not immediately renormalizable. Let e be the unique edge in $[\alpha, -\alpha] \subset H$ such that e is attached to α . By Lemma 6.6 we have $e \in A_n$. Therefore, there is a $k \geq 0$ such that $p_c^{ok}(e) = H$, see Lemma 6.5. Thus e contains as many pre-critical points of generation $n+k$ as the number of pre-critical point of generation n in the entire H . Since $e \subset [\alpha, -\alpha]$ we get $\sup_{i \leq n+k} N_c(i) \leq N_c(n)$, hence

$$\limsup_n \frac{1}{n} \log N_c(n) = \lim_n \frac{1}{n} \log N_c(n).$$

If c is not postcritically finite and not an endpoint of \mathcal{M} (that is, c is associated to two external angles in \mathcal{M}), then there are two non-immediately-renormalizable parameters c_1 and c_2 with $N_{c_1}(n) \leq N_c(n) \leq N_{c_2}(n)$ and $0 \leq \tilde{h}(c_2) - \tilde{h}(c_1) \leq \varepsilon$ for arbitrary $\varepsilon > 0$ and the result holds as well (here we use continuity of entropy).

Finally, if c is a non-immediately-renormalizable endpoint, then by continuity of \tilde{h} , for any $\varepsilon > 0$ there exists a postcritically finite non-immediately-renormalizable parameter $c_1 < c$ so that $\tilde{h}(c) - \tilde{h}(c_1) \leq \varepsilon$. By monotonicity, we have

$$\liminf_n \frac{1}{n} N_{c_1}(n) \leq \liminf_n \frac{1}{n} N_c(n),$$

and

$$\begin{aligned} \limsup_n \frac{1}{n} N_c(n) &= \tilde{h}(c) \leq \tilde{h}(c_1) + \varepsilon = \liminf_n \frac{1}{n} N_{c_1}(n) + \varepsilon \\ &\leq \liminf_n \frac{1}{n} N_c(n) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the claim follows in this case too.

Lemma 6.8 (Immediate renormalization and $A_n = \emptyset$) *A polynomial p_c has $A_n = \emptyset$ if and only if either p_c is immediately renormalizable or it has a superattracting fixed point.*

Proof If p_c is immediately n -renormalizable, let K_0, \dots, K_{n-1} be the little Julia sets. By Lemma 6.4, we have $H \subset \bigcup_i K_i$ and all K_i touch at the α fixed point. Since the α fixed point is a vertex of H every edge is contained in some K_i , so the corresponding state is in A_r , hence $A_n = \emptyset$. Also, if p_c has a superattracting fixed point, then H is trivial and $A_n = \emptyset$.

For the converse, if $A_n = \emptyset$ but H is not trivial, then p_c must be renormalizable (otherwise $A_r = \emptyset$ by Lemma 6.5), and by Lemma 6.4 we must have $H \subset \bigcup_i K_i$, so p_c is immediate renormalizable.

Remark Let us note that there are no arrows from A_r to A_n , so within A there is no escape from the set of renormalization states A_r . However, in $A' \supset A$, if $A_r \neq \emptyset$, then there are two arrows from A_r to $[c', x]$, which is a state in $A' \setminus A$ (see Lemma 6.3).

We will consider the following special states of A and $A' \supset A$:

- the 0-state in A' contains the critical point. Denoting this state by e'_0 , we set the 0-state of A to be $J^{-1}(e'_0)$. This convention is compatible with the inclusion $J: A \hookrightarrow A'$ from Lemma 6.3.
- the $[c', x]$ -state of $A' \setminus A$;
- states of A and of A' that belong to the interval $[\alpha, -\alpha]$.

6.2 Counting Precritical Paths

A *path* in A or A' is a sequence of arrows so that each arrow starts where the previous arrow ends. The *length* of a path is the number of arrows it contains. We can also think of a path as a sequence of states so that there is an arrow from every state to the subsequent one (that is, a sequence of edges in the Hubbard tree so that each edge covers the next one under the map). When a path connects two states that are connected by multiple arrows, then there are accordingly multiple paths along this sequence of states (as an example, in A' there are two paths of length 1 from the 0-state to the $[c', x]$ -state).

We define a *relevant precritical path* in A' or in A as a path that starts at a state in $[\alpha, -\alpha]$ and terminates at the 0-state. By basic properties of symbolic dynamics, relevant precritical paths in A' are in bijection with precritical points of $p_{c'}$ in $[\alpha, -\alpha]$ because the critical point of $p_{c'}$ is not a vertex of H' . Different relevant precritical paths in A encode different relevant precritical points of p_c in $[\alpha, -\alpha]$.

Every relevant precritical path s in A' has the form

$$s = b_0c_0a_1b_1c_1a_2b_2c_3 \dots a_p b_p c_p \tag{6.1}$$

such that (roughly: a_i, b_i, c_i are the sub-paths in $A' \setminus A$, in A_n , and in A_r respectively)

- a_i is an (almost) “choiceless” path that starts at the 0-state, then goes to $[x, c']$, then travels outside of states in $[\alpha, -\alpha]$, and terminates at the first state reached in $[\alpha, -\alpha] \subset A$;

- if a_i terminates at a state in A_r , then $b_i = \emptyset$; otherwise, b_i is a path that starts at the state in $[\alpha, -\alpha]$ where a_i terminates and continues while states in A_n are visited (if $i = 0$, then instead of the terminal state of a_i we take the initial state of s);
- if $A_r = \emptyset$, then $c_i = \emptyset$; otherwise: if $b_i \neq \emptyset$, then c_i is a path that starts in A_n where b_i terminates, and immediately moves into A_r , and otherwise it starts at a state in A_r where a_i terminates. The end of c_i is the 0-state, and until then the path remains in A_r (again, if $i = 0$, then instead of the terminal state of a_i we take the initial state of s).

Observe that paths in A' that are not in A start on the edge $[x, c']$, so they are described by the a_i that are long and have almost no choice, hence contribute little additional entropy. Indeed, every a_i has length at least $m + 1$ because $[x, c']$ needs m iteration to reach $[\beta, \alpha]$, and might need further iterations to land in $[\alpha, -\alpha]$ (Lemma 3.10). Once it lands there, we are either in A_n and we continue with a path b_i as long as we stay in A_n , or we are already in A_r and $b_i = \emptyset$, and c_i continues until the next visit of the 0 state. In particular, if $A_n = \emptyset$, then $b_i = \emptyset$. We will refer to a_i as *excursions* (long almost choice-less parts).

Defining ℓ_i, t_i, k_i as the lengths of a_i, b_i, c_i respectively, we say that s has *combinatorial pattern* $P = (t_0, k_0, \ell_1, t_1, k_1, \ell_2, t_2, k_2, \dots, \ell_p, t_p, k_p)$.

Lemma 6.9 (Almost choiceless paths) *For every length $\ell \geq m + 1$, there are at most two possible paths in A' that*

- start at the 0-state;
- then immediately go to the $[c', x]$ -state;
- then travel outside $[\alpha, -\alpha]$;
- terminate at a given state in $[\alpha, -\alpha]$
- and have a length $\ell \geq m + 1$.

Proof. First, the edge of H' associated with the 0-state covers $[c, x]$ with degree 2 under $p_{c'}$. Then $[x, c]$ maps injectively for at least $m + 1$ iterations until $[x, c]$ starts to partially cover $[\alpha, -\alpha]$. But this is, by definition, when the path under discussion terminates. □

Lemma 6.10 (Number of paths c_i) *Suppose c is renormalizable. Let g be the minimal renormalization period of c . Then there are at most $2^{k/g}$ paths in A that*

- start at a given state in A_n or in A_r ;
- all subsequent states are within A_r ;
- terminate at the 0-state; and
- have length k .

Proof. Let H_0, H_1, \dots, H_{g-1} be the cycle of small Hubbard trees associated with the largest renormalizable Hubbard trees (corresponding to the minimal renormalization period of c). Then the degree $p_c^{\text{ogt}} : H_0 \rightarrow H_0$ is at most 2^t for all t . Therefore, there are at most 2^t paths in A_r with length $k \in \{gt, gt + 1, \dots, g(t + 1) - 1\}$ that terminate at the 0-state; so for given length k , the number of such paths is at most $2^t = 2^{\lfloor k/g \rfloor}$ (and we have not even counted the first step from a given state of A to A_r). □

Fix a combinatorial pattern $P = (t_0, k_0, \ell_1, t_1, k_1, \ell_2, t_2, k_2, \dots, \ell_p, t_p, k_p)$ and let $n = |P| = t_0 + k_0 + \sum_{i \geq 1}^p (\ell_i + t_i + k_i)$. When comparing entropy in A' and in A , we will consider the additional relevant precritical paths in A' and show that they correspond to relevant precritical paths in A of bounded length, so that there are not too many additional paths in A' . More precisely, if an excursion has length $\ell_i \geq 3m$ then the new path within A will be shorter (or have equal length) than before.

We thus introduce a quantity κ , called *uncertainty of P* , that measures the possible increase of length as follows:

$$\kappa(P) := \frac{1}{n} \sum_{\ell_i < 3m} (3m - \ell_i) = \frac{1}{n} \sum_i \max(0, 3m - \ell_i)$$

(the first sum is taken over all ℓ_i that are less than $3m$). Higher values of κ create problems as the paths in A might be longer than the corresponding paths in A' . Since all $\ell_i > m$ and $\sum \ell_i \leq n$, we have $3m - \ell_i \leq 2m < 2\ell_i$ and $\kappa(P) \in [0, 2]$.

Denote by $N'(P)$ the number of all precritical paths in A' with pattern P . For $\kappa \in [0, 2]$ define

$$N'(\kappa, n) := \sum_{\kappa(P) \leq \kappa, |P|=n} N'(P),$$

the numbers of precritical paths in A' with small uncertainty. We define $S'(\kappa, n)$ to be the corresponding set of relevant precritical paths with small uncertainty, so that $N'(\kappa, n) = |S'(\kappa, n)|$.

Lemma 6.11 (Replacing a path in A' by a path in A) *Suppose $A_n \neq \emptyset$. If $s = b_0c_0a_1b_1c_1a_2b_2c_3 \dots a_p b_p c_p$ is a precritical path in A' with uncertainty κ , then there are paths a_i^* in A_n with lengths in $\{m, \dots, 3m\}$ such that $s^* := b_0a_1^*b_1a_1^*b_2a_2^* \dots a_p^*b_p c_p$ is a path in A . If n is the length of s , then s^* has length at most $n + \kappa n$.*

Proof Recall that some b_i might be empty paths. Choose any state $a \in A_n$. For convenience, we say that a is the beginning and the end of every empty b_i with $i < p$. If b_p is empty but c_p is not, then we say that the beginning of b_p is the beginning of c_p . If $b_p c_p$ is empty, then the beginning of b_p is the 0 state.

Since A_n is irreducible and has less than $2m$ vertices (Corollary 6.2) we may replace every $c_i a_{i+1}$ by a path a_{i+1}^* in A_n of length at most $2m$ so that a_{i+1}^* connects the end of b_i with the beginning of b_{i+1} (which are by definition both in A_n); by adding up to m arbitrary steps at the beginning, we may arrange things so that a_i^* has length in $\{m, \dots, 3m\}$.

Since the length of each a_i is at least m , this procedure increases the length of s by at most κn (and even shortens it whenever $c_i a_{i+1}$ has length greater than $3m$).

Lemma 6.12 (Counting patterns) *The quantity*

$$\limsup_n \frac{1}{n} \log (\#\{\text{patterns of length } n\})$$

tends to 0 as m tends to infinity.

Proof. Every pattern P is uniquely characterized by a non-decreasing sequence of positive integers

$$t_0, t_0 + k_0, t_0 + k_0 + \ell_1, t_0 + k_0 + \ell_1 + t_1, \dots$$

Since $\ell_i \geq m + 1$, for every $q \in \{0, \dots, \lfloor n/m \rfloor\}$ the interval $[qm, (q + 1)m)$ contains at most 3 elements of the above sequence; and the same is true for the final interval $[\lfloor n/m \rfloor m + 1, n]$. Therefore, the number of all patterns is bounded by $Z(n, m) := ((m + 1)^3)^{\lfloor n/m \rfloor + 1} = e^{3(n+m)\log(m+1)/m}$. For fixed m , we have $\limsup_n (1/n) \log Z(n, m) = 3 \log(m + 1)/m$, and indeed this tends to 0 as $m \rightarrow \infty$. □

Recall that c' is a dyadic parameter of generation m and c is the parameter where the vein of c' terminates.

Proposition 6.13 (Uniformity) *For every $\varepsilon > 0$ there are $\bar{g}, \bar{m}, \bar{\kappa} > 0$ depending only on ε but not on c and c' such that the following holds. If*

- $m \geq \bar{m}$,
- either c is non-renormalizable, or the minimal renormalization period of c is at least \bar{g} ,
- $\kappa \leq \bar{\kappa}$, and
- h is the entropy of the parameter c ,

then

$$N'(\kappa, n) \leq C e^{(h+\varepsilon)n}$$

for some constant $C > 0$ depending on c .

Proof Let $N(n)$ count the number of relevant precritical paths in A of generation n . Since h is the entropy of c , there is a constant $C_1 > 0$ such that

$$N(n) \leq C_1 e^{(h+\varepsilon/3)n}.$$

Suppose first that $A_n \neq \emptyset$. Our first claim is that there are at most $2^{n/g+n/m}$ precritical paths

$$s = b_0 c_0 a_1 b_1 c_1 a_2 b_2 c_3 \dots a_p b_p c_p$$

with fixed $(b_i)_{i \leq p}$ of a given pattern P of length n . Indeed, the beginning of the c_i is fixed by b_i , or by a_i if b_i is empty, and the end is at the 0 state, so by Lemma 6.10 there are at most $2^{k_i/g}$ choices for each c_i and in total at most $2^{n/g}$ choices for all c_i combined (in the non-renormalizable case, the c_i are empty and there is no choice at all). Each a_i has at most two choices by Lemma 6.9, and since their length is at least m , there are no more than $2^{n/m}$ such choices.

By Lemma 6.11 we may substitute $c_i a_i$ by a_i^* with $m \leq |a_i^*| \leq 3m$ and get a precritical path

$$s^* = b_0 a_1^* b_1 a_2^* b_2 \dots a_p^* b_p c_p$$

in A with length at most $n + \kappa(P)n$. Denoting the length of a_i^* by ℓ_i^* , we call the numbers $P^* = (t_0, 0, \ell_1^*, t_1, 0, \ell_2^*, \dots, \ell_p^*, t_p, k_p)$ the pattern of s^* .

Our next claim is that for fixed patterns P and P^* , the number of triples $(s, (b_i)_{i \leq p}, s^*)$ is at most

$$2^{n/g+n/m} C_1 e^{(h+\varepsilon/3)(n+\bar{\kappa}n)} = C_1 e^{((h+\varepsilon/3)(1+\bar{\kappa})+1/g+1/m)n}.$$

Indeed, the number of paths s for a given pattern P with fixed b_i is at most $2^{n/g+n/m}$. The length of s^* is specified by P^* ; denote it by μ . Since each s^* is a precritical path in A and we have $\mu \leq n + \kappa(P)n$ by Lemma 6.11, the number of different s^* is bounded by

$$N(\mu) \leq C_1 e^{(h+\varepsilon/3)(n+\kappa(P)n)} \leq C_1 e^{(h+\varepsilon/3)(n+\bar{\kappa}n)}.$$

Since the b_i are determined by the paths s^* , the claim is proved.

If g and m are sufficiently large and $\bar{\kappa}$ is sufficiently small, then $(h + \varepsilon/3)(1 + \bar{\kappa}) + 1/g + 1/m \leq h + 2\varepsilon/3$, and the number of triples $(s, (b_i)_{i \leq p}, s^*)$ (still for fixed patterns P and P^*) is bounded by

$$C_1 e^{(h+2\varepsilon/3)n}.$$

Since every $s \in S'(\kappa, n)$ is a part of at least one triple $(s, (b_i)_{i \leq p}, s^*)$ for some patterns P and P^* with $\kappa(P) \leq \kappa$ we get the estimate

$$N'(\kappa, n) \leq |\{(P, P^*)\}| C_1 e^{(h+2\varepsilon/3)n},$$

where $|\{(P, P^*)\}|$ denotes the number of pairs of patterns P and P^* with $|P| = n$ and $|P^*| \leq n + \kappa(P)n$. By Lemma 6.12, we have $|\{(P, P^*)\}| \leq C_2 e^{(\varepsilon/3)n}$ for some constant $C_2 > 0$ when m is sufficiently large. We get

$$N'(\kappa, n) \leq C_1 C_2 e^{(h+\varepsilon)n};$$

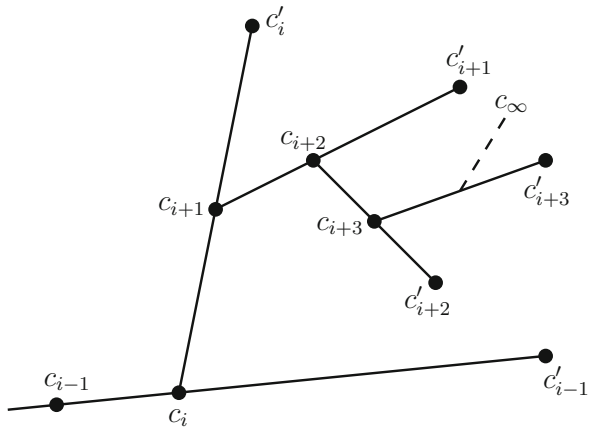
this finishes the proof if $A_n \neq \emptyset$.

The case $A_n = \emptyset$ is simpler: here we have all $b_i = \emptyset$. Therefore, every $s \in A'$ is of the form

$$s = c_0 a_1 c_1 a_2 c_2 \dots a_p c_p$$

and it is easy to see that there are at most $2^{n/g+n/m}$ precritical paths in A' . (By Lemma 6.10 there are at most $2^{k_i/g}$ choices for each c_i and in total at most $2^{n/g}$ choices for all c_i combined. Each a_i has at most two choices by Lemma 6.9, and since

Fig. 4 The relative (combinatorial) positions of the parameters $c_i, c'_i,$ and c_∞ used in the proof



their length is at least m , there are no more than $2^{n/m}$ such choices.) Therefore, if $g \geq \bar{g}$ and $m \geq \bar{m}$ are sufficiently big, then

$$N'(\kappa, n) \leq C e^{\epsilon n}.$$

6.3 Continuity at Non-Renormalizable Irrational Endpoints

Let c_∞ be a non-dyadic endpoint of \mathcal{M} that is not renormalizable; the case that c_∞ is renormalizable will be treated in Sect. 6.4.

There is a sequence of dyadic veins $[c_i, c'_i]$ approximating c_∞ in the following way

- $c_1 < c_2 < \dots < c_\infty$;
- $c_{i+1} \in [c_i, c'_i]$; and
- $\dots c_i \triangleleft \dots \triangleleft c'_2 \triangleleft c'_1$.

Figure 4 illustrates the arrangement of these points. Note that once c_1 is chosen, the remaining parameters are uniquely determined: c'_i is the dyadic of least generation with $c'_i > c_i$ in the same subwake of c_i as c_∞ , and c_{i+1} is the branch point in the vein of c'_i where the vein to c_∞ branches off.

It may be that individual c_i are renormalizable, but only finitely many of them are n -renormalizable for any fixed n , so the renormalization periods of c_i (if any) tend to ∞ ; hence the prerequisite of Proposition 6.13 is satisfied for sufficiently large i ; we will use this in Proposition 6.17.

Similar to the previous discussion we specify the following objects:

- H'_i is the Hubbard tree of c'_i with dynamics $p_{c'_i} : H'_i \rightarrow H'_i$;
- $S'_i(n) \subset [\alpha, -\alpha] \subset H'_i$ is the set of relevant precritical points of generation n in $[\alpha, -\alpha]$ for the parameter c'_i ;
- $N'_i(n) := |S'_i(n)|$ is its cardinality;
- $h_i = \tilde{h}(c_i)$ and $h'_i = \tilde{h}(c'_i)$ are the entropies;
- m_i is the generation of the dyadic parameter c'_i ; and

- for $j \leq i$ we denote by $x_j^{(i)} \in H_i$ the dynamical counterpart of c_j in $p_{c'_i} : H'_i \rightarrow H'_i$ as in Definition 3.5.

We will do our considerations for a fixed Hubbard tree H'_i and often suppress i from the notation. We will now introduce the sets $S'_i(j, \kappa, n)$ for all $j \leq i$; these are the sets of relevant precritical points of generation n with uncertainty at most κ (in analogy to $S'(\kappa, n)$ in Sect. 6.2), but subject to a certain relation with respect to the points $x_j^{(i)}$. The special case $j = i$ is exactly the case that was considered in Sect. 6.2.

Specifically, consider a precritical point $y \in S'_i(n) \subset H'_i$. Let

$$0 = \tilde{\ell}_0 < \tilde{t}_0 < \tilde{\ell}_1 < \tilde{t}_1 < \dots < \tilde{t}_p = n \tag{6.2}$$

be the iteration times of y (depending on j) uniquely specified as follows:

- $\tilde{t}_k > \tilde{\ell}_k$ is the first time so that $p_{c'_i}^{\circ \tilde{t}_k}(y) \in [c'_i, x_j^{(i)}]$;
- $\tilde{\ell}_k > \tilde{t}_{k-1}$ is the first time so that $p_{c'_i}^{\circ \tilde{\ell}_k}(y) \in [\alpha, -\alpha]$.

We also set $\ell_k := \tilde{\ell}_k - \tilde{t}_{k-1}$ and $t_k := \tilde{t}_k - \tilde{\ell}_k$. Clearly, the sequence (6.2) is uniquely specified by $(t_0, \ell_1, \dots, t_p)$. We define the *uncertainty of y with respect to $x_j^{(i)}$* as

$$\kappa_j(y) := \frac{1}{n} \sum_{\ell_k < 3m_j} (3m_j - (\ell_k + 1)) = \frac{1}{n} \sum_k \max(0, 3m_j - (\ell_k + 1))$$

(note that here we start counting at the interval $[c'_i, x_j^{(i)}]$, while in Sect. 6.2 we have $\ell_i = |a_i|$, which starts at the 0-state before going to $[c', x]$; hence in order to be consistent here we have to use $\ell_k + 1$).

We denote by $S'_i(j, \kappa, n)$ the set of all $y \in S'_i(n)$ such that $\kappa_j(y) < \kappa$.

Let us also define

$$I_j(y) := \bigcup_{\ell_k < 3m_j} \{ \tilde{t}_{k-1}, \tilde{t}_{k-1} + 1, \dots, \tilde{\ell}_k \} = \bigcup_{\ell_k < 3m_j} [\tilde{t}_{k-1}, \tilde{\ell}_k] \cap \mathbb{Z};$$

this is the set of iteration times without choice (from $[c'_i, x_j^{(i)}]$ to $[\alpha, -\alpha]$).

Lemma 6.14 (Blocks of iteration times) *In the Hubbard tree H'_i , consider some $y \in S'_i(n)$. If $m_j > 3m_{j'}$ for some $j > j'$, then the corresponding sets $I_j(y)$ and $I_{j'}(y)$ are disjoint.*

Furthermore, $\kappa_j(y) \leq \frac{2}{n} |I_j(y)|$ for all $j \geq i$.

Proof Let us first show that for all $j \geq i$, the set $I_j(y)$ is a union of blocks of consecutive numbers so that each block has length $\ell_k + 1 \in [m_j, \dots, 3m_j]$, and

- its first number \tilde{t}_{k-1} is the unique number t in the block that satisfies $p_{c'_i}^{\circ t}(y) \in [c'_i, \alpha]$;

- its last number $\tilde{\ell}_k$ is the unique number ℓ in the block that satisfies $p_{c'_i}^{\circ\ell}(y) \in [\alpha, -\alpha]$.

We have $\ell_k < 3m_j$ by definition of $I_j(y)$ and we need to prove the lower bound $\ell_k \geq m_j - 1$.

For every integer $k \geq 0$, let $T_k \subset H'_i$ be the minimal tree connecting the α fixed point to all dyadic endpoints of generation at most k . In particular, $[\alpha, -\alpha] \subset T_0$. By (3.1) we have $p_{c'_i}^{-1}(T_{k-2}) \subset T_{k-1}$.

By construction, the parameters c'_i and c'_{j-1} are in different sublimbs of c_j . Thus, by Lemma 3.7, in the tree H'_i the arc $[c'_i, x_j^{(i)})$ is disjoint from T_{m_j-1} ; and we get $[c'_i, x_j^{(i)}) \cap p_{c'_i}^{-k}[\alpha, -\alpha] = \emptyset$ for all $k < m_j$. This shows that $\ell_k \geq m_j - 1$.

By definition of \tilde{t}_{k-1} we certainly have $p_{c'_i}^{\circ\tilde{t}_{k-1}}(y) \in [c'_i, x_j^{(i)}] \subset [c'_i, \alpha]$, and since $p_{c'_i}^{-1}([c'_i, \alpha]) = [\alpha, -\alpha]$, the block would end before the orbit could enter $[c'_i, \alpha]$ again. The claim about the last number is obvious.

Since any block $I_j(y)$ describes a trajectory from $[c'_i, \alpha]$ to $[\alpha, -\alpha]$, the given properties imply that any two blocks $I_j(y)$ and $I_{j'}(y)$ are either disjoint or identical. If $m_j > 3m_{j'}$, then any block of $I_j(y)$ has greater length than any block of $I_{j'}(y)$, and consequently $I_j(y)$ and $I_{j'}(y)$ are disjoint.

Finally, $3m_j - (\ell_k + 1) \leq 2m_j \leq 2(\ell_k + 1)$, and taking the sum we conclude $\kappa_j(y) \leq \frac{2}{n}|I_j(y)|$ as claimed.

Lemma 6.15 (Surgery respects uncertainty) *For $j < i$ the injection $B : S'_i(n) \rightarrow S'_{i-1}(n)$ of Proposition 3.12 injects $S'_i(j, \kappa, n)$ into $S'_{i-1}(j, \kappa, n)$.*

Proof We will show that the injection B respects the sequence (6.2): the orbits of y and $B(y)$ visit the intervals defining this sequence at the same times. This immediately implies that the uncertainty κ is preserved.

Recall that $x_j^{(i)}$ and $x_j^{(i-1)}$ are the dynamical counterparts of c_j in the dynamical planes of $p_{c'_i}$ and $p_{c'_{i-1}}$ respectively. Consider $y \in S'_i(j, \kappa, n)$. It follows from Proposition 3.12 part (B), applied to $c_* = 0$ and thus $x_* = \alpha$ and $x'_* = \alpha$, that

$$p_{c'_i}^{\circ t}(y) \in [\alpha, -\alpha] \text{ if and only if } p_{c'_{i-1}}^{\circ t}(B(y)) \in [\alpha, -\alpha].$$

This takes care of the first pair of corresponding intervals. For the second pair, it follows from Proposition 3.12 part (A) that for all $t \geq 0$

$$p_{c'_i}^{\circ t}(y) \in [c'_i, -\alpha] \text{ if and only if } p_{c'_{i-1}}^{\circ t}(B(y)) \in [c'_{i-1}, -\alpha].$$

By part (B), applied to $c_* = c_j$, it follows that

$$p_{c'_i}^{\circ t}(y) \in [x_j^{(i)}, -\alpha] \text{ if and only if } p_{c'_{i-1}}^{\circ t}(B(y)) \in [x_j^{(i-1)}, -\alpha].$$

Since $[x_j^{(i)}, -\alpha] \subset [c'_i, -\alpha]$ and $[x_j^{(i-1)}, -\alpha] \subset [c'_{i-1}, -\alpha]$, we conclude that

$$p_{c'_i}^{ot}(y) \in [c'_i, x_j^{(i)}] \text{ if and only if } p_{c'_{i-1}}^{ot}(B(y)) \in [c'_{i-1}, x_j^{(i-1)}].$$

Therefore, our surgery respects the sequence (6.2), and thus $\kappa(y) = \kappa(B(y))$ and $B(y) \subset S_{i-1}(j, \kappa, n)$.

Lemma 6.16 (Small uncertainty) *For every $\delta > 0$ and for every $i' \geq 0$ there is an $i'' > i'$ such that for every $n \geq 0$ we have*

$$S'_{i''}(n) \subset \bigcup_{j \in \{i', \dots, i''\}} S'_{i''}(j, \delta, n).$$

Proof Choose a subsequence of indices $i' = i_1 < i_2 < \dots < i_\nu$ so that $3m_{i_r} < m_{i_{r+1}}$ and $\nu > 2/\delta$. We show that $i'' := i_\nu$ satisfies the claim of the lemma. It is sufficient to show that for every $y \in S'_{i_\nu}(n)$ there is a $\nu' \leq \nu$ such that $\kappa_{i_{\nu'}}(y) < \delta$.

By Lemma 6.14 the sets $I_{i_1}(y), I_{i_2}(y), \dots, I_{i_\nu}(y)$ are pairwise disjoint because $3m_{i_r} < m_{i_{r+1}}$. Hence

$$\frac{1}{n} (|I_{i_1}(y)| + |I_{i_2}(y)| + \dots + |I_{i_\nu}(y)|) \leq 1.$$

Since $\kappa_j(y) \leq \frac{2}{n} |I_j(y)|$ (Lemma 6.14 again), we have

$$\kappa_{i_1}(y) + \kappa_{i_2}(y) + \dots + \kappa_{i_\nu}(y) \leq 2 < \nu\delta;$$

this implies that $\kappa_{i_{\nu'}}(y) < \delta$ for some $\nu' \leq \nu$.

The next lemma is a corollary of Proposition 6.13.

Lemma 6.17 (Bound on precritical points) *For every $\varepsilon > 0$ there are $i' \geq 0$ and $\bar{\kappa} > 0$ such that if $j \geq i'$ and $\kappa \leq \bar{\kappa}$, then*

$$|S'_j(j, \kappa, n)| \leq C_j e^{(\varepsilon+h_j)n}$$

for all $n > 0$ and some constant $C_j > 0$.

Proof For given $\varepsilon > 0$ fix $\bar{g}, \bar{m}, \bar{\kappa} > 0$ as in Proposition 6.13 so that $N'(\kappa, n) \leq C e^{(h+\varepsilon)n}$. Since c_∞ is not renormalizable, we may choose i' large enough such that for all $j \geq i'$

- $m_j \geq \bar{m}$; and
- c_j is either non-renormalizable or the renormalization period of c_j is at least \bar{g} (we can make this assumption because c_∞ is non-renormalizable).

We will now apply Proposition 6.13 to the pair $c := c_j$ and $c' := c'_j$; then $h_j = h$ is the entropy of $c = c_j$.

Observe first that $|S'_j(j, \kappa, n)| = |S'(\kappa, n)| = N'(\kappa, n)$ after the substitution. Indeed, every relevant pre-critical point of $p_{c'_j}$ is uniquely characterized by a pre-critical path in A' (again by a fundamental property of the symbolic dynamics because the critical point is in the interior of the 0-state of A'). This bijection preserves the uncertainties: if $y \in S'_j(j, \kappa, n)$ with sequence (6.2) is identified with a relevant pre-critical path s with decomposition (6.1), then $\ell_k + 1 = \tilde{\ell}_k - \tilde{t}_k + 1 = |a_k|$. Hence y and s have the same uncertainties, and $S'_j(j, \kappa, n)$ and $S'(\kappa, n)$ are in bijection.

Now the claim immediately follows from Proposition 6.13.

Theorem 6.18 (Continuity on vein to c_∞ , non-renormalizable case). *If c_∞ is a non-renormalizable combinatorial endpoint of \mathcal{M} , then core entropy is continuous along the combinatorial arc $[0, c_\infty]$.*

Proof We have parameters c_i and c'_i as introduced at the beginning of the section, and we have

$$\tilde{h}(c_i) \leq \tilde{h}(c_{i+1}) \leq \tilde{h}(c_\infty) \leq \tilde{h}(c'_{i+1}) \leq \tilde{h}(c'_i)$$

for all i by radial monotonicity of core entropy (for the first two inequalities), and by Theorem 4.1 (for the last two). Therefore, it suffices to prove the following claim: *for every $\varepsilon > 0$ there is an $\bar{i} \geq 1$ such that $h'_i - h_i \leq \varepsilon$ for all $i \geq \bar{i}$.*

Choose $\bar{\kappa}$ and i' as in Lemma 6.17. By Lemma 6.16 there is an $i'' \geq i'$ such that

$$S'_{i''}(n) \subset \bigcup_{j \in \{i', \dots, i''\}} S'_{i''}(j, \bar{\kappa}, n).$$

By Lemma 6.15 there is an injection from $S'_{i''}(j, \bar{\kappa}, n)$ into $S'_j(j, \bar{\kappa}, n)$ for all $j \in \{i', \dots, i''\}$. Therefore,

$$N'_{i''}(n) = |S'_{i''}(n)| \leq \sum_{j=i'}^{i''} |S'_j(j, \bar{\kappa}, n)|.$$

By Lemma 6.17 we have $|S'_j(j, \bar{\kappa}, n)| \leq C_j e^{(\varepsilon+h_j)n}$. Thus

$$N'_{i''}(n) \leq \sum_{j=i'}^{i''} C_j e^{(\varepsilon+h_j)n} \leq \left(\sum_{j=i'}^{i''} C_j \right) e^{(\varepsilon+h_{i''})n}$$

because $h_{i''} \geq h_j$ by monotonicity. This proves that $h'_{i''} - h_{i''} \leq \varepsilon$, and since the sequence $h'_i - h_i$ is decreasing we have $h'_i - h_i \leq \varepsilon$ for all $i \geq i''$.

6.4 The Renormalizable Case

In this section, we assume that c_∞ is renormalizable. In order to formulate our statements, we need to elaborate on our description of renormalization given earlier in Sect. 6.1. If p_c is simple m -renormalizable, then there exists a “little Mandelbrot set” $\mathcal{M}' \subset \mathcal{M}$ consisting of m -renormalizable parameters with $c \in \mathcal{M}'$ and a straightening homeomorphism $\chi : \mathcal{M}' \rightarrow \mathcal{M}$ so that $p_{\chi(c)}$ in a neighborhood of its filled-in Julia set is hybrid equivalent to p_c^{om} on a neighborhood of the little filled-in Julia set (except possibly when p_c is the root point $\chi^{-1}(1/4)$). The little Mandelbrot set has a main center $c_0 := \chi^{-1}(0)$ with a superattracting orbit of period m . We say that “the parameter c is c_0 tuned with $\chi(c)$ ”. Dynamically, the filled-in Julia set K_c equals K_{c_0} in which every Fatou component is replaced by a copy of $K_{\chi(c)}$.

If p_c is immediate crossed m -renormalizable, then there exists a copy $\mathcal{L}' \subset \mathcal{M}$ with $c \in \mathcal{L}'$ of a primary limb $\mathcal{L}_{a/b}$ together with a straightening homeomorphism $\chi : \mathcal{L}' \rightarrow \mathcal{L}_{a/b}$ so that $p_{\chi(c)}$ in a neighborhood of its filled-in Julia set is hybrid equivalent to p_c^{om} (except at the root of \mathcal{L}'). In general, every crossed renormalization is a composition of a simple renormalization followed by an immediate crossed renormalization.

We will now show that renormalization changes the entropy continuously. Suppose first that $\chi : \mathcal{L}' \rightarrow \mathcal{L}_{a/b}$ or $\chi : \mathcal{M}' \rightarrow \mathcal{M}$ is an immediate (crossed or simple) m -renormalization. Let K_0, K_1, \dots, K_{m-1} be the cycle of small filled-in Julia sets normalized so that $K_0 \ni 0$. Then K_0 is the only small filled-in Julia set intersecting $(\alpha, -\alpha]$, and it follows from $H \subset \bigcup_{i=0}^{m-1} K_i$ (Lemma 6.4) that

$$N_c(n) = \begin{cases} N_{\chi(c)}(n/m) & \text{if } m|n \\ 0 & \text{if } m \nmid n \end{cases} \tag{6.3}$$

From this easily follows that an immediate renormalization changes the entropy continuously. The non-immediate case is in the following lemma:

Lemma 6.19 (Renormalization and entropy) *If \mathcal{M}' is a small copy of \mathcal{M} consisting of m -renormalizable parameters, then the straightening map $\chi : \mathcal{M}' \rightarrow \mathcal{M}$ satisfies*

$$\tilde{h}(c) = \max \left(\tilde{h}(c_0), \frac{1}{m} \tilde{h}(\chi(c)) \right)$$

for all $c \in \mathcal{M}'$, where $c_0 = \chi^{-1}(0)$.

Proof For convenience, in this proof we will count the number of relevant pre-critical points of *strict generation* n ; i.e. relevant pre-critical points of p_c , and similar for other polynomials, that are in

$$p_c^{o(-n)}(c) \setminus \bigcup_{i=1}^{n-1} p_c^{o(-i)}(c).$$

Clearly, this count also gives the entropy of p_c . (Indeed, suppose $N_c(n)$ counts relevant pre-critical points of strict generation n while $N'_c(n)$ counts pre-critical points of

“non-strict” generation n ; these numbers are different only when a precritical point of generation n , i.e. an element of $p_c^{\circ(-n)}(c)$, also is precritical of generation $n' < n$, i.e. when the critical point is periodic of period dividing $n - n'$. So if 0 is periodic, say of minimal period m , then $N'_c(n) - N'_c(n - m) \leq N_c(n) \leq N'_c(n)$. Therefore, $N'_c(n)$ and $N_c(n)$ have the same growth rate.)

Let c_0 be the center of \mathcal{M}' . Denote by $N_{c_0}(n)$ and $N_c(n)$ the numbers of relevant pre-critical points on $[\alpha, -\alpha]$ of p_{c_0} and p_c of strict generation n . Let us denote by $\tilde{N}_{\chi(c)}$ the number of pre-critical points on $[\beta, -\beta]$ of $p_{\chi(c)}$ of strict generation n . We will prove the lemma by relating $N_{c_0}(n)$, $\tilde{N}_{\chi(c)}$, and $N_c(n)$, see (6.4) below.

First let us note that the growth rate of $\tilde{N}_{\chi(c)}$ is the entropy of $\chi(c)$. Indeed, by Lemma 5.4, the precritical points on $[\alpha, -\alpha]$ and those on $[\alpha, \beta]$ give the same entropies; and on $[\beta, -\beta]$ there can be at most twice as many precritical points of any generation n as on $[\alpha, \beta]$.

Denote by F_0 the unique periodic Fatou component of p_{c_0} containing the critical point. A pre-image F' of F_0 under $p_{c_0}^{\circ(n-1)}$ will be called *relevant of strict generation n* if the center of F' is on $[\alpha, -\alpha]$ and F' is not a pre-image of F_0 under $p_{c_0}^{\circ i}$ for any $i < n - 1$. Then $N_{c_0}(n)$ counts the number of relevant pre-images of F_0 of strict generation n .

For a relevant pre-image F' of F_0 , say of strict generation n , the intersection of $\overline{F'}$ with the Hubbard tree of p_{c_0} is equal to $\overline{F'} \cap [\alpha, -\alpha]$ because the Hubbard tree of p_{c_0} has no branch points in F' (see Claim (B) in the proof of Lemma 6.5). Therefore, $p_{c_0}^{\circ(n-1)}$ maps $\overline{F'} \cap [\alpha, \alpha]$ homeomorphically onto $\overline{F_0} \cap [\alpha, \alpha']$.

Let $K_{\chi(c)}$ be the filled in Julia set of $p_{\chi(c)}$. Then the filled in Julia set K_c of p_c is obtained from the filled in Julia set K_{c_0} of p_{c_0} by replacing the closure of each Fatou component F' of K_{c_0} with $K_{\chi(c)}$. And, moreover, if F' is a relevant pre-image of F_0 , then $\overline{F'} \cap [\alpha, \alpha']$ is replaced by the arc $[\beta, -\beta] \subset K_{\chi(c)}$. The inserted copies of $K_{\chi(c)}$ are small (periodic and pre-periodic) filled in Julia sets of p_c . A small filled in Julia set K' of p_c will be called *relevant* if the intersection $K' \cap [\alpha, -\alpha]$ has more than two points. Equivalently, K' is obtained by inserting $K_{\chi(c)}$ into a relevant Fatou component of p_{c_0} . Let us denote by K_0 the small filled in Julia set containing the critical point; i.e. K_0 is obtained from F_0 . By construction, $N_{c_0}(n)$ counts the number of relevant pre-images of K_0 of strict generation n .

Let us now count relevant pre-critical points of p_c . If x is such point, say of strict generation $n \geq 0$, then x is within a relevant pre-image K' of K_0 . Suppose K' is of strict generation $i \geq 0$. Then after $i - 1$ iteration x is within K_0 and $p_c^{\circ(i-1)}(x)$ is a relevant pre-critical point of strict generation $n + 1 - i$. Since the dynamics of $p_c^{\circ m} : K_0 \rightarrow K_0$ is identified with $p_{\chi(c)} : K_{\chi(c)} \rightarrow K_{\chi(c)}$, there are $\tilde{N}_{\chi(c)} \left(\frac{n-i}{m} + 1 \right)$ relevant pre-periodic points of strict generation $n + 1 - i$ in K_0 , where we use the convention $\tilde{N}_{\chi(c)}(t) = 0$ if $t \notin \mathbb{N}$. We get:

$$N_c(n) = \sum_{i=0}^n N_{c_0}(i) \tilde{N}_{\chi(c)} \left(\frac{n-i}{m} + 1 \right), \tag{6.4}$$

where $N_{c_0}(i)$ counts pre-images K' of K_0 and $\tilde{N}_{\chi(c)}\left(\frac{n-i}{m} + 1\right)$ counts the number of relevant pre-critical points of strict generation n within K' . Since the growths of $N_c(n)$, $N_{c_0}(n)$, and $N_{\chi(c)}(n)$ are the entropies of p_c , p_{c_0} , and $p_{\chi(c)}$ we get $\tilde{h}(c) = \max\left(\tilde{h}(c_0), \frac{1}{m}\tilde{h}(\chi(c))\right)$.

Corollary 6.20 (Renormalization and entropy) *If \mathcal{M}' is a small copy of \mathcal{M} consisting of m -renormalizable parameters, then the straightening map $\chi : \mathcal{M}' \rightarrow \mathcal{M}$ satisfies*

$$\tilde{h}(c) = \begin{cases} \tilde{h}(c_0) & \text{if } \tilde{h}(c_0) > 0 \\ \frac{1}{m}\tilde{h}(\chi(c)) & \text{if } \tilde{h}(c_0) = 0 \end{cases}$$

for all $c \in \mathcal{M}'$, where $c_0 = \chi^{-1}(0)$. In particular, \tilde{h} is constant on all small copies for which the center has positive entropy.

Proof We may assume that \mathcal{M}' is a maximal copy; the general case easily follows by induction.

If \mathcal{M}' is maximal satellite, then $\tilde{h}(c_0) = 0$ and $\tilde{h}(c) = \frac{1}{m}\tilde{h}(\chi(c))$. We may thus assume that \mathcal{M}' is maximal primitive. We claim that $\tilde{h}(c_0) \geq \frac{\log 2}{m}$; this implies that

$$\tilde{h}(c) = \max\left(\tilde{h}(c_0), \frac{1}{m}\tilde{h}(\chi(c))\right) = \tilde{h}(c_0)$$

for every $c \in \mathcal{M}'$ because $\tilde{h}(\chi(c)) \leq \log 2$.

In order to prove the claim, we work in the dynamical plane of p_{c_0} and set $v_i := p_{c_0}^i(0)$ for $i \in \{0, 1, \dots, m - 1\}$. Since \mathcal{M}' is primitive, there is a $j \leq m - 1$ such that $[\alpha, -\alpha] \subset [\alpha, v_j]$. Then $p_{c_0}^{\circ j}[v_1, \alpha] \supset [\alpha, -\alpha]$. Since $p_{c_0} : [-\alpha, \alpha] \rightarrow [v_1, \alpha]$ has degree 2, the image of $[\alpha, -\alpha]$ under $p_{c_0}^{\circ(j+1)}$ covers every point in $[\alpha, -\alpha]$ at least twice. This implies that $N_{c_0}(n + j + 1) \geq 2N_{c_0}(n)$ for every $n \geq 0$. Therefore, $\tilde{h}(c_0) \geq \frac{\log 2}{j+1} \geq \frac{\log 2}{m}$.

Corollary 6.21 (Continuity along vein to c_∞ , general case) *For every non-dyadic endpoint c_∞ of \mathcal{M} , core entropy is continuous along the vein $[c_\infty, 0]$.*

Proof Theorem 6.18 proves the case when c_∞ is non-renormalizable.

The second case is when c_∞ is finitely many times renormalizable. Suppose first that the last renormalization of c_∞ is simple. Then there is a renormalization $\chi : \mathcal{M}' \rightarrow \mathcal{M}$ such that $c_\infty \in \mathcal{M}'$ and $c'_\infty := \chi(c_\infty)$ is non-renormalizable. Entropy is continuous along the vein $[0, c'_\infty]$ by Theorem 6.18, and continuous along the image $\chi_*^{-1}([0, c'_\infty]) = [c_0, c_\infty]$ by Lemma 6.19 (where c_0 is the main center of \mathcal{M}'), and continuous along $[0, c_0]$ by Jung (2014, Theorem 4.9) (or Theorem 6.18). Therefore, entropy is continuous along $[0, c_\infty]$. If the last renormalization of c_∞ is of crossed type, then the renormalization is a composition of a simple renormalization and an immediate crossed renormalization (Riedl and Schleicher 1998); the latter changes the entropy continuously by (6.3).

The final case is when c_∞ is infinitely renormalizable. In this case, c_∞ is also infinitely renormalizable of simple type and there is an infinite sequence $\chi_1 : \mathcal{M}_1 \rightarrow$

\mathcal{M} , $\chi_2 : \mathcal{M}_2 \rightarrow \mathcal{M}$, ... of renormalizations such that $\mathcal{M}_n \subsetneq \mathcal{M}_{n-1}$ and $c_\infty \in \mathcal{M}_n$ for all $n \geq 0$. Let c_n be the center of \mathcal{M}_n ; i.e. $c_n = \chi_n^{-1}(0)$. Then $c_1 < c_2 < \dots < c_\infty$, and it is sufficient to show that $\lim_n \tilde{h}(c_n)$ is equal to $\tilde{h}(c_\infty)$.

Suppose that $\chi_m : \mathcal{M}'_n \rightarrow \mathcal{M}$ is an m_n -renormalization. Using Lemma 6.19 we get

$$\tilde{h}(c_\infty) = \max\left(\tilde{h}(c_n), \frac{1}{m_n}\tilde{h}(\chi(c_\infty))\right) \leq \tilde{h}(c_n) + \frac{1}{m_n}h(\chi(c_\infty)).$$

But $\tilde{h}(c_n) + \frac{1}{m_n}h(\chi(c_\infty)) \leq \tilde{h}(c_n) + \frac{1}{m_n} \log 2$ has the same limit as $\tilde{h}(c_n)$ because $\lim_n m_n = +\infty$. Thus entropy is continuous along $[0, c_\infty]$.

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Appendix A: Core Entropy and Biaccessibility Dimension (by Wolf Jung)

Here various definitions of core entropy shall be discussed and related to the biaccessibility dimension. On a compact metric space, the topological entropy of a continuous map is defined by a growth rate, which is referring to preimages of covers, or to ε -shadowing sets. See Brin and Stuck (2002), de Melo and van Strien (1993) for details. When the underlying space is a compact interval, a finite tree, or graph, several equivalent characterizations are due to Alsedà et al. (2000). These include the growth rate of horse shoes, laps (monotonic branches), periodic points, and preimages of a general point. For real and complex quadratic polynomials, core entropy was defined by Li (2007) and Lei (2014), Baik et al. (2015) as the topological entropy of $p_c(z)$ on the Hubbard tree; this definition applies to the postcritically finite case in particular, and more generally to finite or infinite compact trees, but it does not work when c is an endpoint with a dense postcritical orbit.

In a fairly general situation, the filled Julia set \mathcal{K}_c is path-connected with empty interior. Then \mathcal{K}_c consists of the Hubbard tree T_c , the countable family of its preimages, and an uncountable family of endpoints. The dynamics on T_c is interesting because this tree is folded over itself, while the iteration does not return to strict preimages of T_c . On the other hand, with the single exception of $c = -2$ (Zdunik 2000; Smirnov 2001; Meerkamp and Schleicher 2013), the endpoints form a set of full harmonic measure, while the external angles of T_c and its preimages form a set of Hausdorff dimension < 1 . Since all biaccessible points are contained in arcs iterated to T_c , these angles are called biaccessible (or biaccessing). More precisely, the biaccessibility dimension is defined as follows:

- For the lamination generated by an angle $\vartheta \in \mathbb{S}^1$ Thurston (2009), consider all angles of non-trivial leaves. Their Hausdorff dimension is the *combinatorial biaccessibility dimension* $B_{\text{comb}}(\vartheta)$. The same dimension is obtained from pairs of angles with the same itinerary, or from pairs not separated by the precritical leaves: the diameter joining $\vartheta/2$ and $(\vartheta + 1)/2$, and its preimages.
- For a parameter $c \in \mathcal{M}$, the *topological biaccessibility dimension* $B_{\text{top}}(c)$ is the Hausdorff dimension of those angles for which the dynamic ray is landing together with another ray.

These definitions are related analogously to Lemma 2.3:

Lemma A.1 (Combinatorial and topological biaccessibility) *Suppose that $\vartheta \in \mathbb{S}^1$ and $c \in \partial\mathcal{M}$ belongs to the impression of the parameter ray with angle ϑ , or $c \in \mathcal{M}$ is hyperbolic and the ray lands at the corresponding root. Then $B_{\text{comb}}(\vartheta) = B_{\text{top}}(c)$.*

Sketch of proof Bruin and Schleicher (2001), Jung (2014) In the locally connected case, and neglecting the countable set of angles at precritical or precharacteristic points, two dynamic rays are landing together if and only if they are not separated by a precritical ray-pair. (This separation line contains additional internal arcs when the interior is not empty.) When \mathcal{K}_c is not locally connected, exceptional sets of angles are shown to be negligible in terms of Hausdorff dimension: a Cremer periodic point might be topologically biaccessible (Schleicher and Zakeri 2000), but the relevant angles have Hausdorff dimension 0 (Bullett and Sentenac 1994). On the other hand, in an infinitely renormalizable situation, rays with combinatorially biaccessing angles may fail to land together, or to land at all. But their Hausdorff dimension is 0 in the case of the main molecule, and it is less than the dimension of non-renormalizable biaccessing angles in the primitive maximal case (Jung 2014). □

The following relation to entropy is due to Lei (2014), relying on earlier work by Furstenberg (1967) and Douady (1995).

Proposition A.2 (Dimension and entropy of the tree) *Suppose that either*

- \mathcal{K}_c is locally connected with empty interior, or
- f_c is parabolic, or
- f_c is hyperbolic so that the attracting orbit has real multiplier.

Using regulated arcs when necessary, define the tree T_c as the path-connected hull of the critical orbit. If T_c is compact, consider the topological entropy of $p_c(z)$ on T_c . Then it is related to the biaccessibility dimension by $h_{\text{top}}(T_c) = B_{\text{top}}(c) \cdot \log 2$.

Proof The proof is found in version 1 of Bruin and Schleicher (2017) and in Tiozzo (2013), Jung (2014): since $\vartheta \mapsto z(\vartheta)$ is a semi-conjugation with finite fibers, we may consider the topological entropy of the angle-doubling map on the compact set of angles of T_c (de Melo and van Strien 1993, Thm. II.7.1). And this equals the Hausdorff dimension (Furstenberg 1967, Proposition III.1), except for the base 2 instead of e in the logarithm of the growth factor λ .

While the definition of $h_{\text{top}}(T_c)$ requires a compact tree T_c , a general notion was given in Definition 2.1 in terms of precritical ray pairs, and its relation to the biaccessibility dimension shall be discussed now:

Theorem A.3 (Dimension and entropy in general) *Entropy and biaccessibility dimension are related as follows for all parameters $c \in \mathcal{M}$ and all angles $\vartheta \in \mathbb{S}^1$:*

$$\tilde{h}(c) = B_{\text{top}}(c) \cdot \log 2 \quad \text{and} \quad h(\vartheta) = B_{\text{comb}}(\vartheta) \cdot \log 2. \quad (\text{A.1})$$

Proof First, suppose that $c = c(\vartheta)$ is postcritically finite or belongs to a dyadic vein. In particular, \mathcal{K}_c is locally connected and T_c is compact with finitely many endpoints. Then the growth rate of monotonic branches on T_c is equal to the growth rate of precritical points on $[\alpha_c, -\alpha_c]$, so $h_{\text{top}}(T_c) = \tilde{h}(c)$, and Proposition A.2 applies.

Second, suppose that c is a non-renormalizable irrational endpoint; approximate it with biaccessible parameters $c_n < c$. Then monotonicity of $B_{\text{top}}(c)$ (Jung 2014, Proposition 4.6) (which in turn is a consequence of monotonicity of characteristic leaves of the lamination, in analogy to Lemma 2.5) and continuity of $\tilde{h}(c)$ give

$$B_{\text{top}}(c) \cdot \log 2 \geq \lim B_{\text{top}}(c_n) \cdot \log 2 = \lim \tilde{h}(c_n) = \tilde{h}(c). \quad (\text{A.2})$$

For the opposite estimate, note that the plane is cut into pieces successively by precritical ray pairs, and the angles of a piece of level n form up to n intervals of total length 2^{-n} according to Bruin and Schleicher (2017, Lemma 4.1). Recall that $N(n)$ is the number of precritical points of generation n on $[\alpha_c, -\alpha_c]$, and denote the number of level- n pieces intersecting the arc $[\alpha_c, -\alpha_c]$ by $V(n)$. Then $V(n) = 1 + N(1) + \dots + N(n)$ is growing by the same factor $\lambda = e^h$ as $N(n)$, and the same holds for $n \cdot V(n)$. The b -dimensional Hausdorff measure of the angles of $[\alpha_c, -\alpha_c]$ is estimated as

$$\mu_b \leq \lim n \cdot V(n) \cdot 2^{-bn}, \quad (\text{A.3})$$

which is 0 when $b > \log \lambda / \log 2 = \tilde{h}(c) / \log 2$. So the Hausdorff dimension is estimated as $B_{\text{top}}(c) \leq \tilde{h}(c) / \log 2$ as well, implying equality.

Finally, both $\tilde{h}(c)$ and $B_{\text{top}}(c)$ are constant on the main molecule and on primitive small Mandelbrot sets: for $B_{\text{top}}(c)$ see Jung (2014, Theorem 4.7), and $\tilde{h}(c)$ is constant on a dense subset and continuous. Moreover, both scale by the period of immediate satellite renormalization, see Lemma 6.19. Now all cases are covered by the Yoccoz Theorem (Milnor 2000a).

Continuity of entropy according to Theorem 5.9 gives:

Corollary A.4 (Continuity of biaccessibility dimension) *The biaccessibility dimension $B_{\text{comb}}(\vartheta)$ is continuous on \mathbb{S}^1 and $B_{\text{top}}(c)$ is continuous on the Mandelbrot set \mathcal{M} .*

Proposition A.2 and Theorem A.3 show that the definition of entropy in terms of precritical points is a generalization of the original definition in terms of a compact core:

Corollary A.5 (Extending the definition of core entropy) *We have $h_{\text{top}}(T_c) = \tilde{h}(c)$ whenever T_c is defined and compact.*

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