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RESEARCH CONTRIBUTION

Uniqueness of a Three-Dimensional Ellipsoid with Given Intrinsic Volumes

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Abstract
Let $\mathcal{E}$ be an ellipsoid in $\mathbb{R}^n$. Gusakova and Zaporozhets conjectured that $\mathcal{E}$ is uniquely (up to rigid motions) determined by its intrinsic volumes. We prove this conjecture for $n = 3$.

Keywords Intrinsic volumes · Ellipsoid · Mixed volume · Mean width

1 Introduction

1.1 Intrinsic Volumes

For a bounded convex set $K \subset \mathbb{R}^n$ the *intrinsic volumes* $V_0(K), \dots, V_n(k)$ are defined as the coefficients in the Steiner formula

$$\mathrm{Vol}(K + tB_n) = \sum_{k=0}^{n} \kappa_{n-k} V_k(K) t^{n-k}, \tag{1}$$

where B_n denotes the Euclidean unit ball in $\mathbb{R}^n$, $\kappa_k = \pi^{k/2}/\,\Gamma(\frac{k}{2}+1)$ denotes the volume of B_k, and Vol denotes the $n-$dimensional volume. Kubota's formula states that

$$V_k(K) = \binom{n}{k} \frac{\kappa_n}{\kappa_k \kappa_{n-k}} \int_{G_{n,m}} \mathrm{Vol}_k(p_\nu(K))\, d\omega(\nu), \quad 1 \le k \le n. \tag{2}$$

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Here $G_{n,k}$ denotes the Grassmannian of all k-dimensional linear subspaces of $\mathbb{R}^n$; $p_\nu(K)$ denotes the orthogonal projection of K to $\nu \in G_{n,k}$; $d\omega$ is the $O(n)$-invariant probabilistic measure on $G_{n,k}$ (see Burago and Zalgaller 1988, 19.3.2).

In particular, if $k = 1, n-1, n$, then $V_k(K)$ coincides up to a constant factor with the so-called mean width, surface area and the n-dimensional volume of K.

It is clear that in general the convex body can not be determined by the sequence of its intrinsic volumes, but this may be expected to be the case for certain natural n-parametric families of convex bodies. For rectangular parallelepipeds, their intrinsic volumes up to constant factor are elementary symmetric functions of the edge lengths. Hence by Vieta theorem the edge lengths are the roots of the corresponding polynomial. Therefore we can uniquely recover the edge lengths of the rectangular parallelepiped by its intrinsic volumes.

Anna Gusakova and Dmitry Zaporozhets conjectured in 2017 the uniqueness in the class of ellipsoids:

Conjecture 1 *If $\mathcal{E}_1, \mathcal{E}_2$ are two ellipsoids in $\mathbb{R}^n$ such that $V_i(\mathcal{E}_1) = V_i(\mathcal{E}_2)$ for all $i = 1, 2, \dots, n$, then $\mathcal{E}_1$ and $\mathcal{E}_2$ are congruent.*

For $n = 1$ and $n = 2$ this conjecture is quite simple. The main goal of this note is to prove it in dimension 3, when it can be formulated as follows.

Theorem 1 *If the volume, surface area and mean width of two ellipsoids in $\mathbb{R}^3$ are the same, then the ellipsoids are congruent.*

1.2 Explicit Formulas for Intrinsic Volumes of Ellipsoids

Using Tsirelson's formula (see Tsirelson 1985 or Theorem 1.9 in Kabluchko and Zaporozhets 2016 for details) one can obtain the following expression for intrinsic volumes of ellipsoid $\mathcal{E}$ with semiaxes $\{a_i\}_{i=1}^n$:

$$V_m(\mathcal{E}) = \frac{(2\pi)^{m/2}}{m!}\mathbb{E}\sqrt{\det\left(MM^\top\right)}, \tag{3}$$

where the random rows $\xi_1, \dots, \xi_m \in \mathbb{R}^n$ are i.i.d. $\sim \mathcal{N}(0, \operatorname{diag}(a_1^2, \dots, a_n^2))$ and M is the $m \times n$ matrix whose rows are $\xi_1, \dots, \xi_m$. In other words, $\sqrt{\det\left(MM^\top\right)}$ is the m-dimensional volume of the parallelepiped with the edge vectors $\xi_1, \dots, \xi_m$.

Taking $n = 3$, $m = 1$ in (3) we obtain the expression for the mean width of $\mathcal{E}$:

$$V_1(\mathcal{E}) = \sqrt{2\pi}\mathbb{E}\sqrt{\langle \xi_1, \xi_1\rangle} = \sqrt{2\pi}\mathbb{E}\sqrt{a_1^2x^2 + a_2^2y^2 + a_3^2z^2}, \quad x, y, z \sim \mathcal{N}(0, 1). \tag{4}$$

The next relation (see Kabluchko and Zaporozhets 2016, prop. 4.8) states a duality between V_k and V_{n-k}. Consider the following ellipsoids in $\mathbb{R}^n$:

$$\mathcal{E} = \left\{x \in \mathbb{R}^n : \sum_{i=1}^n a_i^2 x_i^2 \le 1\right\}, \quad \mathcal{E}^* = \left\{x \in \mathbb{R}^n : \sum_{i=1}^n \frac{x_i^2}{a_i^2} \le 1\right\}.$$

Then

$$V_k(\mathcal{E}) = \frac{\kappa_k}{\kappa_n \kappa_{n-k}} V_n(\mathcal{E}) V_{n-k}(\mathcal{E}^*).$$

Again, taking $n = 3$ and $k = 2$, we obtain

$$V_2(\mathcal{E}) = \frac{\pi}{2} a_1 a_2 a_3 V_1(\mathcal{E}^*) = \frac{\pi^{3/2}}{\sqrt{2}} a_1 a_2 a_3 \mathbb{E} \sqrt{\frac{1}{a_1^2} x^2 + \frac{1}{a_2^2} y^2 + \frac{1}{a_3^2} z^2}, \quad x, y, z \sim \mathcal{N}(0, 1). \tag{5}$$

2 Proof of Theorem 1

Now we have the explicit formulas for intrinsic volumes of ellipsoids in $\mathbb{R}^3$. From now on we will use the notation (a, b, c) for semiaxes of ellipsoids instead of (a_1, a_2, a_3).

We parametrize the family of three-dimensional ellipsoids by semiaxes and so we identify it with $\mathbb{R}^3_+$.

Definition 1 The function V is given by

$$V(a, b, c) = \Big(V_1(a, b, c), V_2(a, b, c), V_3(a, b, c)\Big), \tag{6}$$

where $V_j(a, b, c)$, $j = 1, 2, 3$, is the jth intrinsic volume of the ellipsoid with semiaxes a, b, c.

This parameterization of ellipsoids by semiaxes is not bijective on $\mathbb{R}^3_+$, but becomes bijective if we restrict it to the set of parameters $\{(a, b, c) \in \mathbb{R}^3_+ : a \geq b \geq c\}$.

Fix a point $(a_0, b_0, c_0) \in \mathbb{R}^3_+$ such that

$$a_0 > b_0 \geq c_0 \;\; or \;\; a_0 \geq b_0 > c_0 \tag{7}$$

For $i \in \{1,\ 3\}$ let $M_{V_i} = M_{V_i}(a_0, b_0, c_0)$ denote the level set of V_i:

$$M_{V_i} = \{(a, b, c) \in \mathbb{R}^3_+ :\ V_i(a, b, c) = V_i(a_0, b_0, c_0)\}.$$

Then M_{V_3} is a smooth two-dimensional manifold given by the equation $abc = \text{const}$, and M_{V_1} is a smooth two-dimensional manifold since V_1 is a smooth function with nonzero gradient (this is clear from differentiating (4), see also the computations below).

Step 1 *The manifolds M_{V_1} and M_{V_3} intersect transversally.*

Proof The reason is that the gradient vectors ∇V_1 and ∇V_3 have the opposite orders of coordinates:

$$\frac{\partial V_1}{\partial a} \geq \frac{\partial V_1}{\partial b} \geq \frac{\partial V_1}{\partial c} \text{ and } \frac{\partial V_3}{\partial a} \leq \frac{\partial V_3}{\partial b} \leq \frac{\partial V_3}{\partial c}, \tag{8}$$

and the inequalities are strict when corresponding semiaxes are not equal. We have

$$\nabla V_3(a,b,c) = \nabla\left(\frac{4\pi}{3}abc\right) = \frac{4\pi}{3}abc\cdot\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right),$$

so the inequality for partial derivatives of V_3 is clear.

We differentiate (4) and obtain

$$\frac{1}{\sqrt{2\pi}}\frac{\partial V_1}{\partial a}(a,b,c) = \mathbb{E}\frac{ax^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} = \mathbb{E}\frac{x^2}{\sqrt{x^2+\frac{b^2}{a^2}y^2+\frac{c^2}{a^2}z^2}}$$
$$\geq \mathbb{E}\frac{x^2}{\sqrt{x^2+\frac{a^2}{b^2}y^2+\frac{c^2}{b^2}z^2}} = \frac{1}{\sqrt{2\pi}}\frac{\partial V_1}{\partial b}(a,b,c),$$

analogously $\frac{\partial V_1}{\partial b}(a,b,c) \geq \frac{\partial V_1}{\partial c}(a,b,c)$, and equalities hold only if $a=b$ or $b=c$, respectively.

Therefore the vectors ∇V_1 and ∇V_3 are collinear if and only if $a=b=c$. But in this case $V_1 = (\frac{48}{\pi}V_3)^{1/3}$ is minimal possible for fixed V_3 (see, for example, Burago and Zalgaller 1988, Section 20.2, or apply the isoperimetric inequality for the ellipsoid with semi-axes $1/a$, $1/b$, $1/c$), and the equality is only possible for a ball. Hence by (7) $M_{V_1}\cap M_{V_3}$ do not contain any point with three equal coordinates. Therefore the manifolds M_{V_1} and M_{V_3} intersect transversally. □

Further we need also the following

Lemma 1 *The intersection $M_{V_1}\cap M_{V_3}$ contains the unique point of the form (a,a,b), $a<b$ and the unique point of the form (c,c,d), $d>c$.*

Proof Consider the curve $\gamma(t) = (t,t,C/t^2)\subset M_{V_3}$, where $C = V_3(a_0,b_0,c_0)$. For $t = C^{1/3}$ the function V_1 takes its minimal value, and this minimum is strictly less than $V_1(a_0,b_0,c_0)$. For large or small $t>0$ it tends to infinity (since the mean width is an inclusion-monotone function of the convex body, and the mean width of the long segment is large.) Thus by continuity it suffices to prove that if the derivative of V_1 along γ equals to zero at point t_0, then $t_0 = C^{1/3}$. Note that the the gradients of both functions V_3 and V_1 have the form (A,A,B) at points of γ. The gradient of V_3 is orthogonal to the tangent vector $(1,1,-2C/t_0^3)$ of the curve $\gamma(t)$ at t_0, since V_3 is constant along γ. This orthogonality rewrites as $A = C\cdot B/t_0^3$. If the gradient of V_1 is also orthogonal to $(1,1,-2C/t_0^3)$, then these two gradients are proportional. But we have already proved that this holds only if $t_0 = C^{1/3}$. □

Step 2 *The set $N := M_{V_1}\cap M_{V_3}$ is diffeomorphic to a union of several circles.*

Proof The set M_{V_1} is bounded by the aforementioned monotonocity argument. By the implicit function theorem, the intersection of two smooth 2-dimensional transversally intersecting manifolds in $\mathbb{R}^3$ is 1-dimensional smooth manifold. Since M_{V_1} is bounded, $M_{V_1}\cap M_{V_3}$ is a compact 1-dimensional smooth manifold. Hence it is diffeomorphic to a union of several circles. □

Step 3 *There does not exist another point* (a_1, b_1, c_1), $a_1 \geq b_1 \geq c_1$, *with the same values of* V_1, V_2, V_3 *as at the point* (a_0, b_0, c_0).

Proof Now we formulate the crucial lemma, whose proof is postponed to Sect. 3.

Lemma 2 *Jacobian of* $V(a, b, c)$ *is non-zero on the set* $\{(a, b, c) \in \mathbb{R}^3 : a > b > c > 0\}$.

By the previous step, the set $N := M_{V_1} \cap M_{V_3}$ is diffeomorphic to the union of several circles, and N contains exactly 6 points with two equal coordinates by Lemma 1.

On the other hand, any connected component γ of N must contain at least two points with equal coordinates. Indeed, by Lemma 2 any point p on γ with locally maximal or locally minimal value of V_2 must have two equal coordinates (since the derivatives of all three functions V_1, V_2, V_3 along γ are equal to 0).

Denote by S_{ij} the transposition of the ith and jth coordinates, say $S_{12}((x, y, z)) = (y, x, z)$. Obviously set N is invariant under all these symmetries.

Let γ be a connected component of N which contains the point $p_0 = (a, a, b)$, $a < b$. Then $S_{12}(\gamma)$ is also a connected component of N containing p_0. So $S_{12}(\gamma) = \gamma$. Analogously, if γ contains a point with another pair of equal coordinates, it gives another symmetry, S_{13} or S_{23}, which preserves γ, and γ is invariant under all the symmetries. In this case all 6 points from N with two equal coordinates belong to γ, and $N = \gamma$.

If not, the second point $q_0 \in \gamma \backslash \{p_0\}$ with two equal coordinates should be $q_0 = (c, c, d)$, $c > d$. But then by continuity there exists a point on γ between p_0 and q_0 with equal second and third coordinates. The contradiction.

Therefore N is a single circle, and six points on N have equal coordinates. The intervals between these six points belong to six Weyl chambers (corresponding to the six orderings of coordinates). Consider two our points (a_0, b_0, c_0) and (a_1, b_1, c_1) in M which belong to the same closed Weyl chamber $\{a \geq b \geq c\}$. If the function V_2 takes the same value at these two points, it has a local maximum or minimum strictly between them. But such a point should have two equal coordinates as noted before. The contradiction. □

3 Proof of Lemma 2

3.1 Explicit Formula for the Jacobian Matrix

It will be more convenient for us to consider functions $\tilde{V}_1$, $\tilde{V}_2$ and $\tilde{V}_3$ given by

$$\tilde{V}_3(a, b, c) = \frac{3}{4\pi} V_3(e^a, e^b, e^c) = e^a e^b e^c = e^{a+b+c},$$

$$\tilde{V}_2(a, b, c) = \frac{4\sqrt{2}}{3\sqrt{\pi}} \cdot \frac{V_2(e^a, e^b, e^c)}{V_3(e^a, e^b, e^c)} = \mathbb{E}\sqrt{e^{-2a}x^2 + e^{-2b}y^2 + e^{-2c}z^2}, \quad x, y, z \sim \mathcal{N}(0, 1),$$

$$\tilde{V}_1(a, b, c) = \frac{1}{\sqrt{2\pi}} V_1(e^a, e^b, e^c) = \mathbb{E}\sqrt{e^{2a}x^2 + e^{2b}y^2 + e^{2c}z^2}, \quad x, y, z \sim \mathcal{N}(0, 1).$$

We first compute gradients of these functions:

$$\nabla \tilde{V}_3(a,b,c) = \nabla e^{a+b+c} = e^{a+b+c}\cdot(1,1,1),$$
$$\tilde{V}_2(a,b,c)'_a = \left(\mathbb{E}\sqrt{e^{-2a}x^2+e^{-2b}y^2+e^{-2c}z^2}\right)'_a = -\mathbb{E}\frac{e^{-2a}x^2}{\sqrt{e^{-2a}x^2+e^{-2b}y^2+e^{-2c}z^2}}, \tag{9}$$

hence

$$\nabla \tilde{V}_2(a,b,c)$$
$$= -\left(\mathbb{E}\frac{e^{-2a}x^2}{\sqrt{e^{-2a}x^2+e^{-2b}y^2+e^{-2c}z^2}}, \mathbb{E}\frac{e^{-2b}y^2}{\sqrt{e^{-2a}x^2+e^{-2b}y^2+e^{-2c}z^2}}, \mathbb{E}\frac{e^{-2c}z^2}{\sqrt{e^{-2C}x^2+e^{-2b}y^2+e^{-2c}z^2}}\right).$$

The same way we obtain:

$$\nabla \tilde{V}_1(a,b,c)$$
$$= \left(\mathbb{E}\frac{e^{2a}x^2}{\sqrt{e^{2a}x^2+e^{2b}y^2+e^{2c}z^2}}, \mathbb{E}\frac{e^{2b}y^2}{\sqrt{e^{2a}x^2+e^{2b}y^2+e^{2c}z^2}}, \mathbb{E}\frac{e^{2c}z^2}{\sqrt{e^{2a}x^2+e^{2b}y^2+e^{2c}z^2}}\right).$$

Now we define auxiliary function, in terms of which the Jacobi matrix can be conveniently written

Definition 2

$$G(a,b,c) = \mathbb{E}\frac{a^2x^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}, \text{ where } x,y,z\sim\mathcal{N}(0,1).$$

Statement 1 *In this notation Jacobi matrix of V has the following form:*

$$J_{\tilde{V}}(a,b,c) = \begin{pmatrix} 1 & 1 & 1 \\ G(e^{-a},e^{-b},e^{-c}) & G(e^{-b},e^{-a},e^{-c}) & G(e^{-c},e^{-a},e^{-b}) \\ G(e^{a},e^{b},e^{c}) & G(e^{b},e^{a},e^{c}) & G(e^{c},e^{a},e^{b}) \end{pmatrix}$$

It is sufficient to prove that the following matrix is nondegenerate for $a > b > c > 0$:

$$\begin{pmatrix} 1 & 1 & 1 \\ G(\frac{1}{a},\frac{1}{b},\frac{1}{c}) & G(\frac{1}{b},\frac{1}{a},\frac{1}{c}) & G(\frac{1}{c},\frac{1}{a},\frac{1}{b}) \\ G(a,b,c) & G(b,a,c) & G(c,a,b) \end{pmatrix}. \tag{10}$$

3.2 Alternative Formula for the Function *G(a,b,c)*

Using the Gaussian integral

$$\int_{\mathbb{R}} e^{-Ts^2}ds = \frac{\sqrt{\pi}}{\sqrt{T}}$$

with $T = a^2x^2 + b^2y^2 + c^2z^2$ we rewrite the formula of function G:

$$\begin{aligned} G(a,b,c) &= \frac{1}{(2\pi)^{3/2}} \cdot \iiint_{\mathbb{R}^3} \frac{a^2x^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} e^{-1/2(x^2+y^2+z^2)} dxdydz \\ &= \frac{1}{2^{3/2}\pi^2} \cdot \int_{\mathbb{R}} \iiint_{\mathbb{R}^3} a^2x^2 e^{-1/2(x^2+y^2+z^2)-s^2(a^2x^2+b^2y^2+c^2z^2)} dxdydzds \\ &= \frac{1}{2^{3/2}\pi^2} \cdot \int_{\mathbb{R}} \int_{\mathbb{R}} a^2x^2 e^{-x^2(1/2+a^2s^2)} dx \int_{\mathbb{R}} e^{-y^2(1/2+b^2s^2)} dy \int_{\mathbb{R}} e^{-z^2(1/2+c^2s^2)} dzds \\ &= \frac{1}{2\sqrt{2\pi}} \cdot \int_{\mathbb{R}} \frac{a^2}{(\sqrt{a^2s^2+1/2})^3} \frac{1}{\sqrt{b^2s^2+1/2}} \frac{1}{\sqrt{c^2s^2+1/2}} ds. \end{aligned}$$

Using the above formulas and applying a change of variables $s \to s/\sqrt{2}$ we obtain

$$G(a,b,c) = \sqrt{\frac{2}{\pi}} \cdot \int_{\mathbb{R}} \frac{1}{s^2+\frac{1}{a^2}} \frac{1}{\sqrt{(a^2s^2+1)(b^2s^2+1)(c^2s^2+1)}} ds.$$

In the same way we see that

$$G\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right) = \sqrt{\frac{2}{\pi}} \cdot \int_{\mathbb{R}} \frac{1}{t^2+a^2} \frac{1}{\sqrt{(\frac{t^2}{a^2}+1)(\frac{t^2}{b^2}+1)(\frac{t^2}{c^2}+1)}} dt.$$

Now we write the determinant of (10) in the following form

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{t^2+a^2} & \frac{1}{t^2+b^2} & \frac{1}{t^2+c^2} \\ \frac{1}{s^2+\frac{1}{a^2}} & \frac{1}{s^2+\frac{1}{b^2}} & \frac{1}{s^2+\frac{1}{c^2}} \end{vmatrix} \frac{1}{\sqrt{(\frac{t^2}{a^2}+1)(\frac{t^2}{b^2}+1)(\frac{t^2}{c^2}+1)}} \frac{1}{\sqrt{(a^2s^2+1)(b^2s^2+1)(c^2s^2+1)}} dtds.$$

The following identity holds:

$$\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{t^2+a^2} & \frac{1}{t^2+b^2} & \frac{1}{t^2+c^2} \\ \frac{1}{s^2+\frac{1}{a^2}} & \frac{1}{s^2+\frac{1}{b^2}} & \frac{1}{s^2+\frac{1}{c^2}} \end{vmatrix} = \frac{(a^2-b^2)(a^2-c^2)(b^2-c^2)(s^2t^2-1)}{(a^2s^2+1)(b^2s^2+1)(c^2s^2+1)(\frac{t^2}{a^2}+1)(\frac{t^2}{b^2}+1)(\frac{t^2}{c^2}+1)}.$$

So the determinant of (10) equals to

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(a^2-b^2)(a^2-c^2)(b^2-c^2)(s^2t^2-1)}{[(a^2s^2+1)(b^2s^2+1)(c^2s^2+1)(\frac{t^2}{a^2}+1)(\frac{t^2}{b^2}+1)(\frac{t^2}{c^2}+1)]^{\frac{3}{2}}} dtds. \tag{11}$$

Note that the integrand is even with respect to s and t. Denote by I the same integral but over the set $\mathbb{R}_+ \times \mathbb{R}_+$. Applying the change of variables $s \to \frac{1}{x},\ t \to \frac{1}{y}$ we get

$$I = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{(a^2-b^2)(a^2-c^2)(b^2-c^2)(\frac{1}{x^2y^2}-1)}{\left[\left(\frac{a^2}{x^2}+1\right)\left(\frac{b^2}{x^2}+1\right)\left(\frac{c^2}{x^2}+1\right)\left(\frac{1}{y^2a^2}+1\right)\left(\frac{1}{y^2b^2}+1\right)\left(\frac{1}{y^2c^2}+1\right)\right]^{\frac{3}{2}}}\frac{1}{x^2y^2}dxdy.$$

This integral is similar to I: the integrands differ by the factor of $(xy)^5$. Indeed,

$$\begin{aligned} I &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{(a^2-b^2)(a^2-c^2)(b^2-c^2)(1-x^2y^2)}{\left[\left(\frac{1}{x^2}+\frac{1}{a^2}\right)\left(\frac{1}{x^2}+\frac{1}{b^2}\right)\left(\frac{1}{x^2}+\frac{1}{c^2}\right)\left(\frac{1}{y^2}+a^2\right)\left(\frac{1}{y^2}+b^2\right)\left(\frac{1}{y^2}+c^2\right)\right]^{\frac{3}{2}}}\frac{1}{x^4y^4}dxdy \\ &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{(a^2-b^2)(a^2-c^2)(b^2-c^2)(1-x^2y^2)}{\left[(xy)^{-6}\left(1+\frac{x^2}{a^2}\right)\left(1+\frac{x^2}{b^2}\right)\left(1+\frac{x^2}{c^2}\right)(1+y^2a^2)(1+y^2b^2)(1+y^2c^2)\right]^{\frac{3}{2}}}\frac{1}{x^4y^4}dxdy \\ &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{(a^2-b^2)(a^2-c^2)(b^2-c^2)(1-x^2y^2)(xy)^5}{\left[\left(1+\frac{x^2}{a^2}\right)\left(1+\frac{x^2}{b^2}\right)\left(1+\frac{x^2}{c^2}\right)(1+y^2a^2)(1+y^2b^2)(1+y^2c^2)\right]^{\frac{3}{2}}}dxdy. \end{aligned}$$

Redenoting the variables we write this as

$$I = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{(a^2-b^2)(a^2-c^2)(b^2-c^2)(1-t^2s^2)(ts)^5}{[(1+\frac{t^2}{a^2})(1+\frac{t^2}{b^2})(1+\frac{t^2}{c^2})(1+s^2a^2)(1+s^2b^2)(1+s^2c^2)]^{\frac{3}{2}}}dtds. \tag{12}$$

Now we consider the sum of the two copies of I, using (11) and (12). The denominators of integrands coincide. Then separately write out the numerators divided by $(a^2 - b^2)(a^2-c^2)(b^2-c^2)$

$$(s^2t^2-1)+(1-s^2t^2)(st)^5 = ((st)^2-1)(1-(st)^5)$$

This observation shows that $I + I$ is an integral of a negative function. Hence I is not equal to zero on the set $\{a > b > c > 0\}$, that finishes the proof of Lemma:

$$I + I = (a^2-b^2)(a^2-c^2)(b^2-c^2)\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{((st)^2-1)(1-(st)^5)}{[(1+\frac{t^2}{a^2})(1+\frac{t^2}{b^2})(1+\frac{t^2}{c^2})(1+s^2a^2)(1+s^2b^2)(1+s^2c^2)]^{\frac{3}{2}}}dtds.$$

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