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RESEARCH CONTRIBUTION

Use of Block Toeplitz Matrix in the Study of Möbius Pairs of Simplexes in Higher-Dimensional Projective Space

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Abstract
A simplex in a projective space of dimension n is expressed by a matrix of order $n + 1$, where each row represents the homogeneous coordinates of a vertex of the simplex with respect to a reference frame. In the present study, a block Toeplitz matrix is used to express a simplex which forms a Möbius pair along with the reference simplex. A pair of mutually inscribed, circumscribed tetrahedrons is called a Möbius pair. The existence of such pairs of simplexes in higher-dimensional (odd) projective spaces is already established. In the present study an existence of an infinite chain of simplexes in a five-dimensional projective space is established where any two simplexes from the chain form a Möbius pair in some order of their vertices. This is studied with the help of powers of a block Toeplitz matrix. Also, attempt has been made to generalize this result to $2n + 1$-dimensional projective space.

Keywords Block toeplitz matrix · Homogeneous coordinates · Möbius pair of simplexes

1 Introduction and Notations

In this study, we use a $(n+1) \times (n+1)$ matrix S to represent a simplex $(B) = B_0 B_1 \ldots B_n$ in a n-dimensional projective space P_n with respect to a reference simplex $(A) = A_0 A_1 \ldots A_n$. Hence, row i, $i = 0, 1, \ldots, n$ of the matrix S represents the homogeneous coordinates of the vertex B_i with respect to the simplex of reference (A).

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A pair of inscribed, circumscribed or interlocked tetrahedrons is called a Möbius pair of tetrahedrons named after their discoverer F. A. Möbius (Maxwell 1961; Todd 1960). Study of Möbius pair is always an interesting topic in projective geometry. Misra and Sanyal (1993) had studied Möbius pair of tetrahedrons using quaternions. They obtained an infinite chain of tetrahedrons where every pair from the chain forms a Möbius pair. It is already established that Möbius pair of simplexes exists in higher-dimensional projective space when the dimension n is odd and $n \geq 3$ (Havlicek et al. 2010). In a P_n, with odd n, $n \geq 3$, two simplexes are mutually inscribed and circumscribed if each vertex of one lies on a face (hyper plane) of the other and vice versa. In the present study, we establish the existence of a chain of Möbius pairs of simplexes in P_5 and some related studies in P_n, where P_n is an n-dimensional projective space over a real field F. If the matrix is nonsingular, it represents a non- degenerate simplex; otherwise the simplex degenerates.

The homogeneous coordinates (x_i) and (tx_i) for $i = 0, 1, 2, \ldots, n$ and t $(\neq 0)$, represent the same point; so for every non-singular diagonal matrix $D = diag(d_0, d_1, \ldots.., d_n)$, DS also represents the coordinates of the vertices of (B) (Panda and Misra 2014). The matrices S and T are defined to be equivalent if $T = DS$ for some non-singular diagonal matrix D. This equivalence relation gives equivalence classes, where rows of any matrix of a particular class represent the vertices of a particular simplex for another. In matrix form, it is written as $(B)_A = S = ((b_{ij}))$ to represent (B) referred to (A). So if $(B)_A = ((b_{ij}))$, $(A)_C = ((a_{ij})$, then $(B)_C = ((d_{ij})) = ((b_{ij}))((a_{ij}))$ (Maxwell 1961).

Note 1 The matrix of a simplex can be multiplied by a nonsingular diagonal matrix from the left without changing the simplex. But this is not permitted, when the matrix is used for the coordinate transformation. However the matrix of a simplex can be multiplied by a nonzero scalar to make the matrix a simpler one, for a coordinate transformation (Herrmann 1952).

If two simplexes (A) and (B) are such that the matrix $(B)_A$ is a zero diagonal matrix, then the vertices of (B) lie on the faces of (A). The converse is also true. In particular, B_i lies on the opposite hyper plane of the vertex A_i in (A). When the vertex B_i of (B) lies on the opposite hyper plane of the vertex A_i in (A) which is the hyper plane containing the vertices $A_0, A_1, \ldots., A_{i-1}, A_{i+1}, \ldots., A_n$, the homogeneous coordinates of B_i can be expressed as a linear combination of the vertices $A_0, A_1, \ldots., A_{i-1}, A_{i+1}, \ldots., A_n$. As the vertex A_i is not present in this linear combination the ith coordinate of the vertex B_i with reference to the simplex (A) will be zero. Hence, the matrix expressing the simplex (B) with reference to the simplex (A) will be a zero diagonal matrix. In general, if $(B)_A$ has exactly one zero in each row and column, then the vertices of (B) will lie exactly on one of the hyper planes of (A). If the inverse of $(B)_A$ also has the same property, then the vertices of (A) will lie exactly on one of the hyper planes of (B) and the pair will form a non-degenerate Möbius pair.

2 Möbius Pairs in P_5

Let M be a 6×6 Block Toeplitz matrix, the rows of which are the homogeneous coordinates of the vertices of simplex (A^1) with respect to (A^0) which are simplexes in P_5.

Here, $(A^0) = A_0^0 A_1^0 A_2^0 \ldots A_5^0$ and $(A^1) = A_0^1 A_1^1 A_2^1 \ldots A_5^1$. This is represented as

$$(A^1)_{A^0} = M = \begin{bmatrix} K & -J & -J \\ J & K & -J \\ J & J & K \end{bmatrix},$$

where $K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. This matrix along with the notations for the blocks K and J are taken from (Havlicek et al. 2010). However unlike in (Havlicek et al. 2010), this matrix is used here to represent one of the simplexes of the Möbius pair. The matrix M is non-singular skew symmetric with determinant equal to 1. As a skew-symmetric matrix is a zero diagonal matrix, the vertices A_i^1 of the simplex (A^1), $i = 0, 1, \ldots, 5$ are lying on the opposite faces (hyper planes) of A_i^0 of the simplex (A^0). As M^{-1} is also skew symmetric, hence zero diagonal, the vertices A_i^0 of (A^0), $i = 0, 1, \ldots, 5$ are lying on the opposite faces (hyper planes) of A_i^1 of the simplex (A^1). So, (A^0) and (A^1) form a Möbius pair of simplexes in P_5 with $(A^1)_{A^0} = M$. Let (A^2) be another simplex with $(A^2)_{A^1} = M$. In that case, (A^1) and (A^2) also form a Möbius pair. To study the relationship between (A^0) and (A^2), let us express (A^2) with reference to (A^0) and obtain

$$(A^2)_{A^0} = (A^2)_{A^1} \times (A^1)_{A^0} = M \times M = M^2.$$

We get $(A^2)_{A^0} = M^2 = \begin{bmatrix} -I-4J & -2K-2J & -2K+2J \\ 2K-2J & -I-4J & -2K-2J \\ 2K+2J & 2K-2J & -I-4J \end{bmatrix}$.

When expressed without any block it becomes

$$\begin{bmatrix} -5 & -4 & -2 & 0 & 2 & 4 \\ -4 & -5 & -4 & -2 & 0 & 2 \\ -2 & -4 & -5 & -4 & -2 & 0 \\ 0 & -2 & -4 & -5 & -4 & -2 \\ 2 & 0 & -2 & -4 & -5 & -4 \\ 4 & 2 & 0 & -2 & -4 & -5 \end{bmatrix}.$$

Here, M^2 is non-singular, symmetric and block Toeplitz but not zero diagonal. Its inverse is given as

$$M^{-2} = \begin{bmatrix} -5 & 4 & -2 & 0 & 2 & -4 \\ 4 & -5 & 4 & -2 & 0 & 2 \\ -2 & 4 & -5 & 4 & -2 & 0 \\ 0 & -2 & 4 & -5 & 4 & -2 \\ 2 & 0 & -2 & 4 & -5 & 4 \\ -4 & 2 & 0 & -2 & 4 & -5 \end{bmatrix},$$

where $(A^0)_{A^2} = M^{-2}$. Both M^2 and its inverse are not zero diagonal. However, there is exactly one zero in every row and column of these matrices. The positions of zeros in these matrices establish that the vertex A_i^2 of the simplex (A^2) lies on the hyper plane opposite to the vertex A_j^0 of the simplex (A^0), where $j = (i + 3) \bmod 6$. Similarly, the vertex A_i^0 of the simplex (A^0) lies on the hyper plane opposite to the vertex A_j^2 of the simplex (A^2), where $j = (i + 3) \bmod 6$. So, the simplexes (A^0) and (A^2) also form a Möbius pair in P_5. Hence, we get the following theorem.

Theorem 2.1 If $(A^1)_{A^0} = M$ and $(A^2)_{A^1} = M$ are two pair of Möbius simplexes in P_5, then (A^0) and (A^2) also form a Möbius pair in P_5.

The above theorem creates possibilities of a chain of Möbius pairs. Suppose we have another simplex (A^3), where $(A^3)_{A^2} = M$, then by Theorem 2.1, the pair (A^2) and (A^3) forms a Möbius pair. But the pair (A^0) and (A^3) also forms a Möbius pair provided the matrix M^3 and its inverse are zero diagonal or having zeros as in M^2. So, to establish a chain of Möbius simplexes in P_5, let us study the properties of the powers of M. All even powers of M are symmetric with determinant equal to 1 and all odd powers of M are skew symmetric with determinant equal to 1. For more clarity in the expressions of even powers of M, we propose the following lemma which will be proved by induction.

Lemma 2.2 If the matrix M is of the form $\begin{bmatrix} K & -J & -J \\ J & K & -J \\ J & J & K \end{bmatrix}$, then

$$M^{2n} = \begin{bmatrix} aI + 2bJ & bK + bJ + cI & bK - bJ - cI \\ -bK + bJ + cI & aI + 2bJ & bK + bJ + cI \\ -bK - bJ - cI & -bK + bJ + cI & aI + 2bJ \end{bmatrix}$$

for some real numbers a, b, and c.

Proof The proof is done by induction using the identities like $K^2 = -I$, $J^2 = 2J$ and $JK + KJ = 2K$.

Let us first prove the lemma for even power of M. The expression for M^2 as obtained earlier is of the form M^{2n} as mentioned above with $a = -1$, $b = -2$ and $c = 0$. Let $M^{2i} = \begin{bmatrix} aI + 2bJ & bK + bJ + cI & bK - bJ - cI \\ -bK + bJ + cI & aI + 2bJ & bK + bJ + cI \\ -bK - bJ - cI & -bK + bJ + cI & aI + 2bJ \end{bmatrix}$ for some a, b and c.
Now multiplying it with M^2 and simplifying it using the identities mentioned above, we get

$$M^{2i+2} = \begin{bmatrix} \alpha I + 2\beta J & \beta K + \beta J + \gamma I & \beta K - \beta J - \gamma I \\ -\beta K + \beta J + \gamma I & \alpha I + 2\beta J & \beta K + \beta J + \gamma I \\ -\beta K - \beta J - \gamma I & -\beta K + \beta J + \gamma I & \alpha I + 2\beta J \end{bmatrix},$$

where $\alpha = -(a + 4b)$, $\beta = -(2a + 13b + 2c)$ and $\gamma = -(c + 2b)$, respectively. It shows that if the proposition in the lemma for even power of M is true for $n = i$, then it is also true for $n = i + 1$. Hence, by mathematical induction, it is true for any n. □.

From Lemma 2.2, it can be observed that the powers of M are also block Toeplitz matrices. As M is skew symmetric, any odd power of M will be skew symmetric, and hence zero diagonal. So if we have a pair of simplexes (P) and (Q) such that $(Q)_P$ can be expressed by an odd power of M, then (P) and (Q) form a pair of Möbius simplexes in P_5, where the vertex Q_j of simplex (Q) lies on the opposite face of the vertex P_j of the simplex (P) and vice-versa.

Let us now consider the even power of M where the middle block of the first row is $bK + bJ + cI$ which is equal to $b\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} b+c & 0 \\ 2b & b+c \end{bmatrix}$. This establishes that the 4th element in the first row is zero. As the matrix is a Toeplitz one, the 5th element of 2nd row and 6th element of the 3rd row are also zero, respectively. Similarly, it can be established that the 1st, 2nd and 3rd elements of the 4th, 5th and 6th rows, respectively, are also zero. So if we have a pair of simplexes (P) and (Q) such that $(Q)_P$ can be expressed by an even power of M, then (P) and (Q) form a pair of Möbius simplexes in P_5, where the vertex Q_i of simplex (Q) lies on the opposite face of the vertex P_j of the simplex (R), where $j = i + 3 \bmod 6$. Hence, two simplexes will form a Möbius pair in some order of their vertices depending on whether the matrix expressing one with reference to the other is odd power of M or an even power of M.

From the coordinate transformation rules for homogeneous coordinates if $(C)_B = M^s$ and $(D)_C = M^t$, then $(D)_B = (D)_C \times (C)_B = M^s \times M^t = M^{s+t}$. The integral powers of M are all distinct; otherwise we must get $M^j = I$ for some j. But this is not possible as the off diagonal blocks of both even and odd powers, which are $bK + bJ + cI$ and $\beta K + \beta J + \gamma I$, are zero only when $b = c = 0$ and $\beta = \gamma = 0$. But in that case, the matrix itself will become zero which is not possible as M is non-singular. Even no two different powers of M are equivalent to each other in the sense that no one is equal to the product of another and a diagonal matrix. So the simplexes represented by these powers of M as expressed with reference to the simplex (A^0) are all distinct.

As M is skew symmetric so also is M^{2n+1}. Hence, the inverse of M^{2n+1} is also skew symmetric. However, the matrix M^{2n} is a symmetric matrix (even power of a skew symmetric matrix). It can be easily verified that the following matrix is scalar equivalent to the inverse of M^{2n} as the product of this matrix with M^{2n} gives a scalar matrix.

$$M^{-2n} \equiv \begin{bmatrix} aI + 2bL & bL - bK + cI & -bL - bK - cI \\ bL + bK + cI & aI + 2bL & bL - bK + cI \\ -bL + bK - cI & bL + bK + cI & aI + 2bL \end{bmatrix}.$$

Here, K is as defined before and $L = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ (Havlicek et al. 2010). The above expression of M^{-2n} is obtained using the identity $KL + LK = 2K$. Like in M^{2n}, the middle block of M^{-2n} has zero which is the fourth element in the first row. So, the position of zeros in M^{-2n} and M^{2n} is same. Hence, two simplexes will form a Möbius pair in some order of their vertices depending on whether the matrix expressing one with respect to the other is odd power of M or an even power of M. The above discussion establishes the following theorem.

Theorem 2.3 If $(A^i) = A_0^i A_1^i A_2^i A_3^i A_4^i A_5^i$ be simplexes with $i = 0, 1, 2, \ldots.$, in P_5 where $(A^{i+1})_{A^i} = M$, then for any pair of simplexes (A^j) and (A^k), $j \neq k$, from this infinite chain of simplexes $(A^j)_{A^k} = M^{j-k}$ and (A^j) and (A^k) form a pair of Möbius simplexes in P_5 in the following way.

(i) If $j–k$ is odd, then the ith vertex of (A^j) lies on the opposite face of then ith vertex of (A^k) and vice-versa.
(ii) If $j–k$ is even, then the ith vertex of (A^j) lies on the opposite face of the lth vertex of (A^k), where $l = i + 3 \bmod 6$ and vice-versa.

While studying the matrix M, its inverse and their different powers the followings are observed.

(i) The matrix of the minors of each of these matrices is the matrix itself. Hence, the inverse of each of these matrix is obtained simply by changing the sign of elements in the position (i, j), where $i + j$ is odd.
(ii) When expressed by blocks the inverse of each of these matrices are obtained by interchanging the block matrix J with L and changing the sign of the coefficient of the block matrix K present in the expression of the matrix.

3 Möbius Pairs in P_{2n+1}

Let us consider the $2n$ + 1-dimensional projective space P_{2n+1} over a field. A non-degenerate simplex of this space will have 2n + 2 vertices and can be expressed by a $(2n+2) \times (2n+2)$ non-singular matrix with respect to a simplex of reference (A^0) and some unit point. Let us generalize the matrix M from article 2 and get the following $(2n + 2) \times (2n + 2)$ non-singular matrix expressed by blocks using the 2×2 matrix blocks K and J as defined in article 2.

$$M = \begin{bmatrix} K & -J & \cdots & -J \\ J & K & \cdots & -J \\ \vdots & \vdots & \ddots & \cdots \\ J & J & \cdots & K \end{bmatrix}.$$

Here, each row and column consist of $n + 1$ number of blocks. If we have simplexes (A^0) and (A^1) in P_{2n+1} where $(A^1)_{A^0} = M$ as mentioned above, then (A^0) and (A^1) form a Möbius pair of simplexes in P_{2n+1}. This follows from the fact that the present M is skew symmetric, and hence a zero diagonal matrix, so also its inverse.

Let (A^2) be another simplex with $(A^2)_{A^1} = M$. In that case, (A^1) and (A^2) also form a Möbius pair. Here (A^2) is a simplex different from (A^0). We have $(A^2)_{A^0} = M^2$. The elements of the matrix M^2, denoted as x_{pq}, are obtained using the following rule.

$$x_{pq} = \begin{cases} -2K - 2[n - 2(q - p) + 1]J & \text{if } p < q \\ -I - 2nJ & \text{if } p = q \\ 2K - 2[n - 2(p - q) + 1]J & \text{if } p > q \end{cases}.$$

M^2 is obtained by two different expressions for odd n and for even n. When n is odd M^2 is equal to

$$\begin{bmatrix} -I-2nJ & \cdots & -2K & \cdots & -2K+2(n-1)J \\ 2K-2(n-1)J & \cdots & -2K-4J & \cdots & -2K+2(n-3)J \\ 2K-2(n-3)J & \cdots & -2K-8J & \cdots & -2K+2(n-5)J \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 2K-4J & \cdots & -2K-2(n-1)J & \cdots & -2K \\ 2K & \cdots & -I-2nJ & \cdots & -2K-4J \\ 2K+4J & \cdots & 2K-2(n-1)J & \cdots & -2K-8J \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 2K+2(n-3)J & \cdots & 2K-8J & \cdots & -2K-2(n-1)J \\ 2K+2(n-1)J & \cdots & 2K-4J & \cdots & -I-2nJ \end{bmatrix}.$$

The above matrix is a symmetric matrix and also a block Toeplitz matrix. The block $-2K$ in first row is the $[(n+3)/2]$th block. There is a zero element in the first row which comes from the block $-2K$. This zero in the first row is in the $(n+2)$th column of the matrix when written without any block. This position shifts by one column each for coming rows. The zero reaches the last column which is the $(2n+2)$th column, in the $(n+1)$th row and comes back to 1st column in the $(n+2)$th row. In the last row, which is $(2n+2)$th, the zero is on the $(n+1)$th column. So, there is a zero in every row and every column of this matrix.

When n is even M^2 is equal to

$$\begin{bmatrix} -I-2nJ & \cdots & -2K-2J & \cdots & -2K+2(n-1)J \\ 2K-2(n-1)J & \cdots & -2K-6J & \cdots & -2K+2(n-3)J \\ 2K-2(n-3)J & \cdots & -2K-10J & \cdots & -2K+2(n-5)J \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 2K-6J & \cdots & -2K-2(n-1)J & \cdots & -2K+2J \\ 2K-2J & \cdots & -I-2nJ & \cdots & -2K-2J \\ 2K+2J & \cdots & 2K-2(n-1)J & \cdots & -2K-6J \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 2K+2(n-3)J & \cdots & 2K-6J & \cdots & -2K-2(n-1)J \\ 2K+2(n-1)J & \cdots & 2K-2J & \cdots & -I-2nJ \end{bmatrix}.$$

The above matrix is also a symmetric matrix and also a block Toeplitz matrix. The block $-2K-2J$ in first row is the $[(n+3)/2]$th block. There is a zero element in the first row which comes from the block $-2K-2J$. This zero in the first row is in the $(n+2)$th column when written without any block. This position shifts by one column each for coming rows. The zero reaches the last column, which is the $(2n+2)$th column, in the $(n+1)$th row and comes back to 1st column in the $(n+2)$th row. In the last row, which is $(2n+2)$th, the zero is on the $(n+1)$th column. So, there is a zero in every row and column of this matrix.

We have M^{-1} equal to

$$M^{-1} = \begin{bmatrix} -K & -L & \cdots & -L \\ L & -K & \cdots & -L \\ \vdots & \vdots & \ddots & \cdots \\ L & L & \cdots & -K \end{bmatrix}.$$

We obtain M^{-2} from M^{-1} where the elements of M^{-2}, denoted as y_{pq}, are obtained using the following rule.

$$y_{pq} = \begin{cases} 2K - 2[n - 2(q - p) + 1]L & \text{if } p < q \\ -I - 2nL & \text{if } p = q \\ -2K - 2[n - 2(p - q) + 1]L & \text{if } p > q \end{cases}$$

When n is odd M^{-2} is equal to

$$\begin{bmatrix} -I - 2nL & \cdots & 2K & \cdots & 2K + 2(n-1)L \\ -2K - 2(n-1)L & \cdots & 2K - 4L & \cdots & 2K + 2(n-3)L \\ -2K - 2(n-3)L & \cdots & 2K - 8L & \cdots & 2K + 2(n-5)L \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -2K - 4L & \cdots & 2K - 2(n-1)L & \cdots & 2K \\ -2K & \cdots & -I - 2nL & \cdots & 2K - 4L \\ -2K + 4L & \cdots & -2K - 2(n-1)L & \cdots & 2K - 8L \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -2K + 2(n-3)L & \cdots & -2K - 8L & \cdots & 2K - 2(n-1)L \\ -2K + 2(n-1)L & \cdots & -2K - 4L & \cdots & -I - 2nL \end{bmatrix}$$

When n is even M^{-2} is equal to

$$\begin{bmatrix} -I - 2nL & \cdots & 2K - 2L & \cdots & 2K + 2(n-1)L \\ -2K - 2(n-1)L & \cdots & 2K - 6L & \cdots & 2K + 2(n-3)L \\ -2K - 2(n-3)L & \cdots & 2K - 10L & \cdots & 2K + 2(n-5)L \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -2K - 6L & \cdots & 2K - 2(n-1)L & \cdots & 2K + 2L \\ -2K - 2L & \cdots & -I - 2nL & \cdots & 2K - 2L \\ -2K + 2L & \cdots & -2K - 2(n-1)L & \cdots & 2K - 6L \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -2K + 2(n-3)L & \cdots & -2K - 6L & \cdots & 2K - 2(n-1)L \\ -2K + 2(n-1)L & \cdots & -2K - 2L & \cdots & -I - 2nL \end{bmatrix}$$

In M^{-2}, the position of zero in each row is same as it is in M^2.

The positions of zeros in the matrix M^2 establish that the vertex A_i^2 of the simplex (A^2) lies on the hyper plane opposite to the vertex A_j^0 of the simplex (A^0), where $j = (i + n + 1) \bmod 2n + 2$. Similarly, the positions of zeros in the matrix M^{-2} establish that the vertex A_i^0 of the simplex (A^0) lies on the hyper plane opposite to the vertex

A_j^2 of the simplex (A^2), where $j = (i + n + 1) \bmod 2n + 2$. So, we have the following theorem.

Theorem 3.1 If $(A^1)_{A^0} = M$ and $(A^2)_{A^1} = M$ are two pairs of Möbius simplexes in $\mathbf{P_{2n+1}}$, then (A^0) and (A^2) also form a Möbius pair in P_{2n+1}.

There will be a chain of simplexes (A^i) in P_{2n+1} just like in $\mathbf{P_5}$ where $(A^{i+1})A^i = M$. As M is a non-singular skew-symmetric matrix any odd power of M will also be a non-singular skew symmetric. So if we have a pair of simplexes (P) and (Q) in P_{2n+1} such that $(Q)_P$ can be expressed by an odd power of M, then (P) and (Q) forms a pair of Möbius simplexes in P_{2n+1} where the vertex Q_j of simplex (Q) lies on the opposite face of the vertex P_j of the simplex (P) and vice versa. Hence, the following theorem follows.

Theorem 3.2 Let (A^i) be simplexes with $i = 0, 1, 2, \ldots$ in P_{2n+1} where $(A^{i+1})A^i = M$, then for any pair of simplexes (A^j) and (A^k) from this infinite chain of simplexes $(A^j)A^k = M^{j-k}$. If j–k is odd, then (A^j) and (A^k) form a pair of Möbius simplexes in P_{2n+1}.

Theorem 3.2 is a generalization of Theorem 2.3 except for the situation when j–k is even.

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