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RESEARCH CONTRIBUTION

Dispersion of the Arnold's Asymptotic Ergodic Hopf Invariant and a Formula for Its Calculation

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Abstract
The main result of the paper is the formula that calculates the dispersion of the asymptotic Hopf invariant of a magnetic field. The paper contain examples, which describe magnetic fields in a conductive medium.

1 Introduction

The main result of the paper is the formula that calculates the dispersion of the asymptotic Hopf invariant of a magnetic field. The paper contain examples, which describe magnetic fields in a conductive medium. The basic equations and the problem can be found in [4]. A new application assumes that the Fourier spectra of magnetic fields are random. This assumption is analogous to the hydrodynamic turbulence introduced by A.N.Kolmogorov, see [5]. The situation with magnetohydrodynamic turbulence is more complicated and Arnold's asymptotic ergodic Hopf invariant is very important. The asymptotic Hopf invariant is called the magnetic helicity; this magnetic helicity is denoted by $\chi_{\mathbf{B}}$. The definition of the magnetic helicity is in [4], we recall it in formula (5). Basic constructions for magnetohydrodynamic turbulence are in [6]. Example 4, Sect. 5 illustrates the importance of magnetic helicity.

Magnetic lines have a complicated geometry. A distribution function of asymptotic linking numbers is said to be random. The approach by V. I. Arnold shows that the helicity is the mean value of the distribution of asymptotic linking numbers of magnetic lines. Dispersions of the distribution are interesting. Dispersions of asymptotic numbers of magnetic lines are called the quadratic helicities. In the paper, we investigate one of the two dispersions, which is denoted by $\chi_{\mathbf{B}}^{[2]}$, with the spectral density,

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which is denoted by $\chi^{[2]}$. A new Example 5 is analogous to Example 4, this example illustrates the importance of the quadratic magnetic helicity for magnetohydrodynamic turbulence.

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2 The Arnold's Asymptotic Hopf Invariant and Its Random Distribution

Let us recall the definition of the asymptotic linking number of a pair of trajectories like you can see in [4], Ch. III. Let $\mathbf{B}$ be a divergent-free (magnetic) field in 3D domain $\Omega \subset \mathbb{R}^3$. We assume that $\mathbf{B}$ is tangent to the boundary $\partial\Omega$ and has no zeros. Denote by $g^t : \Omega \to \Omega$ the phase flow of $\mathbf{B}$.

Definition 1 1) The linking number $lk_{\mathbf{B}}(x_1, x_2; T)$ is defined as follows:

$$lk_{\mathbf{B}}(x_1, x_2; T) = \frac{1}{4\pi}\int_0^T\int_0^T \frac{(\dot\gamma_1, \dot\gamma_2, \gamma_1 - \gamma_2)}{||\gamma_2 - \gamma_1||^3}\mathbf{d}t_1\mathbf{d}t_2, \quad \dot\gamma_i(t_i) = \mathbf{B}(g^{t_i}(x_i)), \quad i = 1, 2 \tag{1}$$

of two segments $\gamma_1 = g^{t_1}(x_1)$, $t_1 \in [0, T]$, $\gamma_2 = g^{t_2}(x_2)$, $t_2 \in [0, T]$ and starting points $x_1, x_2 \in \Omega$.

2) The asymptotic linking number $\lambda_{\mathbf{B}}(x_1, x_2)$ of a pair of trajectories $g^t(x_1), g^t(x_2)$ is defined as the limit

$$\lambda_{\mathbf{B}}(x_1, x_2) = \lim_{T\to+\infty} \frac{lk_{\mathbf{B}}(x_1, x_2; T)}{T^2}. \tag{2}$$

The linking number $lk_{\mathbf{B}}(x_1, x_2; T)$ can be expressed as follows:

$$lk_{\mathbf{B}}(x_1, x_2; T) = \int_0^T\int_0^T G(x_1(t_1), x_2(t_2))\mathbf{d}t_1\mathbf{d}t_2, \quad \dot\gamma_i(t_i) = \mathbf{B}(g^{t_i}(x_i)), \quad i = 1, 2, \tag{3}$$

where the integral in formula (3) is called the Gauss integral. If trajectories γ_1, γ_2 are parametrized by closed circles of the unit length, then limit (1) coincides with the linking number of these two circles, which is a topological invariant.

Let us denote the points $g^{t_i}(x_i)$ by $x_i(t_i)$, $i = 1, 2$. In the right-hand side of formula (3) by

$$G(x_1(t_1), x_2(t_2)) = \frac{1}{4\pi}\frac{(\dot\gamma_1, \dot\gamma_2, \gamma_1 - \gamma_2)}{||\gamma_2 - \gamma_1||^3} \tag{4}$$

is denoted the kernel of the Gauss integral. The denotation $G(x_1, x_2) = (\mathbf{B}(x_1), \mathbf{A}(x_2; x_1))$, where

$$\mathbf{A}(x_2; x_1) = \frac{1}{4\pi} \frac{\mathbf{B}(x_2) \times (x_1 - x_2)}{||x_1 - x_2||^3}$$

is the Biot–Savart potential. It will be used below in formula (10).

Let us consider the subset of such points (x_1, x_2) in $\Omega \times \Omega$ that $\lambda_{\mathbf{B}}(x_1, x_2)$ is well defined. If the trajectory issued from x_1 contains the point x_2, the function $\lambda_{\mathbf{B}}(x_1, x_2)$ is not defined. The domain of $\lambda_{\mathbf{B}}$ is a measurable subset in $\Omega \times \Omega$. The ergodic theorem implies that the function $\lambda_{\mathbf{B}} : \Omega \times \Omega \to \mathbb{R}$ is well defined almost everywhere, and belongs to the space L^1. This follows from the fact that the function (1) belongs to L^1.

A dimension of $\lambda_{\mathbf{B}}$ is $G^2 \cdot cm^{-2}$. This means that the transformation $\mathbf{B} \mapsto l\mathbf{B}$, $\mathbf{x} \mapsto m\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^3$ determines the transformation $\lambda_{\mathbf{B}} \mapsto l^2 m^{-2} \lambda_{\mathbf{B}}$ of the asymptotic linking number. In the CGS system, magnetic field is measured in Gaussian units G. The average self-linking number

$$\chi_{\mathbf{B}} = \int\int \lambda_{\mathbf{B}}(x_1, x_2) \mathbf{d}\Omega \mathbf{d}\Omega \tag{5}$$

of a magnetic field $\mathbf{B}$ in Ω is called the asymptotic Hopf invariant or the helicity. We mean that the integral (5) is a Lebesgue integral over domain $\Omega \times \Omega$ by 6-dimensional Lebesgue measure. The helicity (5) is a lower bound of the magnetic energy by the Arnold's inequality (see [4] Sect. 3, Theorem 1.4). For a divergence-free vector field (a magnetic field) $\mathbf{B}$,

$$\int_{\Omega} (\mathbf{B}(x), \mathbf{B}(x)) \mathbf{d}V \geq C|\chi_{\mathbf{B}}|, \tag{6}$$

where C is a positive constant dependent on the shape and size of the compact domain Ω with a magnetic field. In the right-hand side of formula (6), we have the invariant of volume-preserved transformation of the domain. In the left-hand side of the formula, we have the magnetic energy. The inequality proves that the absolute value of the magnetic helicity $\chi_{\mathbf{B}}$ determines a lower bound of the magnetic energy. Example 4 for magnetohydrodynamic turbulence is an analogous one.

In [3] (the bottom remark in Example 5.2) is mentioned that the function $m(\lambda_0)$, defined as the measure of the set $\{(x_1, x_2) \in \Omega \times \Omega | \lambda_{\mathbf{B}}(x_1, x_2) < \lambda_0\}$, is the much stronger invariant of volume-preserved transformations than the helicity. A lower bound of the magnetic energy, which is calculated using this distribution function, is more sharp than the bound, which is calculated using the magnetic helicity $\chi_{\mathbf{B}}$ in the Arnold's inequality.

The function $m(\lambda_0)$ is a distribution function of asymptotic linking numbers. Using ergodic theorem for the function $G(x_1, x_2)$ with respect to the flow $g^{t_1} \times g^{t_2}$ in $\Omega \times \Omega$ one may say only that $m(\lambda_0)$ admits a mean value: the helicity. But, what to do if dispersion of this distribution is well-defined? We give a positive answer to this question.

3 Quadratic Helicity: A Local Formula

Formally, the dispersion of the asymptotic self-linking number $\lambda_{\mathbf{B}}(x_1, x_2)$ is defined by the integral:

$$D\lambda_{\mathbf{B}} = \iint (\lambda_{\mathbf{B}}(x_1, x_2) - \frac{\chi_{\mathbf{B}}}{Vol(\Omega)})^2 \quad \mathbf{d}\Omega\mathbf{d}\Omega, \tag{7}$$

where $Vol(\Omega)$ is the volume of the domain Ω, $\frac{\chi_{\mathbf{B}}}{Vol(\Omega)}$ is the average value of the linking numbers of magnetic line in Ω. Obviously, (7) is equivalent to the integral

$$D\lambda_{\mathbf{B}} = \iint \lambda^2_{\mathbf{B}}(x_1, x_2) \quad \mathbf{d}\Omega\mathbf{d}\Omega - \chi^2_{\mathbf{B}}. \tag{8}$$

The quadratic helicity $\chi^{[2]}_{\mathbf{B}}$ is defined as the first term of the integral (8):

$$\chi^{[2]}_{\mathbf{B}} = 2 \iint \lambda^2_{\mathbf{B}}(x_1, x_2) \quad \mathbf{d}\Omega\mathbf{d}\Omega = m_{x_1,x_2}[\lambda^2_{\mathbf{B}}(x_1, x_2)]. \tag{9}$$

In [2], it is proved that $\chi^{[2]}_{\mathbf{B}}$ (a dimension of $\chi^{[2]}_{\mathbf{B}}$ is $G^4 cm^2$) is well defined (this fact is easy: we know that $\lambda_{\mathbf{B}}$ is integrable by ergodic theorem, but we have to prove that the integral of the square $\lambda^2_{\mathbf{B}}$ of this measurable function is finite). Also, in this paper, an inequality between $\chi_{\mathbf{B}}$ and $\chi^{[2]}_{\mathbf{B}}$ is proved. The goal of this section is a generalization a local formula from [1] for the quadratic helicity $\chi^{(2)}$ (see Sect. 4 for a brief definition and [2] for definition) for $\chi^{[2]}$.

Let us recall definition of $\delta^{[2]}_{\mathbf{B}}$, which is called (a component of) the magnetic correlation tensor:

$$\delta^{[2]}_{\mathbf{B}}(x_1, x_2) = G^2(x_1, x_2) = (\mathbf{B}(x_1), \mathbf{A}(x_2; x_1))^2 = (\mathbf{A}(x_1; x_2), \mathbf{B}(x_2))^2, \tag{10}$$

where $\mathbf{A}(x; y)$ is the Biot–Savart potential as in formula (1), see [4] Ch 3 Paragraph 4; $G^2(x_1, x_2)$ is the square of the Gaussian kernel. Let us recall that $(\mathbf{B}(x_1), \mathbf{A}(x_2; x_1))$ is the kernel $G(x_1, x_2)$ of the Gauss integral (4). The inequality

$$\chi^{[2]}_{\mathbf{B}} \leq \iint \delta^{[2]}_{\mathbf{B}} \mathbf{d}\Omega\mathbf{d}\Omega \tag{11}$$

is proved in [2]. This proof follows from the fact that the function $\delta^{[2]}_{\mathbf{B}}(x_1, x_2)$ is integrable over $\Omega \times \Omega$.

The correlation tensor

$$G^2(\mathbf{B}_1(x_1), \mathbf{B}_2(x_2)) = (\mathbf{B}_1(x_1), \mathbf{A}_2(x_2; x_1))^2 = (\mathbf{A}_1(x_1; x_2), \mathbf{B}_2(x_2))^2$$

for a pair $\mathbf{B}_1, \mathbf{B}_2$ of magnetic fields is not integrable over $\Omega \times \Omega$, because the formal asymptotic of $G^2(\mathbf{B}_1(x_1), \mathbf{B}_2(x_2)) \simeq ||x_1 - x_2||^{-4}$, when $x_1 \to x_2$.

For an arbitrary function $E(x_1, x_2)$ which is singular, when $x_1 \to x_2$, we define the symmetrization $E^{sim}(x_1, x_2) = \frac{1}{2}[E(x_1, x_2) + E(x_2, x_1)]$.

Theorem 1 *With assumptions below,*

$\odot$ (1) $\mathbf{B}$ *is smooth in* Ω *everywhere, except points on the boundary* $\partial\Omega$ *of the domain;*

⊙ (2) *the function* $G(x_1, x_2)$, $x_1 \in l_1, x_2 \in l_2$ *in* $\Omega \times \Omega \backslash diag$ (*the kernel of the Gauss integral*) *contains a Fourier spectrum with wavenumbers k in a finite interval* $\{0\} \cup [\Delta'; \Delta]$, $0 < \Delta' << \Delta < +\infty$*;*

the following equation is satisfied:

$$\chi_{\mathbf{B}}^{[2]} = \lim_{a \to +\infty} \sum_{s=0}^{\infty} (-a)^s m_{x_1,x_2} \left[\left[\sum_{i,j;i+j=s} \frac{1}{i!j!} \frac{\partial^i}{\partial t_1^i} \frac{\partial^j}{\partial t_2^j} G(\mathbf{B}(x_1(t_1)), \mathbf{B}(x_2(t_2))) \right]^{\text{sim}} \right]^2 . \tag{12}$$

Remark 2 1. Obviously, the main term of formula (12) for $i = 0$, $j = 0$ is given by $\iint G^2(x_1, x_2) \mathbf{d}\Omega \mathbf{d}\Omega$. In formula (12), the parameter a has the dimension $G^{-1}cm$; with this assumption, all terms in formula (12) have the dimension $G^4 cm^2$.

2. In formula (12), there are three operations: symmetrization (on each partial derivative), squaring and average m_{x_1,x_2} over the phase space. These operations are not commuted, in particular, in the case we take the average before the squaring, we get the quadratic helicity $\chi^{(2)}$ instead of $\chi^{[2]}$, see Sect. 4.

Proposition 3 *Terms*

$$\left[\left[\sum_{i,j;i+j=s} \frac{1}{i!j!} \frac{\partial^i}{\partial t_1^i} \frac{\partial^j}{\partial t_2^j} G(\mathbf{B}(x_1(t_1)), \mathbf{B}(x_2(t_2))) \right]^{\text{sim}} \right]^2 , \tag{13}$$

which are defined as symmetrizations of corresponding terms in the right-hand side of formula (12) *belong to the space* $L^2(\Omega \times \Omega)$.

3.1 Proof of Proposition 3

When $x_1 \mapsto x_2$, the kernel of integral (1) becomes singular, but correlation tensor (12) is absolutely integrable. After we take derivatives, as in (12), the integrability of terms is not obvious. Let us prove that the symmetrization of terms keeps the integrability.

Take expression (13) for $s = 0$. When $x_1 \mapsto x_2$, the function $G(\mathbf{B}(x_1), \mathbf{B}(x_2))$ has the formal asymptotic $||x_1 - x_2||^{-2}$ and the asymptotic $||x_1 - x_2||^{-1}$. As the result, $[G(\mathbf{B}(x_1), \mathbf{B}(x_2))]^2$ converges (with no symmetrization). To pass to the next step, we take the term $[\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}] G(\mathbf{B}(x_1), \mathbf{B}(x_2))$, which has the formal asymptotic $||x_1 - x_2||^{-3}$ and the asymptotic $||x_1 - x_2||^{-2}$ before the symmetrization. After symmetrization, we get the formal asymptotic $||x_1 - x_2||^{-2}$ and the asymptotic $||x_1 - x_2||^{-1}$.

A meaning of the calculation above is as follows. The magnetic flow $g^{t_1+dt}(x_1)$, $g^{t_2+dt}(x_2)$ of a pair of closed points $x_1(t_1), x_2(t_2)$ keeps the asymptotic of the correlator $G(x_1, x_2)$. This gives the induction for the estimation of terms in (12). □

3.1.1 Proof of Theorem 1

Take an ordered marked pair of magnetic lines l_1, l_2. Take the natural measure $\mathbf{d}t_1\mathbf{d}t_2$ on $l_1 \times l_2$, where t_1, t_2 are magnetic parameters, see Definition 1. Then the bottom term $i = 0$, $j = 0$ in (12), restricted to the standard $[0, T] \times [0, T]$-segments in the Cartesian product $l_1 \times l_2$, coincides with the integral (1).

Integral (12) is a family of asymptotic integrals over pairs of magnetic lines. Take the Cartesian product $\Omega \times \Omega \times [0, T]^2$ and define a small parameter δ and a large parameter T as follows. Consider a subspace $[\Omega \times \Omega \times [0, T]^2]_{\delta,T} \subset \Omega \times \Omega \times [0, +\infty]^2$, which consists of all pairs of magnetic T-lines with δ-disjoin in $\Omega \times \Omega$. Take the limit $\delta \to +0, T \to +\infty$. Formula (9) is an asymptotic integral over $\Omega \times \Omega \times [0, +\infty]^2$ of the kernel $G(x_1, x_2)$, which is extended to this Cartesian product as $G(x_1(t_1), x_2(t_2))$. By the ergodic theorem, formula (9) for a subset $[\Omega \times \Omega \times [0, T]^2]_{\delta,T}$ tends to $\chi^{[2]}$ in the limit.

Consider the Fourier basis $\aleph$ in $\Omega \times \Omega\backslash\text{diag}$, as in Condition $\odot$ (2). This basis is extended to the Cartesian product $[(\Omega \times \Omega\backslash\text{diag}) \times [0, T]^2]$. Restrict this basis to the subspace $[\Omega \times \Omega \times [0, T]^2]_{\delta,T} \subset [(\Omega \times \Omega\backslash\text{diag}) \times [0, T]^2]$, and denote the restriction by $\aleph^T$. Take another basis $\aleph_0^T$ in $[\Omega \times \Omega \times [0, T]^2]_{\delta,T}$, which is the tensor product of the basis $\aleph$ with the standard Fourier basis over the plane $[0, T]^2$. Take the decomposition of $G(x_1(t_1), x_2(t_2))$ in $\aleph_0^T$. We cut this decomposition to the segment $k \in [0, \Delta]$ of the wavenumbers with fixed upper bound Δ. When (the exterior limit) $\Delta \to +\infty$, we get the total decomposition. By this assumption, the function $G(x_1(t_1), x_2(t_2))$ satisfies the analogous Condition $\odot$ (2) in $\aleph_0^T$.

Define $m_{x_1,x_2}[\dots]$ as the integration of a function, which depends on $x_1, t_1, x_2, t_2 \in [\Omega \times \Omega \times [0, T]^2]_{\delta,T}$, over all points $x_1(t_1), x_2(t_2)$ with prescribed t_1, t_2.

A preliminary formula (12) is the following:

$$\chi_{\mathbf{B}}^{[2]} = T^{-2} \iint_{[0,+T]\times[0,+T]} \lim_{a\to+\infty} \sum_{s=0}^{\infty} \frac{(-a)^s}{s!} m_{x_1,x_2} \left[\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right)^s [G(x_1(t_1), x_2(t_2))] \right]^2 \mathbf{d}t_1\mathbf{d}t_2. \tag{14}$$

The main term in the right-hand side of formula (14) is

$$T^{-2} \iint_{[0,+T]\times[0,+T]} m_{x_1,x_2}[G^2(x_1(t_1), x_2(t_2))]\mathbf{d}t_1\mathbf{d}t_2. \tag{15}$$

The function $G^2(x_1(t_1), x_2(t_2))$ is absolutely integrable in the largest space $[\Omega \times \Omega \times [0, T]^2]$. The limit of integrals over $[\Omega \times \Omega \times [0, T]^2]_{\delta,T}$, $\delta \to +0$, convergences uniformly and absolutely. The limit of integral (15) coincides with the main term in (12); this limit does not depend on T.

Consider a pair of magnetic lines l_1, l_2 of the length T, which are issued from fixed points $x_1 = x_1(0)$, $x_2 = x_2(0)$. To prove (14) we assume that the Fourier spectrum of $G(t_1, t_2)$, $(t_1, t_2) \in [0, +T]^2$ is of the form:

$$G(t_1, t_2) = \lambda_0 + \lambda \sin(\alpha t_1 + \theta_1) \sin(\beta t_2 + \theta_2), \tag{16}$$

$\alpha, \beta \in \mathbb{R}$; $\theta_0, \theta_1 \in [0, \pi]$ are shifts of the coordinate from the starting points $x_1(0), x_2(0)$ along the magnetic lines l_1, l_2. Then formula (14) for these two magnetic lines is (the only terms with non-zero mean values along magnetic the pair of magnetic lines are presented):

$$T^{-2} \iint \left[\lambda_0^2 + \frac{\lambda^2}{4} + \frac{\lambda^2}{4}(\exp^2(-\infty) - \exp^2(0)) \right] \mathbf{d}t_1 \mathbf{d}t_2.$$

The formula for the quadratic helicity is

$$\iint \lambda_{\mathbf{B}}^2(x_1, x_2) \mathbf{d}\Omega \mathbf{d}\Omega = \chi_{\mathbf{B}}^{[2]},$$

where $\lambda_{\mathbf{B}} = \lambda_0(x_1, x_2)$ is the mean value of the main term in (16), which depends only on a pair of starting points of magnetic lines (l_1, l_2). This proves preliminary formula (14) in a particular case. A general case assuming the Fourier basis contains only a finite number of harmonics that follow from linearity, orthogonality of harmonics, and the fact that the limit $a \to +\infty$ commutes with the Fourier integral. □

The following calculation will be used in Sect. 5. Let us prove that the term $s = 1$ with first-order partial derivatives $\frac{\partial G(x_1(t_1), x_2(t_2))}{\partial t_1}$, $\frac{\partial G(x_1(t_1), x_2(t_2))}{\partial t_2}$ are simplified.

We get

$$\begin{aligned} \frac{\partial \tilde{G}(t_1, t_2)}{\partial t_1} &= \frac{\partial \langle \mathbf{B}(x_1(t_1)), \mathbf{B}(x_2(t_2)), x_1(t_1) - x_2(t_2) \rangle}{\partial t_1} \\ &= \left\langle \nabla_{\mathbf{B}_1} \mathbf{B}_1(t_1), \mathbf{B}_2(t_2), x_1 - x_2 \right\rangle, \end{aligned}$$

$\tilde{G}(x_1(t_1), x_2(t_2)) = ||x_1 - x_2||^3 G(x_1(t_1), x_2(t_2))$. because $\frac{\partial x_1}{\partial t_1} = \mathbf{B}_1(t_1)$. For $\frac{\partial \hat{G}(t_1, t_2)}{\partial t_2}$ the formula is analogous.

4 Cubic Helicities

All helicities in this section are invariants for the group of volume-preserved diffeomorphisms of domains with magnetic fields. There exists three quadratic magnetic helicities $\chi_{\mathbf{B}}^{(2)}$ $G^4 cm^5$; $\chi_{\mathbf{B}}^{[2]}$ $G^4 cm^2$; $\chi_{\mathbf{B}}^2$, $G^4 cm^8$. Only $\chi_{\mathbf{B}}^{(2)}$, $\chi_{\mathbf{B}}^{[2]}$ determine second momenta (dispersions) of the asymptotic self-linking number, the square of the helicity $\chi_{\mathbf{B}}^2$ is the second momentum, which is the square of the first-order momentum. The quadratic magnetic helicity $\chi_{\mathbf{B}}^{(2)}$, certainly, is deduced from the distribution function of Arnold's asymptotic ergodic Hopf invariant. The same time the helicity $\chi_{\mathbf{B}}^{(2)}$ is interesting by itself, this is the L^2-norm of the "'Field line helicity"' (see [8]).

There exist eight different third-order momenta of the asymptotic self-linking number, which are called the cubic magnetic helicities, let us list them and indicate dimensions. The following diagram explains how to define the corresponding cubic helicity as a sum of corresponding products of three pairwise linking coefficients (denoted by $---$) for a collection of magnetic lines (denoted by $\odot$).

$$\chi_{\mathbf{B}}^{3} \quad G^{6}cm^{12} \quad \odot - - \odot \ \odot - - \odot \ \odot - - \odot$$

$$\chi_{\mathbf{B}}^{(2)}\chi_{\mathbf{B}} \quad G^{6}cm^{9} \quad \odot - - \odot - - \odot \ \odot - - \odot$$

$$\chi^{(3,1)} \quad G^{6}cm^{8} \quad \begin{array}{ccccc} \odot & -- & \odot & -- & \odot \\ & & | & & \\ & & \odot & & \end{array}$$

$$\chi^{(3,2)} \quad G^{6}cm^{8} \quad \odot - - \odot - - \odot - - \odot$$

$$\chi_{\mathbf{B}}^{[2]}\chi_{\mathbf{B}} \quad G^{6}cm^{6} \quad \odot == \odot \ \odot - - \odot$$

$$\chi^{((3,1))} \quad G^{6}cm^{3} \quad \odot == \odot - - \odot$$

$$\chi^{(3,2)} \quad G^{6}cm^{3} \quad \begin{array}{ccc} \odot & -- & \odot \\ \diagdown & & \diagup \\ & \odot & \end{array}$$

$$\chi^{[3]} \quad G^{6} \quad \odot \equiv\equiv \odot$$

4.1 Explanations

$\chi_{\mathbf{B}}^{3}$ is the cube of the magnetic helicity; $\chi_{\mathbf{B}}^{(2)}\chi_{\mathbf{B}}$ is the product of the quadratic magnetic helicity and the magnetic helicity; $\chi_{\mathbf{B}}^{[2]}\chi_{\mathbf{B}}$ is the product of the quadratic momentum of magnetic helicity and the magnetic helicity; $\chi^{[3]}$ is the cubic momentum of the magnetic helicity, which is analogous to $\chi_{\mathbf{B}}^{[2]}$. The difference between $\chi^{[3]}$ and $\chi_{\mathbf{B}}^{[2]}$ is as follows: for $\chi^{[3]}$, the correlation tensor is unlimited. Only five cubic helicities determine independent third-order momenta of the asymptotic self-linking number; the cubic helicities $\chi_{\mathbf{B}}^{3}$, $\chi_{\mathbf{B}}^{(2)}\chi_{\mathbf{B}}$, $\chi_{\mathbf{B}}^{[2]}\chi_{\mathbf{B}}$ are functions of quadratic helicity and helicity; they are central momenta of independent cubic helicities.

Let us consider an arbitrary connected graph with n edges (multiple edges are admissible, edges from a vertex to itself are not admissible), graphs for $n = 3$ are on the picture. The correlation tensor of a momenta of magnetic helicity is well defined for the corresponding graph. Assume a graph satisfies the following property: for an arbitrary k vertexes, there are strongly less than $3k - 3$ edges between them. In this case, the correlation tensor is limited. In particular, the graph for $\chi^{[3]}$ has 2 vertexes and 3 edges. The inequality $(2 - 1)3 < 3$ is not satisfied and the correlation tensor is unlimited. In the case $k = 6$, consider the graph with the only edge between an arbitrary pair of vertexes. Then the number of edges is 15 and $3(k - 1) = 15$; for this graph, the correlation tensor is unlimited.

4.1.1 Problem

Estimate the asymptotic of independent n-momenta of the helicity (of the Arnold's asymptotic linking number), and n-momenta for which the correlation tensor is limited, $n \to +\infty$.

5 The Spectrum of Magnetic Fields

By the spectrum of magnetic fields, we mean the following expression:

$$\mathbf{B}(\mathbf{x}) = \int_{\mathbf{k}} \mathbf{B}(\mathbf{k}) \exp(\mathbf{ik} \cdot \mathbf{x})\mathbf{dk}, \tag{17}$$

where $\mathbf{k} \cdot \mathbf{B}(\mathbf{k}) = 0$ ($\mathbf{B}$ is divergent free), $\mathbf{B}(-\mathbf{k}) = \mathbf{B}^*(\mathbf{k})$ ($\mathbf{B}$ corresponds to a real solution), and $*$ is the complex conjugation. In formula (17) $\mathbf{B}$, we also assume that random amplitudes $||\mathbf{B}(\mathbf{x})||$ of elementary harmonics $\mathbf{B}(\mathbf{k})$ satisfy the power low: $||\mathbf{B}(\mathbf{x})|| \sim k^{-\alpha}$ (for short: ${}_{\mathbf{k}}\mathbf{B} \sim k^{-\alpha}$), where α is a real parameter.

For the turbulence with no magnetic fields the definition can be found in [5], the MHD turbulence is analogous (and more complicated). The goal is to explain a basic exercise "helicity is a lower bound of magnetic energy", Example 4 [compare with the Arnold inequality (6)]. An analogous exercise, Example 5, is defined with quadratic magnetic helicity instead of magnetic helicity. A question with such a generalization was formulated by D.Sokoloff.

Example 4 Assume ${}_{\mathbf{k}}\mathbf{B} \sim k^{-\alpha}$. Recall $\mathbf{A}$ is the vector-potential for $\mathbf{B}$: $rot\mathbf{A} = \mathbf{B}$. In the case $\mathbf{B}$ is a proper vector of the operator rot, one gets $\mathbf{A} = k^{-1}\mathbf{B}$, where k is the proper value of $\mathbf{B}$. Then we get ${}_{\mathbf{k}}\mathbf{A} \sim k^{-\alpha-1}$. We get ${}_{\mathbf{k}}(\mathbf{A}, \mathbf{B}) \sim k^{-2\alpha-1}$. We assume that for a fundamental domain $vol(\Omega) = 1$. We get ${}_{k}\int(\mathbf{A}, \mathbf{B})d\Omega \sim k^{-2\alpha-1}$. The helicity integral is uniformly distributed over the k-line in the case $\alpha = \frac{-1}{2}$.

This example can be interpreted in the following way: the distribution of the linking number of magnetic lines in Ω does not depend on a scale when $\alpha = \frac{-1}{2}$. The magnetic energy $U = \int \mathbf{B}^2\mathbf{d}\Omega$ (dimension is G^2cm^3) in this case is distributed over the k-line as $\sim k^{-2\alpha} = k$. The spectrum admits an upper bound $|\mathbf{k}| \leq \Delta$, because the magnetic energy is finite. Assume that the magnetic helicity $\chi_{\mathbf{B}}$ is sufficiently large, then the magnetic energy has to be large, and the upper bound of the spectrum has to be sufficiently large.

To formulate a new example, let us considered Example 4 from the point of view of Gaussian kernel G in (1). With assumptions of Example 4, we get the following $\mathbf{k}_1 \times \mathbf{k}_2 \times (x_1, x_2)$-distribution, $x_1 \in \Omega$, $x_2 \in \mathbb{R}^3$, of the kernel $G(x_1, x_2) = (\mathbf{B}_1(x_1), \mathbf{A}_2(x_2; x_1))$:

$$_{\mathbf{k}_1 \times \mathbf{k}_2 \times (x_1 x_2)}(\mathbf{B}_1(x_1), \mathbf{A}_2(x_2; x_1)) \sim .k^{-2\alpha+2}.$$

Passing to the average over (x_1, x_2), using $vol(\Omega) = 1$, we get a distribution

$$_{\mathbf{k}_1 \times \mathbf{k}_2} m_{x_1, x_2}[(\mathbf{B}_1(x_1), \mathbf{A}_2(x_2; x_1))] \sim k^{-2\alpha-1}.$$

This gives the distribution of the helicity integral over the k-line: $_k\chi_{\mathbf{B}} \sim k^{-2\alpha-1}$, $k = |\mathbf{k}_1| = |\mathbf{k}_2|$ as above.

Example 5 A distribution of the integral kernel $G^2(x_1, x_2)$ of the main term in formula (12) is well defined over the Cartesian product $\mathbf{k}_1 \times \mathbf{k}_2 \times \mathbf{k}'_1 \times \mathbf{k}'_2$. Proper vectors give contribution to $G^2(x_1, x_2)$ only with its square, this gives $\mathbf{k}_1 \times \mathbf{k}_2$-distribution. With assumptions of Example 4, we get the following distribution of the kernel

$$G^2(x_1, x_2) = (\mathbf{B}_1(x_1), \mathbf{A}_2(x_2; x_1))$$

at a prescribed point $(x_1, x_2) \in \Omega \times \mathbb{R}^3$:

$$_{\mathbf{k}_1 \times \mathbf{k}_2 \times (x_1, x_2)} G^2(x_1, x_2) \sim k^{-4\alpha+4}.$$

After the average of the distribution over x_2, we get the following $\mathbf{k}_1 \times \mathbf{k}_2$-distribution:

$$_{\mathbf{k}_1 \times \mathbf{k}_2} \delta^{[2]}(\mathbf{B}_1, \mathbf{B}_2) \sim k^{-4\alpha+1}.$$

This describes the distribution

$$_k\delta^{[2]}(\mathbf{B}_1, \mathbf{B}_2) \sim k^{-4\alpha+1}$$

of the main term in (12), where k is the module of the vector $\mathbf{k}_1 \times \mathbf{k}_2$.

Let us describe distribution (12). We have to take two collections of random vectors $\{\mathbf{B}_{\mathbf{k}_{1,a}}; \mathbf{B}_{\mathbf{k}_{1,1}}, \ldots, \mathbf{B}_{\mathbf{k}_{1,i}}\}$ $\{\mathbf{B}_{\mathbf{k}_{2,a}}, \mathbf{B}_{\mathbf{k}_{2,1}}, \ldots, \mathbf{B}_{\mathbf{k}_{2,j}}\}$ in the spectrum. This collections determine random distribution of the term $\delta^{[2]}(\mathbf{B}_1^{(i)}, \mathbf{B}_2^{(j)})$, $i + j = s$ on the right-hand side of formula (12).

$$\mathbf{B}_1^{(i)}(\mathbf{k}_{1,a}, \mathbf{k}_{1,1}, \ldots, \mathbf{k}_{1,i}) = \nabla_{\mathbf{B}_{\mathbf{k}_{1,i}}} \ldots \nabla_{\mathbf{B}_{\mathbf{k}_{1,1}}} \mathbf{B}_{\mathbf{k}_{1,a}},$$

$$\mathbf{B}_2^{(j)}(\mathbf{k}_{2,a}, \mathbf{k}_{2,1}, \ldots, \mathbf{k}_{2,j}) = \nabla_{\mathbf{B}_{\mathbf{k}_{2,j}}} \ldots \nabla_{\mathbf{B}_{\mathbf{k}_{2,1}}} \mathbf{B}_{\mathbf{k}_{2,a}}.$$

In the case $i = 0$, $j = 0$, we get the distribution above for $\mathbf{B}_1 = \mathbf{B}_{1,a}$, $\mathbf{B}_2 = \mathbf{B}_{2,a}$.

For $s = 1$ (we consider one of the two similar distributions with $i = 1$, $j = 0$), using calculations from the last part of Sect. 2, we get

$$_k\delta^{[2]}(\mathbf{B}_1^{(1)}, \mathbf{B}_2) \sim_k \delta^{[2]}(\mathbf{B}_{1,a}, \mathbf{B}_{2,a})\overline{\sin^2(\theta)}C,$$

where θ is a random angle on the unit sphere between vectors $\mathbf{B}_{1,1}$ and $\mathbf{B}_{1,a}$, a positive constant C is dimensionless, $\overline{g(\theta)}$ is the mean value of a distribution $g(\theta)$.

By induction for $s \geq 2$ (we assume that $i \geq 1$ to get an inductive step $i - 1 \mapsto i$), we get

$$_k\delta^{[2]}(\mathbf{B}_1^{(i)}, \mathbf{B}_2^{(j)}) \sim \tag{18}$$

$$\frac{_k\delta^{[2]}(\mathbf{B}_1, \mathbf{B}_2)\overline{|\sin(\theta_{1,1})|}^2 \dots}{\overline{|\sin(\theta_{1,i-1})|}^2 \quad \sin^2(\theta_{1,i})} \overline{|\sin(\theta_{2,1})|}^2 \dots \overline{|\sin(\theta_{2,j})|}^2 2^{s-1}C^s,$$

where $\theta_{1,1}, \dots \theta_{1,i}, \theta_{2,i}, \dots, \theta_{2,j}$ are latitudes on the coordinate unit sphere, pointed by the vectors $\mathbf{B}_{1,1}, \dots, \mathbf{B}_{1,i}$ with $\mathbf{B}_{1,a}$, and by $\mathbf{B}_{2,1}, \dots, \mathbf{B}_{2,j}$ with $\mathbf{B}_{2,a}$. The angles have a common distribution and the convolution of (18) is distributed as

$$\sim_k \delta^{[2]}(\mathbf{B}_1, \mathbf{B}_2) \quad \frac{\overline{\sin^2(\theta)}}{2(\overline{|\sin(\theta)|})^2}C^s.$$

The value of expression (12) is distributed as $\frac{1}{3}$ of the main term (this follows from the formulas: $\int_{S^2} \sin^2(\theta)\mathbf{d}S^2 = \frac{1}{3}\int_{S^2} \mathbf{d}S^2$; $\int_{S^2} |\sin(\theta)|\mathbf{d}S^2 = \frac{1}{2}\int_{S^2} \mathbf{d}S^2$). After we pass to the k-line, terms in (12) are distributed as the main term by the formula:

$$_k\chi_{\mathbf{B}}^{[2]} \sim k^{-4\alpha+1}. \tag{19}$$

The uniform distribution for $\chi_{\mathbf{B}}^{[2]}$ is in the case $\alpha = \frac{1}{4}$.

An elementary magnetic vector in (17) admits the complex and the real component. As the result, for a given $\mathbf{k}$ we get two magnetic harmonics with positive (right) and negative (left) helicity. Assume that the contribution of left and right harmonics for all $\mathbf{k}$ in (17) is opposite. Then Example 4 gives us no estimate of the magnetic energy from above, because $\chi_{\mathbf{B}} = 0$. The quadratic helicity is an invariant of ideal MHD; assume its value is sufficiently large. In this case, the lower bound of the spectra can be estimated.

Cut-out wave vectors with $\Delta'||k|| < \Delta$, $\Delta >> 1, 0 < \Delta' << 1$. The interior limit $a \to +\infty$, $a >> \Delta$, and the exterior limit $\Delta \to +\infty$, $\Delta' \to 0+$ are defined as the two variables' limits.

6 Applications

Assume that a mean velocity field in a liquid conductive domain $\bar{\mathbf{u}}(t)$ is given, in this case, the dynamo mean field equation is following:

$$\begin{array}{c} rot(\eta rot\bar{\mathbf{B}}) - rot(\bar{\mathbf{u}} \times \bar{\mathbf{B}} + \mathbf{E}) + \frac{\partial \bar{\mathbf{B}}}{\partial t} = 0, \\ \mathbf{E} = \alpha\bar{\mathbf{B}} - \beta rot(\bar{\mathbf{B}}, \quad \mathrm{div}(\bar{\mathbf{B}}) = 0. \end{array} \tag{20}$$

We assume for simplicity that $\eta = 0$, $\alpha > 0$, $\beta > 0$. Assume that the magnetic helicity of $\bar{\mathbf{B}}$ is uniformly distributed over the wavenumbers in the interval $\delta_1 - \delta_0 = \Delta$, $0 < \delta_0 << 1$. We get $\chi(k)(t) = \chi(t)$, $\chi_{\bar{\mathbf{B}}}(t) = \Delta\chi(t)$:

$$\frac{\mathrm{d}\chi_{\bar{\mathbf{B}}}(t)}{\mathrm{d}t} = \alpha\Delta\chi_{\bar{\mathbf{B}}}(t) - \frac{2\beta\Delta^2}{3}\chi_{\bar{\mathbf{B}}}(t). \tag{21}$$

Assume that the magnetic helicity spectrum of $\bar{\mathbf{B}}^{[2]}$ is trivial, more precisely, we assume that the right and left harmonics have a common distribution, such that the quadratic helicity spectrum is uniformly distributed over the wavenumbers in the interval $\delta_1 - \delta_0 = \Delta$, $0 < \delta_0 << 1$ with the denominator $\alpha = -\frac{1}{4}$. Then by dimensional reasons, we get

$$U_{\bar{\mathbf{B}}^{[2]}} = \sqrt{\chi^{[2]}_{\bar{\mathbf{B}}^{[2]}}}\Delta^{-2}. \tag{22}$$

Generalized formula (22) is as follows:

$$\frac{\mathrm{d}\chi_{\bar{\mathbf{B}}}(t)}{\mathrm{d}t} = 2\alpha\sqrt{\chi^{[2]}_{\bar{\mathbf{B}}^{[2]}}}\Delta^{-2}. \tag{23}$$

The β-term is trivial, because the current helicity is trivial. We give no assumption about the density $\chi^{(2)}$ of the quadratic helicity $\chi^{(2)}_{\mathbf{B}^{[2]}}$ (see [1] and Sect. 4) for the magnetic field $\mathbf{B}^[2]$.

Appendix by the First Author: 40 Years Ago

Rafail Kalmanovich remembers the following theorem [7].

Theorem 6 *Assume we get in the plane congruent copies of a given figure* Φ, *which is homeomorphic to the standard segment, and which does not contain a small segment or a small circle arc. The number of the pairwise disjoint figures can be more than countable if and only if one of the following two conditions is satisfied:*

$\odot$ *(A) there exists a point* O *and a positive number* ε, *such that all the figures* $R^{\varphi}_{O}(\Phi)$ (R^{φ}_{O} *are denoted the rotation with the fixed point* O *though the angle* φ), *where* $0 < \varphi < \varepsilon$ *are pairwise disjoint;*

$\odot$ (B) *there exists a parallel translation* A *such that all the figures* $(\lambda A)(\Phi)$, *where* $0 < \lambda < 1$ *are pairwise disjoint.*

As an illustration of the theorem, one may take an ellipse $E = \{(x, y) | \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$, $a, b > 0$, $a \neq b$. Then let us consider the intersection E with the half-plane $x \geq -\varepsilon$, where $\varepsilon > 0$ is a small number, $\varepsilon << a, b$. One may prove that the conditions (A), (B) are satisfied. Then a set of pairwise disjoint copies of E is not more than countable.

Rafail Kalmanovich explained me a much more interesting example concerning my theorem.

Example 7 Let Φ be the union of the graphics $f(x) = x^2$, $g(x) = ax^2$, $a > 1$, and $x \in [0, x_0]$. Then Φ satisfies the statement of Theorem 6 (the number of the pairwise disjoint figures can be greater then countable) iff $0 < x_0 \leq \frac{\sqrt{a-1}}{\sqrt{2}a}$.

In publication [7], the assumption that the figure Φ does not contain a segment or an arc is missed, but this condition is required in a proof. The situation is similar to Theorem 1. In Theorem 6, the extra assumption is also required for the proof. In both theorems, the additional assumptions give restrictions for applications.

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