



Complex Caustics of the Elliptic Billiard

Corentin Fierobe¹

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Abstract

The article studies a generalization of the elliptic billiard to the complex domain. We show that the billiard orbits also have caustics, and that the number of such caustics is bigger than for the real case. For example, for a given ellipse E , there exist exactly two confocal ellipses such that the triangular orbits of E are circumscribed about one of them, and each tangent line to one of those ellipses is a side of a triangular orbit. In the case of 4-periodic orbits, we get generically three caustics. We also give an upper bound on the number of caustics for orbits with a fixed number of sides, and explain how to compute its exact value.

Keywords Billiard · Ellipse · Caustics · Complex reflexion law · Elliptic billiard · Complex caustics

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✉ Corentin Fierobe
corentin.fierobe@ens-lyon.fr

¹ École Normale Supérieure de Lyon, Unité de Mathématiques Pures et Appliquées, UMR CNRS 5669, 46, allée d'Italie, 69364 Lyon Cedex 07, France

1 Introduction

It is a well-known fact that on a real ellipse, a billiard trajectory remains tangent to the same confocal conic (ellipse or hyperbola) after successive bounces. This is true in particular for periodic orbits. See for example [22], Chapter 4.

In this present paper, we study the caustics of a complex reflection law, which will be defined later (in Sect. 2). For now, one should just know that it is a natural generalization of the real law and that it was used to prove important results. More precisely, let q be the non-degenerate complex quadratic form $q(x, y) = x^2 + y^2$ on \mathbb{C}^2 . Then to each complex line L passing through a point A , one can associate its orthogonal line (for q) L' passing through A . Then in the case when $L \neq L'$ (non-isotropic case), we can define the reflection of lines passing through A as the usual way. In the case when $L = L'$, the reflection of lines is defined as a limit of lines reflected in non-isotropic cases. We give more details in Sect. 2.

It was introduced by Glutsyuk in order to study Ivrii's conjecture on periodic orbits together with its analogues for pseudo-billiards and complex billiards (cf [10–12]). Another use of it was made by Romaskevich in Ref. [19], to prove that the set of incenters of triangular orbits in an ellipse is also an ellipse.

The idea to approach a real problem by studying its extension to complex domain is not new. For example, a proof of Poncelet theorem was given in Ref. [13] by generalizing it to complex caustics in $\mathbb{C}\mathbb{P}^2$. Note that Poncelet's theorem was also studied on other fields, see for example [5] and [14]. Since it is a key point in this article, let us recall it.

Theorem 1.1 (Poncelet, [13]) *Let C and D be two conics of $\mathbb{C}\mathbb{P}^2$ intersecting transversally. If there is a n -sided polygon inscribed in C and circumscribed about D , then for each point p of C there is such a polygon having p as a vertex.*

Billiards with other reflection laws were already studied, see [15] for an introduction to this topic, and [1] for a precise study in the Minkowski settings. Furthermore, new discoveries about the elliptic billiard are made even recently, see for example Reznik's github page [18], and [2,3] for mathematical proofs of some of its experimental results.

We denote by \mathcal{C}_λ the (complex) conic given in the affine chart (x, y) by equation

$$\mathcal{C}_\lambda : \frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1,$$

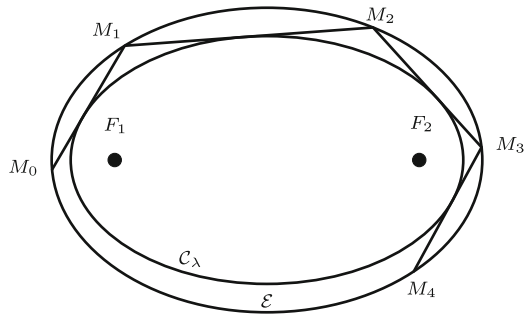
where $x, y \in \mathbb{C}^2$ and $\lambda \in \mathbb{C} \setminus \{-a^2, -b^2\}$; let us also define $\mathcal{E} = \mathcal{C}_0$ that is

$$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We first prove the following theorem (see Fig. 1).

Theorem 1.2 (Complex caustics) *Let $T = (M_0, \dots, M_n)$ be a $(n + 1)$ -uplet of points of \mathcal{E} such that any two consecutive points are distinct. The following statements are equivalent:*

Fig. 1 The confocal caustic C_λ inscribed in a piece of billiard trajectory



- there is a λ , such that $a^2 + \lambda, b^2 + \lambda \neq 0$ and for each $j \in \{1, \dots, n - 1\}$ the lines $M_{j-1}M_j$ and M_jM_{j+1} realize the two tangent lines to C_λ going through M_j ;
- T is a non-degenerate non-isotropic piece of complex billiard trajectory whose sides do not contain the real nor the complex foci.

If this λ exists, then it is unique.

The definition of non-degenerate and non-isotropic complex billiard orbits will be defined later, in Sect. 2.2. For now, the reader can think about them as “good” orbits. Thus Theorem 1.2 generalizes the theorem of the existence of caustics in the elliptic billiard to its complexification.

For periodic orbits, we improve Theorem 1.2 as follows.

Theorem 1.3 Fix $n \geq 3$. There exist N confocal conics $\gamma_1^n, \dots, \gamma_N^n$ satisfying the following properties:

1. any n -sided polygon inscribed in \mathcal{E} , circumscribed about a γ_j^n and having its consecutive vertices distinct, is a periodic billiard orbit (which is non-degenerate non-isotropic and non-flat);
2. any non-degenerate non-isotropic periodic billiard orbit which does not pass through a (complex or real) focus of \mathcal{E} is circumscribed about one of the γ_j^n s;
3. for each point p of \mathcal{E} and each j , one can find an n -sided polygon inscribed in \mathcal{E} and circumscribed about a γ_j^n , having p as a vertex.

Furthermore, $N \leq n^2/4 - 1$ if n is even, and $N \leq (n^2 - 1)/4$ if n is odd. For $n = 3$ one has $N = 2$. For $n = 4$, $N = 3$ in the case when $a \neq \sqrt{2}b$ and $N = 2$ otherwise.

Remark 1.4 Theorem 1.3 gives a meaning to another result which can be found in Ref. [7]. In this paper, at pages 17–18, the authors use a similar method to ours coming from elliptic functions theory to give a classification of the real caustics of the elliptic billiard. Similarly to us, the caustics are related to the roots of a specific polynomial, and in their case some of these roots do not correspond to any real caustics. For example, in the case of triangular orbits, one can find two roots: one of them corresponds to the usual smaller confocal ellipse, the other one defines a confocal ellipse which is bigger than the billiard table and cannot be a caustic for real triangular orbits by convexity. Our result allows to understand this other root also as another caustic, but for the reflection law extended to the complex domain.

Remark 1.5 In the proof of Theorem 1.3, for a given $n \geq 3$ we exhibit a polynomial \mathcal{B}^n whose roots are the λ corresponding to all complex caustics of the n -periodic orbits of the elliptic billiard. The coefficients of \mathcal{B}^n depend on a and b . We show in generic cases that the degree of \mathcal{B}^n is the same as the upper bound given in Theorem 1.3 (see Proposition 5.8) and that $-a^2$ and $-b^2$ are not roots of \mathcal{B}^n (Proposition 5.11). In order to compute the exact number of complex caustics for the n -periodic orbits (in generic cases), one could try to show that generically \mathcal{B}^n has simple roots: this fact seems to be true on computer simulations, but we were unable to confirm it.

A rather interesting application of Theorem 1.3 is the classification of 3-periodic degenerate orbits (defined as the limits of non-degenerate orbits), which is related to a result in Ref. [19].

Corollary 1.6 *There are exactly 8 degenerate triangular orbits, given by an isotropic tangency point α of \mathcal{E} and the point β of \mathcal{E} such that the line $\alpha\beta$ is tangent to a conic γ_j^3 and non-isotropic.*

1.1 Structure of the Document and Main Arguments

We exclude the case when \mathcal{E} is a circle. The document has the following structure:

1. Section 2 gives more details about the complex reflection law and what complex billiard trajectories are.
2. We present a natural and short proof of Theorem 1.2 in Sect. 3: it uses Zariski topology and classical results on real billiards.
3. In Sect. 4, we present another proof of Theorem 1.2 that simultaneously yields new results on Joachimsthal invariant. The latter is a quantity $P(M, v)$ depending on a vertex $M \in \mathcal{E}$ of a polygon T inscribed in \mathcal{E} and a vector v directing a side of this polygon starting at M . We show that this quantity: (a) does not depend on the vertex M chosen or v if and only if T is a billiard trajectory ; (b) is directly related to the existence of a conic inscribed in T .
4. In Sect. 5, we prove Theorem 1.3. We use a theorem of Cayley to establish the existence of confocal conics inscribed in an n -sided polygon which is itself inscribed in \mathcal{E} , and also to bound their number. This theorem reduces the proof to the computation of a certain polynomial's degree.
5. The cases of 3- and 4-periodic orbits are studied in Sects. 6 and 7, where we compute the exact value of the N appearing in Theorem 1.3, getting similar results as in [7].

1.2 Notations and Usual Properties of Conics

In the whole article, we are dealing with the complexification of the real ellipse \mathcal{E} and with the complex ellipses \mathcal{C}_λ . Thus we recall some results about these objects.

The ellipse \mathcal{E} has four isotropic tangent lines (which are tangent lines directed by the vectors $(1, \pm i)$, cf [4], Volume 2, Section 17.4.3). The name isotropic is due to the fact that $(1, \pm i)$ are isotropic vectors of the complex quadratic form $q(x, y) = x^2 + y^2$, i.e., q vanishes on them.

A simple computation shows that their corresponding tangency points have coordinates

$$\left(\pm \frac{a^2}{\sqrt{a^2 - b^2}}, \pm \frac{ib^2}{\sqrt{a^2 - b^2}} \right), \tag{1}$$

where the signs \pm are independent. This allows us to extend the definition of a focus of an ellipse as an intersection point of its non-parallel isotropic tangent lines, cf [4], Volume 2, Section 17.4.3. In our case, the ellipse \mathcal{E} has four foci:

- two real foci of coordinates $(\pm c, 0)$, where $c = \sqrt{a^2 - b^2}$;
- two complex foci of coordinates $(0, \pm ic)$.

The foci lines of \mathcal{E} are defined as two distinct lines: the line joining the complex foci and the line joining the real foci.

Remark 1.7 We see here that \mathcal{E} and \mathcal{C}_λ have the same foci. Hence they have the same isotropic tangent lines!

The following result will be needed, which generalizes a well-known result in the real case (concerning the intersection of an ellipse with a confocal hyperbola). To state it, we recall that $q(x, y) = x^2 + y^2$ is a complex quadratic form, whose associated bilinear form is b defined by

$$b(u, v) = u_x v_x + u_y v_y$$

for all $u = (u_x, u_y), v = (v_x, v_y) \in \mathbb{C}^2$. Two vector spaces F, G of \mathbb{C}^2 are said to be orthogonal if

$$b(u, v) = 0$$

for all $u \in F$ and $v \in G$, and this definition extends naturally to lines in \mathbb{C}^2 : two lines are orthogonal if their directions are orthogonal.

Lemma 1.8 *Let $\lambda \neq 0$. Then \mathcal{E} and \mathcal{C}_λ have four common points, whose coordinates (x, y) are such that*

$$x^2 = \frac{a^2(a^2 + \lambda)}{a^2 - b^2} \quad \text{and} \quad y^2 = \frac{b^2(b^2 + \lambda)}{b^2 - a^2}.$$

The tangent lines to \mathcal{C}_λ and \mathcal{E} at these points are orthogonal (for q).

Proof The coordinates of the common points are obtained by solving the system composed by the equations of \mathcal{C}_λ and \mathcal{E} .

Then, since the equations of the tangent lines of \mathcal{E} and \mathcal{C}_λ in one of these common points can be computed, it is not difficult to check that both tangent lines are orthogonal.

Finally, we will name by S the point of coordinates $(-a, 0)$ of \mathcal{E} .

2 Complex Reflection Law

Here we introduce the notion of complex billiards, which is somewhat similar to pseudo-euclidean billiards studied by Dragovic and Radnovic in [6].

Considering an affine chart whose coordinates will be denoted by (x, y) , we have the inclusion $\mathbb{R}^2 \subset \mathbb{C}^2 \subset \mathbb{C}\mathbb{P}^2$, and $\mathbb{C}\mathbb{P}^2 = \mathbb{C}^2 \sqcup \overline{\mathbb{C}}_\infty$, where $\overline{\mathbb{C}}_\infty$ is the infinity line. As introduced and explained in Ref. [10], and studied in Ref. [19], the reflection law on an algebraic (analytic) curve in \mathbb{R}^2 can be extended to $\mathbb{C}\mathbb{P}^2$ by considering the complexified version of the canonical euclidean quadratic form, that is the complex-bilinear non-degenerate quadratic form q . It leads to construct a notion of symmetry with respect to lines of \mathbb{C}^2 . Just note that q has two isotropic subspaces of dimension 1 (namely $\mathbb{C}(1, i)$ and $\mathbb{C}(1, -i)$).

2.1 Definition

Definition 2.1 Define the cyclic points of $\mathbb{C}\mathbb{P}^2$ as the points $I = (1 : i : 0)$ or $J = (1 : -i : 0)$.

Definition 2.2 ([10], Definition 1.2) A line in $\mathbb{C}\mathbb{P}^2$ is said to be *isotropic* if it contains either I or J and non-isotropic if not. (Thus, the infinity line is automatically isotropic.)

Definition 2.3 ([10], Definition 2.1) The symmetry with respect to a line L is defined by the two following points:

- The symmetry acting on \mathbb{C}^2 (on points and on lines): it is the unique non-trivial complex-isometric involution (isometric for the form q) fixing the points of the line L , if L is non-isotropic;
- The symmetry acting on lines: if L is an isotropic line passing through a point $x \notin \overline{\mathbb{C}}_\infty$, two lines ℓ and ℓ' going through x are called symmetric with respect to L if there exist sequences of lines $(L_n)_n, (\ell_n)_n, (\ell'_n)_n$ through points x_n such that L_n is non-isotropic, ℓ_n and ℓ'_n are symmetric with respect to L_n , $\ell_n \rightarrow \ell, \ell'_n \rightarrow \ell', L_n \rightarrow L$ and $x_n \rightarrow x$, when $n \rightarrow \infty$.

We recall now Lemma 2.3 [10] which gives an idea of this notion of symmetry in the case of an isotropic line through a finite point.

Lemma 2.4 ([10], Lemma 2.3) *If L is an isotropic line through a finite point x and ℓ, ℓ' are two lines passing through x , then ℓ and ℓ' are symmetric with respect to L if and only if either $\ell = L$, or $\ell' = L$.*

2.2 General Billiard Orbits

Definition 2.5 A piece of non-degenerate complex billiard trajectory is an ordered set of points (M_0, M_1, \dots, M_n) on \mathcal{E} such that

1. $M_j \neq M_{j+1}$ for all $j \in \{0, \dots, n-1\}$;
2. for all $j \in \{1, \dots, n-1\}$ the line $T_{M_j}\mathcal{E}$ is non-isotropic;

- for all $j \in \{1, \dots, n-1\}$, the lines $M_{j-1}M_j$ and M_jM_{j+1} are symmetric with respect to the tangent line $T_{M_j}\mathcal{E}$.

We say that a piece of non-degenerate trajectory is a non-degenerate periodic orbit or just an orbit when $M_n = M_0$ and when the above statements are also true for M_0 .

The M_j 's are called *vertices* of the trajectory, and the lines M_jM_{j+1} *sides* of the trajectory.

Computations will be easier with the following definitions.

Definition 2.6 A piece of non-degenerate trajectory is said to be:

- finite* if none of its vertices belong to the infinity line, and *infinite* if one of them belong to \mathbb{C}_∞ ;
- isotropic* if all of its sides are isotropic, and *non-isotropic* if none of its sides is isotropic.
- flat* if all of its sides coincide with one of both foci lines of \mathcal{E} .

Remark 2.7 It is not difficult to see that a piece of trajectory has a side on a foci line if and only if it has all its sides on this foci line.

Then we will use the following proposition, which follows from the fact that the reflection law with respect to a non-isotropic line permutes two isotropic lines, [11], corollary 2 (there exist exactly two distinct isotropic lines passing through any point $x \notin \overline{\mathbb{C}_\infty}$).

Proposition 2.8 *A non-degenerate periodic orbit with an odd number of sides is non-isotropic and none of its sides lie on a foci line.*

When $n = 3$, periodical orbits are called triangular. The proposition 2.8 implies that there are no non-degenerate isotropic triangular orbits.

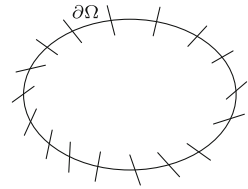
Definition 2.9 ([10]) *A degenerate triangular complex orbit on a complex conic \mathcal{E} is an ordered triple of points in \mathcal{E} which is the limit of non-degenerate periodic triangular orbits and which is not a non-degenerate triangular orbit. We define the sides of a degenerate orbit as the limits of the sides of the non-degenerate orbits which converge to it. Note that the sides of a degenerate orbit are uniquely defined, see the following more precise proposition.*

Proposition 2.10 ([19], Lemma 3.4) *A degenerate triangular orbit of an ellipse \mathcal{E} has an isotropic side A which is tangent to \mathcal{E} , and two coinciding non-isotropic sides B .*

2.3 Digression: Comparison with the Projective Reflection Law

Here we discuss about another generalization of the usual reflection law in the euclidean plane, the so-called projective reflection law, and we compare it to the complex reflection law. This subsection is independent from the rest of the paper and its reading can be skipped without any consequences. If the reader is interested, we refer to Refs. [8,20,21] for more details and recent results about this reflection law.

Fig. 2 A projective billiard Ω



The projective reflection law in the euclidean plane is defined as follows: a quadruple of lines (ℓ, ℓ', L, T) going through the same point p in the euclidean plane is said to be *harmonic* if their cross-ratio is equal to -1 . We recall that the cross-ratio of four distinct lines going through the same point is the cross-ratio of their intersection points with a fifth line d not containing p (and this does not depend on the line d).

Definition 2.11 (*Projective law of reflection*) Let $\gamma \subset \mathbb{R}^2$ be a plane curve, $p \in \gamma$, and L be a transverse line to γ at p . A line ℓ , distinct from L and $T_p\gamma$, passing through p is said to be *reflected into a line ℓ'* through p by the *projective reflection law* with respect to L if the quadruple of lines $(\ell, \ell', T_p\gamma, L)$ is harmonic.

Remark 2.12 This definition can be naturally extended to the case when $L = T_p\gamma$, but we will not detail this further.

Definition 2.13 A *projective billiard* is a domain $\Omega \subset \mathbb{R}^n$ with boundary endowed with a transverse line field L , called the *projective field of lines* (see Fig. 2). The reflection in planar projective billiard with a boundary curve $\partial\Omega$ acts on lines according to the projective reflection law.

Remark 2.14 Notice that the projective law of reflection and projective billiards can also be defined in the complexified euclidean plane, that is in \mathbb{C}^2 (in exactly the same way). In this case we write specifically *complex* projective billiards.

Now let us show that this law of reflection generalizes the (real or complex) reflection law. Denote by T a (real or complex) line through a point p and by T^\perp the (real or complex) line through p which is orthogonal to T (for the standard euclidean metric in the real case, and for q in the complex case). Suppose that T is non-isotropic, so that $T \neq T^\perp$. Two lines ℓ and ℓ' through p are symmetric with respect to T (for the real or complex law of reflection) if and only if the quadruple of lines $(\ell, \ell', T, T^\perp)$ is harmonic, and the projective reflection law is satisfied. In the case when T is isotropic ($T = T^\perp$), we can get the same conclusion. Therefore

Proposition 2.15 A real (resp. complex) billiard Ω is a real (resp. complex) projective billiard whose projective field of lines is defined by the normal lines to $\partial\Omega$ (resp. for the quadratic form q).

3 An Algebraic Proof of Theorem 1.2

In this section, we give a short algebraic proof of Theorem 1.2, noticing that the theorem is true for real orbits and using Zariski topology to conclude.

Proposition 3.1 *For every projective line L passing through neither real, nor complex foci there exists a unique $\lambda \notin \{-a^2, -b^2\}$ such that the conic \mathcal{C}_λ is tangent to L .*

Proof We prove this result by taking projective duality given by polar duality with respect to the Euclidean metric.

Denote by F_1 and F_2 the real foci of all \mathcal{C}_λ , and by G_1 and G_2 their complex foci: F_1, F_2 are on the x -axis, G_1, G_2 on the y -axis (see Sect. 1.2, or [4], Volume 2, Section 17.4). Hence the dual line F_i^* of F_i is vertical and the dual line G_i^* of G_i is horizontal.

Then note that any line $F_i G_j$ is tangent to any \mathcal{C}_λ (again, see Sect. 1.2, or [4], Volume 2, Section 17.4.3). It induces a point x_{ij} defined as the dual of the line $F_i G_j$, which belongs to all dual conics \mathcal{C}_λ^* of \mathcal{C}_λ . Hence, the family \mathcal{C}_λ^* is a pencil of conics through 4 distinct points x_{ij} . The conics \mathcal{C}_λ^* of the dual pencil are given by the equation $(a^2 + \lambda)\tilde{x}^2 + (b^2 + \lambda)\tilde{y}^2 = 1$. They are smooth for every $\lambda \notin \{-a^2, -b^2\}$.

Now, to any line L passing through neither real, nor complex foci corresponds a dual point y_L different from the x_{ij} . By polar duality, Proposition 3.1 is equivalent to say that there is a unique conic passing through y_L in the pencil of conics defined by the dual conics of the \mathcal{C}_λ . And this is a classical result on pencils of conics which is not difficult to prove, see for example [9].

To see that $\lambda \notin \{-a^2, -b^2\}$, suppose the contrary, for example $\lambda = -b^2$. The conic \mathcal{C}_λ^* is a pair of vertical lines passing through the x_{ij} , hence both lines are the dual lines F_1^* and F_2^* of the real foci. But L do not contain any foci, thus its dual point L^* do not belong to $F_1^* \cup F_2^* = \mathcal{C}_\lambda^*$.

The same statement holds for complex foci and $-b^2$ replaced by $-a^2$.

Remark 3.2 By proof of Proposition 3.1, we see in fact that if L goes through a real focus, its dual point L^* belongs to a \mathcal{C}_λ^* only when $\lambda = -b^2$. Hence L is never tangent to a \mathcal{C}_λ for $\lambda \neq -b^2$.

Corollary 3.3 *The collections of (A, L, λ) such that A lies in the given ellipse \mathcal{E} and L is a line through A that is tangent to \mathcal{C}_λ form an irreducible two-dimensional algebraic surface, in which the real part (real lines tangent to real confocal ellipses) is Zariski dense. The image of L under reflection from the tangent line $T_A \mathcal{E}$ is again tangent to \mathcal{C}_λ .*

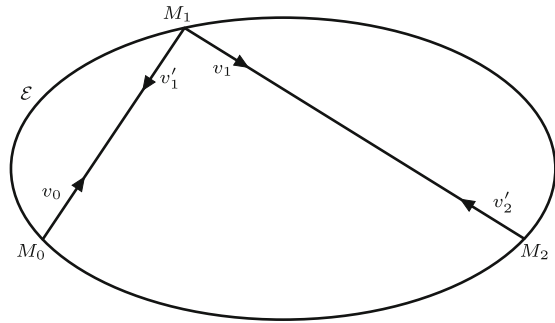
Proof The first statement of Corollary 3.3 follows immediately from Proposition 3.1. The second statement follows from the fact that it obviously holds on the Zariski dense real part and from algebraicity of the tangency condition for the reflected line.

Proof of Theorem 1.2 If the first statement of Theorem 1.2 is realized, then for each j , by Corollary 3.3 the line $M_{j-1}M_j$ is reflected into the $M_{j+1}M_j$ under reflection from the tangent line $T_{M_j} \mathcal{E}$. Hence T is a billiard orbit.

If the second statement of Theorem 1.2 is true, by Proposition 3.1 there is a unique $\lambda \notin \{-a^2, -b^2\}$ such that the conic \mathcal{C}_λ is tangent to the line M_0M_1 . By Corollary 3.3, each M_jM_{j+1} is again tangent to \mathcal{C}_λ .

Note that if you fix j , $M_{j-1}M_j$ and $M_{j+1}M_j$ are the two tangent lines to \mathcal{C}_λ passing through M_j : indeed, suppose $M_{j-1}M_j = M_{j+1}M_j$ and take any tangent line T to \mathcal{C}_λ passing through M_j . By Corollary 3.3, it is reflected in a tangent line T' to

Fig. 3 In Proposition 4.2, we consider all quantities $P(M_0, v_0)$, $P(M_1, v'_1)$, $P(M_1, v_1)$ and $P(M_2, v'_2)$



\mathcal{C}_λ . In the case when $T = T'$ the line T is orthogonal to $T_{M_j}\mathcal{E}$, and so is the line M_jM_{j+1} for the same reasons. If $T \neq T'$, we have T or $T' = M_jM_{j+1}$ or we would have three distinct tangent lines to \mathcal{E} passing through M_j which is impossible. Hence $T = T' = M_jM_{j-1} = M_jM_{j+1}$.

4 Theorem 1.2 and Joachimsthal Invariant

Remark 4.1 This section was inspired by the study of billiards in conics conducted in Chapter 4—*Billiards inside Conics and Quadrics* of [22]. In this book, Theorem 4.4 shows that for a set of points and directions defined as successive billiard reflections on a real conic \mathcal{E} , there is an *invariant* quantity. Known as Joachimsthal invariant, it is defined by

$$\frac{xv_x}{a^2} + \frac{yv_y}{b^2},$$

where (x, y) are the coordinates of a vertex of an orbit, and v a unitary vector having this vertex as starting point and pointing toward the next vertex. The author, Tabachnikov, further explains, without proving it, that one can find such an invariant if and only if one can find a conic tangent to the orbit. Let us mention another reference about Joachimsthal invariant, which was given to us by the referee, which we are grateful to: see [16], chapter IV.

In our case, Joachimsthal invariant does not work anymore and we need to change it a little bit: a square power appears, and we have to handle the case of isotropic directions, for which unitary vectors cannot be found (that are vectors v such that $q(v) = 1$).

4.1 A Billiard Invariant

Proposition 4.2 *Let $T = (M_0, M_1, M_2)$ be a piece of non-degenerate and non-isotropic trajectory on \mathcal{E} with M_0 finite. Then the quantity*

$$P(M_j, v) = \frac{\left(\frac{x_j v_x}{a^2} + \frac{y_j v_y}{b^2}\right)^2}{q(v)},$$

where (x_j, y_j) are the coordinates of a finite vertex M_j and $v = (v_x, v_y)$ is a directing vector of $M_{j-1}M_j$ or M_jM_{j+1} , is independent on the index $j \in \{0, 1, 2\}$ of the finite vertex chosen and on v (see Fig. 3).

Remark 4.3 For periodic orbits with an odd number of sides, one can remove the non-isotropic assumption, see Proposition 2.8.

Proof Since a non-isotropic piece of trajectory has non-isotropic sides by definition, $q(v) \neq 0$ for all v taken like in the proposition we want to prove.

First case If M_0 and M_1 are finite, write $M_0 = (x_0, y_0)$, $M_1 = (x_1, y_1)$. Take v_0 a vector such that $q(v_0) = 1$ and directing M_0M_1 and v_1 vector such that $q(v_1) = 1$ and directing M_1M_2 . Define the matrix

$$A = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}.$$

Then since ${}^tM_jAM_j = 1$ and since A is symmetric, we get

$${}^t(M_1 - M_0)A(M_1 + M_0) = {}^tM_1AM_0 - {}^tM_0AM_1 = 0.$$

Since v_0 is collinear to $M_1 - M_0$ we have further ${}^tv_0A(M_1 + M_0) = 0$, thus

$${}^tv_0AM_1 = -{}^tv_0AM_0. \tag{2}$$

But since M_0M_1 and M_1M_2 are symmetric with respect to the tangent line of \mathcal{E} in M_1 which is also orthogonal to AM_1 (the gradient in M_1 of the bilinear form defining \mathcal{E}), we only have two possibilities: either $v_0 + v_1$ or $v_0 - v_1$ is orthogonal to AM_1 . Hence

$${}^t(v_0 + v_1)AM_1 = 0$$

or

$${}^t(v_0 - v_1)AM_1 = 0.$$

In both cases

$$({}^tv_0AM_1)^2 = ({}^tv_1AM_1)^2$$

and using equality (2), we get

$$({}^tv_0AM_0)^2 = ({}^tv_1AM_1)^2, \tag{3}$$

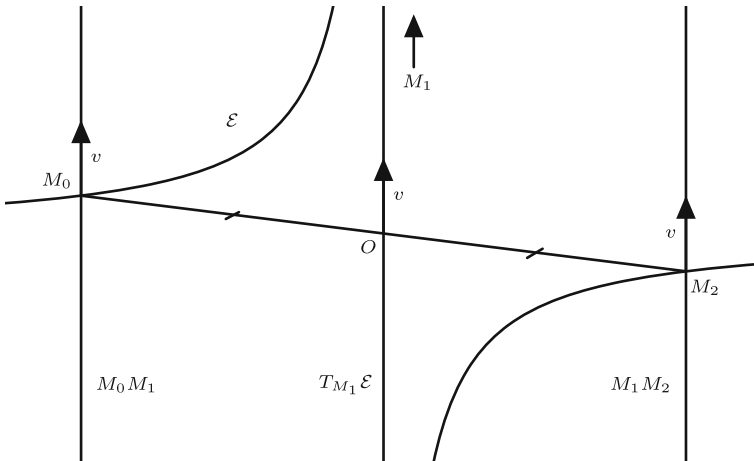


Fig. 4 A piece of billiard trajectory (M_0, M_1, M_2) on \mathcal{E} with M_1 infinite as in the proof of Proposition 4.2. The points M_0 and M_2 are symmetric across O , hence $M_2 = -M_0$ and $P(M_0, v) = P(M_2, v)$. Here \mathcal{E} is represented as an hyperbola which allows us to view the tangent line at the infinity point M_1 as the vertical asymptote

which proves Proposition 4.2 for unitary vectors. For general vectors, it is enough to divide them by a square root of $q(v)$, which explains why there is a $1/q(v)$ in the invariant formula.

Second case If M_0 is finite and M_1 infinite (see Fig. 4), then M_2 is finite. Indeed, M_0M_1 is a finite line and $T_{M_1}\mathcal{E}$ is not isotropic. Hence the line symmetric to M_0M_1 with respect to $T_{M_1}\mathcal{E}$ is finite and parallel to M_0M_1 and to $T_{M_1}\mathcal{E}$. Thus M_2 is finite. And therefore, we need to prove that

$$P(M_0, v) = P(M_2, v),$$

with v a vector directing $T_{M_1}\mathcal{E}$ (because v directs the lines M_0M_1, M_1M_2 and $T_{M_1}\mathcal{E}$). But $M_2 = -M_0$ since $T_{M_1}\mathcal{E}$ goes through the origin $O = (0 : 0 : 1)$ (by property of a tangent line at an point of \mathcal{E} on $\overline{\mathbb{C}}_\infty$) and the ellipse \mathcal{E} is symmetric across O (see Fig. 4). This implies that $P(M_0, v) = P(M_2, v)$.

Corollary 4.4 *Let $T = (M_0, \dots, M_n)$ be a piece of non-degenerate and non-isotropic trajectory on \mathcal{E} . Then the quantity $P(M_j, v)$ defined as before does not depend on the choice of a finite vertex M_j or on v , a directing vector of $M_{j-1}M_j$ or M_jM_{j+1} . Thus we will write $P(M_j, v) = P(T)$.*

Now we see that the invariant property implies a billiard reflection property.

Lemma 4.5 *Let M be a finite point on \mathcal{E} such that the line $T_M\mathcal{E}$ is non-isotropic. Let ℓ_1, ℓ_2 two non-isotropic lines passing through M and directed by the vectors v_1, v_2 . If*

$$P(M, v_1) = P(M, v_2), \tag{4}$$

then one of the following cases holds:

1. $\ell_1 = \ell_2$;
2. ℓ_1 and ℓ_2 are symmetric with respect to $T_M\mathcal{E}$.

Proof Suppose that case 1 is not true. Let us prove case 2.

We can suppose $q(v_1) = q(v_2) = 1$. By the equality (4), we have

$${}^t v_1 AM = \pm {}^t v_2 AM,$$

hence

$${}^t (v_2 \pm v_1) AM = 0.$$

Thus we get that $v_1 + v_2$ or $v_2 - v_1$ is orthogonal to AM which is orthogonal to the tangent line of \mathcal{E} in M . Hence $v_1 + v_2$ or $v_1 - v_2$ is tangent to \mathcal{E} in M . This implies that one of these vectors is fixed by the reflection with respect to $T_M\mathcal{E}$. Therefore this means that the components of the v_j 's along the direction of $T_M\mathcal{E}^\perp$ are the same or have opposite signs. Since the v_j 's are unit vectors, their components along the direction of $T_M\mathcal{E}$ are also the same or have opposite signs.

Hence we have only three possibilities: (a) v_1 and v_2 are symmetric with respect to $T_M\mathcal{E}$, (b) v_1 and v_2 are symmetric with respect to $T_M\mathcal{E}^\perp$, (c) $v_2 = \pm v_1$. Possibility c) cannot happen, otherwise $\ell_1 = \ell_2$. Hence case 2 is proven.

Lemma 4.6 *Let M_0, M_1, M_2 be points on \mathcal{E} such that M_0, M_2 are finite and M_1 infinite. Let v_j be a vector directing the line M_1M_j . If*

$$P(M_0, v_0) = P(M_2, v_2), \tag{5}$$

then one of the following cases holds:

1. $M_0 = M_2$;
2. M_0M_1 and M_1M_2 are symmetric with respect to $T_{M_1}\mathcal{E}$.

Proof Suppose that case 1 is not true. Let us prove case 2.

Since $M_1 \in \mathcal{E}$ is infinite, $M_1 = (a : \pm ib : 0)$. Thus M_0M_1 and M_1M_2 are directed by $v = (a : \pm ib)$. To simplify, suppose $v = (a, ib) = v_1 = v_2$. Thus since $P(M_0, v) = P(M_2, v)$ by equality (5), we have

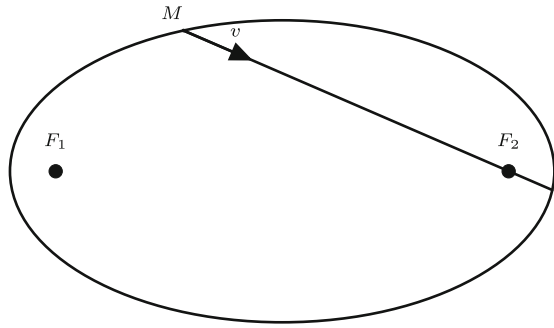
$$\frac{x_0}{a} + i \frac{y_0}{b} = \varepsilon \left(\frac{x_2}{a} + i \frac{y_2}{b} \right),$$

with $\varepsilon \in \{1, -1\}$, $M_j = (x_j, y_j)$. Hence

$$\frac{x_2 - \varepsilon x_0}{a} + i \frac{y_2 - \varepsilon y_0}{b} = 0.$$

We show that $\varepsilon = -1$. Indeed, if $\varepsilon = 1$, the latter equality means that $\overrightarrow{M_0M_2}$ is orthogonal to $v' = (1/a, i/b)$, which is orthogonal to v . Therefore, $\overrightarrow{M_0M_2}$ is colinear

Fig. 5 The non-isotropic line passing through M and directed by v goes through a real focus if and only if $P(M, v) = a^{-2}$, see Proposition 4.7



to v : but this is impossible, otherwise M_0, M_1, M_2 would be three distinct points of \mathcal{E} on the same line.

Thus $\varepsilon = -1$. Then, applying the same arguments, we have $M_0 = -M_2$. Hence M_0M_1 reflects into M_1M_2 .

4.2 Particular Values of the Invariant

The question we consider here is: *for which non-isotropic v do we have $P(M, v) = b^{-2}$ or $P(M, v) = a^{-2}$?*

Proposition 4.7 *If M is not a point of isotropic tangency of \mathcal{E} , we have:*

- $P(M, v) = a^{-2}$ if and only if v has the same direction as the line going through M and one of the real foci of \mathcal{E} (Fig. 5).
- $P(M, v) = b^{-2}$ if and only if v has the same direction as the line going through M and one of the complex foci of \mathcal{E} .

Proof We just prove the first point, the second one is analogous.

First, for a fixed $M \in \mathcal{E}$, and a $k \in \mathbb{C}$, there are at most two directions v such that

$$P(M, v) = k.$$

(Two collinear vectors have the same direction.) Indeed, the equation $\frac{xv_x}{a^2} + \frac{yv_y}{b^2} = k'$ of unknowns v_x, v_y defines a complex line which intersects the affine set $v_x^2 + v_y^2 = 1$ in at most two points (weak form of Bezout theorem). And considering the same equation but with $-k'$ instead of k' , we get at most four unitary vectors such that $P(M, v) = k$, two of them being the opposite of the others. Hence there are at most two directions such that $P(M, v) = a^{-2}$.

We give here the different possibilities for those directions. The real foci of \mathcal{E} have coordinates $(\pm c, 0)$, where $c = \sqrt{a^2 - b^2}$. Hence $v_{\pm} = (x \pm c, y)$ are directing the lines going through M and the real foci. Then we have

$$\frac{xv_{+,x}}{a^2} + \frac{yv_{+,y}}{b^2} = \frac{x(x+c)}{a^2} + \frac{y^2}{b^2} = 1 + \frac{xc}{a^2} = \frac{a^2 + xc}{a^2}$$

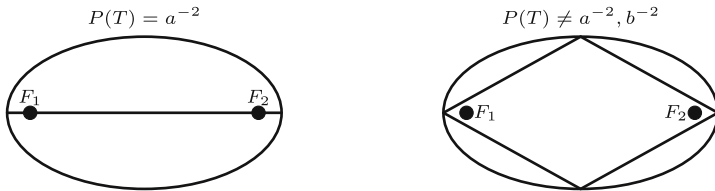


Fig. 6 Trajectories with their respective $P(T)$

and using the fact that $M \in \mathcal{E}$, we have

$$\begin{aligned} a^2q(v_+) &= a^2(x + c)^2 + a^2y^2 \\ &= a^2(x + c)^2 + (ab)^2 - b^2x^2 \\ &= c^2x^2 + 2a^2xc + a^4 \\ &= (a^2 + cx)^2. \end{aligned}$$

Hence, since M is not an isotropic tangency point of \mathcal{E} and by (1), we get that

$$q(v_+) \neq 0.$$

Thus,

$$P(M, v_+) = a^{-2}$$

and the same is true with v_- . There is one case when v_- and v_+ are colinear: when M is one vertex of the ellipse. But this case can be solved easily.

Corollary 4.8 (Forbidden values of $P(T)$) *Let $T = (M_0, \dots, M_n)$, with $n \geq 3$, be a non-degenerate and non-isotropic piece of trajectory. If T has none of its sides passing through a real or a complex focus of \mathcal{E} , then $P(T) \neq a^{-2}$ and $P(T) \neq b^{-2}$ (Fig. 6).*

4.3 Proof of Theorem 1.2

Here we prove that the invariant $P(T)$ characterizes pieces of trajectories which are tangent to the same conic. We first recall the following elementary fact.

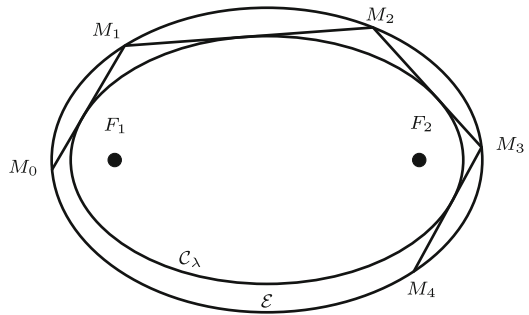
Lemma 4.9 *let C be a conic in $\mathbb{C}\mathbb{P}^2$ given by the equation ${}^tXAX = 0$, where A is a 3×3 invertible matrix, and $v = (a', b', c') \in \mathbb{C}^3$ defining the line ℓ_v of equation $a'x + b'y + c'z = 0$. Then ℓ_v is tangent to C if and only if*

$${}^tvA^{-1}v = 0.$$

We are now ready to prove the first part of Theorem 1.2.

Proposition 4.10 *Let $T = (M_0, \dots, M_n)$ be a non-degenerate non-isotropic piece of billiard trajectory. We suppose that none of its sides pass through a (real or complex)*

Fig. 7 The confocal caustic C_λ inscribed in a piece of billiard trajectory



focus. Then there is a unique $\lambda \in \mathbb{C}$ such that $a^2 + \lambda \neq 0$, $b^2 + \lambda \neq 0$ and T is circumscribed about the conic C_λ (Fig. 7).

We have in this case $\lambda = -(ab)^2 P(T)$.

Proof For $s \in \mathbb{C}$, let us define the matrix

$$B_s = \begin{pmatrix} a^2 + s & 0 & 0 \\ 0 & b^2 + s & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Since the orbit is non-isotropic, two consecutive sides cannot be infinite at the same time. Hence, we suppose without loss of generality that M_0 is finite. Then the line M_0M_1 is defined by the equation in \mathbb{CP}^2 $v_yx - v_x y + (v_x y_0 - v_y x_0)z = 0$. Here $M_0 = (x_0, y_0)$ and $v = (v_x, v_y)$ is a directing vector of M_0M_1 in \mathbb{C}^2 . Hence we have $M_0M_1 = \ell_w$ (in the notations of Lemma 4.9), where

$$w = (v_y, -v_x, v_x y_0 - v_y x_0).$$

It allows us to compute

$${}^t w B_s w = (a^2 + s)v_y^2 + (b^2 + s)v_x^2 - (v_x y_0 - v_y x_0)^2.$$

Using the fact that M_0 lies on the ellipse, that is, substituting

$$a^2 = x_0^2 + \frac{a^2 y_0^2}{b^2}, \quad b^2 = y_0^2 + \frac{b^2 x_0^2}{a^2}$$

to the above formula, we get

$${}^t w B_s w = sq(v) + (ab)^2 \left(\frac{x_0 v_x}{a^2} + \frac{y_0 v_y}{b^2} \right)^2.$$

Hence the previous quantity $P(M_0, v)$ defined in Proposition 4.2 appears here again, since

$${}^t w B_s w = q(v) \left(s + (ab)^2 P(M_0, v) \right).$$

Hence if $\lambda = -(ab)^2 P(M_0, v)$, it is the only λ for which ${}^t w B_\lambda w = 0$. And since $P(M_0, v) = P(T)$ does not depend on the choice of the index j of a M_j , the same computations are true for all lines $M_j M_{j+1}$. Thus, since $P(T) \neq a^{-2}, b^{-2}$ by corollary 4.8, we have $a^2 + \lambda \neq 0$ and $b^2 + \lambda \neq 0$, B_λ is invertible and B_λ^{-1} defines the conic \mathcal{C}_λ . The above equality ${}^t w B_\lambda w = 0$ implies that all $M_j M_{j+1}$ are tangent to \mathcal{C}_λ .

Now let us prove the second part of Theorem 1.2. It is a consequence of Proposition 4.12 which comes later. But first we will need the following

Lemma 4.11 *Let $\lambda \in \mathbb{C}$ be such that $a^2 + \lambda \neq 0$ and $b^2 + \lambda \neq 0$. Then each line $M_0 M_1$ which is tangent to \mathcal{C}_λ , where $M_0 \neq M_1 \in \mathcal{E}$, is such that*

1. M_0 or M_1 is finite;
2. the line $M_0 M_1$ is non-isotropic;
3. $\lambda = -(ab)^2 P(M_j, v)$, where v is directing $M_0 M_1$ and M_j is a finite point among M_0, M_1 .

Proof First notice that this line is non-isotropic because otherwise it would be tangent to \mathcal{C}_λ , hence to \mathcal{E} , and we could not have $M_0 \neq M_1$. □

Furthermore, it is not the infinity line (which is not tangent to \mathcal{C}_λ), hence M_0 or M_1 is finite. Therefore, the lemma results from the computation analogous to that of the proof of Proposition 4.10 for the computation of ${}^t w B_s w$.

Proposition 4.12 *Let $\lambda \in \mathbb{C}$ be such that $a^2 + \lambda \neq 0$ and $b^2 + \lambda \neq 0$. Then each n -uplet of points $T = (M_0, \dots, M_n) \in \mathcal{E}^n$, two consecutive points being distinct, such that for all $j \in \{1, \dots, n - 1\}$ the sides $M_{j-1} M_j$ and $M_j M_{j+1}$ realize the two tangent lines to \mathcal{C}_λ going through M_j , is a non-degenerate and non-isotropic piece of billiard trajectory, with $\lambda = -(ab)^2 P(T)$, whose sides avoid the foci of \mathcal{E} .*

Proof Lemma 4.11 implies that the sides $M_j M_{j+1}$ of T are non-isotropic and for each j , at least one point among M_j or M_{j+1} is finite.

Furthermore, the quantity $P(M_j, v)$ does not depend on the finite point M_j of T or on the vector v directing $M_{j-1} M_j$ or $M_j M_{j+1}$.

Therefore for each j , we have two possibilities by Lemmas 4.5 and 4.6: either $M_{j-1} M_j = M_j M_{j+1}$ or both lines are symmetric with respect to $T_{M_j} \mathcal{E}$.

Let us show that the former case is a subcase of the latter case. Indeed, if there is a j such that $M_{j-1} M_j = M_j M_{j+1}$, then by properties of conics $M_{j-1} = M_{j+1}$. This implies that there exists only one tangent line to \mathcal{C}_λ going through M_j . Hence $M_j \in \mathcal{C}_\lambda \cap \mathcal{E}$ and $M_{j-1} M_j$ is the tangent line $T_{M_j} \mathcal{C}_\lambda$, which is orthogonal to the tangent line $T_{M_j} \mathcal{E}$ by Lemma 1.8. Thus $M_{j-1} M_j$ and $M_j M_{j+1}$ are symmetric with respect to $T_{M_j} \mathcal{E}$.

Hence for each j we have a billiard reflection. Finally, the sides avoid the foci, since $\lambda \neq -a^2, -b^2$ and by Corollary 4.8. This concludes the proof.

5 Proof of Theorem 1.3

The finiteness of the number of conics γ_j^n , which we will call *caustics*, is not difficult to prove. For a fixed integer $n \geq 3$, the set \mathcal{T}_n of non-degenerate n -periodic orbits is an

open set of an algebraic curve of $\mathcal{E}^n \simeq (\mathbb{C}\mathbb{P}^1)^n$ (otherwise we could find an open set of initial conditions $(M_0, M_1) \in \mathcal{E}^2$ corresponding to n -periodic orbits, contradicting the real case). This curve has then a finite number of irreducible components. Now, for a fixed caustic, the set of n -periodic orbits circumscribed about it is an irreducible algebraic curve included in \mathcal{T}_n (this follows from the fact that each n -periodic orbit is uniquely defined by its initial condition (ℓ, M_0) , where $\ell = M_0M_1$ is a line through M_0 that is tangent to the given caustic, and the space of initial conditions is an elliptic curve, see [13]). But two different caustics cannot have the same set of circumscribed orbits: otherwise their corresponding constant $P(T)$ would be the same (Proposition 4.10), which is impossible. Hence there is a finite number of caustics: $\gamma_1^n = \mathcal{C}_{\lambda_1}, \dots, \gamma_N^n = \mathcal{C}_{\lambda_N}$, with pairwise distinct λ_j all different from $-a^2$ and $-b^2$.

Our goal now is to estimate the number N of caustics. To do so, we will use Cayley’s theorem, proven for example in [13]. We will just give an upper bound on N (Propositions 5.4 and 5.8) and explain how we can compute its exact value (Corollary 5.6 and Propositions 5.8 and 5.11).

The following theorem of Cayley is needed. We will say that two conics C and D are *in general position* if they intersect (transversally) at four points. Note that if $\lambda \neq 0, C_\lambda$ and \mathcal{E} are in general position (see Lemma 1.8).

Theorem 5.1 (Cayley) *Fix $n \geq 3$. Let C and D be quadratic forms defining two regular conics in $\mathbb{C}\mathbb{P}^2$ in general position. Write*

$$\sqrt{\det(tC + D)} = A_0 + A_1t + A_2t^2 \dots,$$

the analytic expansion in 0 of the holomorphic function $t \mapsto \sqrt{\det(tC + D)}$. Then there is an n -sided polygon inscribed in C and circumscribed about D if and only if

$$\begin{vmatrix} A_2 & \dots & A_{m+1} \\ \vdots & \ddots & \vdots \\ A_{m+1} & \dots & A_{2m} \end{vmatrix} = 0, \quad \text{with } m = \frac{n-1}{2} \quad (n \text{ odd}),$$

or

$$\begin{vmatrix} A_3 & \dots & A_{m+1} \\ \vdots & \ddots & \vdots \\ A_{m+1} & \dots & A_{2m-1} \end{vmatrix} = 0 \quad \text{with } m = \frac{n}{2} \quad (n \text{ even}).$$

This condition is reduced to $A_2 = 0$ when $n = 3$ and to $A_3 = 0$ when $n = 4$.

Remark 5.2 Note that the determinants we are considering in Theorem 5.1 can be rewritten for n odd and n even, respectively, as

$$\det(A_{i+j})_{1 \leq i, j \leq m} \quad \text{and} \quad \det(A_{i+j+1})_{1 \leq i, j \leq m-1}.$$

Remark 5.3 We recall that an element of $\mathbb{C}\mathbb{P}^2$ can be represented by a triple of the form $(x : y : z)$, where $x, y, z \in \mathbb{C}$ are not all equal to 0. Be careful that this representation

is not unique since $(x : y : z) = (tx : ty : tz)$ for any $t \in \mathbb{C}^*$. Then any polynomial $P(x, y)$ of degree d can be associated to a homogeneous polynomial

$$P^\sharp(x, y, z) = z^d P\left(\frac{x}{z}, \frac{y}{z}\right).$$

Hence the zeros of P in \mathbb{C}^2 can be extended in \mathbb{CP}^2 to the set of zeros of $P^\sharp(x, y, z)$. In our case, the conic \mathcal{C}_λ can be viewed in \mathbb{CP}^2 as the set of $(x : y : z)$ such that

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} - z^2 = 0.$$

Proposition 5.4 *Let $n \geq 3$. There is a polynomial $\mathcal{B}^n(\lambda)$ such that any of its roots $\lambda \notin \{-a^2, -b^2\}$ has the following property: there exists an n -sided polygon inscribed in \mathcal{E} and circumscribed about \mathcal{C}_λ .*

The degree of \mathcal{B}^n is such that

$$\deg \mathcal{B}^n \leq \begin{cases} \frac{n^2-1}{4} & \text{if } n \text{ is odd} \\ \frac{n^2}{4} - 1 & \text{if } n \text{ is even.} \end{cases}$$

Proof Suppose first that $n = 2m + 1$ is odd and fix a λ with $\lambda + a^2, \lambda + b^2 \neq 0$. To understand if there is an n -sided polygon inscribed in \mathcal{E} and circumscribed about \mathcal{C}_λ , we apply Cayley’s theorem: there is such a polygon if and only if the determinant

$$\mathcal{A}^n(\lambda) = \begin{vmatrix} A_2(\lambda) & \dots & A_{m+1}(\lambda) \\ \vdots & \ddots & \vdots \\ A_{m+1}(\lambda) & \dots & A_{2m}(\lambda) \end{vmatrix}$$

vanishes, where the $A_i(\lambda)$ are the coefficients in the analytic expansion of

$$f : t \rightarrow \sqrt{\det(tC + D_\lambda)},$$

with C and D_λ being quadratic forms, respectively, associated to \mathcal{E} and to \mathcal{C}_λ . Thus, to prove the result we want, we have to show that the determinant $\mathcal{A}^n(\lambda)$ vanishes for a finite number of λ . Let us give a more precise formula of $\mathcal{A}^n(\lambda)$. We have by Remark 5.3,

$$tC + D_\lambda = \left(\frac{t}{a^2} + \frac{1}{a^2 + \lambda}\right)x^2 + \left(\frac{t}{b^2} + \frac{1}{b^2 + \lambda}\right)y^2 - (t + 1)z^2,$$

hence

$$\det(tC + D_\lambda) = -\left(\frac{t}{a^2} + \frac{1}{a^2 + \lambda}\right)\left(\frac{t}{b^2} + \frac{1}{b^2 + \lambda}\right)(t + 1),$$

which we factorize in

$$\det(tC + D_\lambda) = -\frac{1}{(a^2 + \lambda)(b^2 + \lambda)} \left(\frac{a^2 + \lambda}{a^2}t + 1 \right) \left(\frac{b^2 + \lambda}{b^2}t + 1 \right) (t + 1).$$

Define the map $g : t \mapsto \sqrt{\left(\frac{a^2+\lambda}{a^2}t + 1\right) \left(\frac{b^2+\lambda}{b^2}t + 1\right) (t + 1)}$ and write its Taylor expansion as

$$g(t) = \sum_{k=0}^{\infty} B_k(\lambda)t^k.$$

Since

$$f(t) = \frac{ig(t)}{\sqrt{(a^2 + \lambda)(b^2 + \lambda)}},$$

we have

$$A_k(\lambda) = \frac{i B_k(\lambda)}{\sqrt{(a^2 + \lambda)(b^2 + \lambda)}}.$$

This shows that $\mathcal{A}^n(\lambda)$ is a function of λ which vanishes if and only if the determinant

$$\mathcal{B}^n(\lambda) = \begin{vmatrix} B_2(\lambda) & \dots & B_{m+1}(\lambda) \\ \vdots & \ddots & \vdots \\ B_{m+1}(\lambda) & \dots & B_{2m}(\lambda) \end{vmatrix}$$

also vanishes. We thus need to compute the B_k s. Write

$$\sqrt{t + 1} = c_0 + c_1t + c_2t^2 + \dots,$$

where

$$c_k = \frac{1}{k!} \left(\frac{1}{2}\right) \left(\frac{1}{2} - 1\right) \dots \left(\frac{1}{2} - k + 1\right) = \frac{(-1)^{k+1}}{4^k(2k - 1)} \binom{2k}{k}. \tag{6}$$

Therefore for any β ,

$$\sqrt{\beta t + 1} = c_0 + c_1\beta t + c_2\beta^2 t^2 + \dots,$$

where

$$B_k(\lambda) = \sum_{u+v+w=k} \frac{c_u c_v c_w}{a^{2u} b^{2v}} (a^2 + \lambda)^u (b^2 + \lambda)^v. \tag{7}$$

Therefore each B_k is a polynomial in λ of degree at least k . Hence $\mathcal{B}^n(\lambda)$ is a polynomial in λ verifying: for any $\lambda \notin \{-a^2, -b^2\}$, $\mathcal{B}^n(\lambda) = 0$ if and only if $\mathcal{A}^n(\lambda) = 0$, which

is true if and only if there exists an n -sided polygon inscribed in \mathcal{E} and circumscribed about \mathcal{C}_λ .

Now for any permutation σ of $\{1, \dots, m\}$, we have

$$\deg \prod_{j=1}^m B_{\sigma(j)+j} = \sum_{j=1}^m \deg B_{\sigma(j)+j} \leq \sum_{j=1}^m (\sigma(j) + j) = m(m + 1)$$

and since $\mathcal{B}^n(\lambda)$ is a sum of $\pm \prod_{j=1}^m B_{\sigma(j)+j}$ over all σ , we have that $\deg \mathcal{B}^n(\lambda) \leq m(m + 1) = \frac{n^2-1}{4}$.

Now if $n = 2m$ is even, the existence of \mathcal{B}^n is treated exactly as in the case when n is odd, but instead, $\mathcal{B}^n(\lambda) = \det(B_{i+j+1})_{1 \leq i, j \leq m-1}$. Hence for any permutation σ of $\{1, \dots, m\}$, we have

$$\deg \prod_{j=1}^{m-1} B_{\sigma(j)+j+1} = m^2 - 1$$

and $\deg \mathcal{B}^n \leq m^2 - 1 = \frac{n^2}{4} - 1$.

Now we are ready for the proof of Theorem 1.3:

Proof (Proof of Theorem 1.3 (without its last statement for $n = 3, 4$)) Let \mathcal{B}^n be the polynomial of Proposition 5.4. By construction, any root λ of \mathcal{B}^n different from $-a^2$ and $-b^2$ corresponds to a \mathcal{C}_λ inscribed in an n -sided polygon P of \mathcal{E} . By Theorem 1.2, P is a n -periodic billiard orbit and \mathcal{C}_λ is its caustic, therefore $\lambda = \lambda_j$ for a certain $1 \leq j \leq N$. Conversely, all λ_j are roots of \mathcal{B}^n since a periodic billiard orbit is a polygon.

The first statement of Theorem 1.3 is obvious by Theorem 1.2. The second statement is a consequence of the argument given at the beginning of Sect. 5.

The third statement comes from Poncelet’s theorem: by definition of the γ_j , there is at least one n -sided polygon inscribed in \mathcal{E} and circumscribed about \mathcal{C}_λ . Now Poncelet’s theorem states that for any $p \in \mathcal{E}$ there exists such a polygon with p as a vertex.

We can try to know if there are periodic orbits passing through the foci. We recall that in the real case, a billiard trajectory going through the real foci accumulates on the (real) foci line (see [22], Exercise 4.3).

Proposition 5.5 *If an n -periodic orbit goes through the real (resp. complex) foci, then n is even and its edges are on the real (resp. complex) foci line.*

Proof Since a line through a focus is reflected into a line through another focus, n has to be even. Now consider the map f from \mathcal{E} to \mathcal{E} defined as follows: for A in \mathcal{E} , let B be the other intersection point of AF_1 with \mathcal{E} and C the other intersection point of BF_2 with \mathcal{E} , where F_1, F_2 are the real foci of \mathcal{E} . A n -periodic orbit through the real foci has a vertex M such that $f^n(M) = M$. Now f is a non-trivial automorphism of $\mathcal{E} \simeq \mathbb{C}\mathbb{P}^1$ hence is a Möbius transform, and so is f^n . But such transform has at least

two fixed points. Since we already know two of them to be the vertices of \mathcal{E} , the latter correspond to the only possible initial points of periodic orbits through the real foci. The same holds with complex foci.

From the proof of Theorem 1.3, we get

Corollary 5.6 *If the roots of \mathcal{B}^n are simple and different from $-a^2$ and $-b^2$, then $N = \deg \mathcal{B}^n$.*

Remark 5.7 In order to compute the exact value of N , we can first try to compute the exact value of $\deg \mathcal{B}^n$ (see Proposition 5.8). Then we can understand when the assumptions of Corollary 5.6 are satisfied (see Proposition 5.11). Unfortunately we still do not know in which cases \mathcal{B}^n has simple roots.

Proposition 5.8 *There exist $r_1, \dots, r_p \in \mathbb{R}^+$ such that for all (a, b) with $a \notin \{r_1 b, \dots, r_p b\}$, we have*

$$\deg \mathcal{B}^n = \begin{cases} \frac{n^2-1}{4} & \text{if } n \text{ is odd} \\ \frac{n^2}{4} - 1 & \text{if } n \text{ is even.} \end{cases}$$

Remark 5.9 We show in Proposition 5.10 that $p \geq 1$, more precisely that 1 belongs to the collection of $\{r_1, \dots, r_p\}$.

Proof Suppose $n = 2m + 1$ is odd. By Eq. (7), B_k is of degree $\leq k$ and the coefficient in front of λ^k is

$$d(B_k) = \sum_{u+v=k} \frac{c_u c_v}{a^{2u} b^{2v}} = \frac{(-1)^k}{4^k} \sum_{u+v=k} \frac{1}{a^{2u} b^{2v} (2u-1)(2v-1)} \binom{2u}{u} \binom{2v}{v}. \tag{8}$$

Fix a permutation σ of $\{1, \dots, m\}$. We have

$$\deg \prod_{j=1}^m B_{\sigma(j)+j} = \sum_{j=1}^m \deg B_{\sigma(j)+j} \leq \sum_{j=1}^m (\sigma(j) + j) = m(m+1)$$

and the coefficient in front of $\lambda^{m(m+1)}$ is $\prod_{j=1}^m d(B_{\sigma(j)+j})$. Since $\mathcal{B}^n(\lambda)$ is a sum of $\pm \prod_{j=1}^m B_{\sigma(j)+j}$ over all σ , we have that $\deg \mathcal{B}^n(\lambda) \leq m(m+1)$, and the coefficient in front of $\lambda^{m(m+1)}$ is

$$d_n(a, b) = \begin{vmatrix} d(B_2) & \dots & d(B_{m+1}) \\ \vdots & \ddots & \vdots \\ d(B_{m+1}) & \dots & d(B_{2m}) \end{vmatrix}.$$

Let us show that $d_n(a, b) \neq 0$ except for specific (a, b) as described in Proposition 5.8. Note first that each $d(B_k)$ is a homogeneous polynomial in (a^{-1}, b^{-1}) of degree $2k$, and by Eq. (8) the coefficient in front of a^{-2k} is

$$\frac{(-1)^{k+1}}{4^k(2k-1)} \binom{2k}{k} = 2 \frac{(-1)^{k+1}}{4^k} \text{Cat}_{k-1},$$

where $\text{Cat}_k = \frac{1}{k+1} \binom{2k}{k}$ is the k th Catalan number.

Now by linearity of the determinant, $d_n(a, b)$ is also a homogeneous polynomial in (a^{-1}, b^{-1}) , and we apply the same procedure as before: for any permutation σ of $\{1, \dots, m\}$, we have

$$\deg \prod_{j=1}^m d(B_{\sigma(j)+j}) = \sum_{j=1}^m \deg d(B_{\sigma(j)+j}) = \sum_{j=1}^m 2(\sigma(j) + j) = 2m(m+1)$$

and the coefficient in front of a^{-2k} is

$$\prod_{j=1}^m 2 \frac{(-1)^{j+\sigma(j)+1}}{4^{j+\sigma(j)}} \text{Cat}_{j+\sigma(j)-1} = \frac{(-1)^{m(m+2)} 2^m}{4^{m(m+1)}} \prod_{j=1}^m \text{Cat}_{j+\sigma(j)-1}.$$

Since $d_n(a, b)$ is a sum of $\pm \prod_{j=1}^m d(B_{\sigma(j)+j})$ over all σ , we have that $\deg d_n(a, b) \leq 2m(m+1)$, and the coefficient in front of $a^{-2m(m+1)}$ is

$$\frac{(-1)^m}{2^{m(2m+1)}} \det H_m,$$

where H_m is a Hankel matrix of the sequence $(\text{Cat}_{k+1})_k$ defined as

$$H_m = \begin{pmatrix} \text{Cat}_1 & \text{Cat}_2 & \cdots & \text{Cat}_m \\ \text{Cat}_2 & \text{Cat}_3 & & \\ \vdots & & \ddots & \vdots \\ \text{Cat}_m & & \cdots & \text{Cat}_{2m} \end{pmatrix}.$$

There are not so easy methods to show that $\det H_m = 1$, see for example [17], Theorem 33. Hence $d_n(a, b)$ is a nonzero homogeneous polynomial in (a^{-1}, b^{-1}) and therefore there exists an at most finite collection of numbers $r_1, \dots, r_p \in \mathbb{R}^+$ such that for all $a, b > 0$, we have $d_n(a, b) = 0$ if and only if $a \in \{r_1 b, \dots, r_p b\}$. \square

We can ask the question if $\deg \mathcal{B}^n$ has always the maximal value (described in Proposition 5.8), or if we can find $a, b > 0$ such that $\deg \mathcal{B}^n$ is less than the value in Proposition 5.8. Proposition 5.10 asserts that we can find indeed such a, b by looking at the case of the circle ($a = b$).

Proposition 5.10 *When $a = b$ (in the case of the circle),*

$$\deg \mathcal{B}^n = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} - 1 & \text{if } n \text{ is even.} \end{cases}$$

Proof Suppose $n = 2m + 1$ is odd. By Eq. 7, when $a = b = R$, for $k \geq 2$

$$B_k = \sum_{w=0}^k c_{k-w} \left(1 + \frac{\lambda}{a^2}\right)^w \sum_{u+v=w} c_u c_v.$$

Let us compute $\sum_{u+v=w} c_u c_v$: it is the Taylor coefficient at t^w of the function $\sqrt{1+t^2} = 1 + t$, therefore we get that

$$\sum_{u+v=w} c_u c_v = \begin{cases} 1 & \text{if } 0 \leq w \leq 1 \\ 0 & \text{if } w \geq 2. \end{cases}$$

Hence $B_k = c_k + c_{k-1}x$, where $x = 1 + \lambda/a^2$. Using the multilinearity of \det , it is not hard to see that B^n is of degree m if n is odd and $m - 1$ if n is even.

Proposition 5.11 *There exist an at most finite collection of numbers $r'_1, \dots, r'_q \in \mathbb{R}^+$ such that for all (a, b) with $a \notin \{r'_1 b, \dots, r'_q b\}$, $-a^2$ and $-b^2$ are not roots of B^n .*

Proof Suppose $n = 2m + 1$ is odd. By Eq. (7), for $k \geq 2$,

$$B_k(-a^2) = \sum_{v+w=k} \frac{c_v c_w}{b^{2v}} (b^2 - a^2)^v = \frac{1}{b^{2k}} \sum_{v+w=k} c_v c_w b^{2w} (b^2 - a^2)^v = \frac{1}{b^{2k}} P_k(a, b), \tag{9}$$

where $P_k(a, b)$ is a homogeneous polynomial in (a, b) of degree $2k$. The coefficient in front of a^{2k} is

$$(-1)^k c_k = -\frac{1}{4^k (2k - 1)} \binom{2k}{k} = -\frac{\text{Cat}_{k-1}}{2^{2k-1}}.$$

As in the proof of Proposition 5.8, for any permutation σ of $\{1, \dots, m\}$,

$$\prod_{j=1}^m B_{\sigma(j)+j}(-a^2) = \prod_{j=1}^m \frac{1}{b^{2(\sigma(j)+j)}} P_{\sigma(j)+j}(a, b) = \frac{Q_\sigma(a, b)}{b^{2m(m+1)}},$$

where $Q_\sigma(a, b)$ is a homogeneous polynomial of degree

$$\sum_{j=1}^m \deg P_{j+\sigma(j)} = \sum_{j=1}^m 2(\sigma(j) + j) = 2m(m + 1)$$

whose coefficient in front of $a^{2m(m+1)}$ is

$$\prod_{j=1}^m \left(-\frac{\text{Cat}_{j+\sigma(j)-1}}{2^{2(j+\sigma(j))-1}}\right) = \frac{(-1)^m}{2^{m(2m+1)}} \prod_{j=1}^m \text{Cat}_{j+\sigma(j)-1}.$$

As in proof of Proposition 5.8, $\mathcal{B}^n(-a^2)$ is a sum of products of the form $\pm \prod_{j=1}^m B_{\sigma(j)+j}(-a^2)$ hence can be written as

$$\frac{R_n(a, b)}{b^{2m(m+1)}}$$

where $R_n(a, b)$ is the sum of $\varepsilon(\sigma) \prod_{j=1}^m Q_{\sigma(j)+j}(a, b)$ and $\varepsilon(\sigma)$ is the parity of σ . Thus $R_n(a, b)$ is a homogeneous polynomial of degree $2m(m + 1)$ whose coefficient in front of $a^{2m(m+1)}$ is

$$\frac{(-1)^m}{2^{m(2m+1)}} \det H_m = \frac{(-1)^m}{2^{m(2m+1)}}$$

as in the proof of Proposition 5.8. Thus $R_n(a, b)$ is a nonzero homogeneous polynomial such that

$$\mathcal{B}^n(-a^2) = \frac{R_n(a, b)}{b^{2m(m+1)}}$$

We can do the same with $\mathcal{B}^n(-b^2)$ to obtain the same conclusion, which finishes the proof.

6 Triangular Orbits

As in the proof of Theorem 1.3, any root $\lambda \notin \{-a^2, -b^2\}$ of $\mathcal{B}^3(\lambda)$ corresponds to a caustic defined by \mathcal{C}_λ for triangular orbits in the complex domain. Therefore we are going to compute $\mathcal{B}^3(\lambda)$ and its roots. This section is very similar to Ref. [7] p. 17, since we get the same results with a similar method (just notice that the conventions adopted in this paper is different from ours: in Ref. [7], a stand for a^2 , b for b^2 and their \mathcal{C}_λ is the same as our $\mathcal{C}_{-\lambda}$). Yet, our conclusion is a little bit more general thanks to our previous study on complex caustics (Theorem 1.2). For the Minkowski plane, similar calculations are given in Ref. [1], p. 469, 470. We have

$$\mathcal{B}^3(\lambda) = -\frac{1}{8a^4b^4} \left((a^2 - b^2)^2 \lambda^2 - 2(a^4b^2 + a^2b^4)\lambda - 3a^4b^4 \right),$$

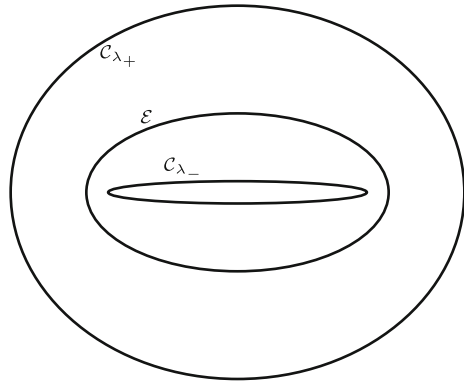
which is a polynomial of second degree. Its roots are

$$\lambda_{\pm} = \frac{a^2b^2(a^2 + b^2) \pm 2a^2b^2\sqrt{a^4 - a^2b^2 + b^4}}{(a^2 - b^2)^2},$$

which correspond to the opposite of the solutions found in Ref. [7] as predicted.

Lemma 6.1 *We have $\lambda_+ > 0$ and $-b^2 < \lambda_- < 0$.*

Fig. 8 The initial ellipse with its two caustics C_{λ_-} and C_{λ_+} when $n = 3, a = 2$ and $b = 1$



Proof The inequality $\lambda_- > -b^2$ is equivalent to

$$2a^2b^2\sqrt{a^4 - 2a^2b^2 + b^4} - a^2b^2(a^2 + b^2) < b^2(a^2 - b^2)^2,$$

which can be simplified in

$$2a^2b^2\sqrt{a^4 - 2a^2b^2 + b^4} < 2a^4 + b^4 - a^2b^2.$$

This last inequality is true as we can see by taking the square of both its right and left sides. The same method can be applied to show that $\lambda_- < 0$.

By Lemma 6.1, the caustics corresponding to $\lambda_{\pm}, C_{\lambda_+}$ and C_{λ_-} , are confocal ellipses which are, respectively, bigger and smaller than $\mathcal{E} = C_0$. As noticed in Ref. [7], C_{λ_-} corresponds to the real motion. And we can add that C_{λ_+} corresponds to no real orbits by convexity. Hence we deduce the following corollary, which is a version of Theorem 1.3 in the case when $n = 3$:

Corollary 6.2 *There exist two distinct real ellipses $\gamma_1^3 := C_{\lambda_-}$ and $\gamma_2^3 := C_{\lambda_+}$ which are confocal to \mathcal{E} by construction, and such that*

- all complex triangles inscribed in \mathcal{E} and circumscribed about a γ_j^3 are billiard orbits;
- any complex triangular billiard orbit of \mathcal{E} is circumscribed about either γ_1^3 , or γ_2^3 ;
- any complex orbit inscribed in \mathcal{E} and circumscribed about a γ_j^3 is a triangular orbit.

Example 6.3 We consider the case when $a = 2$ and $b = 1$; see Fig. 8. We compute that

$$\lambda_- = \frac{20 - 8\sqrt{13}}{9} \approx -0.9827 \quad \text{and} \quad \lambda_+ = \frac{20 + 8\sqrt{13}}{9} \approx 5.4272.$$

We can apply Corollary 6.2 to classify the degenerate triangular orbits:

Proposition 6.4 *There are exactly 8 degenerate triangular orbits, given by an isotropic tangency point α of \mathcal{E} and a point β of \mathcal{E} such that $\alpha\beta$ is tangent to γ_j^3 for some $j = 1, 2$ and non-isotropic.*

Proof Degenerate triangular orbits are limits of non-degenerate (non-isotropic) triangular orbits, hence are circumscribed about a γ_j^3 . We apply Proposition 2.10: one side A is isotropic and tangent to a γ_j^3 . This gives only 4 possible positions. The other side is non-isotropic, which gives two other possibilities (B should be tangent to the same γ_j^3), once A is fixed. □

7 4-Periodic Orbits

We apply in this section the same ideas as in Sect. 6 for $n = 4$. Again these calculations repeat those from Ref. [7], p. 18, 19. For the Minkowski plane, similar calculations are given in Ref. [1], p. 469, 470. In the following, we compute \mathcal{B}^4 and its roots:

$$\mathcal{B}^4(\lambda) = \frac{1}{16a^6b^6} \left((a^2 - b^2)^2(a^2 + b^2)\lambda^3 + (a^3b - ab^3)^2\lambda^2 - (a^6b^4 + a^4b^6)\lambda - a^6b^6 \right).$$

We can check that its roots are

$$\lambda_1 = -\frac{a^2b^2}{a^2 - b^2}, \quad \lambda_2 = -\frac{a^2b^2}{a^2 + b^2}, \quad \lambda_3 = \frac{a^2b^2}{a^2 - b^2}.$$

They satisfy

$$\lambda_1 < -b^2 < \lambda_2 < 0 < \lambda_3. \tag{10}$$

Note also that

$$\lambda_1 \begin{cases} < -a^2 & \text{when } a < \sqrt{2}b \\ = -a^2 & \text{when } a = \sqrt{2}b \\ > -a^2 & \text{when } a > \sqrt{2}b \end{cases} \tag{11}$$

Denote by \mathcal{C}_i the caustic \mathcal{C}_{λ_i} , where $i = 1, 2, 3$, with \mathcal{C}_i not defined when $a = \sqrt{2}b$. By Inequations (10), \mathcal{C}_2 and \mathcal{C}_3 are confocal ellipses, respectively, smaller and bigger than $\mathcal{E} = \mathcal{C}_0$. By Inequation (11), the conic \mathcal{C}_1 is a hyperbola if and only if $a > \sqrt{2}b$.

Corollary 7.1 *When $a \neq \sqrt{2}b$ (resp. when $a = \sqrt{2}b$), there exist three (resp. two) distinct conics $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ (resp. $\mathcal{C}_2, \mathcal{C}_3$) which have the previous described properties, and such that*

- all complex quadrilaterals inscribed in \mathcal{E} and circumscribed about a \mathcal{C}_j are billiard orbits;
- any complex quadrilateral billiard orbit of \mathcal{E} , which do not have its edges on a foci line, is circumscribed about a \mathcal{C}_j ;
- any complex orbit inscribed in \mathcal{E} and circumscribed about a \mathcal{C}_j is a quadrilateral orbit.

Fig. 9 The initial ellipse with its three caustics in the case when $n = 4, a = 2$ and $b = 1$

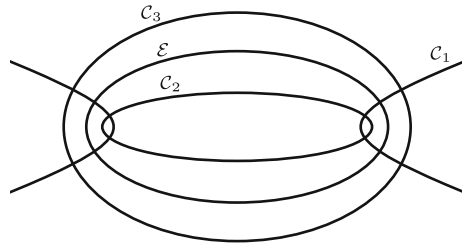
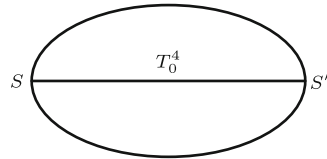


Fig. 10 The flat 4-periodic orbit T_0^4



Example 7.2 We consider the case when $a = 2$ and $b = 1$; see Fig. 9. We compute that

$$\lambda_3 = -\lambda_1 = \frac{4}{3} \approx 1,333 \quad \text{and} \quad \lambda_2 = \frac{4}{5} = 0,8.$$

Let us investigate to which type of orbits corresponds each of the C_i . To do so, we fix a point S to be the vertex $(-a, 0)$ of the initial ellipse \mathcal{E} , and we want to determine the 4-periodic orbits having S as a vertex. One is already known: it is the *flat* orbit whose vertices are on the foci-line and denoted by T_0^4 ; see Fig. 10.

By Theorem 1.3, each one of the other orbits is tangent to a C_i : denote by T_i^4 the 4-periodic orbit tangent to C_i , where $i = 1, 2, 3$. Let $v_i = (v_{x,i}, v_{y,i})$ be a vector directing a side of T_i^4 starting from S , and such that $q(v_i) = 1$. By Proposition 4.12, one has

$$\lambda_i = -(ab)^2 P(T_i^4) = -b^2 v_{x,i}^2. \tag{12}$$

Thus we can determine v_i for each i .

Case $i = 1$: We compute that

$$v_1 = \frac{1}{\sqrt{a^2 - b^2}} (\pm a, \pm ib).$$

A line passing through S and directed by one of the solution for v_2 intersects the ellipse in S and a point at infinity $M_{\pm} = (a : \pm ib : 0)$. We get the orbit (S, M_+, S', M_-) , where $S' = -S$; see Fig. 11.

Case $i = 2$: By Eq. (12) and $q(v_2) = 1$ we get

$$v_2 = \frac{1}{\sqrt{a^2 + b^2}} (\pm a, \pm b).$$

Fig. 11 The infinite 4-periodic orbit T_1^4 . A change of coordinates $y \rightarrow iy$ to represent the orbit properly; this operation changes \mathcal{E} into a hyperbola. The infinite points of the orbit, M_+ and M_- , lie “at the end” of each branch of the hyperbola

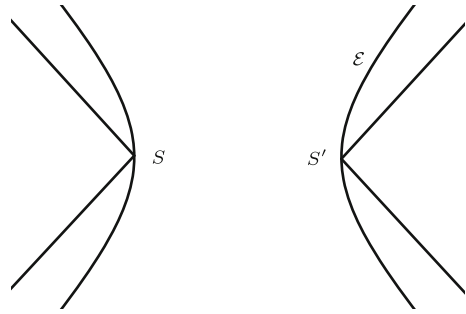
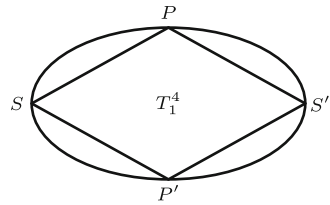


Fig. 12 The 4-periodic orbit T_2^4



It corresponds to the well-known real orbit (S, P, S', P') , where $P = (0, b)$, $P' = -P$ and $S' = -S$; see Fig. 12.

Case $i = 3$: Here we have

$$v_3 = \frac{1}{\sqrt{a^2 - b^2}} (\pm a, \pm i\sqrt{2a^2 - b^2}).$$

A line passing through S and directed by one of the solutions for v_3 intersects the ellipse in S and in one of the points

$$N_{\pm} = \frac{1}{a^2 - b^2} (-a^3, \pm ib\sqrt{2a^2 - b^2})$$

depending on the signs we choose for v_3 's coordinates.

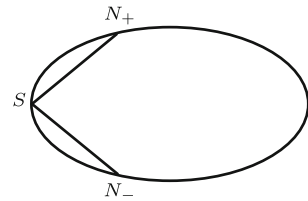
The question is: why are the points N_{\pm} not on the x - or y -axis by symmetry of the 4-periodic orbit? The reason is that the line passing through S and directed by v_3 is reflected in the same line at one of the points N_{\pm} . Indeed, the system of equations in v

$$P(N_+, v) = P(T_3^4) \quad \text{and} \quad q(v) = 1$$

(see Lemma 4.6) has a unique solution (up to multiplication by -1), which is in fact $v = v_3$. Thus the corresponding orbit is (S, N_+, S, N_-) ; see Fig. 13.

Remark 7.3 Observe that the points N_{\pm} belong to \mathcal{E} and to \mathcal{C}_3 . Indeed, SN_+ and N_+S realize the two tangent lines to \mathcal{C}_3 (by Theorem 1.2), thus there is only one such tangent line and therefore $N_+ \in \mathcal{C}_3$. The same argument is true with N_- .

Fig. 13 The 4-periodic orbit T_3^4 . Its edges SN_+ and SN_- are reflected into themselves in N_+ and N_- , respectively. Here the drawing is biased, because we cannot represent T_3^4 which has complex vertices



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