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RESEARCH CONTRIBUTION

On Nörlund–Voronoi Summability and Instability of Rational Maps

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Abstract
We investigate the connection between the instability of rational maps and summability methods applied to the spectrum of a critical point belonging to the Julia set of a rational map.

Keywords Holomorphic Dynamics · Rational maps · Summability methods

1 Motivation and Main Results

Let Rat_d be the space of all rational maps R of degree $d > 0$ defined on the Riemann sphere $\bar{\mathbb{C}}$. Let $Crit(R)$ be the set of critical points of R. For $c \in Crit(R)$, the *individual postcritical* set is:

$$P_c(R) = \overline{\bigcup_{n>0} R^n(c)}.$$

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The *postcritical set* of R is:

$$P(R) = \overline{\bigcup_{c \in Crit(R)} P_c(R)}.$$

The *Julia set* $J(R)$ is the accumulation set of all repelling periodic cycles. A rational map R is called *hyperbolic* if the postcritical set does not intersect the Julia set.

The Fatou conjecture states that hyperbolic rational maps of degree d form an open and dense subset of Rat_d. A rational map $R \in Rat_d$ is called *structurally stable* if there exists a neighborhood U of R in Rat_d, such that for every Q in U, there is a quasiconformal conjugation between R and Q. A theorem of Mañé, Sad, and Sullivan shows that the set of structurally stable maps forms an open and dense subset of Rat_d (see [18]). Hence, the Fatou conjecture can be reformulated in the following way.

If $R \in Rat_d$ has a critical point c in the Julia set $J(R)$, then R is not structurally stable or, equivalently, is an unstable map.

Recall that a critical point c of a rational map R is called *summable* if the series $\sum_{n=0}^{\infty} \frac{1}{(R^n)'(R(c))}$ is absolutely convergent.

Among other results, Avila [1], Levin [11], and Makienko [16] proved the following statement.

Theorem 1.1 *If $c \in J(R)$ is a summable critical point with non-zero sum, then R is not J-stable, whenever $P_c(R) \neq \overline{\mathbb{C}}$.*

The condition $\sum_{n=0}^{\infty} \frac{1}{(R^n)'(R(c))} = 0$ is a special case which requires additional considerations (see again [1,11,12,16] and compare with the discussion in Section 3.1). However, in [12], it was shown that, with additional conditions on the postcritical set, among all critical points in $J(R)$ with absolutely convergent series $\sum_{n=0}^{\infty} \frac{1}{(R^n)'(R(c))}$, there is at least one critical point with $\sum_{n=0}^{\infty} \frac{1}{(R^n)'(R(c))} \neq 0$.

In this paper, we go along with the approach initiated by Avila, Levin, and Makienko to the cases where the series $\sum_{n \geq 0} \frac{1}{(R^n)'(R(c))}$ is absolutely divergent, but has radius of convergence at least 1 with respect to the Nörlund–Voronoi summability method (see definitions and discussion below). Also, we present a class of measures defining directions in the tangent space of Rat_d at R which are not generated by a quasiconformal deformation of R.

Given a point a, we call the sequence:

$$\sigma(a) = \{\sigma_n(a)\} = \left\{ \frac{1}{(R^n)'(R(a))} \right\}_{n=0}^{\infty}$$

the spectrum of a.

To avoid technicalities, in this article, we always assume that:

- *the rational map R fixes* 0,1 *and* ∞;
- *there are no critical relations*;
- *the individual postcritical sets under discussion are bounded.*

For example, if there are no critical relations and $P_c(R)$ does not contains all fixed points of R, then a suitable rotation of R satisfies our assumption. However, in many cases, our arguments do not need all of these restrictions.

Let c be a critical point for R. Let us consider the following trichotomy:

Trichotomy (1) The sequence $\sigma_n(c)$ converges to ∞.
(2) There is a subsequence $\{n_i\}$, such that $\sigma_{n_i}(c)$ converges to 0.
(3) The number $\liminf_n |\sigma_n(c)|$ is neither 0 nor ∞.

Let us comment on the trichotomy above.

- Conjecturally, the case (1) of the trichotomy should imply that the critical point c belongs to the Fatou set. For example, let R be an unimodal polynomial whose critical point c has spectrum $\sigma(c)$ with bounded multiplicative oscillation ($\sup \left|\frac{\sigma_{n+1}(c)}{\sigma_n(c)}\right| < \infty$). Then, c belongs to the Fatou set by Mañé's theorem (see [17]).
- Theorem 1.1 deals with the case (2) for $\sigma(c) \in \ell_1$ (see [1,11,12,16]).
- In case (3), assume $\sigma(c) \in \ell_\infty$, and then, $\sigma(c)$ has bounded multiplicative oscillation. If, for instance, R is unimodal, again by Mañé's theorem, the map R is parabolic and, hence, is not structurally stable.

Where ℓ_1 and ℓ_∞ are, as usual, the Banach spaces of absolutely summable and bounded sequences, respectively.

The following proposition and its subsequent corollaries are the main motivations to consider the trichotomy above.

Proposition 1.2 *Let R be a stable rational map in the complex one-parameter family $R_\lambda = R + \lambda$ and c be a critical point on the Julia set $J(R)$ with bounded individual postcritical set $P_c(R)$. Consider the partial sums $S_n = \sum_{i=0}^{n}\sigma_i(c)$. Then, for all n, we have:*

$$|S_n| \leq C|\sigma_n(c)|,$$

for some constant C.

The proof is contained in Lemma 5 in [16] (see also Avila [1]). The quadratic polynomial case was also noted by Levin in [11]. However, the proof is elementary and we include it in the next section for the sake of completeness.

Corollary 1.3 *Let R be a rational map and c be a critical point on the Julia set $J(R)$ with bounded $P_c(R)$. Each of the following criteria implies that the map R is unstable in the family $R_\lambda = R + \lambda$, for $\lambda \in \mathbb{C}$.*

- *In case* (2) *of the trichotomy, a subsequence $\sigma_{n_i}(c)$ converges to* 0 *and satisfies* $\limsup |S_{n_i}| > 0$.
- *In case* (3) *of the trichotomy, a subsequence $\sigma_{n_i}(c)$ converges to a non-zero finite limit but* $\limsup |S_{n_i}| = \infty$.

Corollary 1.4 *Let R be a rational map, such that $R' \geq 0$ on $P_c(R)$ for a critical point $c \in J(R)$ with $\sigma(c)$ satisfying either one of the conditions* (2) *or* (3) *in the trichotomy, then R is unstable in the family $R_\lambda = R + \lambda$, for $\lambda \in \mathbb{C}$.*

The following theorem generalizes Corollary 1.4.

Theorem 1.5 *Let R be a rational map. Assume there exists a point $x \in J(R)$, such that $R'(z) \geq 0$ for $z \in \bigcup_{n=1}^{\infty}\{R^n(x)\}$ and satisfying the following conditions*

(1) *The set $\bigcup_{n=1}^{\infty}\{R^n(x)\}$ is bounded and does not intersect the set of critical points of R.*
(2) *$R(x)$ has non-negative lower Lyapunov exponent.*
(3) *$\sigma(x) \notin \ell_1$.*

Then, R is unstable in the family $R_\lambda = R + \lambda$, for $\lambda \in \mathbb{C}$.
Even more, if x is a critical point, then condition (3) is redundant.

This theorem will be proved after Theorem 4.4. Theorem 1.5 leads to the following question.

Does there exists a rational map R such that $\sigma(x) \in \ell_1$ for Lebesgue almost every point of $J(R)$?

It seems that the non-negative condition of R' on $P_c(R)$ is extremal and rare within rational dynamics. However, there are many examples of critical circle maps conjugated to rational maps satisfying this condition. Here, a critical circle map is a map which leaves the circle invariant with a critical point in the circle. For entire or meromorphic maps, there are simple maps holding this condition. For example, consider λe^{z^2+c}, where λ and c are real numbers with $\lambda > 0$, which have non-negative derivative on the orbit of 0.

In the next section, we will give a family of examples of rational maps with real coefficients and non-negative derivative on an individual postcritical set $P_c(R)$. Thus, Corollary 1.4 produces new examples of unstable real rational maps, compare with the class of real rational maps discussed by Shen in [23].

The following corollary is the main motivation for the constructions in the present work. To study the spectrum $\sigma(c)$ of a critical point c, we will consider the Abel averages $A_\lambda = (1-\lambda)\sum_{n=0}^{\infty}\sigma_n(c)\lambda^n$. Recall that a sequence $\{a_n\}$ is *Abel convergent* to L, which is allowed to be infinity, whenever $A_\lambda = (1-\lambda)\sum_{n=0}^{\infty} a_n\lambda^n$ is finite for $\lambda < 1$ and:

$$\lim_{\lambda \to 1} A_\lambda = L.$$

Corollary 1.6 *Let R be a rational map and c be a critical point, such that $\sigma_n(c) = o(n)$. Suppose further that the individual postcritical set $P_c(R)$ is bounded. If $\sigma(c)$ is not Abel convergent to* 0*, then R is an unstable map in the one-parameter family $R_\lambda = R + \lambda$.*

Let us note that if a sequence $\{a_n\}$ converges, then the sequence of Abel averages converges to the same limit. The reciprocal fails to be true. Many unbounded sequences are Abel convergent.

According to results by Bruin and van Strien [4] and, independently, by Rivera-Letelier [22], the Julia set $J(R)$ has Lebesgue measure 0 whenever $J(R)$ contains only critical points c satisfying $\sigma(c) \in \ell_1$. Besides, examples given by Buff and Cheritat [5] and by Avila and Lyubich [2] of quadratic polynomials with Julia set of positive measure show that there are rational maps R with a critical point $c \in J(R)$ with $\sigma(c) \notin \ell_1$. Perhaps, $\sigma(c)$ is not in ℓ_1 for every infinitely renormalizable unimodal map. The referee kindly pointed out that $\sigma(c) \notin \ell_1$ for every Feigenbaum quadratic polynomial.

If $J(R)$ contains several critical points and one of them, say c, has $\sigma(c) \in \ell_1$, then R is unstable whenever $P_c(R)$ satisfies additional restrictions (see Avila [1], Levin [11,12], and Makienko [16]).

To formulate our main results, we need the following construction.

Voronoi measures. First, we discuss some weak conditions on the behavior of $\sigma(c)$, for $c \in J(R)$, which include many instances of the cases (1), (2), and (3) of the trichotomy above. To do so, we consider averaged sequences of the spectrum, that is, the Nörlund–Voronoi averages. The construction of such averages uses the so-called Nörlund summability method which was first published by Voronoi in 1902. An English version of Voronoi's work can be found at [24].

Fix a sequence of non-negative real numbers $q_n \geq 0$, such that $q_0 > 0$ and $\lim_n \frac{q_n}{Q_n} = 0$, where $Q_n = q_0 + \cdots + q_n$ are the partial sums. For a sequence of complex numbers $\{x_n\}$, the values

$$t_n = \frac{q_n x_0 + q_{n-1} x_1 + \cdots + q_0 x_n}{Q_n}$$

are called the *Nörlund averages* with respect to the sequence $\{q_n\}$.

If

$$\limsup \sqrt[n]{|t_n|} \leq 1,$$

then we say that $\{x_n\}$ is *Nörlund regular*, or *N-regular* for short.

Indeed, given a sequence $\{a_n\}$, the number $\liminf \frac{1}{\sqrt[n]{|a_n|}}$ is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$. In what follows, we will call the number $\liminf \frac{1}{\sqrt[n]{|a_n|}}$ the *radius of convergence of the sequence* $\{a_n\}$.

A convenient way to think about the Nörlund method is to regard the sequence $\{q_n\}$ as a linear operator $N : \Theta \to \Theta$, where Θ is the linear space of all complex sequences, and N is the infinite matrix with coordinates $N_{m,n} = \frac{q_{m-n}}{Q_m}$ for $n \leq m$ and $N_{m,n} = 0$ for $n > m$. We call N the *Nörlund matrix associated to* $\{q_n\}$. Hence, a sequence $\{x_n\}$ is N-regular if and only if $N(\{x_n\})$ has radius of convergence at least 1.

It is known (see, for example, [3]) that N defines a continuous linear endomorphism of ℓ_∞. If $\mathcal{C} \subset \Theta$ is the subspace of all convergent sequences, then $N(\mathcal{C}) \subset \mathcal{C}$ and, furthermore, $\lim_n x_n = \lim_n t_n$, for $\{x_n\} \in \mathcal{C}$.

The next lemma is not difficult to prove and appears as Lemma 3.3.10 in [3].

Lemma 1.7 *Let N be the Nörlund matrix associated with $\{q_n\}$ and $\{x_n\}$ be a Nörlund regular sequence. Then, the following properties hold.*

(1) *The series $q(\lambda) = \sum_{n=0}^{\infty} q_n \lambda^n$ and $Q(\lambda) = \sum_{n=0}^{\infty} Q_n \lambda^n$ converge for $|\lambda| < 1$.*

(2) *The series* $x(\lambda) = \sum_{n=0}^{\infty} x_n \lambda^n$ *converges in a neighborhood of* 0.
(3) *The convolution series*

$$[C_N(\{x_n\})](\lambda) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} q_i x_{n-i} \right) \lambda^n = \sum_{n=0}^{\infty} t_n Q_n \lambda^n$$

converges for $|\lambda| < 1$.

According to the lemma above, if $\{x_n\}$ is N-regular, then from the relations

$$[C_N(\{x_n\})](\lambda) = x(\lambda) q(\lambda)$$

and $q_0 \neq 0$, we conclude two facts. First, that the radius of convergence of $\{x_n\}$ is non-zero and $x(\lambda)$ can be continued to a meromorphic function $X(\lambda)$ on the unit disk. Second, as $q(\lambda)$ has non-negative Taylor coefficients, there are no zeros on the interval [0, 1), and so, the poles of $X(\lambda)$ lay outside [0, 1).

Reciprocally, if the sequence $\{x_n\}$ has non-zero radius of convergence and $x(\lambda) = \sum_{n=0}^{\infty} x_n \lambda^n$ can be extended to a meromorphic function $X(\lambda) = \frac{\psi(\lambda)}{\phi(\lambda)}$, such that ϕ and ψ are holomorphic and ϕ has non-negative Taylor coefficients with $\phi(0) > 0$. Then, the Taylor coefficients of ϕ define a Nörlund operator N, such that $\{x_n\}$ is an N-regular sequence. For instance, if $\{x_n\}$ itself has radius of convergence at least 1, then the choice $q_0 = 1$ and $q_n = 0$ for all $n > 0$ gives the identity matrix as a Nörlund matrix or, equivalently, $t_n = x_n$.

Another example of a Nörlund matrix is when $q_n = 1$ for every n. Then, the Nörlund averages of $\{x_n\}$ becomes the Cesàro averages. Indeed, the Nörlund averages are a generalization of both iterated Cesàro and Abel averages for suitable sequences $\{q_n\}$. Moreover, Cesàro and Abel convergent sequences have non-zero radius of convergence. In our situation, the spectrum $\sigma(c)$ has non-zero radius of convergence if and only if the lower Lyapunov exponent of $R(c)$ is bounded below. In fact, we believe that the following conjecture holds true.

Conjecture *For every rational map* R*, such that* $J(R)$ *has positive Lebesgue measure, there exists a critical value* $v \in J(R)$ *with finite lower Lyapunov exponent.*

In [21], Przytycki proved that the Lyapunov exponents are non-negative for almost every point of $J(R)$ with respect to any finite invariant measure supported on the Julia set. In the case of unimodal polynomials, Levin, Przytycki, and Shen [13] showed that the radius of convergence of the spectrum $\sigma(c)$ is always at least 1 whenever c belongs to the Julia set. Moreover, they proved the stronger statement that the spectrum $\sigma(z)$ has radius of convergence at least 1 not only for $z = c$ but for Lebesgue almost every point z of the Julia set $J(R)$.

On the other hand, let us note that geometrically divergent sequences are not Nörlund regular. In particular, for a structurally stable R, if $\sigma(c)$ is Nörlund regular, then $c \in J(R)$.

Given the data:

(1) A sequence of points $\{z_n\} \subset \mathbb{C}$.

(2) A Nörlund matrix associated with a sequence $\{q_n\}$.
(3) A sequence $\{x_n\}$ such that $N(\{|x_n|\}) = \{t_n\}$ has radius of convergence at least 1.

We associate the measure:

$$\nu_\lambda = (1-\lambda)\sum_{n=0}^{\infty} T_n \lambda^n,$$

where $T_n = q_n x_0 \delta_{z_0} + q_{n-1} x_1 \delta_{z_1} + \cdots + q_0 x_n \delta_{z_n}$. Then, by Part (3) of Lemma 1.7, the measures ν_λ form an analytic family of finite measures over the open unit disk.

Definition 1.8 Given a regular sequence $\{x_n\}$, we call a finite complex-valued measure ν a *Voronoi measure with respect to* N *and the sequence* $\{z_n\}$ if there exists a complex sequence $\{\lambda_n\}$ with $\lambda_n \to 1$ (and $|\lambda_n| < 1$), such that the sequence $\frac{\nu_{\lambda_n}}{\|\nu_{\lambda_n}\|}$ converges to ν in the $*$-weak topology; here, $\|\nu_{\lambda_n}\|$ denotes the total variation of ν_{λ_n}. In other words, the complex projective classes $[\nu_{\lambda_n}]$ converge to the complex projective class $[\nu]$, whenever $\nu \neq 0$. If $\{x_n\} = \sigma(c)$ and $z_n = R^n(R(c))$ for a suitable critical point $c \in J(R)$, then we call ν *a Voronoi measure associated with* c.

Let us mention that, in [11], complex projective classes of related measures were discussed.

Note that the support of a Voronoi measure associated with c belongs to the individual postcritical set $P_c(R)$.

Voronoi measures always exist for every regular sequence $\{x_n\}$ and every sequence of points $\{z_n\}$. However, a Voronoi measure may not be uniquely determined by fixing either one of the sequences $\{x_n\}$ or $\{z_n\}$.

Fix $\{x_n\}$ and $\{z_n\}$, according to the Consistence Theorem (see Theorem 17 in page 65 of the Hardy's book [9]), if, for two given Nörlund matrices N and N', the Nörlund averages $\{t_n\}$ and $\{t'_n\}$ of $\{x_n\}$ converge then $\lim t_n = \lim t'_n$. Assume that N and N' send the sequence $\{x_n\phi(z_n)\}$ to convergent sequences, for every $\phi \in C(\overline{\bigcup\{z_n\}})$ the space of continuous functions on $\overline{\bigcup_{n=0}^{\infty}\{z_n\}}$. Then, by the Consistence Theorem, the set of Voronoi measures for N coincides with the set of Voronoi measures for N'.

Since there are Abel convergent sequences that are not Cesàro convergent, then it is possible that different Nörlund matrices have different sets of Voronoi measures.

Now, fix a Nörlund matrix N, a regular N-sequence $\{x_n\}$ and a sequence of points $\{z_n\}$, then the correspondence $\phi \mapsto \{\phi(z_n)\}$, for $\phi \in C(\overline{\bigcup_{n=0}^{\infty}\{z_n\}})$, is a continuous linear operator $T : C(\overline{\bigcup_{n=0}^{\infty}\{z_n\}}) \to \ell_\infty$. Hence, the family of measures $\omega_\lambda = \frac{\nu_\lambda}{\|\nu_\lambda\|}$ induces a family of functionals $L_\lambda(T(\phi)) = \int \phi \, d\omega_\lambda$ which extends to an analytic uniformly bounded family of functionals on ℓ_∞. Using arguments from functional analysis, we have the following observation.

Lemma 1.9 *Let* $\{x_n\}$ *be an* N*-regular sequence for the identity matrix* N. *Let* L_λ *be the functionals on* ℓ_∞ *constructed above; assume that the* $*$*-weak accumulation set of* L_λ *is a single-point set, and then* $\{x_n\} \in \ell_1$, *for* $\lambda \to 1$ *and* $|\lambda| < 1$,

For convenience of the reader, we prove this lemma in the following section.

From Lemma 1.9, even in the simplest case, the accumulation set of $\{\omega_\lambda\}$, regarding each ω_λ as a functional in ℓ_∞, is not unique whenever the sequence $\{x_n\} \notin \ell_1$.

For instance, it is possible that there exists a bounded sequence $\{x_n\}$, such that the function given by $\phi(\lambda) = \frac{\sum_{n=0}^{\infty} x_n \lambda^n}{\sum_{n=0}^{\infty} |x_n \lambda^n|}$ is not continuous at 1 from the interior of the unit disk. Then, the set of Voronoi measures for $\{x_n\}$ has more than one point regardless of the choice of the sequence $\{z_n\}$.

Nevertheless, it is possible to show that the space of all Voronoi measures, which are of Mergelyan type (see definition below), associated with a critical point $c \in J(R)$ with bounded individual postcritical set, is finitely dimensional.

We call a finite complex-valued measure μ a *Mergelyan type measure*, or an *M-measure* for short, when its Cauchy transform $f_\mu(z) = \int_{\mathbb{C}} \frac{d\mu(t)}{t-z}$ is not identically 0 outside the support of μ.

For $K \subset \mathbb{C}$ compact, let $C(K)$ be the space of continuous functions on K and $Rat(K)$ be the space of rational functions restricted to K with poles outside of K. If $Rat(K)$ is dense in $C(K)$ with respect to uniform convergence, then any complex-valued finite measure supported on K is an M-measure. When either $m(K) = 0$ or $\inf\{diam(W) : W \text{ component of } \mathbb{C}\backslash K\} > 0$, by classical results, we get $\overline{Rat(K)} = C(K)$. For a more deep treatment of the theory, see, for example, the book by Gamelin [7].

As a consequence, if R is a map with a completely invariant Fatou component, then every finite complex-valued non-zero measure supported on $J(R)$ is an M-measure. In particular, this is the case when R is a polynomial.

Now, we are ready to formulate our first main theorem.

Theorem 1.10 *Suppose that for a critical point $c \in J(R)$, with bounded $P_c(R)$, the sequence $\{|\sigma_n(c)|\}$ is N-regular for a Nörlund matrix N. If a Voronoi measure ν associated with c is an M-measure, then R is unstable.*

The following theorem deals with the situation when the sequence of measures $\{\nu_{\lambda_i}\}$ converges to zero in the $*$-weak topology for a suitable sequence $\lambda_i \to 1$. However, as in the case discussed by Avila, Levin, and Makienko, we need additional restrictions on the individual postcritical set. We say that a critical point c and the individual postcritical set $P_c(R)$ are *separated by the Fatou set* if there exists a Jordan curve in $F(R)$ separating c and $P_c(R)$.

Theorem 1.11 *Under the conditions of Theorem 1.10, assume that λ_i is a sequence of real numbers with $\lambda_i < 1$ and $\lambda_i \to 1$ for which ν_{λ_i} is $*$-weakly convergent to 0. Then, R is unstable whenever c and $P_c(R)$ are separated by the Fatou set.*

The separation condition is stronger than non-recurrence of critical points and is void for connected Julia sets. However, for disconnected Julia sets, the two conditions are closely related. For instance, if $J(R)$ is a Cantor set, then every non-recurrent critical point in $J(R)$ satisfies the separation condition.

Also, if the Fatou set $F(R)$ contains a completely invariant component and $J(R)$ is disconnected, then generically $J(R)$ contains uncountably many wandering single-point components. If a non-recurrent critical point c is one of those wandering single-point components, then c also satisfies the separation condition.

On the other hand, if a critical point c belongs to a preperiodic (non-periodic) non single-point component, then c satisfies the separation condition. This happens, for example, when c belongs to the boundary of a preperiodic but not periodic Fatou component.

Finally, let us note that, in general, if there is no completely invariant component but there is an infinitely connected Fatou component, then the set of buried points contains uncountably many single-point components. Then, again, if a non-recurrent critical point c is a buried single-point component of $J(R)$, then also the separation condition holds.

Nevertheless, the separation condition applies to a single critical point regardless of the behavior of other critical points.

Let us note that Theorem 1.11 is complementary to Theorem 1.10 in the sense that if $\{x_n\} \in \ell_\infty$, then $\|\nu_\lambda\| = O(\|x_n\|_\infty)$ for real λ and the assumption that all Voronoi measures are null implies that $\nu_\lambda \to 0$ for $\lambda \to 1$ in the $*$-weak topology. However, it is possible that ν_λ converges to 0 even when there are non-zero Voronoi measures; for instance, consider a sequence $\{x_n\} \in \ell_1$.

Among other facts, in [16], it was established that if $c \in J(R)$ is non-recurrent and the series $\sum_{n=0}^{\infty} \sigma_n(c)$ is absolutely convergent, then R is unstable. However, it is yet unclear whether a structurally stable map may have a critical point $c \in J(R)$ with $\sigma(c) \in \ell_\infty$.

As a consequence of the previous theorem, we answer this question positively by replacing the non-recurrent condition by a separation condition on the individual postcritical set P_c. Recall that a sequence $\{x_n\} \in \ell_\infty$ is *Abel summable* to L (maybe ∞) whenever the limit

$$\lim_{\lambda \to 1} \sum_{n=0}^{\infty} \lambda^n x_n$$

for $\lambda < 1$ exists and is equal to L.

Theorem 1.12 *Let c be a critical point in $J(R)$ with bounded $P_c(R)$ and spectrum $\sigma(c) \in \ell_\infty$. Then:*

(1) *The map R is unstable whenever $P_c(R)$ has zero Lebesgue measure and $\sigma(c)$ is not Abel summable to* 0.

Even more, if c and P_c are separated by the Fatou set, then each of the following conditions implies that R is unstable.

2. *The individual postcritical set $P_c(R)$ has measure* 0.
3. *The spectrum $\sigma(c)$ is convergent.*

Moreover, the condition that the Lebesgue measure of $P_c(R)$ is zero can be dropped in the case of maps with a completely invariant Fatou domain.

2 Preliminary Results

In this section, we prove Proposition 1.2 and Lemma 1.9 together with its corollaries.

Proof of Proposition 1.2 Given a stable map R in the family $R_\lambda = R + \lambda$ for $|\lambda| < 1$. According to Mañé, Sad, and Sullivan [18], there exists a holomorphic family f_λ of quasiconformal automorphisms of $\mathbb{C}$, such that for ϵ small enough and λ with $|\lambda| < \epsilon$, we have:

$$R_\lambda = f_\lambda \circ R \circ f_\lambda^{-1}.$$

Let $F(z) = \frac{\partial f_\lambda}{\partial \lambda}|_{\lambda=0}(z)$ be the variation of f_λ. Then, F is a continuous function on $\mathbb{C}$. By a straightforward computation of the variation of R_λ, we obtain:

$$F(R(z)) - R'(z)F(z) = 1.$$

Hence, $F(R(c)) = 1$, and by induction:

$$F(R^n(R(c))) = (R^n)'(R(c))S_n.$$

We end up with $|S_n| \leq \sup_{z \in P_c(R)} |F(z)\sigma_n(c)|$. Since F is bounded on compact subsets of the plane, we are done. □

Corollary 1.3 is an immediate consequence of Proposition 1.2. Now, we prove Corollary 1.4.

Proof of Corollary 1.4 By contradiction, assume that R is structurally stable in $\{R_\lambda\}$. By assumption, the partial sums $S_n = \sum_{i=0}^{n} \sigma_i(c)$ satisfy $S_n < S_{n+1}$ with $\limsup_n S_n > 0$. Suppose that $\sigma(c)$ satisfies the condition (2) of the trichotomy. Immediately, we have $\sigma(c) \in \ell_1$ which contradicts Corollary 1.3. Now, suppose that $\sigma(c)$ satisfies the condition (3) of the trichotomy. Since $\liminf_n \sigma_n(c) \neq 0, \infty$, then $\lim_{n\to\infty} S_n = \infty$ which again contradicts Corollary 1.3. □

Now, we briefly describe a large set of rational maps with real coefficients and non-negative derivative on the individual postcritical set $P_c(R)$ for a suitable critical point c.

Let f be a real rational function which attains a finite minimum m_0 in the extended real line, and then, $f(z) - m_0$ is non-negative with a zero in the extended real line. If $R(z)$ is a real primitive of $f(z) - m_0$, then $R(z)$ has a real critical point c and $R'(z) = f(z) - m_0$ is non-negative on $P_c(R)$. Note that, for $\gamma, \tau \in PSL(2, \mathbb{R})$, the map $\gamma \circ R \circ \tau$ also has the desired property. To construct a real rational function f as above, fix the following data:

(1) A real polynomial of even degree $P(z)$, with positive leading coefficient. Here, $P(z)$ may be a positive constant.
(2) A finite set $\{a_i\}$ of real points.
(3) A finite set $\{b_i\}$ of positive real numbers.
(4) A finite set $\{n_i\}$ of even integers numbers.
(5) A finite set $\{k_i\}$ of integers greater than 1.
(6) A finite set $\{d_i\}$ of real numbers.
(7) A finite set $\{w_i\}$ of non-real complex numbers.

Then, the function:

$$f(z)=P(z)+\sum\frac{b_i}{(z-a_i)^{n_i}}+\sum d_i\left(\frac{1}{(z-w_i)^{k_i}}+\frac{1}{(z-\overline{w}_i)^{k_i}}\right)$$

is the desired real rational function. The set of rational maps for which $R'\geq 0$ on $P_c(R)$ for a suitable critical point c includes real maps with non-real critical points.

Proof of Corollary 1.6 By contradiction, if R is a stable rational map in the family $\{R_\lambda\}$, then by assumption and Proposition 1.2, there exists a constant C, such that:

$$\frac{|S_n|}{n}\leq\frac{C}{n}|\sigma_n(c)|.$$

As the right-hand side of the latter inequality converges to 0, the Cesàro averages of the spectrum converge to 0. This implies that the Abel averages of the spectrum converge to 0, which is a contradiction. □

We finish the section with the proof of Lemma 1.9.

Proof of Lemma 1.9 Fix an arbitrary sequence of real numbers $\lambda_i\to 1$ and $\lambda_i<1$. For each i, define the sequence $\alpha^i=\{\alpha^i_n\}$ given by $\alpha^i_n=\frac{\lambda_i^n x_n}{\sum_{n=0}^{\infty}|\lambda_i^n x_n|}$, so $\alpha^i\in\ell_1$. If $b\in\ell_\infty$ is the sequence $\{b_n\}$, where:

$$b_n=\begin{cases}\frac{|x_n|}{x_n}, & x_n\neq 0\\ 0, & x_n=0\end{cases},$$

then $L_{\lambda_i}(b)=\sum_{n=0}^{\infty}\alpha^i_n b_n=1$. Therefore, we identify each functional L_{λ_i} with the sequence α^i. Thus, every accumulation point of L_{λ_i} in the $*$-weak topology as functionals on ℓ_∞ is a continuous non-zero functional.

By assumption, the sequence L_{λ_i} converges to a non-zero functional in the $*$-weak topology on ℓ_∞. That is the same that the sequence $\{\alpha^i\}$ converges in the weak topology on ℓ_1. Since ℓ_1 is weakly complete and the weak topology coincides with the strong topology, the sequence $\{\alpha^i\}$ converges in norm to a non-zero element $\beta=\{\beta_n\}\in\ell_1$. In particular, we have:

$$\lim_i\alpha^i_n=\beta_n.$$

Let k be such that $\beta_k\neq 0$; then:

$$\lim_{i\to\infty}\sum_{n=0}^{\infty}\lambda_i^n|x_n|=\frac{x_k}{\beta_k}.$$

Since $\{\lambda_i\}$ is arbitrary, we conclude that the limit $\lim_{\lambda\to 1}\sum_{n=0}^{\infty}\lambda^n|x_n|$ exists and is finite; therefore, $\sum_{n=0}^{\infty}|x_n|<\infty$. □

3 Some Background in Dynamics and Poincaré Series

Most of the material in this section can be found in [16] (see also [15]).

A rational map R defines a complex push-forward map on $L_1(\mathbb{C})$ with respect to the Lebesgue measure. This contracting endomorphism is called the *complex Ruelle–Perron–Frobenius*, or the *Ruelle operator* for short. The Ruelle operator is explicitly given by the formula:

$$R_*(\phi)(z) = \sum_{y \in R^{-1}(z)} \frac{\phi(y)}{R'(y)^2} = \sum_i \phi(\zeta_i(z))(\zeta_i'(z))^2,$$

where ζ_i is any local complete system of branches of the inverse of R. The *Beltrami operator* $Bel : L_\infty(\mathbb{C}) \to L_\infty(\mathbb{C})$ given by:

$$Bel(\mu) = \mu(R)\frac{\overline{R'}}{R'}$$

is dual to the Ruelle operator acting on $L_1(\mathbb{C})$. The fixed point space $Fix(B)$ of the Beltrami operator is called the *space of invariant Beltrami differentials*.

Every element $\mu \in L_\infty(\mathbb{C})$ defines a continuous function on $\mathbb{C}$ via:

$$F_\mu(a) = a(a-1)\int_{\mathbb{C}} \frac{\mu(z)}{z(z-1)(z-a)}|\mathrm{d}z|^2,$$

which is called the *normalized potential for* μ. By convenience, we write $\gamma_a(z) = \frac{a(a-1)}{z(z-1)(z-a)}$, so we get $F_\mu(a) = \int \gamma_a(z)\mu(z)|\mathrm{d}z|^2$.

The following statement appears as Lemma 5 and Remark 6 in [16].

Lemma 3.1 *Let R be a structurally stable rational map. Then, for every critical value v_i, there exists an invariant Beltrami differential μ_i, such that $F_{\mu_i}(v_j) = \delta_{ij}$, the Kroenecker delta function.*

Now, we present the formal relation of the Poincaré–Ruelle series.

Definition 3.2 The *Poincaré–Ruelle* series are:

- $$B_a(z) = \sum_{n=0}^{\infty}(R_*)^n(\gamma_a(z)),$$

- $$A_a(z) = \sum_{n=0}^{\infty} \frac{1}{(R^n)'(a)}\gamma_{R^n(a)}(z).$$

The lemma below gives a formal relation between both Poincaré–Ruelle series. The proof can be found in Proposition 7 of [16] and further details are contained in [15].

Lemma 3.3 *Let R be a rational map with simple critical points c_i fixing $0, 1,$ and ∞. Set $v_i = R(c_i)$ and let a be a value, such that $\bigcup_n \{R^n(a)\}$ does not contains critical points. Then, we have the following formal relation between the above series:*

$$B_a(z) = A_a(z) + \sum_i \frac{1}{R''(c_i)} A_a(c_i) \otimes B_{v_i}(z); \tag{*}$$

where $\otimes$ is the formal Cauchy product.

Hence, we have (see again Proposition 7 in [16]):

$$(R_*)^n(\gamma_a(z)) = \frac{1}{(R^n)'(a)}\gamma_{R^n(a)}(z) + \sum_i \frac{1}{R''(c_i)}\left[\frac{\gamma_{R^{n-1}(a)}(c_i)}{(R^{n-1})'(a)}\gamma_{v_i}(z) + \frac{\gamma_{R^{n-2}(a)}(c_i)}{(R^{n-2})'(a)}R_*(\gamma_{v_i}(z)) + \cdots + \gamma_a(c_i)(R_*)^{n-1}(\gamma_{v_i}(z))\right].$$

To the formal series $A_a(z)$ and $B_a(z)$ involved in Eq. (*), we associate a formal Abel series parameterized by the unit disk as follows. For $|\lambda| < 1$, write:

$$A_a(z, \lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{(R^n)'(a)}\gamma_{R^n(a)}(z)$$

and

$$B_a(z, \lambda) = \sum_{n=0}^{\infty} \lambda^n (R_*)^n(\gamma_a(z)).$$

We have the following lemma.

Lemma 3.4 *Let R be a structurally stable rational map, $c \in J(R)$ be a critical point with bounded $P_c(R)$, and $v = R(c)$. Assume that the spectrum $\sigma(c)$ has radius of convergence $r > 0$, and then for any complex number λ with $|\lambda| < r$, we have the following.*

(1) *The series $A_v(z, \lambda)$ is absolutely convergent almost everywhere with respect to z and is an integrable function holomorphic off $P_c(R) \cup \{0, 1, \infty\}$.*
(2) *Let $\tilde{c}$ be a critical point. The numerical series $A_v(\tilde{c}, \lambda)$ is absolutely convergent.*
(3) *The series $B_a(z, \lambda)$ is absolutely convergent almost everywhere and is an integrable function for every $|\lambda| < 1$ and every $a \in \mathbb{C}$.*

Furthermore, each of the series above defines a holomorphic function with respect to λ for $|\lambda| < r$.

Proof By assumption the series $\sum \lambda^n \sigma_n(c)$ is absolutely convergent for $|\lambda| < r$ and defines a holomorphic function with respect to λ in the disk $|\lambda| < r$.

Part (1). As there exists a constant C, such that $\int |\gamma_a(z)||\mathrm{d}z|^2 \le C|a \ln |a||$ (see, for example, the books by Gardiner–Lakič [8] and Krushkal [10]), we get:

$$\int_{\mathbb{C}} |A_v(z,\lambda)||\mathrm{d}z|^2 \le \sum |\lambda^n \sigma_n(c)| \int_{\mathbb{C}} \left|\gamma_{R^n(v)}(z)\right| |\mathrm{d}z|^2 \\ \le C \sum |\lambda^n \sigma_n(c) R^n(v) \ln |R^n(v)||.$$

Since $P_c(R)$ is bounded for every $|\lambda| < r$, the last expression is absolutely convergent. Hence, $A_v(z,\lambda)$ converges in the L_1 norm and is a holomorphic integrable function outside $P_c(R) \cup \{0, 1, \infty\}$. By the mean value theorem, the partial sums of the series $A_v(z,\lambda)$ are uniformly bounded on compact sets outside $P_c(R) \cup \{0, 1, \infty\}$. Hence, the series $A_v(z,\lambda)$ converges uniformly on compact sets off $P_c(R) \cup \{0, 1, \infty\}$.

Part (2). If $\tilde{c} \notin P_c(R)$, then by Part (1), we are done. Otherwise, we have $\tilde{c} \in P_c(R)$. Given $\epsilon > 0$, let U_ϵ be the ϵ neighborhood of $\tilde{c}$. Let n_i be such that $R^{n_i}(v) \in U_\epsilon$. As in the arguments in Part(1), it is enough to estimate the expression:

$$\left| \sum_i \lambda^{n_i} \sigma_{n_i}(c) \gamma_{R^{n_i}(v)}(\tilde{c}) \right|.$$

Note that for $z \in U_\epsilon$, we have $R'(z) = (z - \tilde{c})R''(\tilde{c}) + O(|z - \tilde{c}|^2)$. Thus, we get:

$$\left| \frac{1}{R^{n_i}(v) - \tilde{c}} \right| \le \left| \frac{R''(\tilde{c}) + O(|R^{n_i}(v) - \tilde{c}|)}{R'(R^{n_i}(v))} \right| \le M \left| \frac{1}{R'(R^{n_i}(v))} \right|$$

and

$$\left|\gamma_{R^{n_i}(v)}(\tilde{c})\right| \le M \left| \frac{1}{R'(R^{n_i}(v))} \cdot \frac{R^{n_i}(v)(R^{n_i}(v) - 1)}{\tilde{c}(\tilde{c} - 1)} \right| \le M_1 \left| \frac{1}{R'(R^{n_i}(v))} \right|,$$

where M and M_1 are suitable constants depending on ϵ and $\tilde{c}$. As a result of the previous computation, we obtain:

$$\left| \sum_i \lambda^{n_i} \sigma_{n_i}(c) \gamma_{R^{n_i}(v)}(\tilde{c}) \right| \le M_1 \sum_i \left| \frac{\lambda^{n_i} \sigma_{n_i}(c)}{R'(R^{n_i}(v))} \right| \\ \le \frac{1}{|\lambda|} M_1 \sum_i |\lambda^{n_i+1} \sigma_{n_i+1}(c)| < \infty,$$

for $0 < |\lambda| < r$.

Part (3). Since $\|R_*(f)\|_{L_1} \le \|f\|_{L_1}$ holds for any $f \in L_1(\mathbb{C})$, then for every $|\lambda| < 1$, the series $B_a(z,\lambda)$ is an integrable function and so converges absolutely almost everywhere for $a \in \mathbb{C}$. □

We have the following immediate consequence.

Corollary 3.5 *Let c be a critical point of a rational map R with bounded $P_c(R)$. If the spectrum $\sigma(c)$ of a critical point c has radius of convergence $r > 0$, then we can rewrite Eq. (*) as:*

$$B_{R(c)}(z,\lambda) = A_{R(c)}(z,\lambda) + \lambda \sum_i \frac{1}{R''(c_i)} A_{R(c)}(c_i,\lambda) \cdot B_{v_i}(z,\lambda), \qquad (**)$$

for every $|\lambda| < r$ and almost every $z \in \mathbb{C}$.

Proof This comes from the previous lemma and the Cauchy product theorem. □

4 Proofs of the Main Theorems

Now, we are ready to proof Theorems 1.5, 1.10, 1.11, and 1.12.

4.1 The Abel Case

Our plan is first to prove these theorems in the case where a spectrum has radius of convergence at least 1 and taking the identity matrix as Nörlund matrix, this is what we call the *Abel case*. We will consider general matrices in the next subsection. Accordingly, a Voronoi measure associated with the identity matrix, a regular sequence $\{x_n\}$, and a sequence of points $\{z_n\}$ will be called an *Abel measure*. Again, as in the Voronoi case, we say that an Abel measure is associated with a point a whenever $\sigma(a) = \{x_n\}$ and $z_n = R^n(a)$. In this situation, the measure ν_λ has the following simple form:

$$\nu_\lambda = (1-\lambda) \sum_{n=0}^{\infty} \sigma_n(a) \lambda^n \delta_{R^n(a)}.$$

Lemma 4.1 *Let c be a critical point with bounded $P_c(R)$, such that the $\sigma(c)$ has radius of convergence at least 1. Let ν_0 be a finite complex valued measure. Let $\{\lambda_i\}$ and $\{r_i\}$ be sequences of complex numbers, with $|\lambda_i| < 1$, so that $r_i \nu_{\lambda_i}$ converges $*$-weakly to ν_0. If R is structurally stable, then for any critical point $\tilde{c} \neq c$, the sequence $\{r_i A_{R(c)}(\tilde{c}, \lambda_i)\}$ is bounded and convergent. Finally, for the critical point c, the sequence $\{r_i(1 - \frac{1}{R''(c)} A_{R(c)}(c, \lambda_i))\}$ is bounded and convergent.*

Proof Let $\{c_j\}_{j=1}^{2deg(R)-2}$ be the set of critical points with $c_1 = c$ and $v_j = R(c_j)$ their respective critical values. For a critical value w, take an invariant Beltrami differential μ_w as in Lemma 3.1.

As μ_w is invariant, we get:

$$\int_{\mathbb{C}} \mu_w(z) B_{v_j}(z,\lambda) |dz|^2 = \frac{1}{1-\lambda} F_{\mu_w}(v_j) = \begin{cases} \frac{1}{1-\lambda}, & w = v_j \\ 0 & \text{otherwise.} \end{cases}$$

Integrating Eq. (**) in Corollary 3.5 with respect to μ_w yields:

$$\int_{\mathbb{C}} \mu_w(z) B_{v_1}(z,\lambda)|\mathrm{d}z|^2 = \int_{\mathbb{C}} \mu_w(z) A_{v_1}(z,\lambda)|\mathrm{d}z|^2 + \lambda \sum_j \frac{1}{R''(c_j)} A_{v_1}(c_j,\lambda) \cdot \int_{\mathbb{C}} \mu_w(z) B_{v_j}(z,\lambda)|\mathrm{d}z|^2.$$

After multiplying on both sides of the latter equation by $r_i(1-\lambda_i)$ and taking $\lambda = \lambda_i$, we obtain:

$$r_i(1-\lambda_i)\int \mu_w(z) A_{v_1}(z,\lambda_i)|\mathrm{d}z|^2 = r_i\left[F_{\mu_w}(v_1) - \frac{\lambda_i}{R''(c)} A_{v_1}(c,\lambda_i)\right]. \quad (1)$$

On the other hand, as we have $(1-\lambda_i)A_{v_1}(z,\lambda_i) = \int \gamma_a(z)\mathrm{d}\nu_{\lambda_i}(a)$, then applying Fubini's theorem, we get:

$$r_i(1-\lambda_i)\int \mu_w(z) A_{v_1}(z,\lambda_i)|\mathrm{d}z|^2 = r_i \int \mathrm{d}\nu_{\lambda_i}(a) \int \gamma_a(z)\mu_w(z)|\mathrm{d}z|^2 = \int F_{\mu_w}(a) r_i \mathrm{d}\nu_{\lambda_i}(a).$$

Since F_{μ_w} is continuous on $\mathbb{C}$ and $P_c(R)$ is bounded, we have:

$$\lim_{i\to\infty} \int F_{\mu_w}(a) r_i \mathrm{d}\nu_{\lambda_i}(a) = \int F_{\mu_w}(a)\mathrm{d}\nu_0(a),$$

and then returning to (1), by the choice of μ_w, we conclude the proof. □

Proposition 4.2 *Let R be structurally stable and ν_0 be as in Lemma 4.1. Then, $\phi(z) = \int \gamma_a(z)\mathrm{d}\nu_0(a)$ is a well-defined integrable function that satisfies $R_*(\phi(z)) = \phi(z)$.*

Proof First, again, since $P_c(R)$ is bounded and using Fubini's theorem, we get:

$$\int |\phi(z)||\mathrm{d}z|^2 \leq \int |\mathrm{d}\nu_0(a)| \int |\gamma_a(z)||\mathrm{d}z|^2 \leq M \int |a||\ln|a|||\mathrm{d}\nu_0(a)| < \infty.$$

Therefore, ϕ is integrable. Applying $r_i(1-\lambda_i)[I - \lambda_i R_*]$ to Eq. (**) in Corollary 3.5 with $\lambda = \lambda_i$, and then using the resolvent equation

$$(Id - \lambda R_*) \circ \left(\sum_{n=0}^{\infty} \lambda^n (R_*)^n\right) = Id,$$

we obtain:

$$r_i(1-\lambda_i)\gamma_{v_1}(z) = r_i(1-\lambda_i)\,[Id-\lambda_i\cdot R_*]\,(A_{v_1}(z,\lambda_i) \\ + r_i(1-\lambda_i)\lambda_i\sum_j \frac{1}{R''(c_j)}A_{v_1}(c_j,\lambda_i)\cdot\gamma_{v_j}(z)).$$

By Lemma 4.1, we have that: $\lim_{i\to\infty}\|r_i(1-\lambda_i)\,[Id-\lambda_i\cdot R_*]\,A_{v_1}(z,\lambda_i)\|_{L_1}=0$.
Rearranging, we get:

$$r_i(1-\lambda_i)[Id-\lambda_i R_*]A_{v_1}(z,\lambda_i) = [Id-R_*][r_i(1-\lambda_i)A_{v_1}(z,\lambda_i)] \\ + R_*[r_i(1-\lambda_i)^2A_{v_1}(z,\lambda_i)].$$

Finally:

$$\|R_*(r_i(1-\lambda_i)^2A_{v_1}(z_1,\lambda_i))\|_{L_1} \le |1-\lambda_i|\|r_i(1-\lambda_i)A_{v_1}(z,\lambda_i)\|_{L_1}$$

However, the latter converges to 0 as i tends to ∞, since $\|r_i(1-\lambda_i)A_{v_1}(z,\lambda_i)\|_{L_1}$ is bounded. Since $\phi(z)=\lim_{i\to\infty}r_i(1-\lambda_i)A_{v_1}(z,\lambda_i)$ holds almost everywhere, we are done. □

Lemma 4.3 *Under conditions of Proposition 4.2, assume that measure ν_0 is an M-measure, and then, $\phi(z)=\int\gamma_z(z)\mathrm{d}\nu_0(a)$ is non-zero identically on $\mathbb{C}\backslash P_c(R)$.*

Proof Since ν_0 is an M-measure, then we can think that ν_0 is a non-zero Abel measure. By assumption, R is structurally stable, and then, we can assume that $\sigma(c)\notin\ell_1$ by Proposition 9 and Proposition 10 (2) in [16]. A direct computation gives $\int\frac{f(R)}{R'}\mathrm{d}\nu_0 = \int f\mathrm{d}\nu_0$ for every continuous function f.

Now, we proceed by contradiction, if $\phi(z)\equiv 0$ on $\mathbb{C}\backslash P_c(R)$, then by assumption:

$$\int\frac{1}{z-a}\mathrm{d}\nu_0(a)=\frac{A}{z}+\frac{B}{z-1},$$

for suitable A and B complex numbers not both equal to 0. In other words, if $\nu_1 = \nu_0-(A\delta_0+B\delta_1)$, where δ_a denotes the delta measure at the point a, the function $\Phi(z)=\int\frac{1}{z-a}\mathrm{d}\nu_1=0$ for $z\in\mathbb{C}\backslash P_c(R)$.

For $z\in\mathbb{C}\backslash P_c(R)$, we have:

$$0=R_*(\Phi(z))=\int R_*\left(\frac{1}{z-a}\right)\mathrm{d}\nu_1(a);$$

by part 1 of Lemma 5 in [16], we have:

$$R_*(\Phi(z)) = \int\left(\frac{1}{R'(a)(z-R'(a))}+\sum_i\frac{1}{R''(c_i)(c_i-a)(z-R(c_i))}\right)\mathrm{d}\nu_1(a) \\ = \int\frac{1}{R'(a)(z-R(a))}\mathrm{d}\nu_1(a)+R_1(z),$$

where R_1 is a rational function with simple poles only in critical values of R. However, since $\int \frac{1}{R'(a)(z-R(a))}\mathrm{d}\nu_0(a) = \int \frac{1}{z-a}\mathrm{d}\nu_0(a)$, we conclude:

$$\int \frac{1}{R'(a)(z-R(a)}\mathrm{d}\nu_1(a) = \int \frac{1}{R'(a)(z-R(a))}\mathrm{d}\nu_0(a) - \frac{A}{R'(0)z} - \frac{B}{R'(1)(z-1)}$$
$$= \frac{A}{z}\left(1-\frac{1}{R'(0)}\right) + \frac{B}{z-1}\left(1-\frac{1}{R'(1)}\right) := R_0(z)$$

is a rational function with $0 = R_*(\Phi(z)) = R_0(z) + R_1(z)$. Then, by the Residue theorem, either $R'(0) = 1$ or $R'(1) = 1$ which contradicts the structural stability of R. □

The following theorem is the Abel version of Theorem 1.10.

Theorem 4.4 *Let $c \in J(R)$ be a critical point, so that $P_c(R)$ is bounded and $\sigma(c)$ has radius of convergence at least* 1. *Then, R is unstable whenever the Abel measure associated with c is an M-measure.*

Proof Let ν_0 be an Abel measure associated with c, and then, there is a sequence of $\lambda_i \to 1$, such that $\frac{\nu_{\lambda_i}}{\|\nu_{\lambda_i}\|}$ converges $*$-weakly to ν_0. Since ν_0 is an M-measure, then $\nu_0 \neq 0$.

Assume that R is structurally stable, and then, by Lemma 4.3, we have that $\phi(z) = \int \gamma_a(z)\mathrm{d}\nu_0(a) \not\equiv 0$ on $\mathbb{C}\backslash P_c(R)$, and $R_*(\phi) = \phi$ by Proposition 4.2. Hence, by Corollary 12 in [16] and Lemma 3.16 of [19] (see also [6]), R is a flexible Lattès map. This is a contradiction with the structural stability of R. □

Now, we are ready to prove Theorem 1.5.

Proof of Theorem 1.5 Since $R(x)$ has non-negative lower Lyapunov exponent, then $\sigma(x)$ has radius of convergence at least 1. Because $R'(z) \geq 0$ on $\bigcup_{n=1}^{\infty} R^n(x)$, then, for each $0 \leq \lambda < 1$, the measure

$$\omega_\lambda = \frac{\sum_{n=0}^{\infty}\lambda^n \sigma_n(x)\delta_{R^n(x)}}{\sum_{n=0}^{\infty}\lambda^n \sigma_n(x)}$$

is a probability measure. Let ω be an accumulation point of $\{\omega_\lambda\}$, and then, ω is a probability measure. Since $\sigma(x) \notin \ell_1$, then a straightforward calculation gives:

$$\int \frac{\phi(R)}{R'}\mathrm{d}\omega = \int \phi \mathrm{d}\omega \tag{2}$$

for every continuous function ϕ on the support of ω.

Assume that R is stable in the family R_λ, and then, by Proposition 1.2, there exist a function F, continuous on the plane, with

$$\frac{F(R)(a)}{R'(a)} - F(a) = \frac{1}{R'(a)},$$

integrating the latter equation with respect to ω leads to a contradiction with Eq. (2) above. Since the right side becomes 0, whereas the left side becomes 1.

If x is a critical point, then $\sigma(x)$ does not belong to ℓ_1 by Corollary 1.3. ◻

Remark Indeed, the non-negative condition on the orbit of x is not necessary whenever ω is non-zero, $\sigma(x) \notin \ell_1$ and $R(x)$ has non-negative lower Lyapunov exponent.

Theorem 1.5 gives examples of rational maps having non-zero Abel measures. Now, we give three criteria for a complex sequence to allow a non-zero Abel measure. The first is elementary and states that if the arguments of the sequences $\lambda_i^n a_n$ are close enough to 0, then this sequence has a non-zero Abel measure with respect to every rational map R and $z \in \bar{\mathbb{C}}$. More precisely, we have the following lemma.

Lemma 4.5 *Let* $\{a_n\}$ *be a complex sequence with radius of convergence at least* 1. *Assume there exist* $\alpha < 1$ *and a complex sequence* $\{\lambda_i\}$ *converging to* 1 *with* $|\lambda_i| < 1$, *such that:*

$$\left|\lambda_i^n a_n - |\lambda_i^n a_n|\right| \leq \alpha |\lambda_i^n a_n|.$$

Then, for every rational map R *and every point* $z \in \mathbb{C}$, *there exists a non-zero Abel measure with respect to* $\{a_n\}$ *and* z.

Proof Fix R and $z \in \mathbb{C}$. Then, every $*$-weak limit of the family of the probability measures

$$w_\lambda = \frac{\sum |\lambda|^n |a_n| \delta_{R^n(R(z))}}{\sum |\lambda^n||a_n|}$$

is a probability measure. Write:

$$u_\lambda = \frac{\sum \lambda^n a_n \delta_{R^n(R(z))}}{\sum |\lambda^n||a_n|},$$

so that u_λ is a family of complex-valued measures absolutely continuous with respect to w_λ. For the sequence $\{\lambda_i\}$, assume by contradiction that u_{λ_i} converges $*$-weakly to 0. Define $X = \overline{\bigcup R^n(z)}$ and let 1_X be the characteristic function on X. Notice that the supports of w_λ and u_λ belong to X. Now, the inequality

$$\begin{aligned} 1 &= \lim_{\lambda_i \to 1} \left| \int 1_X du_{\lambda_i} - \int 1_X dw_{\lambda_i} \right| \\ &= \lim_{\lambda_i \to 1} \frac{|\sum_n \lambda_i^n a_n - \sum_n |\lambda_i^n a_n||}{\sum_n |\lambda_i^n a_n|} \\ &\leq \lim_{\lambda_i \to 1} \frac{\sum_n |\lambda_i^n a_n - |\lambda_i^n a_n||}{\sum_n |\lambda_i^n a_n|} \leq \alpha < 1 \end{aligned}$$

establishes a contradiction. ◻

The second criterion is connected with the L_1 norm of the function $A_z(\lambda)$ on $\mathbb{C}\backslash P_c(R)$. Indeed, we prove a more general statement.

Lemma 4.6 *Let $K \subset \mathbb{C}$ be compact and let ν_i be a bounded sequence of complex-valued measures on K. If we assume*

$$\limsup_i \int_{\mathbb{C}} \left| \int_{\mathbb{C}} \gamma_a(z) \mathrm{d}\nu_i(a) \right| |\mathrm{d}z|^2 > 0,$$

then there exists a $*$-weak accumulation point ν_0 which is not null. The reciprocal is also true.

Proof According to a well-known result on quasiconformal theory (see, for example, Gardiner and Lakič [8] or Krushkal [10]), the operator $T : L_\infty(\bar{\mathbb{C}}) \to C(K)$ given by:

$$T(\mu)(a) = \int_{\mathbb{C}} \gamma_a(z)\mu(z)|\mathrm{d}z|^2,$$

which maps μ to $F_\mu|_K$, is continuous and compact. The same is true for the dual operator $T^* : M(K) \to (L_\infty(\mathbb{C}))^*$ given by:

$$T^*(m)(z) = \int_K \gamma_a(z)\mathrm{d}m(a),$$

which is continuous and compact.

By Fubini's theorem, we have $rank(T^*) \subset L_1(\mathbb{C})$ and each $T^*(m)(z)$ is holomorphic off of $K \cup \{0, 1, \infty\}$. Hence, $T^* : M(K) \to L_1(\mathbb{C})$ is a compact operator. Now, by assumption, passing to a subsequence, we have that:

$$f_i(z) = \int_K \gamma_a(z)\mathrm{d}\nu_i(z) = T^*(\nu_i)(z)$$

converges in norm to a non-zero f in $L_1(\mathbb{C})$.

Note that we have $\partial_{\bar{z}} f_i = \nu_i$ in the sense of distributions. If ν_i converges $*$-weakly to 0, then, by continuity, the integrability of f, and an application of Weyl's lemma, we get $f = 0$, which is a contradiction, so ν_i cannot converge $*$-weakly to 0.

Reciprocally, if a measure $\nu_0 \neq 0$ is a $*$-weak limit of ν_i, then $T^*(\nu_0) \not\equiv 0$ almost everywhere on $\mathbb{C}$ and $T^*(\nu_i)$ converges to $T^*(\nu_0)$ in norm. □

The last criterion is formulated for bounded sequences and follows from Proposition 1.2. In this situation, we consider the Abel sum $S(\lambda) = \sum a_n \lambda^n$. If the Abel averages $A(\lambda) = (1 - \lambda)S(\lambda)$ are not continuous at 1 from the left, then there exists a non-zero Abel measure. However, if $\limsup_{\lambda<1} |S(\lambda)|$ is bounded, then $\lim_{\lambda\to 1^-} A(\lambda) = 0$. In this situation, we have the following existence lemma.

Lemma 4.7 *If R is structurally stable and $z_0 \in \mathbb{C}$ has infinite bounded forward orbit, such that $\sigma(z_0) \in \ell_\infty$. Then, there is a non-zero Abel measure whenever $\sigma(z_0)$ is* *not Abel summable to a finite limit. Moreover, if z_0 is a critical value, then the Abel measure is non-zero whenever $\sigma(z_0)$ is not Abel summable to zero.*

Proof Let F be the continuous function on the plane constructed in the proof of Proposition 1.2, and then for every n we have:

$$\frac{F(R^n(z_0))}{(R^n)'(z_0)} = F(z_0) - 1 + (1 + \sigma_1(z_0) + \cdots + \sigma_n(z_0)) = F(z_0) - 1 + S_n(z_0).$$

Multiplying by λ^n and adding with respect to n, we get:

$$\sum_{n=0}^{\infty} \lambda^n F(R^n(z_0))\sigma_n(z_0) = \frac{F(z) - 1}{1 - \lambda} + \sum_{n=0}^{\infty} \lambda^n S_n(z_0),$$

and so

$$(1 - \lambda) \sum_{n=0}^{\infty} \lambda^n F(R^n(z_0))\sigma_n(z_0) = F(z_0) - 1 + \sum_{n=0}^{\infty} \lambda^n \sigma_n(z_0).$$

Since $\sigma(c) \in \ell_\infty$ and the only Abel measure is 0, then the left-hand side of the previous equation converges to 0 as $\lambda \to 1^-$, which contradicts the fact that the right-hand side is not continuous at 1 from the left. If z_0 is a critical value, then by Proposition 1.2, we have $F(z_0) = 1$, and now, we apply the formulae above to finish the proof. □

To prove the Abel version of Theorem 1.11, we need some additional preparation. Recall that a positive Lebesgue measurable subset W of $\bar{\mathbb{C}}$ is called *wandering* if $R^{-n}(W)$ forms a family of pairwise almost disjoint sets with respect to the Lebesgue measure. The union $D(R)$ of all wandering sets is called the *dissipative set*, its complement $\bar{\mathbb{C}} \backslash D(R)$ is the *conservative set*. A Fatou component U belongs to $D(R)$ precisely when U is not an invariant rotational component.

We start the proof of Theorem 1.11 with the following lemma which is reminiscent of Lemma 3.1 and the arguments of the proof of Lemma 4.1.

Lemma 4.8 *Let R be a structurally stable map. If the measures ν_{λ_i} converge $*$-weakly to* 0 *for a suitable sequence of reals $\lambda_i \to 1$ and a critical point $c \in J(R)$, then:*

$$\lim_{\lambda_i \to 1} A_{R(c)}(\tilde{c}, \lambda_i) = \begin{cases} 0 & \text{if } \tilde{c} \neq c \\ R''(c) & \text{if } \tilde{c} = c. \end{cases}$$

Proof By assumption, ν_{λ_i} converges $*$-weakly to 0, and then, $\sup \|\nu_{\lambda_i}\| < \infty$, where $\|\nu\|$ is the total variation of ν. Since $P_c(R)$ is bounded, by Fubini's theorem, we have:

$$\|(1 - \lambda) A_{R(c)}(z, \lambda_i)\|_{L_1} \leq M \sup_{a \in P_c(R)} |a| |\ln |a|| \sup_i \|\nu_{\lambda_i}\| \leq \infty$$

for a suitable constant M depending on $P_c(R)$. For every critical point c_j, we define:

$$D(c_j,\lambda_i)=\begin{cases}\left|\lambda_i\frac{A_{R(c)}(c_j,\lambda_i)}{R''(c_j)}\right| & \text{if } c_j\neq c\\ \left|\lambda_i\frac{A_{R(c)}(c_j,\lambda_i)}{R''(c_j)}-1\right| & \text{if } c_j=c.\end{cases}$$

By structural stability, Eq. (**) of Corollary 3.5, and Lemma 3.1, there are constants M_1 and M_2, such that for every i:

$$M_1\max_j\big(D(c_j,\lambda_i)\big)\leq \|(1-\lambda)A_{R(c)}(z,\lambda_i)\|_{L_1}\leq M_2\left(\sum_j D(c_j,\lambda_i)\right).$$

Since ν_{λ_i} converges $*$-weakly to 0, then by Lemma 4.6, we have $\lim_{i\to\infty}\|(1-\lambda)A_{R(c)}(z,\lambda_i)\|_{L_1}=0$, and thus for every j, the limit $\lim_{i\to\infty}D(c_j,\lambda_i)=0$ which finishes the proof. □

We have the following.

Proposition 4.9 *Let R be as in Lemma 4.8. If the measures ν_{λ_i} converge $*$-weakly to* 0 *for a suitable sequence of reals $\lambda_i\to 1$ and a critical point $c\in J(R)$, then:*

$$\lim_{i\to\infty}\frac{1}{1-\lambda_i}\int\gamma_a(z)\mathrm{d}\nu_{\lambda_i}(a)=0$$

for almost every $z\in D(R)$. Moreover, on the Fatou set, the limit above is uniform on compact subsets outside the postcritical set $P(R)$.

Proof First, let us show that, for any $\phi\in L_1(\mathbb{C})$, the series $\sum R_*^n(\phi)$ is finite and converges absolutely almost everywhere on $D(R)$. It is enough to show that $\sum|R_*^n(\phi)|$ is integrable on any wandering set W. Direct computations show:

$$\int_W|R_*^n(\phi)(z)||\mathrm{d}z|^2\leq\int_{R^{-n}(W)}|\phi(z)||\mathrm{d}z|^2,$$

which yields:

$$\int_W\sum_{n=0}^{\infty}|R_*^n(\phi)(z)|\leq\sum_{n=0}^{\infty}\int_{R^{-n}(W)}|\phi(z)||\mathrm{d}z|^2\leq\int_{\mathbb{C}}|\phi(z)||\mathrm{d}z|^2.$$

In particular, $R_*^n(\phi)(z)$ converges to 0 almost everywhere on $D(R)$.

Second, if $z_0\in F(R)\setminus P(R)$, we can find a disk $D_0\subset D(R)$ centered at z_0. Now, suppose that ϕ is holomorphic on $\mathbb{C}\setminus P(R)$. Since, we have:

$$\int_{D_0}|R_*^n(\phi)(z)|\mathrm{d}z|^2\leq\int_{\mathbb{C}}|\phi(z)||\mathrm{d}z|^2;$$

by the mean value theorem, $\{R_*^n(\phi)\}$ forms a normal family of holomorphic functions on D_0. By the discussion above, $R_*^n(\phi)$ and $\sum R_*^n(\phi)(z)$ converge to their respective limits uniformly on compact subsets of D_0. Then, by the Abel theorem, we get:

$$\lim_{\lambda_i \to 1} \sum \lambda_i^n R_*^n(\phi)(z) = \sum R_*^n(\phi)(z)$$

almost everywhere on $D(R)$ and uniformly on compact subsets of $F(R)\backslash P(R)$.

To finish the proof, we take $\lambda_i \to 1$ and apply Lemma 4.8 and the discussion above to Eq. (**) of Corollary 3.5 to get:

$$\lim_{i\to\infty} A_{R(c)}(z, \lambda_i) = \lim_{i\to\infty} \frac{1}{1-\lambda_i} \int_{\mathbb{C}} \gamma_a(z) \mathrm{d}\nu_{\lambda_i}(a) = 0$$

almost everywhere on the dissipative set $D(R)$ and uniformly on compacts subsets of $F(R)\backslash P(R)$. □

Now, we are ready to prove the Abel version of Theorem 1.11.

Theorem 4.10 *Assume that for a critical point $c \in J(R)$, the sequence measure $\{\nu_{\lambda_i}\}$ converges $*$-weakly to 0 for a suitable sequence $\lambda_i < 1$ converging to 1. Then, R is an unstable map whenever c and the individual postcritical set $P_c(R)$ are separated by the Fatou set.*

Proof By contradiction, assume that R is structurally stable. Then, by Lemma 4.8, we get:

$$\lim_{\lambda_i \to 1} A_{R(c)}(c, \lambda_i) = R''(c) \neq 0.$$

On the other hand, by Proposition 4.9, we have that:

$$A_{R(c)}(z, \lambda_i) = \frac{1}{1-\lambda_i} \int \gamma_a(z) \mathrm{d}\nu_{\lambda_i}(a)$$

converges to 0 uniformly on compact subsets of $F(R)\backslash P(R)$. By assumption, we can select a Jordan curve $\gamma \subset F(R)\backslash P(R)$ separating c and $P_c(R)$. Since $A_{R(c)}(z, \lambda_i)$ is holomorphic for z in the interior of γ, by Cauchy's theorem, we have:

$$\lim_{\lambda_i \to 1} A_{R(c)}(c, \lambda_i) = \frac{1}{2\pi i} \int_\gamma \frac{A_{R(c)}(z, \lambda_i)}{z - c} \mathrm{d}z = 0,$$

a contradiction. □

Proof of Theorem 1.12 The first part of the Theorem is a consequence of Lemma 4.7 and Theorem 4.4.

For the last part, we proceed by contradiction. Assume that R is structurally stable. Since the measures ν_λ form a uniformly bounded family of measures for $0 \leq \lambda < 1$, then by Theorem 1.11, every $*$-weak limit of ν_λ for $\lambda \to 1$ is a non-zero Abel measure.

(2) Since $P_c(R)$ has measure zero, then we have a contradiction with Theorem 4.4.

(3) Assume that $\sigma(c)$ is convergent. Let $\tau \neq 0$ be an Abel measure which is not an M-measure. Then:

$$f(z) = \int_{\mathbb{C}} \gamma_a(z) \mathrm{d}\tau(a)$$

is a non-zero integrable function on $\mathbb{C}$ supported on $P_c(R)$ satisfying $R_*(f) = f$. By Lemma 11 in [16], there exists an invariant Beltrami differential μ with $\mu(z) = \frac{|f(z)|}{f(z)}$ almost everywhere on the support of f. Computations give rise to:

$$\begin{aligned}
0 \neq \int_{\mathbb{C}} |f||\mathrm{d}z|^2 &= \int_{\mathbb{C}} \mu(z) f(z)|\mathrm{d}z|^2 \\
&= \lim_{\lambda_i \to 1} (1 - \lambda_i) \int_{\mathbb{C}} \mu(z) A_{R(c)}(z, \lambda_i)|\mathrm{d}z|^2 \\
&= \lim_{\lambda_i \to 1} (1 - \lambda_i) \left[\left(\frac{1}{R''(c)} A_{R(c)}(c, \lambda_i) - 1 \right) \int_{\mathbb{C}} \mu(z) B_{R(c)}(z, \lambda_i)|\mathrm{d}z|^2 \right. \\
&\quad \left. + \sum_{\tilde{c} \in Crit(R) \setminus \{c\}} \frac{1}{R''(\tilde{c})} A_{R(c)}(\tilde{c}, \lambda_i) \int_{\mathbb{C}} \mu(z) B_{R(\tilde{c})}(z, \lambda)|\mathrm{d}z|^2 \right].
\end{aligned}$$

By the invariance of μ, this reduces to:

$$0 \neq F_\mu(v) \lim_{\lambda_i \to 1} \left[\frac{1}{R''(c)} A_{R(c)}(c, \lambda_i) - 1 \right] + \sum_{\tilde{c} \in Crit(R) \setminus \{c\}} \frac{F_\mu(\tilde{v})}{R''(\tilde{c})} \lim_{\lambda_i \to 1} A_{R(c)}(\tilde{c}, \lambda_i). \tag{3}$$

Now, as R is stable and since $\sigma(c)$ is convergent, then by Corollary 1.3, we have that $\sigma(c)$ converges to 0. Let $s_n(\bar{c})$ be the partial sums of the formal series $A_{R(c)}(\bar{c})$, and here, $\bar{c}$ is any critical point. Then, the assumptions, Lemma 3.1, and the formula after Lemma 3.3 together with the fact that $\sigma(c)$ converge to 0 yield:

$$\lim_{n \to \infty} s_n(\bar{c}) = \begin{cases} 0 & \text{if } \bar{c} \neq c \\ R''(c) & \text{if } \bar{c} = c. \end{cases}$$

Abel's theorem then gives:

$$\lim_{\lambda \to 1} A_{R(c)}(\bar{c}, \lambda) = (1 - \lambda) \sum_n s_n(\bar{c}) \lambda^n = \begin{cases} 0 & \text{if } \bar{c} \neq c \\ R''(c) & \text{if } \bar{c} = c. \end{cases}$$

After replacing these values in (3), we achieve the desired contradiction. □

Note that the condition that $P_c(R)$ has measure zero is used to guarantee that the measure τ is an M-measure. As mentioned in the introduction, this condition can be dropped for maps with a completely invariant Fatou domain.

With small modifications, the theorems above can be extended to the case of entire or meromorphic functions with finitely many critical and asymptotic values.

4.2 The Voronoi Case

Now, we prove Theorem 1.10 which extends the ideas of the Abel versions of Theorems 4.4 and 4.10 to the Voronoi case.

Proof of Theorem 1.10 As $|\sigma_n(c)|$ is Nörlund regular with respect to the matrix $N = \{\frac{q_{n-m}}{Q_n}\}$, it has radius of convergence $r > 0$. Hence:

$$e_\lambda = (1-\lambda)\sum \lambda^n \sigma_n(c)\delta_{R^n(c)}$$

is a finite measure for $|\lambda| < r$. By Part (3) of Lemma 1.7, we have:

$$q(\lambda)\cdot e_\lambda = \nu_\lambda.$$

Thus, e_λ can be extended to the open unit disk as a meromorphic family of measures which is holomorphic on a neighborhood of $[0, 1)$. Let $E_\lambda = \frac{\nu_\lambda}{q(\lambda)}$ be the induced extension. In this way, the function

$$A_v(z,\lambda) = \frac{1}{1-\lambda}\int_{\mathbb{C}} \gamma_a(z) de_\lambda(a)$$

extends to

$$\begin{aligned} E_v(z,\lambda) &= \frac{1}{q(\lambda)(1-\lambda)}\int_{\mathbb{C}} \gamma_a(z) \mathrm{d}\nu_\lambda(a) \\ &= \frac{1}{(1-\lambda)}\int_{\mathbb{C}} \gamma_a(z) dE_\lambda(a). \end{aligned}$$

In other words, for z outside $P_c(R)\cup\{0,1,\infty\}$, the sequence $\{\sigma_n(c)\gamma_{R^n(v)}(z)\}$ is Nörlund regular with respect to N.

Like in the proofs of Lemma 3.1 and Corollary 3.5, we extend $A_v(\tilde{c},\lambda)$ to a meromorphic function $E_v(\tilde{c},\lambda)$ for every critical point $\tilde{c}$, so that:

$$B_v(z,\lambda) = E_v(z,\lambda) + \lambda \sum_{c_i \in Crit(R)} \frac{1}{R''(c_i)} E_v(c_i,\lambda) B_{v_i}(z,\lambda) \qquad (***)$$

holds on a neighborhood of $[0, 1)$.

Under the assumptions, the arguments of Proposition 4.2 apply. □

Proof of Theorem 1.11 We proceed as in Theorem 4.10, but we only need to apply Lemma 3.1 and Proposition 4.9 to Eq. (***). If R is structurally stable, we get a contradiction to the assumptions proceeding as in Theorem 4.10. □

So far, we have consider the Nörlund–Voronoi method to produce finite non-zero measures. A general averaging mechanism can be hinted to establish the Fatou conjecture for a wider class of sequences.

Let Θ be the space of all complex sequences. An infinity matrix M is *regular* if, when restricted to $\mathcal{C}$, the space of converging sequences, it defines a continuous operator, such that if $\{t_n\} = M\{a_n\}$, then $\lim a_n = \lim t_n$. A basic fact here is Agnew's theorem which states that given $x, y \in \Theta$ if either:

(1) $x \in \ell_\infty \backslash \mathcal{C}$ and $y \in \ell_\infty$ or
(2) $x \in \Theta \backslash \ell_\infty$ and $y \in \Theta$,

then there is regular matrix M with $Mx = y$ (see Theorem 2.6.4 in [3]). The matrix M is not unique. Moreover, the space of all regular matrices sending x to y is infinitely dimensional.

Let us recall that ℓ_1 acts on Θ by convolutions. Take an infinite matrix M subject to the following two conditions.

(i) If $c \in J(R)$, then M transforms the measures $\rho_k = \sum_{n=0}^{k} \delta_{R^n(v)} \sigma_n(c)$ onto measures t_k with $\sup_k \|t_k\| < \infty$, and there is a non-zero α which is a $*$-weak accumulation point of the sequence $\{t_k\}$. (Agnew's theorem guarantees the existence of such a matrix for non-convergent complex sequences).
(ii) The matrix M commutes with the action of ℓ_1 by convolutions.

When this happens, using the formal Eq. (*), we can show the instability of the corresponding rational map whenever α is an M-measure. Although the algebra of linear operators on Θ commuting with the action of ℓ_1, by convolutions, is non-separable and the space of regular matrices mapping x to y is infinitely dimensional, the authors were not able to find a matrix satisfying the conditions (i) and (ii) above. On the other hand, most of the summation methods discussed, for instance, in the book by Boos [3] satisfy both conditions. In this paper, we used one of the most general summation methods.

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