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RESEARCH CONTRIBUTION

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On Lagrangian and Legendrian Singularities

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Abstract

We describe the topology of stable simple multisingularities of Lagrangian and Legendrian maps. In particular, the tables of adjacency indices of monosingularities to multisingularities are given for generic caustics and wave fronts in spaces of small dimensions. The paper is an extended version of the author's talk in the International Conference "Contemporary mathematics" in honor of the 80th birthday of V. I. Arnold (Moscow, Russia, 2017).

Keywords Lagrangian and Legendrian maps · Caustics · Wave fronts · (Multi)singularities · Adjacency index

1 Introduction

"Caustic" is a term from Optics. It refers to a place where light is concentrated. A caustic is the envelope of a system of light rays. For example, we can see a caustic on the bottom of a cup in a Sunny day (Fig. 1).

A caustic is the set of critical values of so-called Lagrangian map (see [1]). A generic caustic is a singular hypersurface in the target space. Singularities of this hypersurface are defined by multisingularities of the corresponding Lagrangian map. In particular, a generic caustic in the plane may have only cusps and transversal intersections of two smooth branches.

A simple example of a wave front is an equidistant of a smooth (of class C^∞) cooriented hypersurface in the Euclidean space $\mathbb{R}^n$ (Fig. 2). A wave front is the image of so-called Legendrian map [1]. A generic wave front is a singular hypersurface in the target space. Singularities of this hypersurface are defined by multisingularities of the corresponding Legendrian map. In particular, a generic wave front in the plane may have only cusps and transversal intersections of two smooth branches.

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Fig. 1 Caustic on the bottom of a cup

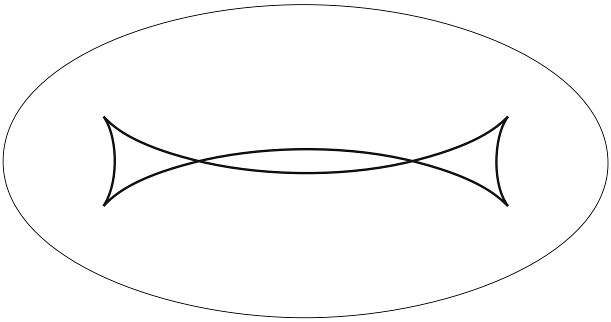

Fig. 2 An equidistant of an ellipse in the plane

Stable simple singularities of caustics and wave fronts were classified by V. I. Arnold [1]. He repeatedly posed the natural problem on the local and global topological properties of caustics and wave fronts (see Problem 1987-5 in [2]). This problem was the subject of numerous works of different authors (see, for example, the book [11]). However, even local problem is not solved completely until now.

In this paper we describe the recent results on the topology of adjacencies of stable simple multisingularities of Lagrangian and Legendrian maps. Adjacency indices of monosingularities to multisingularities are given for generic caustics in the spaces of dimension $n \leq 5$ and for generic wave fronts in the spaces of dimension $n \leq 6$.

2 Lagrangian Singularities

Let E be a smooth manifold. A *symplectic structure* on E is a closed non-degenerate 2-form ω. We recall that ω is a closed form if $d\omega = 0$; it is non-degenerate if for every $x \in E$ and $v \in T_x E$ there exists $w \in T_x E$ such that $\omega(v, w) \neq 0$.

Remark Symplectic structures may exist only on even-dimensional manifolds.

Let $\dim E = 2n$. We fix a symplectic structure ω on E. A smooth submanifold L in E is said to be *isotropic* if $\omega|_L = 0$. Isotropic submanifolds having highest dimension (equal to n) are called *Lagrangian submanifolds* of E. A smooth bundle

$$\varrho : E \to V$$

over a smooth n-dimensional manifold V is said to be *Lagrangian* if all its fibers are Lagrangian submanifolds of E.

Let ϱ be a Lagrangian bundle and L be a smooth Lagrangian submanifold of E. Then the diagram

$$f : L \overset{i}{\hookrightarrow} E \overset{\varrho}{\to} V,$$

where $i : L \hookrightarrow E$ is the identity embedding, is said to be a *Lagrangian map*. The set of critical values of a Lagrangian map is called a *caustic*.

We shall consider only proper Lagrangian maps.

Example Let us consider the cotangent bundle T^*V of a smooth n-dimensional manifold V. This is a smooth $2n$-dimensional manifold. It is formed by pairs (ξ, y) where $y \in V$ and $\xi \in T^*_y V$. The natural projection

$$\varrho : T^*V \to V, \quad (\xi, y) \mapsto y$$

is a smooth bundle over V.

The *canonical* 1 *-form* α on T^*V is defined as follows: if v is a tangent vector to T^*V at a point (ξ, y), then

$$\alpha(v) = \xi(d\varrho|_{(\xi,y)}(v)).$$

The form $\omega = d\alpha$ defines the *canonical* symplectic structure on T^*V. We claim that ϱ is a Lagrangian bundle with respect to this structure. Indeed, the velocity of y along the fibre of ϱ is equal to 0.

Example Let us consider the cotangent bundle $\varrho : T^*\mathbb{R}^2 \to \mathbb{R}^2$ of the plane $\mathbb{R}^2$ containing the bottom of a cup. Light rays transversally intersecting the bottom define a domain in $T^*\mathbb{R}^2$. Light rays reflected from the side surface of the cup define a Lagrangian submanifold in this domain. Its caustic with respect to ϱ is a light caustic on the bottom of the cup.

Example Consider the space $\mathbb{R}^{2n}$ with the coordinates $(\mathbf{t}, \mathbf{x}, \mathbf{p}, \mathbf{q})$, where

$$\mathbf{t} = (t_1, \dots, t_k), \quad \mathbf{x} = (x_1, \dots, x_k), \quad \mathbf{p} = (p_{k+1}, \dots, p_n), \quad \mathbf{q} = (q_{k+1}, \dots, q_n).$$

The form $\omega = d\mathbf{t} \wedge d\mathbf{x} + d\mathbf{p} \wedge d\mathbf{q}$ determines a symplectic structure on $\mathbb{R}^{2n}$. The map

$$\varrho : \mathbb{R}^{2n} \to \mathbb{R}^n, \quad \varrho : (\mathbf{t}, \mathbf{x}, \mathbf{p}, \mathbf{q}) \mapsto (\mathbf{x}, \mathbf{q})$$

is a Lagrangian bundle over the space $\mathbb{R}^n = \{(\mathbf{x}, \mathbf{q})\}$.

Let $S = S(\mathbf{t}, \mathbf{q})$ be a family of smooth functions of $\mathbf{t}$ smoothly depending on $\mathbf{q}$. Then the system of the equations

$$\mathbf{x} = -\frac{\partial S(\mathbf{t}, \mathbf{q})}{\partial \mathbf{t}}, \quad \mathbf{p} = \frac{\partial S(\mathbf{t}, \mathbf{q})}{\partial \mathbf{q}}$$

determines a smooth Lagrangian submanifold

$$L = \{(\mathbf{t}, \mathbf{q}) \in \mathbb{R}^n\} \overset{i}{\hookrightarrow} \mathbb{R}^{2n}.$$

The Lagrangian map $f : L \overset{i}{\hookrightarrow} \mathbb{R}^{2n} \overset{\varrho}{\to} \mathbb{R}^n$ is given by the formula

$$f : \mathbb{R}^n \to \mathbb{R}^n, \quad (\mathbf{t}, \mathbf{q}) \mapsto \left(-\frac{\partial S(\mathbf{t}, \mathbf{q})}{\partial \mathbf{t}}, \mathbf{q} \right). \tag{1}$$

Let us endow the space W of all embeddings $i : L \hookrightarrow E$ with Whitney's fine C^∞-topology. We say that an assertion about Lagrangian maps holds for a *generic* map $f : L \overset{i}{\hookrightarrow} E \overset{\varrho}{\to} V$ (or, for a *generic* caustic of f) if there is an open dense subset $U \subset W$ such that this assertion holds for any $i \in U$.

Definition Two Lagrangian maps

$$L_1 \overset{i_1}{\hookrightarrow} E_1 \xrightarrow{\varrho_1} V_1, \quad L_2 \overset{i_2}{\hookrightarrow} E_2 \xrightarrow{\varrho_2} V_2$$

are said to be *(Lagrange) equivalent* if there are diffeomorphisms $\Phi : E_1 \to E_2$, $\varphi : V_1 \to V_2$, $\varepsilon : L_1 \to L_2$ such that Φ takes the symplectic structure on E_1 to the symplectic structure on E_2 and the diagram

$$\begin{array}{ccccc} L_1 & \xrightarrow{i_1} & E_1 & \xrightarrow{\varrho_1} & V_1 \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \varphi} \\ L_2 & \xrightarrow{i_2} & E_2 & \xrightarrow{\varrho_2} & V_2 \end{array}$$

is commutative.

Remark The caustics of equivalent Lagrangian maps are diffeomorphic.

The equivalence classes of Lagrangian map germs under Lagrange equivalences are called *(Lagrangian) singularities*.

Definition The germ of a Lagrangian map $f : L \overset{i}{\hookrightarrow} E \overset{\varrho}{\to} V$ at a point $x \in L$ is said to be *(Lagrange) stable* if for any Lagrangian map $\widetilde{f}$ close to f (in Whitney's fine C^∞-topology) there is a point $\widetilde{x} \in L$ close to x such that the germ of $\widetilde{f}$ at $\widetilde{x}$ is Lagrange equivalent to the germ of f at x. The germ of f at x is said to be *simple* if germs of all close Lagrangian maps at all close points belong to a finite number of Lagrange equivalence classes.

Arnold's Theorem on Lagrangian singularities [1]. *Lagrange stable simple germs of Lagrangian maps to a smooth n-manifold are Lagrange equivalent to the germs of the map* (1) *at the origin where* $S = S(\mathbf{t}, \mathbf{q})$ *is a function of one of the following types with an integer* $\mu \leq n + 1$:

$$A_\mu^\pm : \quad S = \pm t_1^{\mu+1} + q_{\mu-1} t_1^{\mu-1} + \ldots + q_2 t_1^2, \qquad \mu \geq 1;$$

$$D_\mu^\pm : \quad S = t_1^2 t_2 \pm t_2^{\mu-1} + q_{\mu-1} t_2^{\mu-2} + \ldots + q_3 t_2^2, \qquad \mu \geq 4;$$

$$E_6^\pm : \quad S = t_1^3 \pm t_2^4 + q_5 t_1 t_2^2 + q_4 t_1 t_2 + q_3 t_2^2, \qquad \mu = 6;$$

$$E_7 : \quad S = t_1^3 + t_1 t_2^3 + q_6 t_1^2 t_2 + q_5 t_1^2 + q_4 t_1 t_2 + q_3 t_2^2, \qquad \mu = 7;$$

$$E_8 : \quad S = t_1^3 + t_2^5 + q_7 t_1 t_2^3 + q_6 t_1 t_2^2 + q_5 t_2^3 + q_4 t_1 t_2 + q_3 t_2^2, \quad \mu = 8.$$

Generic Lagrangian maps to a manifold of dimension $n \leq 5$ *may have only stable simple singularities.*

Here the number $\mu - 1$ is called the *codimension* of the singularity. If μ is even or $\mu = 1$, then singularities of types A_μ^+ and A_μ^- are Lagrange equivalent and are referred as singularities of type A_μ. Other singularities of types listed above are pairwise Lagrange inequivalent.

Remark Caustic germs with singularities of types A_μ^+ and A_μ^- are diffeomorphic for any μ. Caustic germs of types D_μ^+ and D_μ^- are diffeomorphic for any odd μ. Caustic germs of types E_6^+ and E_6^- are diffeomorphic as well. Caustic germs with singularities of other types are pairwise non-diffeomorphic.

Example Cusps of a generic caustic in the plane are defined by the Lagrangian singularities of types $A_3^\pm$.

3 Legendrian Singularities

Let E be a smooth manifold. A *contact hyperplane* on E at a point $x \in E$ is a codimension 1 vector subspace $\pi \subset T_x E$. The pair (π, x) is said to be a *contact element* on E.

Consider a smooth field of contact hyperplanes on E. Locally such a field is given by a smooth 1-form α as the field of its zeros. A field of contact hyperplanes is called a *contact structure* if $d\alpha$ is a non-degenerate 2-form on each hyperplane of the field. This condition is independent of the choice of α.

Remark Contact structures may exist only on odd-dimensional manifolds.

Let $\dim E = 2n - 1$. Then a field $\alpha = 0$ of contact hyperplanes on E is a contact structure if and only if $\alpha \wedge (d\alpha)^{n-1} \neq 0$.

We fix a contact structure on E. Smooth integral manifolds of this structure having the highest dimension (equal to $n - 1$) are called *Legendrian submanifolds* of E. A smooth bundle

$$\varrho : E \to V$$

over a smooth n-dimensional manifold V is said to be *Legendrian* if all its fibres are Legendrian submanifolds of E.

Let ϱ be a Legendrian bundle and L be a smooth Legendrian submanifold of E. Then the diagram

$$f : L \overset{i}{\hookrightarrow} E \overset{\varrho}{\to} V,$$

where $i : L \hookrightarrow E$ is the identity embedding, is said to be a *Legendrian map*. The image of a Legendrian map is called a *(wave) front*.

We shall consider only proper Legendrian maps.

Example Let PT^*V be the projectivization of the cotangent bundle T^*V of a smooth n-dimensional manifold V. This space is a smooth $(2n - 1)$-dimensional manifold. It is formed by contact elements (π, y) where $y \in V$ and $\pi \subset T_yV$.

The space PT^*V has a natural contact structure: the velocity of (π, y) belongs to a hyperplane of the contact structure if the velocity of y lies in π. The projection

$$\varrho : PT^*V \to V, \quad (\pi, y) \mapsto y$$

is a Legendrian bundle with respect to this structure (the velocity of y along the fibre of ϱ is equal to 0).

Example Let M be a smooth cooriented hypersurface in $\mathbb{R}^n$. We fix a real t and consider the set L_t of contact elements $(\pi_y, y + t\mathbf{n}_y)$ in $PT^*\mathbb{R}^n$ such that $y \in M$, $\pi_y = T_yM$ and $\mathbf{n}_y$ is the unit normal to M at y.

The set L_t is a Legendrian submanifold in $PT^*\mathbb{R}^n$. Its front with respect to the Legendrian projection $\varrho : PT^*\mathbb{R}^n \to \mathbb{R}^n$ is the t-equidistant of M.

Example Consider the space $\mathbb{R}^{2n-1}$ with the coordinates $(y, \mathbf{t}, \mathbf{x}, \mathbf{p}, \mathbf{q})$, where

$$\mathbf{t} = (t_1, \ldots, t_k), \quad \mathbf{x} = (x_1, \ldots, x_k), \quad \mathbf{p} = (p_{k+1}, \ldots, p_{n-1}), \quad \mathbf{q} = (q_{k+1}, \ldots, q_{n-1}).$$

The form $\alpha = dy + \mathbf{t}d\mathbf{x} + \mathbf{p}d\mathbf{q}$ determines a contact structure on $\mathbb{R}^{2n-1}$. The mapping

$$\varrho : \mathbb{R}^{2n-1} \to \mathbb{R}^n, \quad \varrho : (y, \mathbf{t}, \mathbf{x}, \mathbf{p}, \mathbf{q}) \mapsto (y, \mathbf{x}, \mathbf{q})$$

is a Legendrian bundle over the space $\mathbb{R}^n = \{(y, \mathbf{x}, \mathbf{q})\}$.

Let $S = S(\mathbf{t}, \mathbf{q})$ be a family of smooth functions of $\mathbf{t}$ smoothly depending on $\mathbf{q}$. Then the system of the equations

$$y = -S(\mathbf{t}, \mathbf{q}) + \mathbf{t}\frac{\partial S(\mathbf{t}, \mathbf{q})}{\partial \mathbf{t}}, \quad \mathbf{x} = -\frac{\partial S(\mathbf{t}, \mathbf{q})}{\partial \mathbf{t}}, \quad \mathbf{p} = \frac{\partial S(\mathbf{t}, \mathbf{q})}{\partial \mathbf{q}}$$

determines a smooth Legendrian submanifold

$$L = \{(\mathbf{t}, \mathbf{q}) \in \mathbb{R}^{n-1}\} \overset{i}{\hookrightarrow} \mathbb{R}^{2n-1}.$$

The Legendrian map $f : L \overset{i}{\hookrightarrow} \mathbb{R}^{2n-1} \overset{\varrho}{\to} \mathbb{R}^n$ is given by the formula

$$f : \mathbb{R}^{n-1} \to \mathbb{R}^n, \quad (\mathbf{t}, \mathbf{q}) \mapsto \left(-S(\mathbf{t}, \mathbf{q}) + \mathbf{t}\frac{\partial S(\mathbf{t}, \mathbf{q})}{\partial \mathbf{t}}, -\frac{\partial S(\mathbf{t}, \mathbf{q})}{\partial \mathbf{t}}, \mathbf{q} \right). \qquad (2)$$

Let us endow the space W of all embeddings $i : L \hookrightarrow E$ with Whitney's fine C^∞-topology. We say that an assertion about Legendrian maps holds for a *generic* map $f : L \overset{i}{\hookrightarrow} E \overset{\varrho}{\to} V$ (or, for a *generic* front $f(L)$) if there is an open dense subset $U \subset W$ such that this assertion holds for any $i \in U$.

Definition Two Legendrian maps

$$f_1 : L_1 \overset{i_1}{\hookrightarrow} E_1 \xrightarrow{\varrho_1} V_1, \quad f_2 : L_2 \overset{i_2}{\hookrightarrow} E_2 \xrightarrow{\varrho_2} V_2$$

are said to be *(Legendre) equivalent* if there are diffeomorphisms $\Phi : E_1 \to E_2$, $\varphi : V_1 \to V_2$, $\varepsilon : L_1 \to L_2$ such that Φ takes the contact structure on E_1 to the contact structure on E_2 and the diagram

$$\begin{array}{ccccc} L_1 & \xrightarrow{i_1} & E_1 & \xrightarrow{\varrho_1} & V_1 \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \varphi} \\ L_2 & \xrightarrow{i_2} & E_2 & \xrightarrow{\varrho_2} & V_2 \end{array}$$

is commutative.

Remark Generic Legendrian maps are equivalent if and only if the fronts of these maps are diffeomorphic.

The equivalence classes of Legendrian map germs under Legendre equivalences are called *(Legendrian) singularities*.

Definition The germ of a Legendrian map $f : L \overset{i}{\hookrightarrow} E \overset{\varrho}{\to} V$ at a point $x \in L$ is said to be *(Legendre) stable* if for every Legendrian map $\widetilde{f}$ close to f (in Whitney's fine C^∞-topology) there is a point $\widetilde{x} \in L$ close to x such that the germ of $\widetilde{f}$ at $\widetilde{x}$ is Legendre equivalent to the germ of f at x. The germ of f at x is said to be *simple* if germs of all close Legendrian maps at all close points belong to a finite number of Legendre equivalence classes.

Arnold's Theorem on Legendrian singularities ([1]). *Legendre stable simple germs of Legendrian maps to a smooth n-dimensional manifold are Legendre equivalent to the germs of the map* (2) *at the origin where $S = S(\mathbf{t}, \mathbf{q})$ is a function of one of the following types with an integer $\mu \le n$:*

$$A_\mu : \quad S = t_1^{\mu+1} + q_{\mu-1}t_1^{\mu-1} + ... + q_2 t_1^2, \qquad \mu \ge 1;$$

$$D_\mu^\pm : \quad S = t_1^2 t_2 \pm t_2^{\mu-1} + q_{\mu-1}t_2^{\mu-2} + ... + q_3 t_2^2, \qquad \mu \ge 4;$$

$$E_6 : \quad S = t_1^3 + t_2^4 + q_5 t_1 t_2^2 + q_4 t_1 t_2 + q_3 t_2^2, \qquad \mu = 6;$$

$$E_7 : \quad S = t_1^3 + t_1 t_2^3 + q_6 t_1^2 t_2 + q_5 t_1^2 + q_4 t_1 t_2 + q_3 t_2^2, \qquad \mu = 7;$$

$$E_8 : \quad S = t_1^3 + t_2^5 + q_7 t_1 t_2^3 + q_6 t_1 t_2^2 + q_5 t_2^3 + q_4 t_1 t_2 + q_3 t_2^2, \qquad \mu = 8.$$

Generic Legendrian maps to a manifold of dimension $n \le 6$ may have only stable simple singularities.

Here the number μ is called the *codimension* of the singularity. If μ is odd, then singularities of types D_μ^+ and D_μ^- are Legendre equivalent and are referred as singularities of *type* D_μ. Other singularities of the types listed above are pairwise Legendre inequivalent.

Example Cusps of a generic wave front in the plane are defined by the Legendrian singularities of type A_2.

4 Lagrangian and Legendrian Multisingularities

Let us fix a Lagrangian or Legendrian map $f : L \to V$ with stable simple singularities. The *multisingularity* of f at a point $y \in V$ is the unordered set of singularities of this map at the distinct points $x \in f^{-1}(y)$. Multisingularities of the map f are labeled by the elements of the free Abelian multiplicative semigroup $\mathbb{S}$ whose generators are the symbols of types of Lagrangian or Legendrian (respectively) singularities from the corresponding Arnold's theorem.

Let us consider an arbitrary element $\mathcal{A} = X_1 \dots X_p$ of the semigroup $\mathbb{S}$, where $X_1, \dots, X_p$ is any p-tuple of its generators.

Definition A map $f : L \to V$ has a *multisingularity of type* $\mathcal{A}$ at a point $y \in V$ if:

1. $f^{-1}(y)$ consists of p distinct points;

2. there exists an order $x_1, \ldots, x_p$ of points from $f^{-1}(y)$ such that f has a singularity of type X_i at the point x_i, $i = 1, \ldots, p$.

In particular, if **1** is the identity element of the semigroup $\mathbb{S}$, then f has a *multisingularity of type* **1** at each point of the complement $V \setminus f(L)$. If the number $\#(\mathcal{A}) = p$ of factors of the element $\mathcal{A} = X_1 \ldots X_p \in \mathbb{S}$ is equal to 1, then a multisingularity of type $\mathcal{A}$ is called a *monosingularity*. If each factor X_i is equal to $A^{\pm}_{\mu_i}$ or A_{μ_i}, then a multisingularity of type $\mathcal{A}$ is called a *multisingularity of corank* 1.

Definition The sum of the codimensions of the singularities of types $X_1, \ldots, X_p$ is called the *codimension* of the multisingularity of type $\mathcal{A}$ and is denoted by $\operatorname{codim}\mathcal{A}$.

For a generic map f, the set $\mathcal{A}_f$ of points $y \in V$ such that f has a multisingularity of type $\mathcal{A} \in \mathbb{S}$ at y is a smooth submanifold in V. Its codimension is equal to $\operatorname{codim}\mathcal{A}$. The closure $\overline{\mathcal{A}_f} \subset V$ of the manifold $\mathcal{A}_f$ is the union of manifolds of the form $\mathcal{B}_f$, where $\mathcal{B} \in \mathbb{S}$. The partition of the set $\overline{\mathcal{A}_f}$ into connected components of these manifolds is a Whitney C^∞-stratification (see [4]).

Let $y \in \overline{\mathcal{A}_f}$ and f have a multisingularity of type $\mathcal{B} \in \mathbb{S}$ at y, where $\operatorname{codim}\mathcal{B} = c$. Choose a neighborhood U of the origin 0 in $\mathbb{R}^c$ and consider a smooth embedding $h : U \to V$ such that $h(0) = y$ and the submanifold $h(U) \subset V$ is transversal to the submanifold $\mathcal{B}_f$ at y. Let $D_\varepsilon \subset \mathbb{R}^c$ be the open c-dimensional ball of radius $\varepsilon > 0$ centered at 0. Then there exists a positive number $\varepsilon_0 = \varepsilon_0(f, y, h)$ such that for any $\mathcal{A} \in \mathbb{S}$ and $\varepsilon < \varepsilon_0$ the intersection $h(D_\varepsilon) \cap \mathcal{A}_f$ is a smooth manifold whose diffeomorphism class depends only on $\mathcal{A}$ and $\mathcal{B}$.

Let us fix an arbitrary representative $\Xi_{\mathcal{A}}(\mathcal{B})$ of this diffeomorphism class for every pair of elements $\mathcal{A}, \mathcal{B} \in \mathbb{S}$. By $J_{\mathcal{A}}(\mathcal{B})$ denote the Euler characteristic of the manifold $\Xi_{\mathcal{A}}(\mathcal{B})$.

Definition A multisingularity of type $\mathcal{B}$ is *adjacent* to a multisingularity of type $\mathcal{A}$ if $\mathcal{A} \neq \mathcal{B}$ and $\Xi_{\mathcal{A}}(\mathcal{B}) \neq \emptyset$. The number $J_{\mathcal{A}}(\mathcal{B})$ is called the *index* of the adjacency. An adjacency is said to be *simple* if all connected components of the manifold $\Xi_{\mathcal{A}}(\mathcal{B})$ are contractible. Otherwise it is said to be *complicated*.

Remark All adjacencies of multisingularities of a generic Legendrian map with stable simple singularities are simple (see [3]). In the case of a generic Lagrangian map, all adjacencies of corank 1 multisingularities are simple as well ([1]).

There are various relations between adjacency indices of multisingularities.

Theorem [5,6],[7,8]. *For all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{S}$ we have:*

1. *the manifold $\Xi_{\mathcal{A}}(\mathcal{B}\mathcal{C})$ is diffeomorphic to the disjoint union of the manifolds $\Xi_X(\mathcal{B}) \times \Xi_Y(\mathcal{C})$ over all pairs $(X, Y) \in \mathbb{S}^2$ such that $XY = \mathcal{A}$. In particular,*

$$J_{\mathcal{A}}(\mathcal{B}\mathcal{C}) = \sum_{(X,Y)\in\mathbb{S}^2 : XY=\mathcal{A}} J_X(\mathcal{B}) J_Y(\mathcal{C}); \tag{3}$$

2.

$$\sum_{X\in\mathbb{S}} (-1)^{\operatorname{codim} X} J_{\mathcal{A}}(X)\, J_X(\mathcal{B}) = \begin{cases} 0 & if\, \mathcal{A} \neq \mathcal{B}, \\ (-1)^{\operatorname{codim}\mathcal{A}} & if\, \mathcal{A} = \mathcal{B}; \end{cases}$$

in particular, if $\mathcal{A} \neq \mathcal{B}$ *and* $\operatorname{codim}\mathcal{A} \equiv \operatorname{codim}\mathcal{B} \pmod{2}$, *then*

$$J_{\mathcal{A}}(\mathcal{B}) = \frac{(-1)^{\operatorname{codim}\mathcal{A}+1}}{2} \sum_{X \in \mathbb{S}:\, X \neq \mathcal{A}, \mathcal{B}} (-1)^{\operatorname{codim} X} J_{\mathcal{A}}(X)\, J_X(\mathcal{B}); \tag{4}$$

3.

$$\sum_{X \in \mathbb{S}} (-1)^{\operatorname{codim} X} J_X(\mathcal{A}) = 1;$$

4.

$$\sum_{X \in \mathbb{S}} (-1)^{\operatorname{codim} X} \#(X) J_X(\mathcal{A}) = (-1)^{\operatorname{codim} A_1} \#(\mathcal{A}).$$

The first statement of the above assertion shows that the topology of the adjacencies of multisingularities of a generic Lagrangian or Legendrian map with stable simple singularities is completely determined by that of the adjacencies of monosingularities to multisingularities.

5 The Adjacency Indices of Lagrangian Monosingularities

In this section, we describe the topology of monosingularities of generic Lagrangian maps with stable simple singularities.

It is clear that a monosingularity of type A_1 is adjacent to no other Lagrangian multisingularity. A monosingularity of type A_2 is adjacent only to multisingularities of types $\mathbf{1}$ and A_1^2. These adjacencies are simple and $J_{\mathbf{1}}(A_2) = J_{A_1^2}(A_2) = 1$.

Example A germ of a Lagrangian map with singularity of type A_3^+ to the plane is the germ at the origin of the projection

$$\mathbb{R}^3 \to \mathbb{R}^2, \quad (t_1, x_1, q_2) \mapsto (x_1, q_2)$$

restricted to the surface $4t_1^3 + 2q_2t_1 + x_1 = 0$ (the Whitney cusp; see Fig. 3). Therefore a monosingularity of type A_3^+ is adjacent only to multisingularities of types A_1, A_1^3 and A_2A_1. These adjacencies are simple and $J_{A_1}(A_3^+) = J_{A_1^3}(A_3^+) = 1$, $J_{A_2A_1}(A_3^+) = 2$. We note that $J_{\mathcal{A}}(A_3^-) = J_{\mathcal{A}}(A_3^+)$ for every $\mathcal{A} \in \mathbb{S}$.

Example [1] Germs of the caustics of generic Lagrangian maps with monosingularities of types A_4, D_4^+, D_4^- to a 3-space are shown in Figs. 4, 5, 6, respectively. The indices of all adjacencies of these monosingularities can easily be found from the given figures:

$\mathcal{A}$	$\mathbf{1}$	A_1^2	A_1^4	A_2	$A_2A_1^2$	$A_3^+A_1$	$A_3^-A_1$	A_2^2
$J_{\mathcal{A}}(A_4)$	1	1	1	1	3	1	1	1

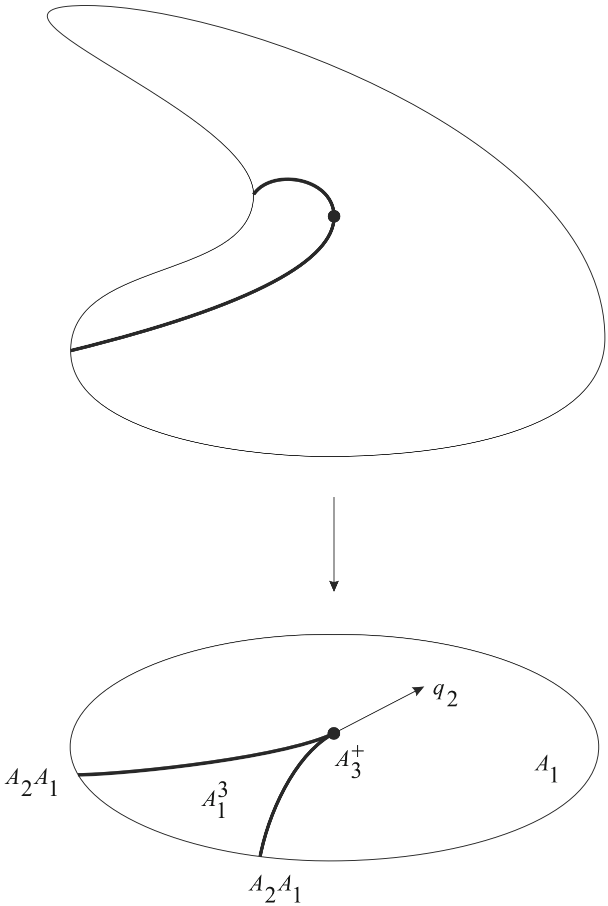

Fig. 3 Lagrangian monosingularity of type A_3^+

$\mathcal{A}$	**1**	A_1^2	A_1^4	A_2	$A_2A_1^2$	$A_3^+A_1$	$A_3^-A_1$	A_2^2
$J_{\mathcal{A}}(D_4^+)$	1	2	1	2	4	1	1	2

$\mathcal{A}$	A_1^2	A_1^4	$A_2A_1^2$	$A_3^+A_1$	$A_3^-A_1$
$J_{\mathcal{A}}(D_4^-)$	0	2	6	3	3

All these adjacencies are simple with one exception. The adjacency of a Lagrangian monosingularity of type D_4^- to a multisingularity of type A_1^2 is complicated: the manifold $\Xi_{A_1^2}(D_4^-)$ is homotopy equivalent to a circle S^1.

Now, we describe the topology of adjacencies of all Lagrangian corank 1 monosingularities. They may be adjacent only to multisingularities of corank 1. All these adjacencies are simple.

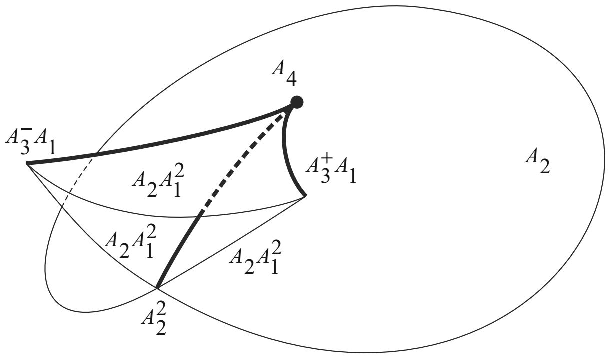

Fig. 4 A caustic germ with singularity of type A_4 (the swallowtail)

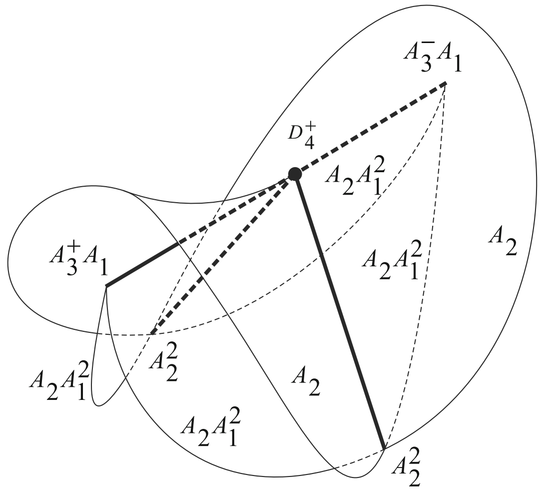

Fig. 5 A caustic germ with singularity of type D_4^+ (the purse)

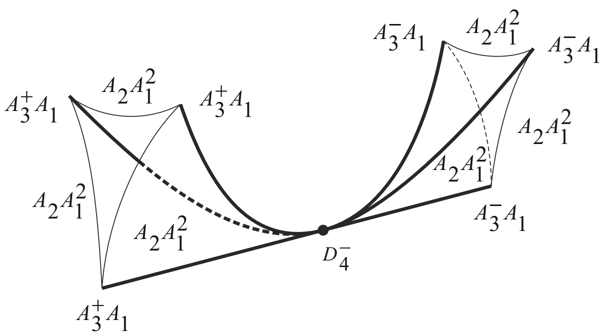

Fig. 6 A caustic germ with singularity of type D_4^- (the pyramid)

Theorem [7,8] *Let* $\delta = \pm 1$,

$$A_\mu^\delta = \begin{cases} A_\mu^+ & if\, \delta = +1, \\ A_\mu^- & if\, \delta = -1, \end{cases}$$

and

$$\mathcal{A} = A_1^k A_{\mu_1}^{m_1} \dots A_{\mu_p}^{m_p} (A_{\mu_1^+}^+)^{m_1^+} \dots (A_{\mu_q^+}^+)^{m_q^+} (A_{\mu_1^-}^-)^{m_1^-} \dots (A_{\mu_r^-}^-)^{m_r^-},$$

where $\mu_1, \dots, \mu_p$ *are pairwise-distinct even positive integers and each of the tuples* $\mu_1^+, \dots, \mu_q^+$ *and* $\mu_1^-, \dots, \mu_r^-$ *consists of pairwise-distinct odd positive integers that are greater than* 1. *We put*

$$d = k + m_1^+ + \dots + m_q^+ + m_1^- + \dots + m_r^-, \quad N = d + m_1 + \dots + m_p,$$
$$k_+^\delta = \left[\frac{2d + \delta + 1}{4}\right] - (m_1^+ + \dots + m_q^+), \quad k_-^\delta = \left[\frac{2d - \delta + 1}{4}\right] - (m_1^- + \dots + m_r^-),$$

and

$$\langle k_1, \dots, k_l \rangle = \frac{(k_1 + \dots + k_l)!}{k_1! \cdot \dots \cdot k_l!}$$

for all non-negative integers $k_1, \dots, k_l$ *and* $\langle k_1, \dots, k_l \rangle = 0$ *otherwise.*

Then a Lagrangian monosingularity of type A_μ^δ *is adjacent to a multisingularity of type* $\mathcal{A}$ *if and only if*

$$\operatorname{codim}\mathcal{A} \le \min(\mu - 2, \mu - N), \quad \operatorname{codim}\mathcal{A} \equiv \mu - N \pmod 2,$$
$$J_\mathcal{A}(A_\mu^\delta) = \langle m_1^+, \dots, m_q^+, k_+^\delta \rangle \langle m_1^-, \dots, m_r^-, k_-^\delta \rangle \langle m_1, \dots, m_p, d \rangle \ne 0.$$

Corollary *The adjacency indices of a Lagrangian monosingularity of type* A_5^+ *are given in the following table:*

$\mathcal{A}$	A_1	A_1^3	A_1^5	A_2A_1	$A_2A_1^3$	A_3^+
$J_\mathcal{A}(A_5^+)$	1	1	1	2	4	1

$A_3^+A_1^2$	$A_3^-A_1^2$	$A_2^2A_1$	A_4A_1	$A_3^+A_2$
2	1	3	2	2

.

The table of the adjacency indices of a Lagrangian monosingularity of type A_5^- *is obtained from the table of indices* $J_\mathcal{A}(A_5^+)$ *by simultaneously changing all the signs in the superscripts of the multisingularity types.*

Corollary *The adjacency indices of a Lagrangian monosingularity of type* A_6 *are given in the following table:*

$\mathcal{A}$	$\mathbf{1}$	A_1^2	A_1^4	A_1^6	A_2	$A_2A_1^2$	$A_2A_1^4$	$A_3^+A_1$	$A_3^-A_1$
$J_\mathcal{A}(A_6)$	1	1	1	1	1	3	5	1	1

$A_3^+A_1^3$	$A_3^-A_1^3$	A_2^2	$A_2^2A_1^2$	A_4	$A_4A_1^2$	$A_3^+A_2A_1$	$A_3^-A_2A_1$
2	2	1	6	1	3	3	3

A_2^3	$A_5^+A_1$	$A_5^-A_1$	A_4A_2	$A_3^+A_3^-$
1	1	1	2	1

Lagrangian monosingularities of types $D_\mu^\pm$ are adjacent only to multisingularities of types $D_\nu^\pm A_{\mu_1}^\pm \dots A_{\mu_p}^\pm$ and $A_{\mu_1}^\pm \dots A_{\mu_p}^\pm$.

Theorem [7,8] *All adjacencies of Lagrangian monosingularities of types $D_\mu^\pm$ to multisingularities of types $D_\nu^\pm A_{\mu_1}^\pm \dots A_{\mu_p}^\pm$ are simple. The manifolds $\Xi_{A_{\mu_1}^\pm \dots A_{\mu_p}^\pm}(D_\mu^\pm)$ may have only connected components of two kinds: contractible components and components which are homotopy equivalent to a circle S^1.*

The combinatorial formulas for the indices $J_{D_\nu^\pm A_{\mu_1}^\pm \dots A_{\mu_p}^\pm}(D_\mu^\pm)$ and for the numbers of connected components of each kind of the manifolds $\Xi_{A_{\mu_1}^\pm \dots A_{\mu_p}^\pm}(D_\mu^\pm)$ are given in [7,8]. Here we give only some corollaries from them.

Corollary *The adjacency indices of a Lagrangian monosingularity of type D_5^+ are given in the following table:*

$\mathcal{A}$	A_1	A_1^3	A_1^5	A_2A_1	$A_2A_1^3$	$A_2^2A_1$	A_3^+	A_3^-
$J_\mathcal{A}(D_5^+)$	1	1	2	3	9	6	1	1

$A_3^+A_1^2$	$A_3^-A_1^2$	$A_3^+A_2$	$A_3^-A_2$	A_4A_1	$D_4^+A_1$	$D_4^-A_1$
4	4	2	2	2	1	1

The table of adjacency indices of a Lagrangian monosingularity of type D_5^- coincides with the table of indices $J_\mathcal{A}(D_5^+)$.

All adjacencies of Lagrangian monosingularities of types $D_5^\pm$ are simple except for the adjacencies to a multisingularity of type A_1^3, which are complicated. Namely, each of the manifolds $\Xi_{A_1^3}(D_5^+)$, $\Xi_{A_1^3}(D_5^-)$ has two connected components, one of which is contractible and the other is homotopy equivalent to a circle.

Corollary *The adjacency indices of a Lagrangian monosingularity of type D_6^+ are given in the following table:*

$\mathcal{A}$	**1**	A_1^2	A_1^4	A_1^6	A_2	$A_2A_1^2$	$A_2A_1^4$	A_2^2	$A_2^2A_1^2$	A_2^3
$J_\mathcal{A}(D_6^+)$	1	2	2	2	2	8	12	3	16	4

$A_3^+A_1$	$A_3^-A_1$	$A_3^+A_1^3$	$A_3^-A_1^3$	$A_3^+A_2A_1$	$A_3^-A_2A_1$	$(A_3^+)^2$	$(A_3^-)^2$
3	3	5	5	7	7	1	1

A_4	$A_4A_1^2$	A_4A_2	D_4^+	$D_4^+A_1^2$	$D_4^-A_1^2$	$D_4^+A_2$	$D_5^+A_1$	$D_5^-A_1$
2	4	4	1	2	1	2	1	1

All adjacencies of a Lagrangian monosingularity of type D_6^+ are simple except for the adjacency to a multisingularity of type A_1^4. The adjacency to a multisingularity of type A_1^4 is complicated: the manifold $\Xi_{A_1^4}(D_6^+)$ has three connected components, two of which are contractible and the third is homotopy equivalent to a circle.

Corollary *The adjacency indices of a Lagrangian monosingularity of type D_6^- are given in the following table:*

$\mathcal{A}$	A_1^2	A_1^4	A_1^6	$A_2A_1^2$	$A_2A_1^4$	$A_2^2A_1^2$	$A_3^+A_1$	$A_3^-A_1$
$J_{\mathcal{A}}(D_6^-)$	0	0	3	2	16	17	2	2

$A_3^+A_1^3$	$A_3^-A_1^3$	$A_3^+A_2A_1$	$A_3^-A_2A_1$	$(A_3^+)^2$	$A_3^+A_3^-$	$(A_3^-)^2$
8	8	11	11	1	2	1

$A_4A_1^2$	$A_5^+A_1$	$A_5^-A_1$	D_4^-	$D_4^+A_1^2$	$D_4^-A_1^2$	$D_4^-A_2$	$D_5^+A_1$	$D_5^-A_1$
6	2	2	1	1	2	2	1	1

All adjacencies of a Lagrangian monosingularity of type D_6^- are simple except for the adjacency to a multisingularities of types A_1^2, A_1^4, $A_2A_1^2$. The adjacencies to multisingularities of types A_1^2, A_1^4, $A_2A_1^2$ are complicated. Namely, the manifold $\Xi_{A_1^2}(D_6^-)$ is homotopy equivalent to a circle. The manifold $\Xi_{A_1^4}(D_6^-)$ is homotopy equivalent to a disjoint union of two circles. The manifold $\Xi_{A_2A_1^2}(D_6^-)$ has four connected components, two of which are contractible and each of the other two is homotopy equivalent to a circle.

At the moment we study the topology of Lagrangian monosingularities of types $E_6^{\pm}$. Recently we have described the topology of adjacencies of these monosingularities to multisingularities of a Lagrangian map at singular points of the caustic which are not points of transversal intersections of its smooth branches.

Theorem [9,10] *Let $\delta = \pm 1$ and*

$$E_6^{\delta} = \begin{cases} E_6^+ & if\, \delta = +1, \\ E_6^- & if\, \delta = -1. \end{cases}$$

Then all adjacencies of a Lagrangian monosingularity of type E_6^{δ} to multisingularities of types $\mathcal{A} \in \mathbb{S}$ such that $\mathcal{A} = X\mathcal{B}$, where $\mathcal{B} \in \mathbb{S}$ and X is a generator of the semigroup $\mathbb{S}$ with $\operatorname{codim} X > 1$, *are simple. The indices of these adjacencies are given in the following table:*

$\mathcal{A}$	$A_3^{\delta}A_1$	$A_3^{-\delta}A_1$	$A_3^{\delta}A_1^3$	$A_3^{-\delta}A_1^3$	$A_3^{\delta}A_2A_1$	$A_3^{-\delta}A_2A_1$
$J_{\mathcal{A}}(E_6^{\delta})$	2	3	6	5	8	6

$(A_3^{\delta})^2$	A_4	$A_4A_1^2$	A_4A_2	$A_5^+A_1$	$A_5^-A_1$
1	2	6	4	1	1

D_4^+	$D_4^+A_1^2$	$D_4^-A_1^2$	$D_4^+A_2$	$D_5^\delta A_1$
1	2	1	2	2

Remark Using relations (3) and (4), we get

$$J_{A_2^3}(E_6^\delta) = \frac{1}{2}\left(J_{A_2^3}(A_4A_2)\,J_{A_4A_2}(E_6^\delta) + J_{A_2^3}(D_4^+A_2)\,J_{D_4^+A_2}(E_6^\delta)\right)$$
$$= \frac{1}{2}\,(1\cdot 4 + 2\cdot 2) = 4.$$

6 The Adjacency Indices of Legendrian Monosingularities

In this section, we describe the topology of monosingularities of generic Legendrian maps with stable simple singularities. We recall that all these adjacencies are simple ([3]).

It is clear that a monosingularity of type A_1 is adjacent only to a multisingularity of type **1** and $J_{\mathbf{1}}(A_1) = 2$. A monosingularity of type A_2 is adjacent only to multisingularities of types **1** and A_1. As before, $J_{\mathbf{1}}(A_2) = J_{A_1}(A_2) = 2$.

Example A germ of the wave front of a generic Legendrian map with a monosingularity of type A_3 to a 3-space is shown in Fig. 7. The indices of all adjacencies of this monosingularity can easily be found from the given figure:

$\mathcal{A}$	**1**	A_1	A_1^2	A_2
$J_{\mathcal{A}}(A_3)$	3	4	1	2

Now, we describe the topology of adjacencies of all Legendrian corank 1 monosingularities. They may be adjacent only to multisingularities of corank 1.

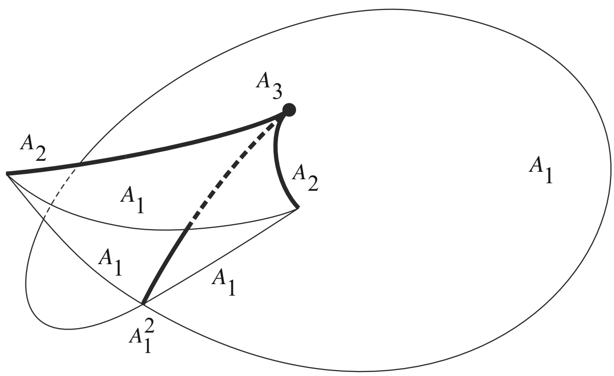

Fig. 7 A wave front germ with singularity of type A_3 (the swallowtail)

Theorem [5,6] *Let* $\mathcal{A} = A_{\mu_1}^{k_1} \dots A_{\mu_p}^{k_p}$, *where* $\mu_1, \dots, \mu_p$ *are pairwise-distinct positive integers. Then a Legendrian monosingularity of type* A_μ *is adjacent to a multisingularity of type* $\mathcal{A}$ *if and only if the number* $N = \mu + 1 - \sum_{i=1}^{p} k_i(\mu_i + 1)$ *is non-negative. The adjacency index is calculated by the formula*

$$J_{\mathcal{A}}(A_\mu) = \sum_{\substack{0 \le k_0 \le N \\ k_0 \equiv N \ (\mathrm{mod}\ 2)}} \frac{(k_0 + k_1 + \cdots + k_p)!}{k_0! \cdot k_1! \cdot \dots \cdot k_p!}.$$

Corollary *The adjacency indices of Legendrian monosingularities of types* A_4, A_5, A_6 *are given in the following tables:*

$\mathcal{A}$	$\mathbf{1}$	A_1	A_2	A_1^2	A_3	A_2A_1
$J_{\mathcal{A}}(A_4)$	3	6	4	3	2	2

,

$\mathcal{A}$	$\mathbf{1}$	A_1	A_2	A_1^2	A_3	A_2A_1	A_1^3	A_4	A_3A_1	A_2^2
$J_{\mathcal{A}}(A_5)$	4	9	6	7	4	6	1	2	2	1

,

$\mathcal{A}$	$\mathbf{1}$	A_1	A_2	A_1^2	A_3	A_2A_1	A_1^3
$J_{\mathcal{A}}(A_6)$	4	12	9	13	6	14	4

A_4	A_3A_1	A_2^2	$A_2A_1^2$	A_5	A_4A_1	A_3A_2
4	6	3	3	2	2	2

.

Theorem [5,6] *The adjacency indices of Legendrian monosingularities of types* $D_4^{\pm}$, D_5, $D_6^{\pm}$, E_6 *are given in the following tables:*

$\mathcal{A}$	$\mathbf{1}$	A_1	A_2	A_1^2	A_3
$J_{\mathcal{A}}(D_4^+)$	3	4	2	1	2

,

$\mathcal{A}$	$\mathbf{1}$	A_1	A_2	A_1^2	A_3	A_1^3
$J_{\mathcal{A}}(D_4^-)$	7	14	6	9	6	2

,

$\mathcal{A}$	$\mathbf{1}$	A_1	A_2	A_1^2	A_3	A_2A_1	A_1^3
$J_{\mathcal{A}}(D_5)$	6	15	9	12	8	6	3

D_4^+	D_4^-	A_4	A_3A_1	$A_2A_1^2$
1	1	2	2	1

,

$\mathcal{A}$	$\mathbf{1}$	A_1	A_2	A_1^2	A_3	A_2A_1	A_1^3	D_4^+	D_4^-
$J_{\mathcal{A}}(D_6^+)$	6	16	10	14	10	10	4	3	1

A_4	A_3A_1	A_2^2	$A_2A_1^2$	D_5	$D_4^+A_1$	A_3A_2
2	6	2	2	2	2	2

,

$\mathcal{A}$	$\mathbf{1}$	A_1	A_2	A_1^2	A_3	A_2A_1	A_1^3	D_4^+	D_4^-
$J_{\mathcal{A}}(D_6^-)$	11	34	20	40	20	26	18	1	3

A_4	A_3A_1	A_2^2	$A_2A_1^2$	A_1^4	D_5	$D_4^-A_1$	A_5	A_3A_2	$A_3A_1^2$
6	18	4	4	3	2	2	4	2	2

,

$\mathcal{A}$	$\mathbf{1}$	A_1	A_2	A_1^2	A_3	A_2A_1	A_1^3
$J_{\mathcal{A}}(E_6)$	5	15	11	17	10	16	6

D_4^+	D_4^-	A_4	A_3A_1	A_2^2	$A_2A_1^2$	D_5	A_5	A_4A_1	$A_2^2A_1$
1	1	6	6	2	5	2	2	2	1

.

Combinatorial formulas for the calculation of indices $J_{\mathcal{A}}(D_\mu^\pm)$, $\mu \geq 7$ in the Legendrian case are not found yet.

Problem Study the topology of Lagrangian and Legendrian monosingularities of types E_7 and E_8.

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