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RESEARCH CONTRIBUTION

Torus Action on Quaternionic Projective Plane and Related Spaces

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Abstract
For an effective action of a compact torus T on a smooth compact manifold X with nonempty finite set of fixed points, the number $\frac{1}{2}\dim X - \dim T$ is called the complexity of the action. In this paper, we study certain examples of torus actions of complexity one and describe their orbit spaces. We prove that $\mathbb{H}P^2/T^3 \cong S^5$ and $S^6/T^2 \cong S^4$, for the homogeneous spaces $\mathbb{H}P^2 = \mathrm{Sp}(3)/(\mathrm{Sp}(2) \times \mathrm{Sp}(1))$ and $S^6 = G_2/\mathrm{SU}(3)$. Here, the maximal tori of the corresponding Lie groups Sp(3) and G_2 act on the homogeneous spaces from the left. Next we consider the quaternionic analogues of smooth toric surfaces: they give a class of eight-dimensional manifolds with the action of T^3. This class generalizes $\mathbb{H}P^2$. We prove that their orbit spaces are homeomorphic to S^5 as well. We link this result to Kuiper–Massey theorem and its generalizations studied by Arnold.

Keywords Torus action · Complexity one · Quaternions · Octonions · Quasitoric manifold · Kuiper–Massey theorem

Mathematics Subject Classification 55R91 · 57S15 · 57S25 · 22F30 · 57R91 · 57S17 · 20G20 · 57M60 · 20G41

1 Introduction

Consider an effective action of the compact torus T^k on a compact smooth manifold $X = X^{2n}$, such that the set of fixed points is finite and nonempty. The number $n - k$ can be shown to be nonnegative; it is called the *complexity of the action*.

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Buchstaber and Terzic [6–8] introduced the theory of $(2n, k)$-manifolds to study the orbit spaces of nonfree actions of a compact torus T^k on $2n$-manifolds. Using this theory, they proved the homeomorphisms $G_{4,2}/T^3 \cong S^5$ and $F_3/T^2 \cong S^4$, where $G_{4,2}$ is the Grassmann manifold of complex 2-planes in $\mathbb{C}^4$, and F_3 is the manifold of complete flags in $\mathbb{C}^3$. The torus actions are naturally induced from the standard torus actions on $\mathbb{C}^4$ and $\mathbb{C}^3$, respectively. In both cases of $G_{4,2}$ and F_3, the complexity of the natural torus action is equal to 1. Karshon and Tolman proved in [15] that, for Hamiltonian actions of complexity one, the orbit space is homeomorphic to a sphere provided that the weights of the tangent representation at each fixed point are in general position (see Definition 2.2). This result covers the cases of $G_{4,2}$ and F_3.

If X^{2n} is a quasitoric manifold with the action of T^n, and $T^{n-1} \subset T^n$ is a subtorus, such that the induced action of T^{n-1} on X^{2n} is in general position, then the orbit space X^{2n}/T^{n-1} is homeomorphic to a sphere [4], as well.

For an action of a torus T on a space X, consider the fibration $X \hookrightarrow X \times_T ET \to BT$ and the corresponding Serre spectral sequence:

$$E_2^{*,*} \cong H^*(BT) \otimes H^*(X) \Rightarrow H^*(X \times_T ET) = H_T^*(X), \tag{1.1}$$

where $ET \xrightarrow{T} BT$ is the universal T-bundle, $X \times_T ET$ is the Borel construction of X, and $H_T^*(X)$ is the equivariant cohomology algebra[1] of X. The space X with a torus action is called *equivariantly formal* in the sense of Goresky–Kottwitz–Macpherson [12] if its Serre spectral sequence (1.1) degenerates at E_2 term. In particular, the spaces with vanishing odd degree cohomology are all equivariantly formal. It is known [16, Prop.5.8] that manifolds with Hamiltonian torus actions are equivariantly formal. Quasitoric manifolds also have vanishing odd degree cohomology, and hence, the restricted actions of T^{n-1} on quasitoric manifolds are equivariantly formal. This leads to the following question.

Problem 1 *Assume the complexity one action of $T = T^{n-1}$ on a compact smooth manifold $X = X^{2n}$ has isolated fixed points and the tangent weights at each fixed point are in general position. Does equivariant formality of X imply that X/T is homeomorphic to a sphere?*

This problem is related to the result of Masuda and Panov [20], which states that complexity zero action is equivariantly formal if and only if its orbit space is a manifold with corners all of whose faces are acyclic.

We have an example supporting the relation between equivariant formality and sphericity of the orbit space from the other side. In [3], the space of isospectral periodic tridiagonal Hermitian matrices of size n was studied. These spaces provide an infinite series of manifolds with torus actions of complexity one satisfying the general assumption of the problem. For $n \geq 4$ these manifolds are not equivariantly formal and their orbit spaces are not spheres.

Besides $G_{4,2}$ and F_3, there exist two other natural examples of actions satisfying the assumptions of the problem: these examples appear in the classification of torus

[1] We assume that all cohomology rings are taken with $\mathbb{Z}$ coefficients unless stated otherwise.

actions of complexity one on homogeneous spaces, see [18]. These are the T^2-action on $S^6 = G_2/SU(3)$ and the T^3-action on $\mathbb{H}P^2 = \mathbb{H}^3/\mathbb{H}^* \cong \mathrm{Sp}(3)/(\mathrm{Sp}(2) \times \mathrm{Sp}(1))$. Here, G_2 is considered as the automorphism group of octonion algebra, S^6 is the sphere of imaginary octonions of unit length, and T^2 is the maximal torus of G_2, acting on G_2 by left multiplication. The torus T^3 acts on $\mathbb{H}^3$ by left multiplication of each homogeneous coordinate; it could as well be understood as a maximal torus of Sp(3) acting on the homogeneous space $\mathrm{Sp}(3)/(\mathrm{Sp}(2) \times \mathrm{Sp}(1))$ from the left. Notice, however, that the action of T^3 on $\mathbb{H}P^2$ has a discrete subgroup $\langle(-1, -1, -1)\rangle \cong \mathbb{Z}_2$ as a noneffective kernel. To make the action effective, we consider the action of the torus $T^3/\langle(-1, -1, -1)\rangle \cong T^3$.

Both $\mathbb{H}P^2$ and S^6 have vanishing odd degree cohomology, and hence, they are equivariantly formal. In this paper we prove

Theorem 1 *Let the maximal compact torus $T^2 \subset G_2$ act on $S^6 = G_2/\,\mathrm{SU}(3)$ from the left. Then, S^6/T^2 is homeomorphic to S^4.*

This statement is proved by the technique, described in [4]. However, to prove the applicability of this technique, we need an explicit description of a maximal torus of the group G_2. This is done in Sect. 2, where we recall the basic ideas of [4] concerning the restrictions of actions of complexity zero to the actions of complexity one. In Sect. 2, we also give an example of a nonequivariantly formal manifold with complexity one torus action, whose orbit space is, however, homeomorphic to a sphere. This example explains why the implication in Problem 1 cannot be reversed.

Our second result is the following.

Theorem 2 *Let a maximal compact torus $T^3 \subset \mathrm{Sp}(3)$ acts on the homogeneous space $\mathbb{H}P^2 = \mathrm{Sp}(3)/(\mathrm{Sp}(2) \times \mathrm{Sp}(1))$ from the left. Then, $\mathbb{H}P^2/T^3$ is homeomorphic to S^5.*

This statement is related to the result of Arnold [1, Example 4] which asserts the homeomorphism:

$$\mathbb{H}P^2/T^1 \cong S^7 \tag{1.2}$$

The proof of Theorem 2 relies on the set of ideas, similar to those used by Arnold. In Sect. 3, we make some preparations related, in particular, to the notion of spectrohedron. Then, in Sect.4, we prove Theorem 2, describe the equivariant skeleton of $\mathbb{H}P^2$, and show its connection to equivariant topology of the Grassmann manifold $G_{4,2}$.

Next, we recall the classical Kuiper–Massey theorem [17,19] which asserts the homeomorphism $\mathbb{C}P^2/\,\mathrm{conj} \cong S^4$. Here, conj is the antiholomorphic involution on the complex projective plane, which conjugates all homogeneous coordinates simultaneously. In [1], Arnold discussed this theorem and noticed its closed relation to the homeomorphism (1.2). These two results were further extended by Atiyah and Berndt [2] who proved their octonionic version, namely, $\mathbb{O}P^2/\,\mathrm{Sp}(1) \cong S^{13}$, where $\mathrm{Sp}(1) \cong \mathrm{SU}(2) \cong S^3$ is the group of unit quaternions.

Note that Kuiper–Massey theorem is not a specific property of the complex projective plane. In Sect. 5, we recall a generalization [11] of Kuiper–Massey theorem, which asserts that $X/\operatorname{conj} \cong S^4$ for any smooth compact toric surface X. This result can be extended to quasitoric manifolds if one defines the involution conj in a natural way, see Proposition 5.7. Due to this observation, we suggested that Theorem 2 and Arnold's homeomorphism (1.2) can also be extended to more general "quaternionic surfaces".

Following the work of Jeremy Hopkinson [14], we consider the class of quaternionic analogues of quasitoric manifolds. In [14], these spaces were called *quoric* manifolds; we borrow this terminology. More specifically, we are interested in compact eight-dimensional manifolds, carrying the action of $S^3 \times S^3$, which is locally standard in certain sense, and whose orbit space is diffeomorphic to a polygon. This class of manifolds naturally contains the spaces $\mathbb{H}P^2$ (the $(S^3)^2$-orbit space is a triangle) and $\mathbb{H}P^1 \times \mathbb{H}P^1$ (the $(S^3)^2$-orbit space is a square). A lot of care and a lot of preparatory work should be made only to define quoric manifolds and their basic properties, since the acting group is noncommutative: the intuition behind many aspects of quasitoric manifolds may fail in quaternionic case. This big work was done in detail in [14]. Since we do not have an opportunity to give all the definitions and statements, we only provide rough ideas of the constructions of quoric manifolds, and specifically restrict our attention to dimension 8.

Our observation, which seems to have not been covered previously, is the following. We noticed that each eight-dimensional quoric manifold carries an effective action of T^3 (not just the induced action of $T^2 \subset S^3 \times S^3$, which is natural to expect). Then, we prove the following generalization of Theorem 2.

Theorem 3 *For any eight-dimensional quoric manifold, its orbit space by the T^3-action is homeomorphic to S^5:*

However, while the quotient $\mathbb{H}P^2/T^3$ can be understood geometrically by gluing two copies of five-dimensional spectrohedra along their boundaries (see Proposition 3.2), the quotients of quoric manifolds lack such a description. This situation is similar to the generalization of Kuiper–Massey theorem: while the quotient $\mathbb{C}P^2/\operatorname{conj}$ can be understood as the boundary of the five-dimensional spectrohedron, the quotient spaces X/conj of general toric surfaces do not have a description in terms of convex geometry. We believe that both toric and quoric surfaces require further study in the same context.

2 Reductions of Half-Dimensional Actions

For a smooth action of a Lie group G on a smooth manifold X, define the partition of X by orbit types:

$$X = \bigsqcup_{H \in S(G)} X^{(H)}. \tag{2.1}$$

Here, H runs over all closed subgroups of G and $X^{(H)} = \{x \in X \mid G_x = H\}$. The decomposition (2.1) is called the fine partition of X and $X^{(H)}$ is called a fine partition element.

Definition 2.1 An effective action of G on a compact smooth manifold X is called *appropriate* if:

- the fixed points set $X^{(G)}$ is finite;
- (adjoining condition) the closure of every connected component of a fine partition element $X^{(H)}$, $H \neq G$, contains a point x' with $\dim G_{x'} > \dim H$.

In the following, we assume that all actions are effective and appropriate.

Assume the compact torus T^k acts smoothly on a manifold $X = X^{2n}$, and let x be an isolated fixed point. The representation of the torus in the tangent space T_xX decomposes into the sum of two-dimensional real representations:

$$T_xX = V(\alpha_1) \oplus \cdots \oplus V(\alpha_n),$$

where $\alpha_1, \dots, \alpha_n \in \mathrm{Hom}(T^k, T^1) \cong \mathbb{Z}^k$, and $V(\alpha_i)$ is the representation $tv = \alpha(t) \cdot v$ for $v \in \mathbb{C} \cong \mathbb{R}^2$. In general, the weights $\alpha_1, \dots, \alpha_n$ are defined uniquely up to sign. One needs to impose a stably complex structure on X for the weights to be defined without sign ambiguity. However, in the following, we do not need stably complex structure; all definitions are invariant under the change of signs of the weight vectors.

Definition 2.2 An appropriate effective action of T^n on X^{2n} is called *locally standard* if the weights of the tangent representation at each fixed point form a basis of the lattice $\mathrm{Hom}(T^n, T^1) \cong \mathbb{Z}^n$. An appropriate effective action of T^{n-1} on X^{2n} is called *a complexity one action in general position* if, for any fixed point, any $n-1$ of n tangent weights $\alpha_1, \dots, \alpha_n$ are linearly independent over $\mathbb{Q}$.

The orbit space of any locally standard action is a manifold with corners (the appropriate condition means that any face of this manifold has a vertex). The orbit space of a complexity one action in general position is a topological manifold (see [4, Thm.2.10]).

Proposition 2.3 *Consider a locally standard action of T^n on $X = X^{2n}$, whose orbit space is homeomorphic to a disc D^n. Assume that $T^{n-1} \subset T^n$ is a subtorus, such that the induced action of T^{n-1} on X is in general position. Then, X/T^{n-1} is homeomorphic to the sphere S^{n+1}.*

Proof In [4], this fact was proved in the case when X/T^n is a simple polytope. The general statement is completely similar, so we only give an idea of the proof. Consider the induced map:

$$p\colon X/T^{n-1} \to X/T^n \cong D^n.$$

The map p is the projection to the orbit space, for the residual action of $T^1 = T^n/T^{n-1}$ on X/T^{n-1}. Interior points of D^n correspond to free orbits, while the boundary points of D^n correspond to point orbits. Hence, X/T^{n-1} is the identification space $D^n \times S^1$, where S^1 is collapsed over points of the boundary ∂D^n. This yields the result. □

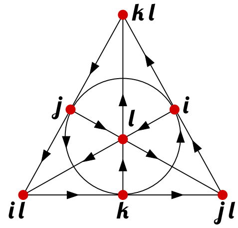

Fig. 1 Fano plane model for the multiplication of octonionic imaginary units

Construction 2.4 *Consider the standard T^3-action on S^6, given by:*

$$(t_1, t_2, t_3)(r, z_1, z_2, z_3) = (r, t_1 z_1, t_2 z_2, t_3 z_3), \tag{2.2}$$

where $S^6 = \{(r, z_1, z_2, z_3) \in \mathbb{R} \times \mathbb{C}^3 \mid r^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$. *The orbit space*

$$S^6/T^3 = \{(r, c_1, c_2, c_3) \in \mathbb{R} \times \mathbb{R}^3_{\geq 0} \mid r^2 + c_1^2 + c_2^2 + c_3^2 = 1\}$$

is a manifold with two corners homeomorphic to a 3-disc (sometimes this space is called the "rugby ball"). The subtorus

$$T^2 = \{(t_1, t_2, t_3) \subset T^3 \mid t_1 t_2 t_3 = 1\}$$

acts on S^6 in general position. Hence, Proposition 2.3 implies:

$$S^6/T^2 \cong S^4. \tag{2.3}$$

This formula already looks like the proof of Theorem 1. However, we need to check that the action of $T^2 \subset T^3$ on S^6 defined by Construction 2.4 is the same as the action in Theorem 1.

Proposition 2.5 *The constructed action of T^2 on S^6 coincides with the action of a maximal torus $T^2 \subset G_2$ on the sphere $G_2/\operatorname{SU}(3)$ of unit imaginary octonions.*

Proof Let $1, l, i, j, k, il, jl, kl$ be the standard basis of the octonion algebra $\mathbb{O}$ over $\mathbb{R}$, with the multiplication given by the standard mnemonic diagram shown of Fig. 1. For each $\alpha, \beta, \gamma \in \mathbb{R}/2\pi\mathbb{Z}$, $\alpha + \beta + \gamma = 0$, consider the automorphism $\sigma_{\alpha,\beta,\gamma}$ of $\mathbb{O}$ given by:

$$\sigma_{\alpha,\beta,\gamma}(1) = 1; \quad \sigma_{\alpha,\beta,\gamma}(l) = l; \quad \sigma_{\alpha,\beta,\gamma}(i) = \mathrm{e}^{\alpha l} i;$$
$$\sigma_{\alpha,\beta,\gamma}(j) = \mathrm{e}^{\beta l} j; \quad \sigma_{\alpha,\beta,\gamma}(k) = \mathrm{e}^{\gamma l} k.$$

Here, we use the notation $\mathrm{e}^{\phi\epsilon} = \cos\phi + \epsilon \sin\phi$ for any imaginary unit $\epsilon \in \{i, j, k\}$ and $\phi \in \mathbb{R}/2\pi\mathbb{Z}$. A direct calculation shows that $\sigma_{\alpha,\beta,\gamma}$ is indeed an automorphism of $\mathbb{O}$. Moreover, we have:

$$\sigma_{\alpha_1+\alpha_2,\beta_1+\beta_2,\gamma_1+\gamma_2} = \sigma_{\alpha_1,\beta_1,\gamma_1} \circ \sigma_{\alpha_2,\beta_2,\gamma_2}.$$

Hence, the torus $T_\sigma^2 = \{\sigma_{\alpha,\beta,\gamma} \mid \alpha+\beta+\gamma = 0\}$ is a maximal torus of G_2. Let us write the automorphism $\sigma_{\alpha,\beta,\gamma}\colon \mathbb{R}^8 \to \mathbb{R}^8$ in the matrix form. In the basis $1, l, i, il, j, jl, k, kl$, we have:

$$\sigma_{\alpha,\beta,\gamma} = \begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && \cos\alpha & \sin\alpha &&&& \\ && -\sin\alpha & \cos\alpha &&&& \\ &&&& \cos\beta & \sin\beta && \\ &&&& -\sin\beta & \cos\beta && \\ &&&&&& \cos\gamma & \sin\gamma \\ &&&&&& -\sin\gamma & \cos\gamma \end{pmatrix},$$

where void spaces denote zeroes. Since $\alpha+\beta+\gamma = 0$, we see that the action of T_σ^2 on the 6-sphere of unit imaginary octonions is exactly the restriction of the standard action (2.2) of T^3 on S^6 to the subtorus $T^2 = \{(t_1, t_2, t_3) \in T^3 \mid t_1t_2t_3 = 1\}$. □

Proposition 2.5 and (2.3) imply Theorem 1.

Now, we use the arguments from the beginning of this section to show that the reverse statement to Problem 1 is false.

Proposition 2.6 *There exists a manifold $X = X^{2n}$ with an appropriate complexity one action of T^{n-1} in general position, such that the orbit space X/T^{n-1} is homeomorphic to S^{n+1}; however, the action is not equivariantly formal.*

Proof In view of Proposition 2.3, the idea of the following construction is very simple: we construct a not equivariantly formal manifold with half-dimensional torus action, whose orbit space is a disc. For locally standard torus action of T^n on X^{2n}, we have the result of Masuda and Panov [20] which asserts the equivalence of the following conditions:

- the action of T^n on X^{2n} is equivariantly formal;
- the orbit space X^{2n}/T^n is a homology polytope (that is a manifold with corners with all faces $\mathbb{Z}$-acyclic);
- $H^{\text{odd}}(X^{2n}) = 0$.

□

Lemma 2.7 (see [22]) *There exists a manifold with corners P^n which satisfies the following properties:*

1. *Every face of P^n has a vertex.*
2. *P^n is homeomorphic to a disc.*
3. *P^n is not a homology polytope.*
4. *P^n is the orbit space of some locally standard torus action.*

Proof The example of such manifold with corners is shown in Fig. 2: it is homeomorphic to 3-disc; however, one of its facets is nonacyclic. We describe the procedure

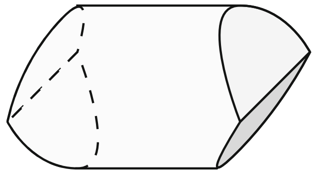

Fig. 2 A manifold with corners P^3, satisfying the properties of Lemma 2.7. It is obtained by taking a connected sum of two rugby balls along points in the interior of their facets

which allows to construct many other examples. The details of this procedure can be found in [22].

Take any two simple polytopes P_1^n and P_2^n, with $n \geq 3$. Let $2 \leq k < n$, and x_1 (resp. x_2) be a point lying in the interior of some k-dimensional face of P_1^n (resp. P_2^n). One can form a connected sum $P^n = P_1^n \#_{x_1,x_2} P_2^n$ of manifolds with corners P_1^n and P_2^n along the points x_1, x_2. The conditions 1–3 are easily checked for P^n. Now, if both P_1^n and P_2^n are the orbit spaces of some locally standard manifolds X_1 and X_2, then P^n is the orbit space of the manifold $X_1 \#_{T^k} X_2$, where the connected sum is taken along some k-dimensional torus orbit. For example, the manifold with corners shown in Fig. 2 is the orbit space of the manifold $S^6 \#_{T^2} S^6$ carrying the locally standard action of T^3. We refer the reader to [22] where these manifolds were studied in detail. □

If P^n is a manifold with corners, given by Lemma 2.7, and X is a torus manifold over P^n, then Proposition 2.3 implies that $X/T^{n-1} \cong S^{n+1}$ for any subtorus $T^{n-1} \subset T^n$ in general position. On the other hand, the result of Masuda and Panov implies that X is not equivariantly formal with respect to the action of T^n. However, we shall check that X is not equivariantly formal with respect to the action of T^{n-1}, as well.

The latter fact is the consequence of the localization theorem. Indeed, assume the converse, that is X is equivariantly formal with respect to T^{n-1}. This implies $H^*_{T^{n-1}}(X) \cong H^*(BT^{n-1}) \otimes H^*(X)$. Lemma 2.7 and the result of Masuda and Panov [20] imply that $H^*(X)$ has nontrivial odd degree components. Hence, $H^*_{T^{n-1}}(X)$ has nontrivial odd degree components, as well. Localization theorem asserts the isomorphism of the localized modules:

$$S^{-1} H^*_{T^{n-1}}(X) \cong S^{-1} H^*_{T^{n-1}}(X^{T^{n-1}}), \tag{2.4}$$

for some homogeneous multiplicative set $S \subset H^*(BT^{n-1})$. However, the fixed point set $X^{T^{n-1}}$ is finite by assumption; therefore, $H^*_{T^{n-1}}(X^{T^{n-1}}) \cong H^*(BT^{n-1})^d$. Hence, the right-hand side of (2.4) does not have odd degree components. This gives a contradiction. □

3 Vector Tuples Up to Orthogonal Transformations

Let $\mathrm{Mat}_{l\times k}$ denote the space of nonzero real $(l \times k)$-matrices and:

$$Y_{l,k} = \mathrm{Mat}_{l\times k}/\mathbb{R}_+ \cong S^{lk-1}$$

denote the sphere of normalized matrices. The orthogonal group $O(l)$ acts on $Y_{l,k}$ by the left multiplication. In [1], Arnold proved.

Proposition 3.1 *The orbit space $Y_{n-1,n}/O(n-1)$ is homeomorphic to the sphere of dimension $(n^2+n-4)/2$.*

Proof We outline the proof. Each matrix $A \in Y_{n-1,n}$ can be thought as a normalized n-tuple of vectors in Euclidean space $\mathbb{R}^{n-1}$ which are not simultaneously zero. With each such tuple, we can associate its Gram matrix, that is the square matrix $G = A^{\top}A$ of size n. It can be seen that two n-tuples of vectors produce the same Gram matrix if and only if the tuples differ by common orthogonal transformation. All Gram matrices produced from $Y_{n-1,n}$ are positive semidefinite symmetric matrices; moreover, they are degenerate and nonzero. Positive semidefinite symmetric matrices form a strictly convex cone C_n in the space of all symmetric matrices of size n (the space and the cone both have dimension $n(n+1)/2$). The boundary of C_n consists of degenerate positive semidefinite matrices. Therefore, the space of rays lying in ∂C_n is homeomorphic to $Y_{n-1,n}/O(n-1)$. On the other hand, the space of rays, lying in the boundary of any strictly convex cone of dimension d, is a sphere of dimension $d-2$. □

It is convenient to intersect the strictly convex cone C_n of positive semidefinite $(n \times n)$-matrices with a generic affine hyperplane. Whenever such an intersection is nonempty and bounded, the intersection is called a *spectrohedron*[2]. A spectrohedron is therefore a compact convex body, defined uniquely up to projective transformations. Denoting the spectrohedron by Spec_n, we have the formula $\dim \mathrm{Spec}_n = n(n+1)/2 - 1$. Therefore, spectrohedra have dimensions 2, 5, 9, 14, etc.

Proposition 3.2 *The orbit space $Y_{n,n}/O(n)$ is a disc of dimension $d = (n^2+n-2)/2$. The orbit space $Y_{n,n}/\mathrm{SO}(n) = S^{n^2-1}/\mathrm{SO}(n)$ is homeomorphic to a sphere of the same dimension d.*

Proof For a matrix $A \in \mathrm{Mat}_{n\times n}$, we consider its polar decomposition $A = QP$, where $Q \in O(n)$ and P is a positive semidefinite symmetric matrix. This decomposition is nonunique if A is degenerate; however, the nonnegative part of the decomposition is uniquely determined by the formula $P = \sqrt{A^{\top}A}$ for any matrix A. The first part of the statement follows easily: the space $Y_{n,n}/O(n)$ is homeomorphic to the set of normalized positive semidefinite symmetric matrices of size n, which is nothing but the spectrohedron Spec_n, hence a disc of the required dimension.

Let $Z_{\geq} = \{A \in Y_{n,n} \mid \det A \geq 0\}$ and $Z_{\leq} = \{A \in Y_{n,n} \mid \det A \leq 0\}$. For $A \in Z_{\geq}$, the matrix Q in the polar decomposition can be chosen from $\mathrm{SO}(n)$; therefore, $Z_{\geq}/\mathrm{SO}(n) \cong D^d$ as in the previous case. On the other hand, there is an $\mathrm{SO}(n)$-equivariant homeomorphism $Z_{\geq} \to Z_{\leq}$, given by the right multiplication by a fixed reflection matrix. Therefore, we also have $Z_{\leq}/\mathrm{SO}(n) \cong D^d$. The boundaries of both discs are formed by (normalized) degenerate n-tuples in $\mathbb{R}^n$ considered up to rotations: in degenerate case, the matrix Q in the polar decomposition can be taken either with positive or negative determinant; therefore, such points belong to both discs. The

[2] In some sources, the term spectrohedron denotes an intersection of the cone C_n with a plane of any dimension, not just hyperplanes.

boundary sphere of both discs is described by Proposition 3.1. Two discs patched along common boundary form a sphere. □

Example 3.3 Taking $n = 2$ in the previous proposition, we get $Y_{2,2}/\operatorname{SO}(2) \cong S^2$. The space $Y_{2,2}$ of normalized 2-tuples in $\mathbb{R}^2$ coincides with the join of two circles $S^1 * S^1 \cong S^3$, and the circle $S^1 = \operatorname{SO}(2)$ acts on the join $S^1 * S^1$ diagonally. Hence, the map $Y_{2,2} \to S^2$ coincides with the Hopf fibration.

Example 3.4 For $n = 3$, we get $Y_{3,3}/\operatorname{SO}(3) \cong S^5$. The geometrical meaning of this homeomorphism is explained in the next section.

4 Quaternionic Projective Plane

Recall that $\mathbb{H}P^2 = \mathbb{H}^3/\mathbb{H}^* = S^{11}/S^3$, where S^{11} is the unit sphere in the space $\mathbb{H}^3 \cong \mathbb{C}^6 \cong \mathbb{R}^{12}$, and S^3 is the sphere of unit quaternions. The group $S^3 = \operatorname{Sp}(1)$ is isomorphic to $\operatorname{SU}(2)$ and its action on $\mathbb{H}^3$ can be identified with the natural coordinate-wise action of $\operatorname{SU}(2)$ on $(\mathbb{C}^2)^3$. In the following, we assume that the groups $\operatorname{SU}(2)$ act from the right, while the torus T^3 acts on $\mathbb{H}^3 \cong (\mathbb{C}^2)^3$ by the left multiplication. Each component of the torus acts on the corresponding copy of $\mathbb{C}^2$ by multiplication.

We now prove Theorem 2, that is the homeomorphism $\mathbb{H}P^2/T^3 \cong S^5$.

Proof of Theorem 2 We need to describe the double quotient:

$$\mathbb{H}P^2/T^3 = T^3\backslash S^{11}/S^3.$$

First note that S^{11} can be represented as the join:

$$S^{11} \cong S^3 * S^3 * S^3,$$

where each S^3 is the sphere of unit quaternions, taken in the corresponding factor of the product $\mathbb{H} \times \mathbb{H} \times \mathbb{H}$. Therefore:

$$T^3\backslash S^{11} \cong (S^1\backslash S^3) * (S^1\backslash S^3) * (S^1\backslash S^3) \cong \mathbb{C}P^1 * \mathbb{C}P^1 * \mathbb{C}P^1,$$

since each factor of the join forms a Hopf bundle $S^3 \xrightarrow{S^1} \mathbb{C}P^1$. Let us describe the quotient:

$$T^3\backslash S^{11}/S^3 \cong (\mathbb{C}P^1 * \mathbb{C}P^1 * \mathbb{C}P^1)/\operatorname{SU}(2),$$

where $\operatorname{SU}(2)$ acts simultaneously on all copies of $\mathbb{C}P^1$ in the standard way. The action of $\operatorname{SU}(2)$ on $\mathbb{C}P^1$ has noneffective kernel $\{\pm 1\}$, which therefore reduces to the action of $\operatorname{SU}(2)/\{\pm 1\} \cong \operatorname{SO}(3)$ on $\mathbb{C}P^1$. This action coincides with the action of the rotation group $\operatorname{SO}(3)$ on the round 2-sphere $S^2 \subset \mathbb{R}^3$. Hence, we need to describe the orbit space:

$$(S^2 * S^2 * S^2)/\operatorname{SO}(3)$$

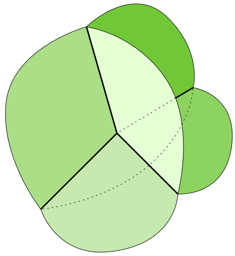

Fig. 3 Local structure of a sponge for a T^3-action on X^8

Note that the join $S^2 * S^2 * S^2 \cong S^8$ coincides with the sphere $Y_{3,3}$ of normalized 3-tuples of vectors in $\mathbb{R}^3$. Proposition 3.2 implies that the quotient $S^8/\operatorname{SO}(3)$ is homeomorphic to S^5. This concludes the proof of Theorem 2. □

Recall that given an action of a compact torus $T = T^k$ on a manifold $X = X^{2n}$, we have an equivariant filtration of X:

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_k = X,$$

where the filtration term X_i consists of torus orbits of dimension at most i. If $Q = X/T$ denotes the orbit space, we obtain the quotient filtration on Q:

$$Q_0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_k = Q, \qquad Q_i = X_i/T.$$

As was mentioned in Sect. 2, whenever all fixed points of a T^{n-1}-action on X^{2n} are isolated, and the weights are in general position, then the orbit space is a topological manifold. Moreover, in this case $\dim Q_i = i$, $\dim X_i = 2i$ for $i \leq n-2$, and the $(n-1)$-skeleton Q_{n-2} has a local topological structure, encoded in the notion of a *sponge* [4].

A sponge is a topological space locally modeled by an $(n-2)$-skeleton of the fan Δ_{n-1} of the toric variety $\mathbb{C}P^{n-1}$. The sponge corresponding to an appropriate action of T^2 on a 6-manifold is a three-valent graph: this is merely the GKM graph of the action. The sponges corresponding to appropriate actions of T^3 on 8-manifolds are locally modeled by the space, as shown in Fig. 3. In [4], we described the sponges for F_3 and $G_{4,2}$, and showed that, in some cases, the question of extendability of the torus action to the action of a larger torus can be reduced to the question of embeddability of the sponge into a low-dimensional sphere.

We now describe the torus action on $\mathbb{H}P^2$ and its equivariant skeleton in detail, to understand its sponge (the result is shown in Fig. 5). Let $[h_0 : h_1 : h_2]$ be the homogeneous coordinates on $\mathbb{H}P^2$, defined up to multiplication by $h \in \mathbb{H}^*$ from the right. We represent any quaternion $h \in \mathbb{H}$ as $h = z + ju$, and write $h = (z, u)$, where $z, u \in \mathbb{C}$, and j is the imaginary unit. The letter t denotes the element of one-dimensional torus: $t \in \mathbb{C}$, $|t| = 1$. As was already mentioned, the torus T^3 acts on $\mathbb{H}P^2$ by multiplication from the left:

$$(t_0, t_1, t_2)[h_0 : h_1 : h_2] = [t_0h_0 : t_1h_1 : t_2h_2].$$

It can be seen that the left action of the circle on $\mathbb{H}$ can be written in complex coordinates as follows: $t(z,u)=(tz,t^{-1}u)$.

The discrete subgroup $\langle(-1,-1,-1)\rangle\subset T^3$ acts trivially. So far, to make the action effective, we consider the induced action of $T^3/\langle(-1,-1,-1)\rangle$.

Construction 4.1 *In $\mathbb{H}P^2$, we have the following torus invariant submanifolds:*

1. *Three copies of $\mathbb{H}P^1\cong S^4$ given by:*

$$M_{01}=\{[*:*:0]\},\quad M_{02}=\{[*:0:*]\},\quad M_{12}=\{[0:*:*]\}.$$

2. *Four copies of $\mathbb{C}P^2$ given by*[3]*:*

$$\begin{aligned}
N_{+++}&=\{[(*,0):(*,0):(*,0)]\}\quad (=N_{---}=\{[(0,*):(0,*):(0,*)]\});\\
N_{++-}&=\{[(*,0):(*,0):(0,*)]\}\quad (=N_{--+}=\{[(0,*):(0,*):(*,0)]\});\\
N_{+-+}&=\{[(*,0):(0,*):(*,0)]\}\quad (=N_{-+-}=\{[(0,*):(*,0):(0,*)]\});\\
N_{+--}&=\{[(*,0):(0,*):(0,*)]\}\quad (=N_{-++}=\{[(0,*):(*,0):(*,0)]\}).
\end{aligned}$$

3. *Six copies of $\mathbb{C}P^1$ given by:*

$$\begin{aligned}
S_{0+1+}&=\{[(*,0):(*,0):0]\}\quad (=S_{0-1-}=\{[(0,*):(0,*):0]\});\\
S_{0+1-}&=\{[(*,0):(0,*):0]\}\quad (=S_{0-1+}=\{[(0,*):(*,0):0]\});\\
S_{0+2+}&=\{[(*,0):0:(*,0)]\}\quad (=S_{0-2-}=\{[(0,*):0:(0,*)]\});\\
S_{0+2-}&=\{[(*,0):0:(0,*)]\}\quad (=S_{0-2+}=\{[(0,*):0:(*,0)]\});\\
S_{1+2+}&=\{[0:(*,0):(*,0)]\}\quad (=S_{1-2-}=\{[0:(0,*):(0,*)]\});\\
S_{1+2-}&=\{[0:(*,0):(0,*)]\}\quad (=S_{1-2+}=\{[0:(0,*):(*,0)]\}).
\end{aligned}$$

4. *The fixed points*

$$v_0=[*:0:0],\qquad v_1=[0:*:0],\quad v_2=[0:0:*].$$

All the listed submanifolds have nontrivial stabilizer subgroups. The four-submanifolds M_{ij} and $N_{\epsilon_0\epsilon_1\epsilon_2}$ consist of at most two-dimensional torus orbits. A submanifold M_{ij} is stabilized by the circle $\{t_i=t_j=1\}$, and $N_{\epsilon_0\epsilon_1\epsilon_2}$ is stabilized by the circle $\{t_0^{\epsilon_0}=t_1^{\epsilon_1}=t_2^{\epsilon_2}\}$. The 2-spheres $S_{i\epsilon_i j\epsilon_j}$ consist of one-dimensional orbits. These 2-spheres are the pairwise intersections of the submanifolds M_{ij} and $N_{\epsilon_0\epsilon_1\epsilon_2}$.

Proposition 4.2 *The equivariant 2-skeleton $(\mathbb{H}P^2)_2$ of $\mathbb{H}P^2$ is the union of the submanifolds $\{M_{ij}\}$ and $\{N_{\epsilon_0\epsilon_1\epsilon_2}\}$. The equivariant 1-skeleton $(\mathbb{H}P^2)_1$ is the union of the submanifolds $\{S_{i\epsilon_i j\epsilon_j}\}$. The action of $T^3/\langle(-1,-1,-1)\rangle$ on $\mathbb{H}P^2$ is free outside $(\mathbb{H}P^2)_2$.*

[3] It is convenient to have two different symbols for the same object.

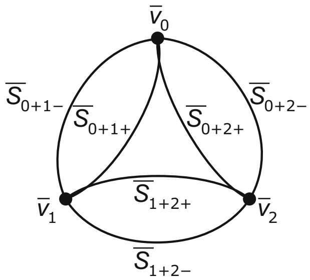

Fig. 4 The unlabeled GKM graph $(\mathbb{H}P^2/T^3)_1$ of $\mathbb{H}P^2$

Proof Consider the element $t = (t_0, t_1, t_2) \in T^3$ stabilizing a point $x = [h_0 : h_1 : h_2] \in \mathbb{H}P^2$. We need to show that whenever $(t_0, t_1, t_2) \notin \langle(-1, -1, -1)\rangle$, the point $[h_0 : h_1 : h_2]$ lies in one of the subsets M_{ij} or $N_{\epsilon_0\epsilon_1\epsilon_2}$. We may assume $h_0 = 1$, since otherwise $x \in M_{12}$, and the statement is proved. Since $tx = x$, we have:

$$[1 : h_1 : h_2] = [t_0 : t_1h_1 : t_2h_2] = [1 : t_1h_1t_0^{-1} : t_2h_2t_0^{-1}].$$

Therefore, $h_1 = t_1h_1t_0^{-1}$ and $h_2 = t_2h_2t_0^{-1}$. In complex coordinates, we have:

$$\begin{cases} (z_1, u_1) = (t_1t_0^{-1}z_1, t_1t_0u_1) \\ (z_2, u_2) = (t_2t_0^{-1}z_2, t_2t_0u_2). \end{cases}$$

If at least three of the complex numbers z_1, u_1, z_2, u_2 are nonzero, the relations on t_i would imply $(t_0, t_1, t_2) = (1, 1, 1)$ or $(-1, -1, -1)$, contradicting the assumption. Hence, we have at least two zeroes among z_1, u_1, z_2, u_2, which shows that x lies in either M_{ij} or $N_{\epsilon_0\epsilon_1\epsilon_2}$.

The GKM graph, that is the structure of the set $(\mathbb{H}P^2/T^3)_1$ of at most one-dimensional orbits, is well known for $\mathbb{H}P^2$, see for example [18]. The GKM graph of $\mathbb{H}P^2$ is given by doubling the edges of a triangle (see Fig. 4). □

Recall that the orbit space of the T^2-action on $\mathbb{C}P^2$ is a triangle, and the orbit space of the T^2-action on $\mathbb{H}P^1 \cong S^4$ is a biangle. If A is one of the subsets M_{ij}, $N_{\epsilon_0\epsilon_1\epsilon_2}$, $S_{i\epsilon_i j\epsilon_j}$ or $\{v_i\}$, we denote its orbit space by $\overline{A}$. Henceforth, $\overline{M_{ij}}$ are biangles, $\overline{N_{\epsilon_0\epsilon_1\epsilon_2}}$ are triangles, and $\overline{S_{i\epsilon_i j\epsilon_j}}$ are closed intervals.

Corollary 4.3 *The sponge $(\mathbb{H}P^2/T^3)_2$ of the T^3-action on $\mathbb{H}P^2$ is obtained by gluing 4 triangles and 3 biangles, as shown in Fig. 5.*

Remark 4.4 There is a simple way to visualize the sponge $(\mathbb{H}P^2/T^3)_2$. First note that 4 triangles in Fig. 5 provide the standard triangulation of $\mathbb{R}P^2$, obtained by identifying pairs of opposite points at the boundary of octahedron. So far, to obtain $(\mathbb{H}P^2/T^3)_2$, one needs to attach three discs along three projective lines in $\mathbb{R}P^2$ in general position.

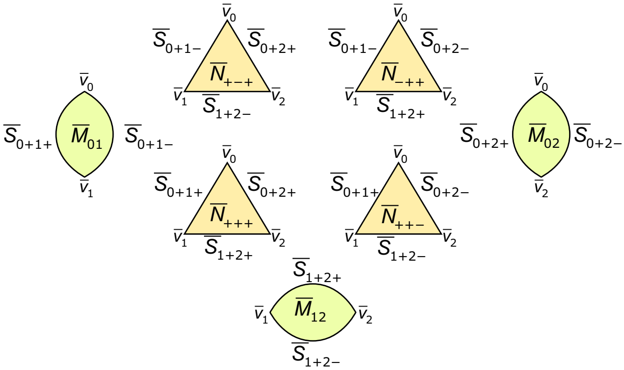

Fig. 5 The sponge of the T^3-action on $\mathbb{H}P^2$ is the cell complex obtained by attaching 4 triangles and 3 biangles

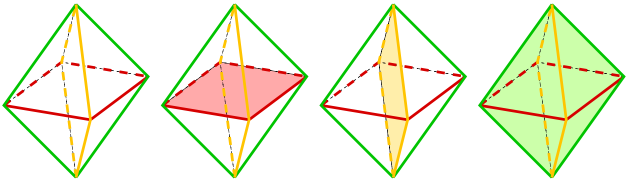

Fig. 6 The sponge of the T^3-action on $G_{4,2}$ is the cell complex obtained by attaching 8 triangles, as in the boundary of an octahedron, and 3 equatorial squares

Remark 4.5 There is an interesting connection of $\mathbb{H}P^2$ to the complex Grassmann manifold $G_{4,2}$. Recall [4], that the sponge of the T^3-action on $G_{4,2}$ is given by attaching 3 squares to the boundary of an octahedron along equatorial circles, see Fig. 6. One can notice that, by identifying pairs of opposite points in this construction, we obtain exactly the sponge $(\mathbb{H}P^2/T^3)_2$ of $\mathbb{H}P^2$.

There is a standard involution σ on $G_{4,2}$ which maps a complex 2-plane in $\mathbb{C}^4$ into its orthogonal complement. This involution induces the antipodal map on the octahedron, the moment map image of $G_{4,2}$. The torus action commutes with σ, therefore there is a T^3-action on the 8-manifold $G_{4,2}/\sigma$. The sponge $((G_{4,2}/\sigma)/T^3)_2$ of this action therefore coincides with the sponge $(\mathbb{H}P^2/T^3)_2$. However, $\mathbb{H}P^2$ is not the same as $G_{4,2}/\sigma$: $\mathbb{H}P^2$ is simply connected, while $G_{4,2}/\sigma$ is not. This example shows that a manifold with a torus action of complexity one cannot be uniquely reconstructed from its sponge.

5 Kuiper–Massey Theorem and Quasitoric Manifolds

We recall the classical Kuiper–Massey theorem.

Theorem 4 (Kuiper [17], Massey [19]) *Let* conj: $\mathbb{C}P^2 \to \mathbb{C}P^2$ *be the antiholomorphic involution of complex conjugation. Then,* $\mathbb{C}P^2/\operatorname{conj} \cong S^4$.

A similar fact holds not only for $\mathbb{C}P^2$ but for many other 4-manifolds as well. First, we recall several particular examples to be used in the following.

Example 5.1 Let $\sigma\colon S^2 \to S^2$ be the reflection in the equatorial plane of a sphere. This involution coincides with $\mathrm{conj}\colon \mathbb{C}P^1 \to \mathbb{C}P^1$. Then, considering the involution $\sigma\colon S^2 \times S^2 \to S^2 \times S^2$, which acts on both coordinates simultaneously, we have:

$$(\mathbb{C}P^1 \times \mathbb{C}P^1)/\,\mathrm{conj} = (S^2 \times S^2)/\sigma \cong S^4. \tag{5.1}$$

See, for example, [13], where a more general collections of involutions on the products of spheres have been studied. The homeomorphism of (5.1) can be considered the "complex version" of the classical homeomorphism:

$$T^2/\sigma \overset{\cong}{\to} S^2, \ \text{where } T^2 = \mathbb{R}^2/\mathbb{Z}^2 \text{ and } \sigma(a) = -a, \tag{5.2}$$

given by the Weierstrass's $\wp$-function.

Example 5.2 Another example is given by the involution σ on S^4, where:

$$S^4 = \{(r, z_1, z_2) \in \mathbb{R} \times \mathbb{C}^2 \mid r^2 + |z_1|^2 + |z_2|^2 = 1\}$$

and $\sigma((r, z_1, z_2)) = (r, \overline{z}_1, \overline{z}_2)$. It can be seen that $S^4 = \Sigma(S^1 * S^1)$, and the involution acts trivially on the first factor of the join (which corresponds to the real parts of z_i) and acts freely on the second factor of the join (which corresponds to the imaginary parts of z_i). Again, we have $S^4/\sigma \cong \Sigma(S^1 * (S^1/\sigma)) \cong \Sigma(S^1 * S^1) \cong S^4$.

Now, according to the result of Orlik and Raymond [21], any simply connected closed 4-manifold X^4, carrying an action of the torus T^2, with no nontrivial finite stabilizers, is diffeomorphic to either S^4 or to an equivariant connected sum of several copies of $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$, and $\overline{\mathbb{C}P}^2$ (the manifold $\mathbb{C}P^2$ with the reversed orientation) along T^2-fixed points. All these spaces carry the natural involutions σ, and it is not difficult to check that their T^2-fixed points have isomorphic tangent σ-representations (up to orientation reversals). Hence, X^4 is represented as a σ-equivariant connected sum, so its orbit space X^4/σ is homeomorphic to $S^4\#\cdots\#S^4 \cong S^4$. This, in particular, proves the result mentioned in [11].

Theorem 5 ([11]) *Let* $\mathrm{conj}\colon X^4 \to X^4$ *be the antiholomorphic involution of complex conjugation on a smooth compact toric surface* X^4. *Then,* $X^4/\,\mathrm{conj} \cong S^4$.

In the focus of toric topology, there lie the notions of a quasitoric manifold and a moment–angle manifold. Next, we recall these basic notions and show that there is a natural "complex conjugation" on a quasitoric manifold. We prove the analogue of Theorem 5 without referring to the result of Orlik and Raymond. In Sect. 6, we extend the analogy to a more interesting eight-dimensional quaternionic case.

Construction 5.3 *A* quasitoric manifold *is a manifold* $X = X^{2n}$ *with a locally standard action of* T^n, *and the orbit space diffeomorphic, as a manifold with corners, to a* *simple polytope. Each quasitoric manifold determines a characteristic pair* (P, λ)*, where* P *is the polytope of the orbit space, and* $\lambda\colon \mathrm{Facets}(P) \to \mathrm{Subgroups}_1(T^n)$ *is the map from the set of facets of* P *to the set of one-dimensional subgroups of* T^n*. The value* $\lambda(\mathcal{F}_i)$ *is the stabilizer of any orbit, lying in the interior of a facet* $\mathcal{F}_i$*. Since the action is locally standard, the characteristic function* λ *satisfies the conditions: (1) each* $\lambda(\mathcal{F}_i)$ *is a circle; (2) whenever facets* $\mathcal{F}_{i_1}, \ldots, \mathcal{F}_{i_n}$ *of* P *intersect in a vertex of* P*, the corresponding values* $\lambda(\mathcal{F}_{i_1}), \ldots, \lambda(\mathcal{F}_{i_n})$ *form a basis of* T^n *(that is the homomorphism* $\lambda(\mathcal{F}_{i_1}) \times \cdots \times \lambda(\mathcal{F}_{i_n}) \to T^n$ *induced by inclusions is an isomorphism). The second condition is called* $(*)$*-condition; it obviously implies the first condition.*

On the other hand, given a characteristic pair (P, λ)*, it is possible to construct the model space* [10]*:*

$$X^{2n}_{(P,\lambda)} = (P \times T^n)/\sim, \tag{5.3}$$

where the equivalence relation is generated by the relations $(x_1, t_1) \sim (x_2, t_2)$*, whenever* $x_1 = x_2 \in \mathcal{F}_i$ *and* $t_1^{-1}t_2 \in \lambda(\mathcal{F}_i)$*. The torus* T^n *acts on* $X^{2n}_{(P,\lambda)}$ *by rotating the second coordinate. The space* $X^{2n}_{(P,\lambda)}$ *is a topological manifold if* λ *satisfies* $(*)$*-condition. According to [10], whenever* X *is a quasitoric manifold,* (P, λ) *its characteristic pair, the model space* $X^{2n}_{(P,\lambda)}$ *is* T^n*-homeomorphic to the original manifold* X*.*

Additional considerations are required to construct a smooth structure on $X^{2n}_{(P,\lambda)}$; we now briefly recall these ideas and refer to [5] for details.

Construction 5.4 *Let* P *be realized as a convex polytope by affine inequalities* $P = \{x \in \mathbb{R}^n \mid Ax + b \geq 0\}$*, where* A *is an* $(m \times n)$*-matrix and* b *is a column* m*-vector. It is assumed that no redundant inequalities appear in the definition, so that each of* m *inequalities corresponds to a facet of* P*. Consider the affine map* $i_A\colon \mathbb{R}^n \to \mathbb{R}^m$*,* $i_A(x) = Ax + b$*. The map* i_A *embeds* P *into the nonnegative cone* $\mathbb{R}^m_{\geq 0}$*. The moment–angle space* $\mathcal{Z}_P$ *is defined as the pullback in the diagram:*

$$\begin{array}{ccc} \mathcal{Z}_P & \hookrightarrow & \mathbb{C}^m \\ \downarrow & & \downarrow \mu \\ P & \xrightarrow{i_P} & \mathbb{R}^m_{\geq 0} \end{array} \qquad \begin{array}{c} (z_1, \ldots, z_m) \\ \downarrow \mu \\ (|z_1|^2, \ldots, |z_m|^2) \end{array} \tag{5.4}$$

that is, $\mathcal{Z}_P = \mu^{-1}(i_P(P))$*. The moment–angle space carries a natural* T^m*-action. The subspace* $\mathcal{Z}_P \subset \mathbb{C}^m$ *is a smooth submanifold, whenever* P *is simple according to [5].*

Using the characteristic function λ *on* P*, one can construct the subgroup* $K_\lambda \cong T^{m-n}$ *of* T^m*, which acts freely on* $\mathcal{Z}_P$*. Then, the model space* $X^{2n}_{(P,\lambda)}$ *can be defined as the quotient* $\mathcal{Z}_P/K_\lambda$*. This construction provides a smooth structure on* $X^{2n}_{(P,\lambda)}$*: this structure is induced from* $\mathcal{Z}_P$*.*

Construction 5.5 *Let us construct an analogue of the complex conjugation on a quasitoric manifold. There is an involution σ on T^n:*

$$\sigma\colon (t_1, \ldots, t_n) \mapsto (\bar{t}_1, \ldots, \bar{t}_n).$$

By extending this involution trivially to P, we get an involution σ on the product $P \times T^n$. This involution can be correctly descended to the identification space (5.3). Indeed, if $t = (t_1, \ldots, t_n)$ lies in a characteristic subgroup $H \subset T^n$, then, obviously, $(\bar{t}_1, \ldots, \bar{t}_n) = (t_1^{-1}, \ldots, t_n^{-1})$ lies in the same subgroup, so the involution respects the equivalence classes in (5.3). We get an involution σ on the model space $X^{2n}_{(P,\lambda)}$. This is the natural analogue of the involution of complex conjugation defined on toric varieties.

Remark 5.6 It is possible to define σ as the smooth involution on a quasitoric manifold. One should first consider the conjugation involution on the moment–angle manifold $\mathcal{Z}_P$, and then descend it to $X^{2n}_{(P,\lambda)} \cong \mathcal{Z}_P/K_\lambda$. Details are left to the reader.

We now prove a particular case of Theorem 5 to demonstrate the main arguments of the next section.

Proposition 5.7 *Let $X^4 = X^4_{(P^2,\lambda)}$ be a quasitoric 4-manifold and $\sigma\colon X^4 \to X^4$ an involution defined above. Then, $X^4/\sigma \cong S^4$.*

Proof We denote the orbit space X^4/σ by Q. From construction, it follows that there is a map $p\colon Q \to P^2$, induced by the projection of X into its T^2-orbit space. For the interior point x of the polygon P^2, the preimage $p^{-1}(x)$ is the quotient T^2/σ, and it is homeomorphic to S^2 according to (5.2). If x is the vertex of P^2, then, obviously, $p^{-1}(x)$ is a point. Finally, if x lies on a side of P^2, then $p^{-1}(x)$ is the quotient of the circle S^1 by a nonfree involution, and hence, $p^{-1}(x)$ is an interval. Therefore, to obtain Q, one needs to start[4] with $P^2 \times S^2$, and pinch the S^2-components over the boundary of P^2 into either intervals or points. The result is homeomorphic to S^4. □

6 Quoric Surfaces

Some parts of the story described in Sect. 5 remain true when the torus T^n is replaced by its quaternionic analogue, the group $(S^3)^n$, where $S^3 = \mathrm{Sp}(1)$ is the sphere of unit quaternions. However, the generalization should be carried with certain care, due to noncommutativity of the group S^3. For the detailed exposition of such generalization, we refer to the work of Jeremy Hopkinson [14]: he introduced both the concept and the terminology, which we use here. In particular, quaternionic analogues of quasitoric manifolds are called quoric manifolds in [14]. Here, we present only general ideas of this theory.

Definition 6.1 A *quoric manifold* is a smooth $4n$-manifold X^{4n} with an action of $(S^3)^n$ satisfying the conditions:

[4] At this point, we essentially use the fact that the quotient map from a quasitoric manifold to its orbit space admits a section.

(1) The action is locally modeled by one of the “standard actions” of $(S^3)^n$ on $\mathbb{H}^n$;
(2) The orbit space is diffeomorphic to a simple polytope.

Here, unlike the toric case, there are several left actions of $(S^3)^n$ on $\mathbb{H}^n$ which can be called “standard”. For example, we have the following two left actions of $(S^3)^2$ on $\mathbb{H}^2$:

$$(s_1, s_2)(h_1, h_2) = (s_1h_1, s_2h_2); \tag{6.1}$$
$$(s_1, s_2)(h_1, h_2) = (s_1h_1s_2^{-1}, s_2h_2). \tag{6.2}$$

In both cases, the orbit space is the nonnegative cone $\mathbb{R}^2_{\geq 0}$, with the quotient map given by $(h_1, h_2) \mapsto (|h_1|^2, |h_2|^2)$. The orbits corresponding to the interior of $\mathbb{R}^2_{\geq 0}$ are free, the orbits corresponding to the sides of $\mathbb{R}^2_{\geq 0}$ are homeomorphic to S^3, and the orbit corresponding to the vertex of $\mathbb{R}^2_{\geq 0}$ is a single point. Therefore, both actions can be considered as the analogues of the standard T^2-action on $\mathbb{C}^2$.

Remark 6.2 The actions (6.1) and (6.2) are not equivalent in any sense, as will be explained in a minute. Notice that there exist three distinguished subgroups of $S^3 \times S^3$, which are isomorphic to S^3: the first coordinate sphere $S^3_{\{1\}}$, the second coordinate sphere $S^3_{\{2\}}$, and the diagonal subgroup $S^3_{\{1,2\}} = \{(s, s) \mid s \in S^3\}$. All other subgroups isomorphic to S^3 lie in the conjugacy class of $S^3_{\{1,2\}}$. There is no automorphism of $(S^3)^2$, which takes the conjugacy class of $S^3_{\{1\}}$ to the conjugacy class of $S^3_{\{1,2\}}$, simply because the conjugacy classes of $S^3_{\{1\}}$ and $S^3_{\{2\}}$ consist of one element each, while the conjugacy class of $S^3_{\{1,2\}}$ contains infinitely many subgroups. The actions (6.1) and (6.2) are nonequivalent, since the subgroup $S^3_{\{1,2\}}$ appears as the stabilizer in the second action, while it does not appear as the stabilizer in the first action.

We are interested mainly in quoric 8-manifolds, so it will be enough to mention that the actions (6.1) and (6.2) give an exhaustive list of the “standard actions” of $(S^3)^2$ on $\mathbb{H}^2$, up to weak equivalence. Recall that two actions of G on X_1 and X_2 are called weakly equivalent if there exist an automorphism $\psi : G \to G$ and a homeomorphism $\varphi : X_1 \to X_2$, such that $\varphi(gx) = \psi(g)\varphi(x)$ for any $x \in X_1$ and $g \in G$. If we consider representations, then φ is required to be a linear isomorphism.

Construction 6.3 *For a quoric manifold X^{4n}, one gets a characteristic pair (P, Λ), where P is the orbit polytope, and Λ is a characteristic functor, defined on the poset category of faces of P and taking values in conjugacy classes of subgroups in $(S^3)^n$. This functor satisfies a collection of technical properties, see [14]. With these properties satisfied, one can reconstruct a quoric manifold out of the characteristic pair as the model space:*

$$X^{4n}_{(P,\Lambda)} = (P \times (S^3)^n)/\sim,$$

with the identification determined by Λ similarly to (5.3): $(x_1, s_1) \sim (x_2, s_2)$ if $x_1 = x_2$ and $s_1^{-1}s_2 \in \hat{\Lambda}(F(x_1))$. Here, $F(x_1)$ is the unique face of P containing x_1 in its *interior; and $\hat{\Lambda}(F(x_1))$ denotes the representative in the conjugacy class $\Lambda(F(x_1))$ (it is possible to take all representatives simultaneously, so that the inclusion order is preserved for the representatives). The group $(S^3)^n$ acts on $X^{4n}_{(P,\Lambda)}$ from the left: $s'[(x,s)] = [(x, s's)]$. Here, $s's$ denotes the product of two elements in the group $(S^3)^n$. This action of $(S^3)^n$ is well defined on $X^{4n}_{(P,\Lambda)}$.*

For a general quoric manifold X^{4n}, one can reconstruct the model space $X^{4n}_{(P,\Lambda)}$ out of characteristic pair of X^{4n}. The model space is $(S^3)^n$-equivariantly homeomorphic to X^{4n}, see [14].

Example 6.4 Let us show how two actions (6.1) and (6.2) emerge in the local model. Consider the angle $\mathbb{R}^2_{\geq 0}$ and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the sides of this angle.

1. Consider the characteristic functor Λ taking values $\Lambda(\mathcal{F}_1) = S^3_{\{1\}}$, $\Lambda(\mathcal{F}_2) = S^3_{\{2\}}$. In this case, $\mathbb{R}^2_{\geq 0} \times (S^3)^2/\sim$ is equivariantly homeomorphic to $\mathbb{H}^2$ with the action (6.1).
2. Now, consider the characteristic functor Λ taking values $\Lambda(\mathcal{F}_1) = S^3_{\{1\}}$, $\Lambda(\mathcal{F}_2) = S^3_{\{1,2\}}$. In this case, $\mathbb{R}^2_{\geq 0} \times (S^3)^2/\sim$ is equivariantly homeomorphic to $\mathbb{H}^2$ with the action (6.2).

Note that, in both examples, we start with the "obvious standard action" of $(S^3)^2$ on $(S^3)^2$, i.e., just the multiplication from the left. The structure of the $(S^3)^2$-action on $\mathbb{R}^2_{\geq 0} \times (S^3)^2/\sim$ is determined solely by the characteristic functor.

For some characteristic functors Λ, it is possible to obtain the model $X^{4n}_{(P,\Lambda)}$ as a smooth manifold, using the same principle as in the quasitoric case. Define the quaternionic moment–angle space as the pullback in the diagram:

$$\begin{array}{ccc} \mathcal{Z}_P^{\mathbb{H}} & \hookrightarrow & \mathbb{H}^m \\ \downarrow & & \downarrow \mu^{\mathbb{H}} \\ P & \xrightarrow{i_P} & \mathbb{R}^m_{\geq} \end{array} \qquad \begin{array}{c} (h_1, \ldots, h_m) \\ \downarrow \mu^{\mathbb{H}} \\ (|h_1|^2, \ldots, |h_m|^2) \end{array} \tag{6.3}$$

Similar to the complex case, $\mathcal{Z}_P^{\mathbb{H}}$ is a smooth submanifold for a simple polytope P. The group $(S^3)^m$ acts on $\mathbb{H}^m$ in one of the standard ways, so that the map $\mu^{\mathbb{H}}$ can be identified with the projection to the orbit space. Hence, $(S^3)^m$ acts on $\mathcal{Z}_P^{\mathbb{H}}$, and its orbit space is the original polytope P. If Λ satisfies some additional conditions (axiomatized in the notion of a *global characteristic functor* in [14]), then it is possible to choose a subgroup $K_\Lambda \subset (S^3)^m$ which is isomorphic to $(S^3)^{m-n}$ and acts freely on $\mathcal{Z}_P^{\mathbb{H}}$. In this case, the smooth manifold $X^{4n}_{(P,\Lambda)}$ can be constructed as the quotient $\mathcal{Z}_P^{\mathbb{H}}/K_\Lambda$.

Example 6.5 The quaternionic projective space $\mathbb{H}P^n$ is the straightforward example of a quoric manifold. Its corresponding polytope is an n-simplex.

Remark 6.6 With the use of the Morse-theoretical argument for a polytope P, it can be shown that cohomology of quoric manifolds is concentrated in degrees divisible by 4, and there holds $H^{4k}(X^{4n}_{(P,\Lambda)}) \cong \mathbb{Z}^{h_k}$, where $(h_0, h_1, \dots, h_n)$ is the h-vector of a simple polytope P.

Construction 6.7 *We will consider eight-dimensional quoric manifolds, that are quoric manifolds over polygons. According to Construction 6.3, one needs a characteristic pair. A pair consists of an m-gon P^2, and a characteristic functor. In case $n = 2$, the characteristic functor assigns, to any side of P^2, one of the three distinguished subgroups of $S^3 \times S^3$: either $S^3_{\{1\}}$ or $S^3_{\{2\}}$ (the coordinate spheres) or $S^3_{\{1,2\}}$ (the diagonal sphere). Obviously, in order for the action to be locally standard, different subgroups should be assigned to neighboring sides of P^2. This condition is also sufficient for $n = 2$. Therefore, a quoric 8-manifold is encoded by polygons with their sides colored in three paints: $\{S^3_{\{1\}}, S^3_{\{2\}}, S^3_{\{1,2\}}\}$, so that the adjacent sides have distinct colors.*

Notice that similar combinatorial objects, 3-colored m-gons, appear in the classification of two-dimensional small covers (see, e.g., [9], where, in particular, these objects were enumerated). Nevertheless, there is an important difference between small covers and quoric manifolds over polygons even from the combinatorial point of view. In two-dimensional small covers, all colors are interchangeable, since the automorphisms of the real torus $\mathbb{Z}_2^2$ form a permutation group Σ_3. In quoric case, the colors $S^3_{\{1\}}, S^3_{\{2\}}$ are interchangeable, while the color $S^3_{\{1,2\}}$ cannot be interchanged with any of $S^3_{\{1\}}, S^3_{\{2\}}$ by an automorphism of $S^3 \times S^3$, as explained in Remark 6.2.

Our interest in quoric 8-manifolds arises from a surprising fact: these manifolds provide examples of torus actions of complexity one.

Proposition 6.8 *There is an effective action of a compact torus T^3 on any quoric 8-manifold $X^8_{(P^2,\Lambda)}$. This action has m isolated fixed points and the weights of the tangent representation at each fixed point are in general position.*

Proof It is straightforward to get the action of T^2: the torus T^2 is naturally a subgroup of $(S^3)^2$. Since $(S^3)^2$ acts on $X^8_{(P^2,\Lambda)}$, the torus T^2 acts as well. The additional circle action is, roughly speaking, "the diagonal action from the other side". Let us construct this action globally. At first, we introduce the action of $T^3 = T^1_1 \times T^1_2 \times T^1_3$ on $(S^3)^2$ by setting:

$$(t_1, t_2, t_3)(s_1, s_2) = (t_1 s_1 t_3, t_2 s_2 t_3), \tag{6.4}$$

where $s_1, s_2 \in S^3$ are unit quaternions, and t_i are the elements of the coordinate circles T^1_i. Note that the discrete subgroup $\langle(-1, -1, -1)\rangle \subset T^3$ is a noneffective kernel of (6.4); therefore, one needs to consider the action of $T^3/\langle(-1, -1, -1)\rangle \cong T^3$ to make it effective. The effectiveness, however, does not affect the topology of the orbit space.

We get a T^3-action on $P^2 \times (S^3)^2$: the torus acts trivially on the first factor, and acts by (6.4) on the second factor.

Now, we need to prove that this action descends correctly to the identification space $(P^2 \times (S^3)^2)/\sim$. It is sufficient to prove the correctness only for the action of the third component T^1_3, since the first two components constitute the part of the natural $(S^3)^2$-action. So far, we need to check that $(x, (s_1, s_2)) \sim (x, (\tilde{s}_1, \tilde{s}_2))$ implies $(x, (s_1 t_3, s_2 t_3)) \sim (x, (\tilde{s}_1 t_3, \tilde{s}_2 t_3))$.

Recalling the definition of the equivalence $\sim$, we have $(s_1^{-1}\tilde{s}_1, s_2^{-1}\tilde{s}_2) \in \hat{\Lambda}(F(x))$, and we need to prove that $(t_3^{-1}s_1^{-1}\tilde{s}_1 t_3, t_3^{-1}s_2^{-1}\tilde{s}_2 t_3) \in \hat{\Lambda}(F(x))$. Hence, we need to check that all possible representative subgroups $\hat{\Lambda}(F(x)) \subset (S^3)^2$ are stable under the conjugation by an element $(t_3, t_3) \in T^2 \subset (S^3)^2$. There are a finite number of possibilities to be checked:

1. If x lies in the interior of P^2, then the subgroup $\hat{\Lambda}(F(x))$ is trivial; therefore $\hat{\Lambda}(F(x))$ is stable under conjugation.
2. If x is the vertex of P^2, then $\hat{\Lambda}(F(x))$ is the whole group $(S^3)^2$, and there[3.] is nothing to prove, as well.
4. If x lies on the side of P, then $\hat{\Lambda}(F(x))$ is one of the groups $S^3_{\{1\}}$, $S^3_{\{2\}}$, or $S^3_{\{1,2\}}$. The coordinate subgroups $S^3_{\{1\}}$, $S^3_{\{2\}}$ are stable under any conjugation, since their conjugacy classes consist of single elements, see Remark 6.2.
5. Finally, if $\hat{\Lambda}(F(x)) = S^3_{\{1,2\}}$, it is a direct check that $S^3_{\{1,2\}}$ is stable:

$$(s, s) \in S^3_{\{1,2\}} \Rightarrow (t_3^{-1}st_3, t_3^{-1}st_3) \in S^3_{\{1,2\}}.$$

Hence, we get a well-defined T^3-action on the model space $X^8_{(P^2,\Lambda)}$. It can be seen that its fixed points correspond to the vertices of the polygon.

The statement about the general position of weights follows from the consideration of the standard actions (6.2). It can be derived that in the first standard action of $(S^3)^2$ on $\mathbb{H}^2$, the corresponding action of T^3 on $\mathbb{H}^2 \cong \mathbb{R}^8$ is given by:

$$(t_1, t_2, t_3)(h_1, h_2) = (t_1 h_1 t_3, t_2 h_2 t_3);$$

and for the second standard action, the corresponding action of T^3 is given by:

$$(t_1, t_2, t_3)(h_1, h_2) = (t_1 h_1 t_2^{-1}, t_2 h_2 t_3).$$

By writing the actions in complex coordinates, we get the weights $(1, 0, 1)$, $(1, 0, -1)$, $(0, 1, 1)$, $(0, 1, -1)$ in the first case, and the weights $(1, -1, 0)$, $(1, 1, 0)$, $(0, 1, 1)$, $(0, 1, -1)$ in the second case. Both vector collections are in general position. □

Now, we prove Theorem 3, which tells that the T^3-orbit space of any quoric 8-manifold is homeomorphic to a sphere S^5.

Proof Given a construction of the quoric manifold as a model $(P^2 \times (S^3)^2)/\sim$, it is natural to prove the theorem by moding out each fiber $(S^3)^2/\sim$ of the map:

$$(P^2 \times (S^3)^2)/\sim \to P$$

independently. To do this, we need several lemmas. □

Lemma 6.9 *Consider the action of T^2 on the sphere S^3 of unit quaternions given by $(t_1, t_2)s = t_1^{\pm 1} s t_2^{\pm 1}$. Then, its orbit space is homeomorphic to the closed interval D^1.*

Proof Assume for simplicity that the action is $(t_1, t_2)s = t_1 s t_2$. In complex coordinates, it has the form:

$$T^2 \circlearrowleft S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}, \qquad (t_1, t_2)(z_1, z_2) = (t_1 t_2 z_1, t_1 t_2^{-1} z_2).$$

Hence, the orbit space can be identified with the interval:

$$D^1 = \{(c_1, c_2) \in \mathbb{R}^2_{\geq 0} \mid c_1 + c_2 = 1\}$$

(this is the action of T^2 on the moment–angle manifold of the interval). □

Lemma 6.10 *Consider the action of* T^3 *on* $(S^3)^2$ *given by* $(t_1, t_2, t_3)(s_1, s_2) = (t_1 s_1 t_3, t_2 s_2 t_3)$ *or by* $(t_1 s_1 t_2^{-1}, t_2 s_2 t_3)$*. Then, its orbit space is homeomorphic to* S^3.

Proof Consider the larger action of $T^4 = T^2 \times T^2$ on $S^3 \times S^3$, where each T^2 acts on the corresponding S^3 as in the previous lemma. Then, $(S^3)^2/T^4 \cong D^1 \times D^1$. The T^3-action on $(S^3)^2$ is given by restricting the T^4-action to a certain three-dimensional subtorus. We have a map:

$$(S^3)^2/T^3 \to (S^3)^2/T^4 \cong (D^1)^2.$$

Similar to Proposition 2.3, the preimages of interior points in the square $(D^1)^2$ are circles, while the preimages of boundary points are single points. Hence, $(S^3)^2/T^3$ is a sphere. □

Now, consider the action of T^3 on $X^8_{(P^2,\Lambda)}$ introduced in the proof of Proposition 6.8. We have a map:

$$p\colon X^8_{(P^2,\Lambda)}/T^3 \to X^8_{(P^2,\Lambda)}/(S^3)^2 = P^2.$$

The preimage of the point $x \in P^2$ is the double quotient:

$$p^{-1}(x) = ((S^3)^2/\hat{\Lambda}(F(x)))/T^3$$

If x is a vertex of P^2, then $p^{-1}(x)$ is a point. If x lies in the interior of a polygon P^2, then $p^{-1}(x) = (S^3 \times S^3)/T^3$ is a sphere S^3 according to Lemma 6.10. If x lies on a side of a polygon P^2, then $(S^3)^2/\hat{\Lambda}(F(x))$ is homeomorphic to a sphere S^3 and its quotient by the residual action of $T^2 = T^3/(T^3 \cap \hat{\Lambda}(F(x)))$ is an interval according to Lemma 6.9. Hence, we are in a similar situation as the one described in Sect. 2: the space $X^8_{(P^2,\Lambda)}/T^3$ is obtained from the product $P^2 \times S^3$ by pinching the 3-spheres over the boundary ∂P^2 into contractible spaces. Hence, $X^8_{(P^2,\Lambda)}/T^3 \cong S^5$. □

We conclude with the proposition, generalizing Arnold's homeomorphism (1.2).

Proposition 6.11 *Let* T^1 *be the diagonal subtorus of* T^3 *acting on a quoric* 8*-manifold* $X^8_{(P^2,\Lambda)}$*. Then,* $X^8_{(P^2,\Lambda)}/T^1 \cong S^7$.

Proof We have already constructed the T^3-action on $S^3 \times S^3$ in the proof of Proposition 6.8. The induced action of the diagonal circle $T^1 \subset T^3$ on the product $S^3 \times S^3 = \{(h_1, h_2) \in \mathbb{H}^2 \mid |h_1| = |h_2| = 1\}$ is given by

$$t(h_1, h_2) = (th_1t, th_2t) \tag{6.5}$$

The proof of the proposition follows the same lines as the proof of Theorem 3: we use a sequence of lemmas. □

Lemma 6.12 *For the action of T^1 on $S^3 = \{h \in \mathbb{H} \mid |h| = 1\}$ given by $t(h) = tht$, there holds $S^3/T^1 \cong D^2$.*

Lemma 6.13 *For the diagonal action of T^1 on $S^3 \times S^3$ given by (6.5), there holds $(S^3 \times S^3)/T^1 \cong S^5$.*

Lemma 6.12 is an exercise similar to Example 5.2. The proof of Lemma 6.13 and the remaining proof of the proposition are completely similar to Lemma 6.10 and Theorem 3, respectively. □

Remark 6.14 It is reasonable to expect that quoric 8-manifolds (at least some of them) can be obtained from $\mathbb{H}P^2$ and $\mathbb{H}P^1 \times \mathbb{H}P^1$ by a sequence of equivariant connected sums. In this case, Theorem 3 will be a simple consequence of Theorem 2 and the corresponding statement for $\mathbb{H}P^1 \times \mathbb{H}P^1$. In a similar way, Proposition 6.11 is the consequence of Arnold's result and the corresponding statement for $\mathbb{H}P^1 \times \mathbb{H}P^1$.

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