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PROBLEM CONTRIBUTION

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Billiard Trajectories in Regular Polygons and Geodesics on Regular Polyhedra

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Abstract
This article is devoted to the geometry of billiard trajectories in a regular polygon and geodesics on the surface of a regular polyhedron. Main results are formulated as conjectures based on ample computer experimentation.

Keywords Billiard trajectories · Parallel trajectories · Short trajectories · Dodecahedron · Hyperbolic dodecaherdon · Geodesics

Introduction

Billiard trajectories in regular polygons have been thoroughly studied, and it is difficult to say something new about them. Known results are usually obtained in a more general context; most of them belong to William A. Veech [18,19] and his followers. Sometimes, it is difficult to attribute these results to concrete authors, and I restrict myself to a comprehensive list of surveys [11–13,15,16].

The goal of this article is to report of a series of detailed computer experimentations that lead to a series of conjectures (Conjectures 1.7, 2.3, 2.4, 2.5, 2.6, 2.7, and 3.2). I believe that these are all true, and in fact, can be proved using the works of Veech, Ward [20], and others. Some results of this kind for regular pentagons are contained in the recent work of D. Davis and S. Lelievre [5], which may provide a model for such proofs.

It should be mentioned that Veech's theory is based mostly on the construction of Fox-Kirshner and Katok-Zemlyakov [10,21] which represents billiard trajectories in polygons with rational (with respect to π) angles as geodesics on flat compact surfaces with conic singularities. In the case of regular polygon, an algebraic construction of this surface is contained in the dissertation of Veech's student Clayton Ward [20] (Theorem C). Since the main goal of this article is rather statements than proofs, I

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prefer to use more elementary language (sufficient for the statements, but probably not for the proofs). Still, some constructions below (in particular, in Sects. 2.1 and 2.4) are related to the Fox–Kirshner–Katok–Zemlyakov–Veech approach.

The plan of the article is as follows.

Section 1 contains a survey of known results with some enlightening illustrations. In Sect. 2, all “short billiard trajectories” (aka “generalized diagonals”) in a regular n-gon are subdivided into $n - 2$ “types,” and the conjectures of Sect. 2 demonstrate the importance of this subdivision for the geometry of billiard trajectories.

It is clear that the behavior of billiard trajectories in a regular n-gon is closely related to the behavior of geodesics on regular polyhedra (it is true even for the case $n = 4$, see a remark in the beginning of Sect. 2.3). In Sect. 3, we apply the main construction of Sect. 2 to geodesics on the surface of the regular dodecahedron. This section contains a report of some computer experiments related to the behavior of these geodesics.

Section 4 shows a possible direction of further research. The faces of Platonic solids have no more than five edges. However, there are polyhedra of hyperbolic origin whose faces are regular n-gons with arbitrarily large n. We describe such a polyhedron with regular heptagonal faces; it has 28 vertices, 42 edges, and 12 faces; thus, its Euler characteristic is -2. Certainly, it cannot be isometrically embedded into a Euclidean space. Its group of isometries is small, but it also has geodesics, which correspond to billiard trajectories in a regular heptagon. So far, there are almost no results, only statements of problems. (It should be mentioned that there are works about geodesics on polyhedra which may be regarded as generalizations of Platonic solids; some of these polyhedra also have hyperbolic origin; see, for example, [14]. The property of the example in Sect. 4, which makes it relevant for us, is that all the faces are regular heptagons.)

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1 Known Results

1.1 Parallel Trajectories

A billiard trajectory in a polygon is fully determined by any small segment on it. It would be fair to say that two such trajectories are parallel, if they contain parallel segments. In the case of a regular n-gon, it is more convenient to say that two billiard trajectories are *parallel*, or *have the same directions*, if, for any two segments on them, the sum or the difference of angles they form with (any) side of the n-gon” is a multiple of $\dfrac{2\pi}{n}$.

At this moment, we first encounter the difference between the cases of odd and even n. If n is odd, then all the sides and diagonals of the regular n-gon have the same directions. However, if n is even, then the sides and diagonals split into two classes (to which we again refer as to even and odd) with a parallelism within each class, but not between representatives of different class. Belonging to an even or odd class is

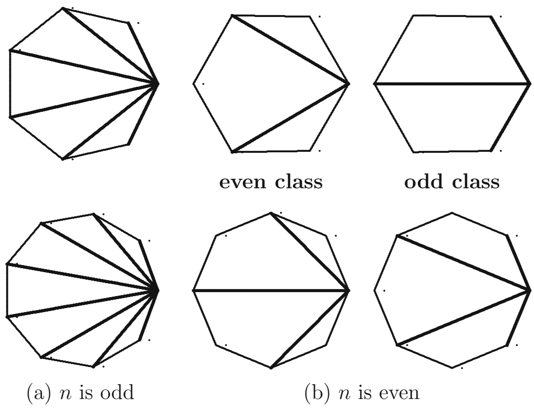


Fig. 1 Trajectories of even and odd classes

determined by the parity of the number of sides between the endpoints of diagonals or sides (see Fig. 1 on the next page).

1.2 Special Directions

We consider billiard trajectories in a regular n-gon. *Short trajectories* are trajectories which go from a vertex to a vertex (the same or other). *Closed trajectories* are trajectories which avoid vertices and, from some moment, repeat themselves. Short trajectories and closed trajectories both have *lengths*.

Proposition 1.1 *The directions of short trajectories and closed trajectories are the same.*

1.3 Ratios of Lengths of Parallel Short Trajectories

In the case of even n, short trajectories (precisely as sides and diagonals) may belong to an *even* or *odd class*: as before, this is determined by the parity of the number of sides between the endpoints of the trajectory.

Proposition 1.2 *The ratios of lengths of short geodesics of a given special direction do not depend on the choice of a special direction. In particular, they are the same as ratios of lengths of sides and diagonals (see Sect. 1.1).*

1.4 A Contribution from the Middle School Trigonometry

Let us assume that the side of our regular n-gon is 1. Then, the lengths of the diagonals are:

$$\frac{\sin((k+1)\pi/n)}{\sin(\pi/n)}, k = 1, \ldots, \left[\frac{n}{2}\right] - 1.$$

Proposition 1.3 *For an* $\alpha \in \mathbb{R}$*, let* $\lambda_k = \dfrac{\sin((k+1)\alpha)}{\sin\alpha}$*. (In particular,* $\lambda_1 = 2\cos\alpha$*.) Then, for all k, the following relation holds:*

$$\lambda_1\lambda_k = \lambda_{k-1} + \lambda_{k+1}.$$

This relation allows to express all λ_k as polynomial of $\lambda = \lambda_1 = 2\cos\alpha$:

$$\begin{aligned}
&\lambda_0 = 1,\\
&\lambda_1 = \lambda,\\
&\lambda_2 = \lambda^2 - 1,\\
&\lambda_3 = \lambda^3 - 2\lambda,\\
&\lambda_4 = \lambda^4 - 3\lambda^2 + 1,\\
&\lambda_5 = \lambda^5 - 4\lambda^3 + 3\lambda,\\
&\lambda_6 = \lambda^6 - 5\lambda^4 + 6\lambda^2 - 1,
\end{aligned}$$

etc. In other words, $\lambda_k = U_k(\lambda/2)$, where U_k is the *k-th Chebyshev polynomial of the second kind*. If $\alpha = \dfrac{\pi}{n}$, then:

$$\begin{aligned}
&\lambda_{m-1} = \lambda_m, \text{ if } n = 2m+1,\\
&\lambda_{m-2} = \lambda_m, \text{ if } n = 2m,
\end{aligned}$$

which provides an algebraic equation for λ.

1.5 The Case of Odd *n*. Examples

If n is odd, $n = 2m+1$, then, for each special direction, the lengths of short trajectories assume m different values with the ratio:

$$1 : \lambda : \lambda^2 - 1 : \ \ldots, \lambda = 2\cos\frac{\pi}{n},$$

or equivalently:

$$\sin\frac{\pi}{n} : \ \sin\frac{2\pi}{n} : \ \ldots : \ \sin\frac{m\pi}{n}.$$

In particular, for $n = 5$, there are two lengths with the ratio $1 : \tau$, $\tau = \dfrac{1+\sqrt{5}}{2}$ is the golden ratio. For $n = 7$, there are three lengths with the ratio $1 : \lambda_1 : \lambda_2$, where $\lambda_1 = 2\cos\dfrac{\pi}{7} \approx 1.801938$, and $\lambda_2 = \lambda_1^2 - 1 \approx 2.2498$. The examples are shown in Fig. 2.

1.6 The Case of Even *n*. Examples

Let n is even, $n = 2m$.

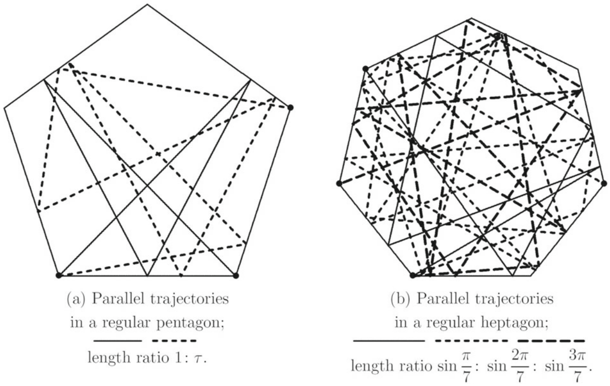


Fig. 2 Parallel trajectories run regular odd- gons

Proposition 1.4 *For every special direction of trajectories in a regular n-gon, all short trajectories of this directions belong either to the odd class, or to the even class.*

Hence, the ratios of lengths of parallel short trajectories are (dependently on the parity of the class) either.

$$\sin\frac{\pi}{m}:\ \sin\frac{2\pi}{m}:\ \sin\frac{3\pi}{m}:\ \dots$$

or

$$\sin\frac{\pi}{n}:\ \sin\frac{3\pi}{n}:\ \sin\frac{5\pi}{n}:\ \dots$$

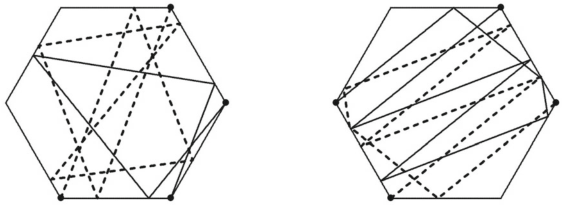
In particular, if $n = 4\ell$, then the lengths of short trajectories of each of the even and odd classes assume ℓ different values; if $n = 4\ell + 2$, then the lengths of trajectories of even class assume ℓ values, while the lengths of trajectories of odd class assume $\ell + 1$ values. Moreover, in the case of $n = 4\ell + 2$, the ratios of lengths of short trajectories of even class are the same as the ratios of lengths of short trajectories in a regular $(2\ell + 1)$-gon, that is $(n/2)$-gon.

For example, consider the cases $n = 6, 8, 10$.

If $n = 6$, then all parallel short trajectories of even class have equal lengths, while all parallel short trajectories of odd class have two lengths with the ratio $1: 2$. See examples in Fig. 3.

If $n = 8$, then all parallel short trajectories have two lengths. The length ratio is $\sqrt{2}$ for even class and is $1 + \sqrt{2}$ for odd class (see Fig. 4).

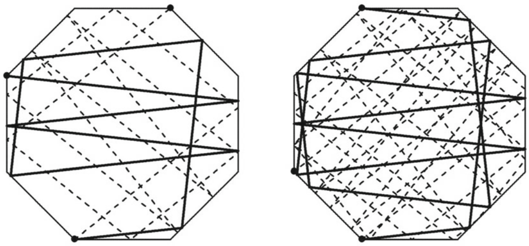
If $n = 10$, then parallel short trajectories of even class have two different lengths with length ratio $1: \tau$ (where τ denotes, as before, the golden ratio), while parallel short trajectories of odd class have three lengths with length ratio $1: 1 + \tau : 2\tau$. See examples in Fig. 5.



(a) Parallel trajectories of odd class in a regular hexagon; length ratio 1: 2

(b) Parallel trajectories of even class in a regular hexagon; the lengths are equal

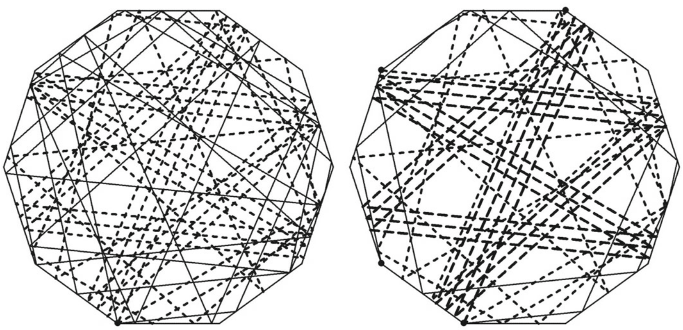
Fig. 3 Parallel trajectories in a regular hexagon



(a) Parallel trajectories of even class in a regular octagon; length ratio 1: $\sqrt{2}$

(b) Parallel trajectories of odd class in a regular octagon; length ratio 1: $(1+\sqrt{2})$

Fig. 4 Parallel trajectories in a regular octagon



(a) Parallel trajectories of even class in a regular decagon; length ratio 1: τ

(b) Parallel trajectories of odd class in a regular decagon; length ratio 1: $(1+\tau)$: 2τ

Fig. 5 Parallel trajectories in a regular decagon

1.7 Closed Trajectories

1.7.1 Closed and Preclosed Trajectories

As we have mentioned before (Proposition 1.1), closed billiard trajectories (in a regular n-gon) are the same as trajectories parallel to short trajectories. To describe the relations between the lengths of parallel short and closed trajectories, we will need some additional terminology.

A trajectory is called *preclosed* if its endpoints belong to (possibly different) edges, divide these edges into the parts of the same length and the angles between the trajectory and these edges are the same (the trajectory and the edges are considered with their natural orientation). Two parallel closed or preclosed trajectories are called *strongly parallel* if their sequences of edges of reflection are identical. Finally, I will call a closed (preclosed) trajectory *simple* if no proper part of it is closed (preclosed). Below, I will drop the adjective *simple* when it is obviously meant.

The following statements are obvious.

Proposition 1.5 *(a) A preclosed trajectory becomes closed if repeated a certain number of times; the smallest number with these properties is a divisor of n. (b) Strongly parallel closed or preclosed trajectories have equal lengths.*

1.7.2 Lengths of Preclosed and Short Trajectories

Proposition 1.6 *(a) If n is odd, then, for any special direction, the lengths of simple preclosed trajectories of this direction are the same, as the lengths of short trajectories of this direction,* ***multiplied by*** $\mathbf{2\cos}\dfrac{\pi}{n}$. *(b) If n is even, then for any special direction of* ***even*** *or* ***odd*** *class, the amount and ratios of different lengths of simple preclosed trajectories of this direction are the same as the amount and ratios of different lengths of short trajectories of the* ***opposite*** *class.*

An explanation of this fact will be given in Sect. 2.6.

1.7.3 Lengths of Closed Trajectories: Experimental Data

It follows from Proposition 1.5 that the lengths of (simple) closed trajectories stay unchanged within the class of strongly parallel trajectories and are equal to the lengths of preclosed trajectories multiplied by some divisor of n. This means that the length ratios of parallel closed trajectories are obtained from the length ratios of parallel preclosed trajectories by multiplication by (possibly, different) divisors of n. Certainly, we can assume that these factors (within a class of parallel trajectories) do not have any proper common factors. However, the problem of finding these factors stays open in many cases.

Conjecture 1.7 *After the cancelation described, the factors distinguishing the length ratios of closed and preclosed trajectories are never equal to n.*

This statement is supported by ample experiments, but, so far, is not proved. Some results of the experiments are shown in the table in Sect. 1.7.3.2.

1.7.3.1. The case of prime n.

Proposition 1.8 *If n is prime, then for any special direction, the lengths of closed trajectories of this direction are either all equal to the length of short trajectories* *multiplied by* $2\cos\dfrac{\pi}{n}$ *or all equal to the length of short trajectories multiplied by* $2n\cos\dfrac{\pi}{n}$. *In particular, if* n *is prime, then the length ratios of closed trajectories of any special direction are the same as the length ratios of short trajectories of this direction.*

This statement follows from Proposition 1.6 and Conjecture 1.7, so must be also considered as a conjecture. For $n = 5$ and 7, a proof is contained in [3].

1.7.3.2. Experimental Data for Composite n ≤ 12. The table table on the next page on the next page contains experimentally found length ratios of parallel closed trajectories for the cases $n = 6, 8, 9, 10, 12$. One can expect that no other ratios are possible. These data may be a ground for general conjectures, but we prefer to refrain from any guessing.

n	class	length ratios of pre-closed trajectories	known length ratios of closed trajectories
6	even	$1\colon 2$	$1\colon 2$ $3\colon 2$
6	odd	1	1
8	even	$1\colon (1+\sqrt{2})$	$1\colon (1+\sqrt{2})$
8	odd	$1\colon \sqrt{2}$	$1\colon \sqrt{2}$
9		$1\colon \lambda = 2\cos\frac{\pi}{9}\colon \lambda^2-1\colon \lambda+1$	$1\colon \lambda\colon \lambda^2-1\colon \lambda+1$ $3\colon 3\lambda\colon \lambda^2-1\colon 3(\lambda+1)$
10	even	$1\colon 1+\tau\colon 2\tau$	$5\colon 5(1+\tau)\colon 2\tau$
10	odd	$1\colon \tau$	$1\colon \tau$
12	even	$1\colon (1+\sqrt{3})\colon (2+\sqrt{3})$	$1\colon (1+\sqrt{3})\colon (2+\sqrt{3})$ $3\colon (1+\sqrt{3})\colon 3(2+\sqrt{3})$
12	odd	$1\colon \sqrt{3}\colon 2$	$3\colon 3\sqrt{3}\colon 2$

2 Types of Short Trajectories

In Sect. 1.1, we introduced for short trajectories in regular even-gons two classes: even and odd. Now, we are introducing a finer subdivision of the set of all short trajectories in regular n-gons. According to a definition we are going to give, every short trajectory in a regular n-gon belongs to one of $n-2$ types, which we will denote as $A_0 = A_{n-2}, A_1, A_2, \ldots, A_{n-3}$.

It will be explained in Sect. 2.4 that, in some informal sense, Conjectures 2.3 and 2.4 in that section mean that the types are orbits of the *Veech group.*

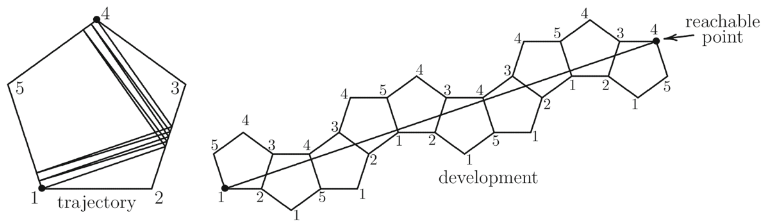
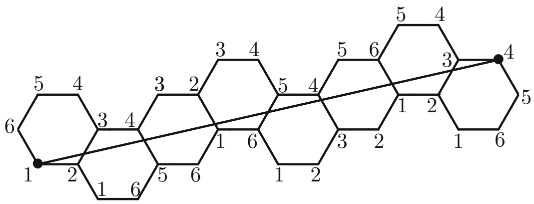


Fig. 6 Development of a short trajectory in a regular pentagon

Fig. 7 Development of a short trajectory in a regular hexagon



2.1 Developments of Trajectories and Reachable Points

Consider a regular n-gon in the (x, y) plane, such that one of the vertices is the origin O, the first side is contained in the positive x semi-axis, and the whole n-gon is contained in the upper half-plane $y \geq 0$. We consider short trajectories emanating from O. Such trajectory gives rise to a *development*, which appears if we replace reflections of the trajectory in sides by reflections of the whole n-gon. It is illustrated (for $n = 5$) in Fig. 6 on the next page. The endpoint of this development is a point within the angle formed by two sides at the vertex O. Such points are called *reachable*. Thus, reachable points form a set within the named angle.

Figure 6 contains also numbers of vertices of the initial pentagon and the pentagons covering the development. We see a chain of pentagons, the ones in the odd-numbered places are parallel translations of the initial pentagon, and the ones in the even-numbered places are parallel translations of the initial pentagon flipped upside down. The numeration of vertices is, respectively, counterclockwise and clockwise. The picture will look similarly for all odd-gons, but for even-gons, it will be different. The picture for hexagons is shown in Fig. 7 on the next page.

The most visible difference between the developments in Figs. 6 and 7 is that all the hexagons covering this development are parallel translates of each other. And although the ordering of numbers of vertices are still alternating counterclockwise and clockwise, the *parity* of the numbers is preserved by the translations. This property, which is shared by all developments of short trajectories in regular even-gons, provides a splitting of the set of reachable points into "even" and "odd." This, obviously, corresponds to the splitting of special directions into even and odd classes, as described in Sect. 1.1.

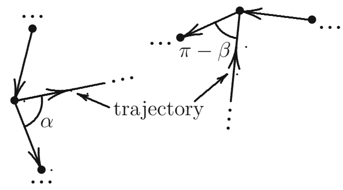


Fig. 8 To definition of types

2.2 Types of Short Trajectories: Definition

We are going to describe now one of the most important constructions of this article. According to this construction, a short trajectory in an oriented regular n-gon belongs to one of $n - 2$ *types*, which we will denote by A_k, where k is a residue modulo $n - 2$. This subdivision of the set of short trajectories into types may be regarded as a refinement of the subdivision of the set of short trajectories in even-gons into classes (see Proposition 2.2 below).

Now, let us give the definition of types. Consider for an oriented N-gonal short trajectory in an oriented regular n-gon its endpoints and define the angles α and β, as shown in Fig. 8. It is easy to understand that either $\alpha + \beta$ or $\alpha - \beta$ is divisible by $\frac{2\pi}{n}$; the sign is $-$, if N is even, and $+$, if N is odd.

Definition 2.1 If $n = 2m$, then a short trajectory belongs to the type A_k, if one of the following triples of conditions is satisfied:

$$k \le m - 1,\ \alpha \le \frac{m-k-1}{m}\pi,\ \beta - \alpha = \frac{k+1}{m}\pi;$$

$$k \le m - 1,\ \alpha \ge \frac{m-k-1}{m}\pi,\ \beta + \alpha = \frac{n-k-1}{m}\pi;$$

$$k \ge m - 1,\ \alpha \ge \frac{n-k-2}{m}\pi,\ \ \beta - \alpha = \frac{k+3-n}{m}\pi;$$

$$k \ge m - 1,\ \alpha \le \frac{n-k-2}{m}\pi,\ \ \beta + \alpha = \frac{n-k-1}{m}\pi.$$

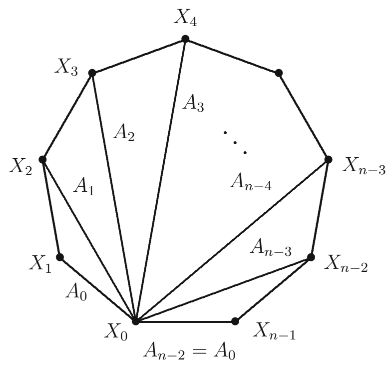


Fig. 9 Types of diagonals of a regular polygon

If $n = 2m + 1$, then a short trajectory belongs to the type A_k, if one of the following conditions is satisfied:

$$k \le m + 1,\ \alpha \le \frac{n - 2(k+1)}{n}\pi,\ \beta - \alpha = \frac{2(k+1)}{n}\pi;$$

$$k \le m - 1,\ \alpha \ge \frac{n - 2(k+1)}{n}\pi,\ \beta + \alpha = \frac{2(n-k-1)}{n}\pi;$$

$$k \ge m, \qquad \alpha \ge \frac{2(n-k-2)}{n}\pi,\ \beta - \alpha = \frac{2(k+3-n)}{n}\pi;$$

$$k \ge m, \qquad \alpha \le \frac{2(n-k-2)}{n}\pi,\ \beta + \alpha = \frac{2(n-k-1)}{n}\pi.$$

This definition, together with remarks in the end of Sect. 2.1, implies the following statement (implicitly promised above).

Proposition 2.2 *If n is even, then short trajectories of the type A_k belong to the even class, if k is even, and belong to the odd class, if k is odd.*

THE MOST OBVIOUS EXAMPLE. Let $X_0 = (0, 0), X_1, \ldots, X_{n-2}, X_{n-1} = (1, 0)$ be the clockwise ordered vertices of the "initial" regular n-gon. Then, for $0 \le k \le n-2$, X_0X_{k+1} is a short trajectory of the type A_k (see Fig. 9).

To make Definition 2.1 more transparent, we will present it in tables.

If n is even, $n = 2m$, then belonging of the short trajectory to the type A_k, $0 \le k \le n - 3$, is determined by the table on the next page.

range for α:	0 $\qquad \frac{\pi}{m}$	$\frac{2\pi}{m}$	$\frac{3\pi}{m}$	$\dots$ $\quad \frac{(m-4)\pi}{m}$	$\frac{(m-3)\pi}{m}$	$\frac{(m-2)\pi}{m}$	$\frac{(m-1)\pi}{m}$
$\beta-\alpha=-(m-3)\pi/m$							A_m
$\beta-\alpha=-(m-4)\pi/m$						A_{m+1}	A_{m+1}
$\beta-\alpha=-(m-5)\pi/m$					A_{m+2}	A_{m+2}	A_{m+2}
$\dots\dots\dots\dots\dots$				$\dots$	$\dots$	$\dots$	$\dots$
$\beta-\alpha=-\pi/m$			A_{2m-4}	$\dots$	A_{2m-4}	A_{2m-4}	A_{2m-4}
$\beta-\alpha=0$		A_{2m-3}	A_{2m-3}	$\dots$	A_{2m-3}	A_{2m-3}	A_{2m-3}
$\beta-\alpha=\pi/m$	$\mathbf{A_0}$	$\mathbf{A_0}$	$\mathbf{A_0}$	$\dots$	$\mathbf{A_0}$	$\mathbf{A_0}$	$\mathbf{A_0}$
$\beta-\alpha=2\pi/m$	A_1	A_1	A_1	$\dots$	A_1	A_1	
$\beta-\alpha=3\pi/m$	A_2	A_2	A_2	$\dots$	A_2		
$\dots\dots\dots\dots\dots$	$\dots$	$\dots$	$\dots$	$\dots$			
$\beta-\alpha=(m-3)\pi/m$	A_{m-4}	A_{m-4}	A_{m-4}				
$\beta-\alpha=(m-2)\pi/m$	A_{m-3}	A_{m-3}					
$\beta-\alpha=(m-1)\pi/m$	A_{m-2}						
$\beta+\alpha=(2m-2)\pi/m$							A_1
$\beta+\alpha=(2m-3)\pi/m$						A_2	A_2
$\beta+\alpha=(2m-4)\pi/m$					A_3	A_3	A_3
$\dots\dots\dots\dots\dots$				$\dots$	$\dots$	$\dots$	$\dots$
$\beta+\alpha=(m+2)\pi/m$			A_{m-3}	$\dots$	A_{m-3}	A_{m-3}	A_{m-3}
$\beta+\alpha=(m+1)\pi/m$		A_{m-2}	A_{m-2}	$\dots$	A_{m-2}	A_{m-2}	A_{m-2}
$\beta+\alpha=\pi$	$\mathbf{A_{m-1}}$	$\mathbf{A_{m-1}}$	$\mathbf{A_{m-1}}$	$\dots$	$\mathbf{A_{m-1}}$	$\mathbf{A_{m-1}}$	$\mathbf{A_{m-1}}$
$\beta+\alpha=(m-1)\pi/m$	A_m	A_m	A_m	$\dots$	A_m	A_m	
$\beta+\alpha=(m-2)\pi/m$	A_{m+1}	A_{m+1}	A_{m+1}	$\dots$	A_{m+1}		
$\dots\dots\dots\dots\dots$	$\dots$	$\dots$	$\dots$	$\dots$			
$\beta+\alpha=4\pi/m$	A_{2m-5}	A_{2m-5}	A_{2m-5}				
$\beta+\alpha=3\pi/m$	A_{2m-4}	A_{2m-4}					
$\beta+\alpha=2\pi/m$	A_{2m-3}						

If n is odd, $n = 2m + 1$, then belonging of the short trajectory to the type A_k is determined by the following table:

$\beta-\alpha=-2(m-2)\pi/n$						A_m
$\beta-\alpha=-2(m-3)\pi/n$					A_{m+1}	A_{m+1}
$\beta-\alpha=-2(m-4)\pi/n$				A_{m+2}	A_{m+2}	A_{m+2}
$\dots\dots\dots\dots$			$\dots$	$\dots$	$\dots$	$\dots$
$\beta-\alpha=0$		A_{n-3}	$\dots$	A_{n-3}	A_{n-3}	A_{n-3}
$\beta-\alpha=2\pi/n$	$\mathbf{A_0}$	$\mathbf{A_0}$	$\dots$	$\mathbf{A_0}$	$\mathbf{A_0}$	$\mathbf{A_0}$

range for α $\quad 0 \quad \frac{2\pi}{n} \quad \frac{4\pi}{n} \quad \dots \quad \frac{(n-7)\pi}{n} \quad \frac{(n-5)\pi}{n} \quad \frac{(n-3)\pi}{n}$

range for α $\quad 0 \quad \frac{\pi}{n} \quad \frac{3\pi}{n} \quad \frac{5\pi}{n} \quad \dots \quad \frac{(n-6)\pi}{n} \quad \frac{(n-4)\pi}{n} \quad \frac{(n-2)\pi}{n}$

$\beta-\alpha=4\pi/n$	A_1	A_1	A_1	$\dots$	A_1	
$\dots\dots\dots\dots$	$\dots$	$\dots$	$\dots$	$\dots$		
$\beta-\alpha=2(m-2)\pi/n$	A_{m-3}	A_{m-3}	A_{m-3}			
$\beta-\alpha=2(m-1)\pi/n$	A_{m-2}	A_{m-2}				
$\beta-\alpha=2m\pi/n$	A_{m-1}					
$\beta+\alpha=2(n-2)\pi/n$						A_1
$\beta+\alpha=2(n-3)\pi/n$					A_2	A_2
$\dots\dots\dots\dots$				$\dots$	$\dots$	$\dots$
$\beta+\alpha=2(m+2)\pi/n$			A_{m-2}	$\dots$	A_{m-2}	A_{m-2}
$\beta+\alpha=2(m+1)\pi/n$		A_{m-1}	A_{m-1}	$\dots$	A_{m-1}	A_{m-1}

range for α $\quad 0 \quad \frac{\pi}{n} \quad \frac{3\pi}{n} \quad \frac{5\pi}{n} \quad \dots \quad \frac{(n-6)\pi}{n} \quad \frac{(n-4)\pi}{n} \quad \frac{(n-2)\pi}{n}$

range for α $\quad 0 \quad \frac{2\pi}{n} \quad \frac{4\pi}{n} \quad \dots \quad \frac{(n-7)\pi}{n} \quad \frac{(n-5)\pi}{n} \quad \frac{(n-3)\pi}{n}$

$\beta+\alpha=2m\pi/n$	A_m	A_m	$\dots$	A_m	A_m	
$\beta+\alpha=2(m-1)\pi/n$	A_{m+1}	A_{m+1}	$\dots$	A_{m+1}		
$\dots\dots\dots\dots$	$\dots$	$\dots$	$\dots$			
$\beta+\alpha=6\pi/n$	A_{n-4}	A_{n-4}				
$\beta+\alpha=4\pi/n$	A_{n-3}					

Notice that the type A_0 is always characterized by the condition $\beta - \alpha = \frac{2}{n}\pi$ (with no restriction for α), and if $n = 2m$, then the "middle" type A_{m-1} is characterized by the condition $\beta + \alpha = \pi$ (again, with no restriction for α).

Let us say, in conclusion, that although the definition of types may appear long and boring, we will demonstrate its importance in the rest of this article.

2.3 The Cases of n $\leq$ 6

If $n = 4$, then reachable points are the points (p, q) in the first quadrant with positive relatively prime p and q (and also points $(1, 0)$ and $(0, 1)$. There are two easily distinguishable types: if p and q are both odd, then the type of (p, q) is A_1; otherwise, it is A_0. This case has zero importance for us now (and the main result of Sect. 2.4 holds only for $n \geq 5$), but it is demonstrated in [7–9] that the behavior of closed geodesic on the surface of a cube depends in a very essential way on the type of the related short trajectory in the square.

The case $n = 6$ is also not especially important to us, but it is instructive, because the description of types becomes very explicit and some conjectures of the remainder of Sect. 2 obtain very elementary and convincing proofs.

Consider the beehive tiling of the sector $x\sqrt{3} \geq -y$ and use the coordinate system with the basis $(1, 0)$, $\frac{1}{2}(-1, \sqrt{3})$. Then, the vertices of the hexagonal tiles will have coordinates (p, q) with non-negative integers p, q, such that $p + q \not\equiv 2 \bmod 3$. Such point (p, q) is reachable if the interval $\big((0, 0), (p, q)\big)$ contains no vertices of the tiles. In other words, the point (p, q) (with $p + q \not\equiv 2 \bmod 3$) is reachable if either $GCD(p, q) = 1$ or $GCD(p, q) = 2$ and $p + q \equiv 1 \bmod 3$.

The description of types becomes very simple: the (reachable) point (p, q) belongs to the type:

$$\begin{aligned}
&A_1, \text{ if } (p, q) \equiv (1, 2) \bmod 3,\\
&A_2, \text{ if } (p, q) \equiv (0, 0) \bmod 2,\\
&A_3, \text{ if } (p, q) \equiv (2, 1) \bmod 3,\\
&A_0, \text{ if } (p, q) \text{ does not belong to } A_1, A_2, A_3.
\end{aligned}$$

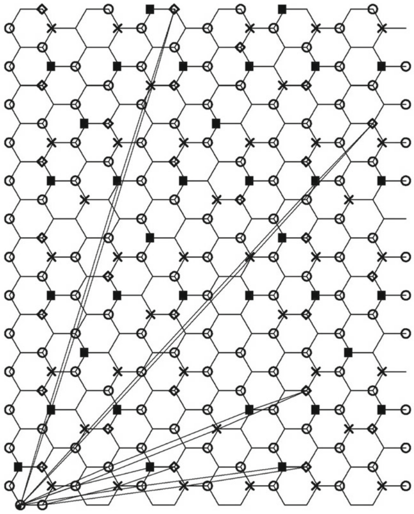
Another description of the type A_0: $p + q \equiv 1 \bmod 3$ and p, q are not both even. Notice also that if the point (p, q) belongs to the type A_2, then $p + q \equiv 1 \bmod 3$. Indeed, otherwise $\frac{p}{2} + \frac{q}{2} \not\equiv 2 \bmod 3$, and hence, the point $\left(\frac{p}{2}, \frac{q}{2}\right)$ is reachable, so the point (p, q) is not reachable. These remarks, together with the descriptions of the types A_0, A_1, A_2, A_3 given above, show that *a reachable point* (p, q) *belongs to the even class, if* $p + q \equiv 1 \bmod 3$ *and belongs to the odd class, if* $p + q \equiv 0 \bmod 3$.

In Fig. 10, we show the beehive tiling (within the "positive quadrant" $q \geq 2p - 1$). Reachable points are marked with special symbols; for the types, we used the following code: $A_0 = \circ$; $A_1 = \blacksquare$; $A_2 = \Diamond$; $A_3 = \times$.

The meaning of violet lines in Fig. 10 will be explained in Sect. 2.4.

In Fig. 11, we show a picture of reachable points for the case $n = 5$. It shows all reachable points at the distance ≤ 60 (where the side of the pentagon is taken for 1)

Fig. 10 Reachable points for a regular hexagon



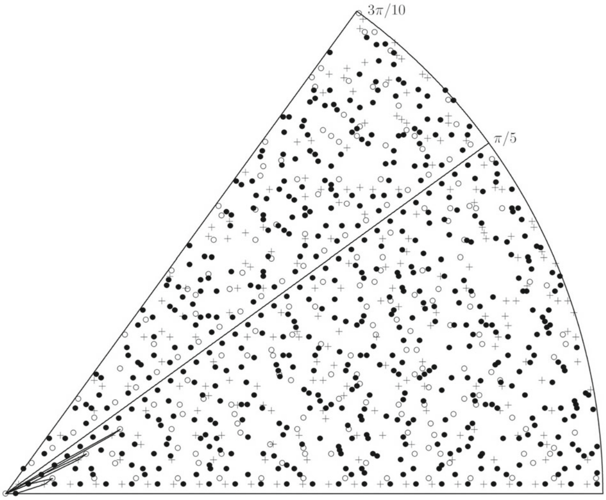


Fig. 11 Reachable points for a regular pentagon

within the sector $0 \leq \theta \leq \dfrac{3\pi}{10}$ (θ is the polar angle). The code used is $A_0 = \bullet,\ A_1 = \circ,\ A_2 = +$.

To get the "full picture" in the angle $\dfrac{3\pi}{5}$, we need to reflect the picture provided in the radius $\theta = \dfrac{3\pi}{10}$ and, in the appearing new part to switch $\circ$ and $+$. The meaning of long and narrow pentagons will be explained in Sect. 2.4.

2.4 Reachable n-Gons

A *reachable n-gon* is defined as an $SL(2,\mathbb{R})$-image of the initial regular n-gon, whose vertices (with the exception of O) are reachable points. Thus, a reachable n-gon must be contained in the sector $0 \le \theta \le \dfrac{n-2}{n}\pi$ and is the union of segments of developments of trajectories emanating from O. Examples of reachable n-gons are shown in Figs. 10 and 11 in violet.

Conjecture 2.3 *Let $n \ge 5$, let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of a reachable n-gon ordered clockwise, and let $v_0 = O$. Then, $v_1, \ldots, v_{n-1}$ belong, respectively, to the types $A_0, A_1, \ldots, A_{n-3}, A_0$.*

This statement reveals a true meaning of the types of short trajectories. In view of "the most obvious example" in Sect. 2.2, it shows that the $SL(2,\mathbb{R})$-transformation from the definition of a reachable polygon preserves the types. This observation may, possibly, pave a way to proving Conjecture 2.3.

Proof of Conjecture 2.3 for n = 6 Let $v_1 = (p, q)$ and $v_5 = (p', q')$. Then, the Fig. 12 shows that $v_2 = (2p+p', 2q+q')$, $v_3 = (2p+2p', 2q+2q')$, $v_4 = (p+2p', q+2q')$. Since the point v_3 is reachable and its coordinates are both even, v_3 is of the type A_2; moreover, $2p + 2p' + 2q + 2q' \equiv 1 \bmod 3$, and hence, $p + p' + q + q' \equiv 2 \bmod 3$. However, $p + q \not\equiv 2 \bmod 3$ and $p' + q' \not\equiv 2 \bmod 3$ (since (p, q) and (p', q') are both reachable); hence, $p + q \equiv p' + q' \equiv 1 \bmod 3$. Next, since our hexagon is $SL(2)$-equivalent to the standard regular hexagon, we have:

$$\det \begin{vmatrix} p' & q' \\ p & q \end{vmatrix} = \det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Hence, neither p, q, nor p', q' can be both even, and v_1 and v_5 are of the type A_0. Finally, the system of congruences

$$\begin{cases} p'q - pq' \equiv 1 \bmod 3 \\ \quad\; p + q \equiv 1 \bmod 3 \\ \quad p' + q' \equiv 1 \bmod 3 \end{cases}$$

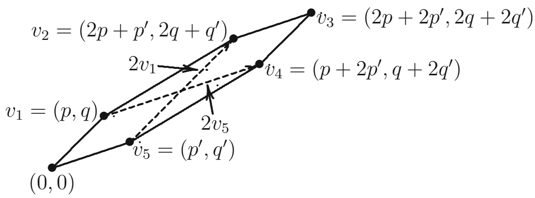


Fig. 12 To proof of Conjecture 2.3 for n= 6

has three solutions:

$$\begin{aligned}(p,q) &\equiv (0,1) \bmod 3,\ (p',q') \equiv (1,0) \bmod 3,\\ (p,q) &\equiv (2,2) \bmod 3,\ (p',q') \equiv (0,1) \bmod 3,\\ (p,q) &\equiv (1,0) \bmod 3,\ (p',q') \equiv (2,2) \bmod 3;\end{aligned}$$

in all these cases, $2(p,q) + (p',q') \equiv (1,2) \bmod 3$ and $(p,q) + 2(p',q') \equiv (2,1) \bmod 3$, so v_2 is of the type A_1 and v_4 is of the type A_3. This completes the proof.

Let us return to the case of an arbitrary n. □

Conjecture 2.4 *Every reachable point is a vertex of (infinitely many) reachable n-gons.*

Together with Conjecture 2.3 (and a remark after it), this means the following. Call a transformation $\varphi \in SL(2,\mathbb{R})$ a *Veech transformation* if $\varphi\{0 < \theta < (n-2)\pi/n\} \cap \{0 < \theta < (n-2)\pi/n\} \neq \emptyset$ and φ takes reachable points from $\varphi^{-1}\{0 < \theta < (n-2)\pi/n\}$ into reachable points. Claim: *for reachable points A, B a Veech transformation φ with $\varphi(A) = B$ exists if and only if A and B belong to the same type.*

Conjecture 2.4 may be refined in many ways. Here is one of them.

We will say that reachable points u, v of type A_0 form a *unitary pair*, if $\det(u, v)$ is equal to $\sin\dfrac{n-2}{n}\pi$ (that is, to $\det(\xi, \eta)$ where $\eta = \left(-\cos\dfrac{n-2}{n}\pi, \sin\dfrac{n-2}{n}\pi\right)$, $\xi = (1,0)$ are the sides of the standard regular n-gon at the vertex O). Equivalently (equivalence follows from Theorem 3.1), a unitary pair is the same as a pair of vertices of a reachable n-gon, joined by sides with the vertex O.

Conjecture 2.5 *Let $n \geq 5$, let u, v be a unitary pair, and let $\lambda = 2\cos\dfrac{\pi}{n}$. Then:*

(a) $u_m = u + m(\lambda + 1)v$ (where m is an integer, such that u_m lies in the upper half-plane) is a reachable point of type A_0 (and hence (u_m, v) is a unitary pair);
(b) $w_m = u + (\lambda + m(\lambda + 1))v$ (where m is an integer, such that w_m lies in the upper half-plane) is a reachable point of type A_1,
(c) there are no other reachable points on the line $u + tv$;
(d) all reachable points on the line $v + tu$ are described in the similar way (they all belong to the types A_0 and A_{n-3}).

Proofs of Conjectures 2.4 and 2.5 for $n = 6$ Let (in the coordinate system of Sect. 2.3) $u = (p', q')$ and $v = (p, q)$. Consider the equation $xp + yq = 1$. Since $\det(u, v) = 1$, $(x, y) = (-q', p')$ is a solution of this equation, and, since $GCD(p, q) = 1$, all solutions have the form $(x, y) = (-q', p') + k(-q, p)$. Thus, all points w with integral coordinates have the form $u + kv$ with integral k. For $u + kv = (p' + kp, q' + kq$, we have $(p' + kp) + (q' + kq) = (p' + q') + k(p + q) \equiv 1 + k \bmod 3$. Also $GCD(x, y) = GCD(p' + kp, q' + kq) = 1$, so the coordinates of $u + kv$ are not both even. Hence, the point $u + kv$ belongs to the type A_0, if $k \equiv 0 \bmod 3$, belongs to the type A_1, if $k \equiv 2 \bmod 3$, and is not reachable, if $k \equiv 1 \bmod 3$. This proves the statement (a), (b), and (c); proof of (d) is similar.

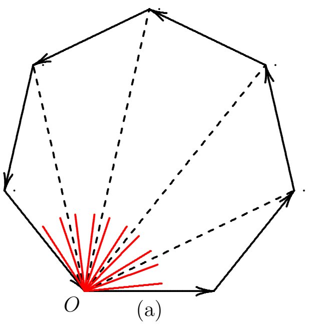


Fig. 13 Directions of parallel trajectories (n = 7)

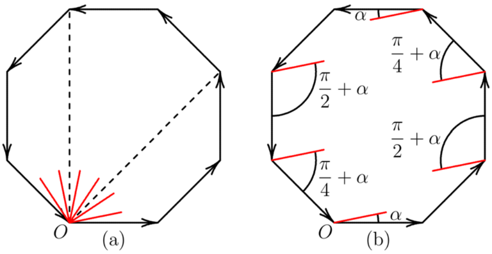


Fig. 14 Directions of parallel trajectories (n = 8)

2.5 Types of Parallel Trajectories

For a short trajectory emanating from O, we will refer to the angle which it forms with the horizontal direction at to the *slope angle*. If n is odd, then for the trajectory with the slope angle $\alpha \in \left(0, \dfrac{\pi}{2n}\right)$, the parallel trajectories (emanating from O) have the slope angles $\alpha, \dfrac{\pi}{n} \pm \alpha, \dots, \dfrac{(n-3)\pi}{n} \pm \alpha, \dfrac{(n-2)\pi}{n} - \alpha$; there are $2(n-2)$ of them (see Fig. 13a). Another description of this trajectories is shown in Fig. 13b.

Appropriate rotations of the n-gon in Fig. 13b transform the trajectories shown into trajectories emanating from O under the slope angles $\alpha, \dfrac{\pi}{n}+\alpha, \dots, \dfrac{(n-3)\pi}{n}+\alpha$. The reflection in the bisector of the angle O transforms these trajectories into trajectories with the slope angles $\dfrac{\pi}{n} - \alpha, \dots, \dfrac{(n-2)\pi}{n} - \alpha$.

If n is even, $n = 2m$, then for the trajectory with the slope angle $\alpha \in \left(0, \dfrac{\pi}{n}\right)$, there are $n-2$ parallel trajectories; they have the slope angles $\alpha, \dfrac{\pi}{m} \pm \alpha, \dots, \dfrac{(m-2)\pi}{m} \pm \alpha, \dfrac{(m-1)\pi}{m} - \alpha$. The picture similar to the previous picture for the case of even n is shown in Fig. 14.

The following conjecture describes the types of parallel trajectories (we assume that, in the notation A_k for the type, k is a residue modulo $n-2$.)

Conjecture 2.6 *Suppose that a short trajectory emanating from O with the slope angle α belongs to the type A_k. Then, the parallel short trajectory emanating from O with the slope angle $\dfrac{\ell\pi}{n} + \varepsilon\alpha$ (where $\varepsilon = \pm 1$) belongs to the type $A_{\varepsilon k-\ell}$.*

Notice that this statement does not make difference between the cases of odd and even n; but if n is even, then ℓ in the statement must be also even.

To make the statement more transparent, we present it in the tables on the next page.

n is odd

α	A_0	A_1	A_2	$\cdots$	A_{n-4}	A_{n-3}
$\frac{\pi}{n} - \alpha$	A_{n-3}	A_{n-4}	A_{n-5}	$\cdots$	A_1	A_0
$\frac{\pi}{n} + \alpha$	A_{n-3}	A_0	A_1	$\cdots$	A_{n-5}	A_{n-4}
$\frac{2\pi}{n} - \alpha$	A_{n-4}	A_{n-5}	A_{n-6}	$\cdots$	A_0	A_{n-3}
$\frac{2\pi}{n} + \alpha$	A_{n-4}	A_{n-3}	A_0	$\cdots$	A_{n-6}	A_{n-5}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\frac{(n-3)\pi}{n} - \alpha$	A_1	A_0	A_{n-3}	$\cdots$	A_3	A_2
$\frac{(n-3)\pi}{n} + \alpha$	A_1	A_2	A_3	$\cdots$	A_{n-3}	A_0
$\frac{(n-2)\pi}{n} - \alpha$	A_0	A_{n-3}	A_{n-4}	$\cdots$	A_2	A_1

n is even, $n = 2m$

α	A_0	A_1	A_2	$\cdots$	A_{n-4}	A_{n-3}
$\frac{\pi}{m} - \alpha$	A_{n-4}	A_{n-5}	A_{n-6}	$\cdots$	A_0	A_{n-3}
$\frac{\pi}{m} + \alpha$	A_{n-4}	A_{n-3}	A_0	$\cdots$	A_{n-6}	A_{n-5}
$\frac{2\pi}{m} - \alpha$	A_{n-6}	A_{n-7}	A_{n-8}	$\cdots$	A_{n-4}	A_{n-5}
$\frac{2\pi}{m} + \alpha$	A_{n-6}	A_{n-5}	A_{n-4}	$\cdots$	A_{n-8}	A_{n-7}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\frac{(m-2)\pi}{m} - \alpha$	A_2	A_1	A_0	$\cdots$	A_4	A_3
$\frac{(m-2)\pi}{m} + \alpha$	A_2	A_3	A_4	$\cdots$	A_0	A_1
$\frac{(m-1)\pi}{m} - \alpha$	A_0	A_{n-3}	A_{n-4}	$\cdots$	A_2	A_1

In each of these tables, the left column contains slope angles, the upper row shows types of trajectories with the slope angle α, and the rest of the tables contain types of all parallel trajectories.

Proof of Conjecture 2.6 for $n = 6$ Consider three linear transformations of the plane which map the vector with the slope angle α into vectors of the same length with the slope angles $\dfrac{\pi}{3} - \alpha$, $\dfrac{\pi}{3} + \alpha$, $\dfrac{2\pi}{3} - \alpha$. In the coordinates described in Sect. 2.3, these transformations are presented by the matrices $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; thus, they take (p, q) into $(p, p-q)$, $(p-q, p)$ and (q, p).

If (p, q) is a reachable point of the type A_0, then $p+q \equiv 1 \bmod 3$ and $GCD(p, q) = 1$. Hence, $p + (p-q) = 2p - q \equiv 2(p+q) \equiv 2 \bmod 3$, so $(p, p-q)$ is not reachable, and, since $GCD(p, p-q) = GCD(p, q) = 1$, to make this point reachable, we need to multiply it by 2. However, $2(p, p-q)$ is of the type A_2. Similarly, $2(p-q, p)$ is of the type A_2, and (q, p) is obviously a reachable point of the type A_0.

If (p, q) is a reachable point of the type A_1, that is, $GCD(p, q) = 1$ and $(p, q) \equiv (1, 2) \bmod 3$, then $GCD(p, p-q) = GCD(p-q, p) = GCD(q, p) = 1$ and $(p, p-q) \equiv (1, 1-2) \equiv (1, 2) \bmod 3$, $(p-q, p) \equiv (1-2, 1) \equiv (2, 1) \bmod 3$, $(q, p) \equiv (2, 1) \bmod 3$. Thus, $(p, p-q)$, $(p-q, p)$ and (q, p) are reachable points of the types, respectively, A_1, A_2, and A_2.

In a similar way, we check that if (p, q) is a reachable point of the type A_2, then $(p, p-q)$, $(p-q, p)$ and (q, p) are reachable points of the types, respectively, A_2, A_1, and A_1.

Finally, if (p, q) is a reachable point of the type A_2, then $p+q \equiv 1 \bmod 3$ and p, q are both even. In this case, all the points $(p, p-q)$, $(p-q, p)$ and (q, p) have even coordinates and $p + (p - q) \equiv (p - q) + p \equiv 2 \bmod p$, Thus, the points $(p, p - q)$ and $(p - q, p)$ are not reachable, and to make them reachable, we need to divide them by 2. The points $\frac{1}{2}(p, p - q)$ and $\frac{1}{2}(p - q, p)$ are of the type A_0.

Thus, dependence of the type of the trajectories on the slope angle is as shown in the table below. This table coincides with the right table above for $m = 3$. This completes the proof of Conjecture 2.6 for $n = 6$.

α	$A_0\ A_1\ A_2\ A_3$
$\frac{\pi}{3} - \alpha$	$A_2\ A_1\ A_0\ A_3$
$\frac{\pi}{3} + \alpha$	$A_2\ A_3\ A_0\ A_1$
$\frac{2\pi}{3} - \alpha$	$A_0\ A_3\ A_2\ A_1$

In conclusion, we formulate a statement, which may be regarded as a sharpening of Proposition 1.2.

Conjecture 2.7 *The length ratio of parallel short trajectories of types A_k and A_ℓ is* $\sin \dfrac{(k+1)\pi}{n} : \sin \dfrac{(\ell+1)\pi}{n}$.

Remark Conjecture 2.7 implies the statement that the shortest of parallel short trajectories belongs to the type A_0. In turn, this shows that *all the sides of a reachable n-gon, are developments of short trajectories of the type A_0*. Indeed, every side of a reachable n-gon is parallel to a sufficient amount of diagonals of this n-gon. These diagonals are also developments of short trajectories, and they all are longer than the side.

2.6 Lengths of Preclosed Trajectories Revisited

In this section, we will explain how Proposition 1.6 from Section 1.7.2 is related to the results of Sect. 2.5. The main geometric idea (for the odd-gonal case) is presented in Figs. 15 and 16.

Since every preclosed trajectory is strongly parallel to a short trajectory, we can restrict ourselves to trajectories which are very close to short trajectories. Consider a development of a short trajectory of the type A_k emanating from O. If we continue it through its endpoint B, then the added line may be considered as a development of a short trajectory in two ways (because two sides of the n-gon contain B); we show them in Figs. 15 and 16.

Below, we present two calculations based on conjectures from Sect. 2.5.

The first one shows that the trajectory BC is the development of a short trajectory of the type A_{k-2} or $A_{-(k+2)}$, which is symmetric with respect to the bisector of the angle O to (and have the same length as) the trajectory of the type A_{k+2}. Indeed, for this trajectory, the slope angle, we denote it by α', is the same as β for the trajectory OB. According to formulas in Definition 2.1 (Sect. 2.2), there are three possibilities for $\beta = \alpha'$; for each of these possibilities, the type $A_{k'}$ of the trajectory BC may be

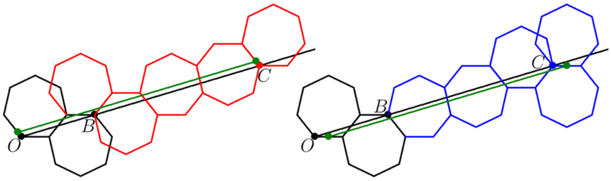
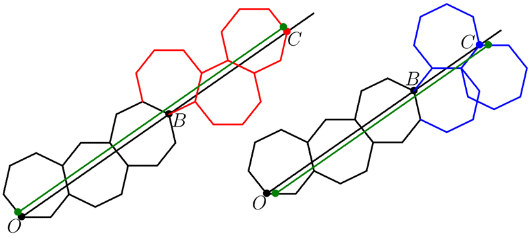


Fig. 15 Developments of parallel short and long trajectories (example)

Fig. 16 Developments of parallel short and long trajectories (another example)



determined by means of Conjecture 2.6:

$$\alpha' = \beta = \alpha + \frac{2(k+1)}{n}\pi, k' = k - 2(k+1) = -k - 2;$$
$$\alpha' = \beta = -\alpha + \frac{2(n-k-1)}{n}\pi, k' = -k - 2(n-k-1) \equiv k - 2 \bmod (n-2);$$
$$\alpha' = \beta = \alpha + \frac{2(k+3-n)}{n}\pi, k' = k - 2(k+3-n) \equiv -k - 2 \bmod (n-2).$$

The second calculation shows that the slope angle α'' of the continuation of the lime BC is always equal to α. This calculation requires considering many cases, but they are all similar, and we restrict ourselves to showing one of them. If the trajectories OB and BC are both covered by the first of the formulas in the odd-gonal case of Definition 2.1, then:

$$\alpha'' = \alpha' + \frac{2(k'+1)}{n}\pi = \alpha + \frac{2(k'+1) + 2(k+1)}{n}$$
$$\pi = \alpha + \frac{2(-k-2+1+k+1)}{n}\pi = \alpha.$$

The length of the preclosed trajectory is the sum of the lengths of the short trajectories OB and BC, so it is equal to the length of the minimal length of the short trajectory of our direction times:

$$\lambda_k + \lambda_{k+2} = \lambda_1\lambda_{k+1} \text{ or } \lambda_k + \lambda_{k-2} = \lambda_1\lambda_{k-1}$$

(we use the notations and the formula from Proposition 1.3). This proves Proposition 1.6(a).

Our computations are confirmed by Figs. 15 and 16. For $n = 7$, $\lambda_1 = \dfrac{\sin(2\pi/7)}{\sin(\pi/7)}$, $\lambda_2 = \dfrac{\sin(3\pi/7)}{\sin(\pi/7)}$. In Fig. 15, $OC = \lambda_1^2 OB$; in Fig. 16, left, $OC = \lambda_1 BC$, and in Fig. 16, right, $OC = \lambda_1^2 BC$.

Our arguments work also in the even-gonal case, but in addition to this, we can notice that the parity of $k-1$ and $k+1$ is opposite to the parity of $k-2$, k, and $k+2$, which explains class reversion in Proposition 1.6(b).

In conclusion, let us notice that if $k = 1$, then λ_{k-2} or λ_{k+2} may be $\lambda_{-1} = 0$ (sere again Proposition 1.3). In this case, the trajectory BC collapses to a point, and the length of the preclosed trajectory is the same as the length of a short trajectory. This happens if the length of the short trajectory is $\lambda_1 = \lambda_1 \cdot 1$.

3 Geodesics on the Surface of a Dodecahedron

Since faces of a regular dodecahedron are regular pentagons, geodesics of the surface of a regular dodecahedron are essentially the same as billiard trajectories in the regular pentagon. In this sense, "short" geodesics, which begin and end at the vertices, correspond to short billiard trajectories. In particular, short geodesics can belong to the types A_0, A_1, and A_2. In addition to that, short geodesics on the regular dodecahedron possess a characteristic not directly related to the properties of billiard trajectories: the vertices of the dodecahedron, where they begin and end.

It is known that the case of the regular dodecahedron is sharply different from the cases of other Platonic solids.

Theorem 3.1 [4,8,17] *No short geodesic on regular polyhedra, besides the dodecahedron, can end at a vertex where it begins.*

In the dodecahedron case, however, it was demonstrated in [1,2,8,17] that short geodesics beginning and ending at the same vertex exist. Moreover, the article [2] contains a full classification of such geodesics.

Below, we present some computer generated results concerning short geodesics on the regular dodecahedron.

We fix a face f of the dodecahedron, an edge e contained in this face, and vertex v contained in this edge. Then, every vertex w of the dodecahedron has a *distance* from v, which is the minimal number of edges forming a path from v to w. Then, all 20 vertices of the dodecahedron are divided into six groups:

1 vertex at the distance 0 from v (this is v),
3 vertices at the distance 1 from v,
6 vertices at the distance 2 from v,
6 vertices at the distance 3 from v,
3 vertices at the distance 4 from v,
1 vertex at the distance 5 from v.

We consider short geodesics beginning at the vertex v, whose first segment is contained in the face f. We compiled a list of all such geodesics of length < 120 (we assumed that the length of an edge of the dodecahedron is 1), whose first segment makes the angle $< \frac{3\pi}{10}$. The total amount of these geodesics is 3,750. The table below shows the numbers of such geodesics of every type depending on the group of vertices containing the end of the geodesic. We combined the types A_1 and A_2, since these types are switched by the reflection of the dodecahedron in the plane passing through the bisector of the angle v of the face f.

The distance of of the endpoint from v	Types		Total
	A_0	$A_1\&A_2$	
0	0	128	128
1	672	211	883
2	0	529	529
3	778	313	1091
4	342	322	664
5	330	125	455
Total	2122	1628	3750

This table has some striking properties. The most visible is appearing two zeroes in the A_0 column. One of them means that no short geodesic of the type A_0 can be closed. This statement follows from the classification of closed short geodesics given in the article [2]. Still, it may be interesting to find a geometric proof of this fact.

Another zero gives rise to the following

Conjecture 3.2 *A short geodesic of the type A_0 beginning at a vertex v never ends at a vertex at the distance 2 from v.*

In addition to that, we can observe that the distribution of endpoints of short geodesics of the type A_0 is sharply different from that for types A_1 and A_2. There arise a temptation to make guesses based on the figures in the table, but we prefer to refrain from that.

In conclusion, remark that some statistical information regarding closed and pre-closed geodesics on the regular dodecahedron is contained in the article [7]. This information also may be a source of exciting conjectures.

4 Regular Polyhedra of Hyperbolic Origin

Platonian polyhedra arise from tessellations of the sphere by identical regular polygons. Are there such tessellations of compact surfaces of constant negative curvature? An answer to this question is contained in the beautiful (and almost forgotten) article [6]. The main result of this article is the following

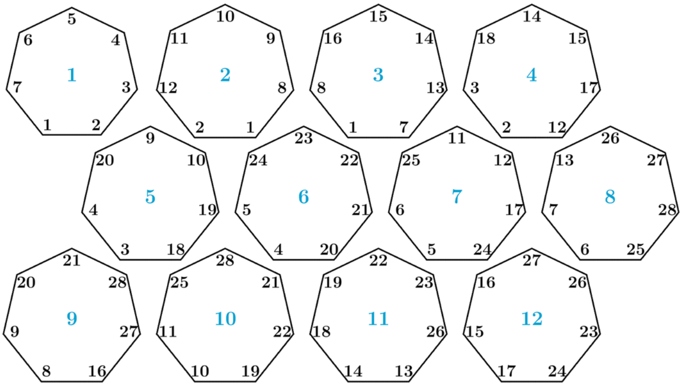


Fig. 17 Faces and vertices of the hyperbolic dodecahedron

Theorem 4.1 [6] *Let p, q, and $g \geq 2$ be positive integers, and let V, E, and F be positive integers with $pF = 2E = qV$ and $V - E + F = 2 - 2g$. Then, there exists a tessellation of the surface of genus 2 furnished with a Riemannian structure of constant curvature -1 by regular p-gons with vertices of valence q with V vertices, E edges, and F faces.*

(The article [6] contains also a classification of such tessellations.)

Using such a tessellation as a pattern, we can form a union of F identical Euclidean regular p-gons, such that every edge is shared by precisely 2 p-gons and every vertex is shared by precisely q p-gons. This is a *regular polyhedron of a hyperbolic origin*. The problems of classification of short, preclosed, and closed geodesics on it looks quite worthwhile. The goal of this section is not to present any results (this may be a subject of subsequent articles), but rather to draw attention to this circle of problems.

Consider the following example. Let $p = 7, q = 3$ and $g = 2$. Take $V = 28$, $E = 42$, and $F = 12$. We get a "hyperbolic dodecahedron" with 12 hexagonal faces, 28 vertices, and 42 edges. A decomposition of this "dodecahedron" into faces with vertices numerated is shown in Fig. 17.

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