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RESEARCH CONTRIBUTION

Element-Building Games on $\mathbb{Z}_n$

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Abstract
We consider a pair of games where two players alternately select previously unselected elements of $\mathbb{Z}_n$ given a particular starting element. On each turn, the player either adds or multiplies the element they selected to the result of the previous turn. In one game, the first player wins if the final result is 0; in the other, the second player wins if the final result is 0. We determine which player has the winning strategy for both games except for the latter game with nonzero starting element when $n \in \{2p, 4p\}$ for some odd prime p.

Keywords Combinatorial games · Finite rings · Two-player games · Modular arithmetic

Mathematics Subject Classification Primary 91A46; Secondary 16P10

1 Introduction

We propose two games named α-Zero and β-Zero. Each game begins with the number 0. Two players, Alpha and Beta, take turns removing numbers from $\{1, 2, 3, \ldots, 10\}$ and choosing to either add or multiply their choice to the current number. For example, the first player, Alpha, might select the number 5 and choose to add, resulting in $0+5 = 5$. The second player, Beta, might then select the number 7 from $\{1, 2, \ldots, 10\}\backslash\{5\}$ and choose to multiply, resulting in $5 \cdot 7 = 35$. Play will continue until there are no numbers left to select. Alpha wins α-Zero exactly when the final result is a multiple of 10, and Beta wins β-Zero exactly when the final result is a multiple of 10.

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These games are effectively played on the ring $\mathbb{Z}_{10}$, and we will generalize them be played on $\mathbb{Z}_n$ for all $n \geq 1$ in the natural way. The games continue the trend of studying games on algebraic structures. Anderson and Harary [1] introduced a game on finite groups, which was studied in [3–8,11]. Brandenburg [9] played a similar game on groups and rings, and [13] describes a game on polynomials with coefficients chosen from a ring.

We will determine winning strategies for α-Zero and β-Zero on $\mathbb{Z}_n$ for $n \geq 1$, where the winner is determined by whether the final number is 0. We also consider similar games with a nonzero starting number. We end with a concise summary of our results, followed by several open questions.

2 Preliminaries

We will use standard notation for $\mathbb{Z}_n$, as found in textbooks such as [10,12]. If n is an even integer, we will say an element $x \in \mathbb{Z}_n$ is *even* when all members of the equivalence class $[x]$ are even integers and *odd* otherwise.

We can now fix an n and formally define α-Zero on $\mathbb{Z}_n$. We set $S_0 = \emptyset$ and $e_0 = 0$. Two players, Alpha and Beta, take turns selecting r_{i+1} to be any element of $\mathbb{Z}_n \backslash S_i$, defining $S_{i+1} := S_i \cup \{r_{i+1}\}$, and defining e_{i+1} to be either $e_i + r_{i+1}$ or $e_i \cdot r_{i+1}$. Alpha defines e_{i+1} if $i+1$ is odd and Beta defines e_{i+1} if $i+1$ is even. Alpha is defined to be the winner if $e_n = 0$. We define the game β-Zero on $\mathbb{Z}_n$ in the same way, except we define Beta to be the winner if $e_n = 0$.

We will sometimes omit the reference to $\mathbb{Z}_n$ when referring to α-Zero on $\mathbb{Z}_n$ or β-Zero on $\mathbb{Z}_n$ if the n is clear from context or unimportant, simply saying α-Zero or β-Zero instead.

We end this section with a couple comments about notation. We will frequently use the notation e_i defined above. There will also be times when we want to refer to an element that could be either $e_i + r$ or $e_i \cdot r$; in this case, we will write $e_i \star r$. All other notation should be standard.

3 The α-Zero Game

We start with the α-Zero game on $\mathbb{Z}_n$. We will see that Alpha has a winning strategy if $n = 2$ or n is odd, and Beta has a winning strategy otherwise.

Proposition 1 *If n is odd, then Alpha has a winning strategy for α-Zero.*

Proof Alpha begins by defining $e_1 := e_0 \cdot 0 = 0 \cdot 0 = 0$. Subsequently, Alpha can employ the strategy in Table 1, which results in $e_{2i+1} = 0$ for all i. Since n is odd, we conclude that $e_n = 0$, which results in a win for Alpha by structural induction.

The α-Zero results for $n = 2$ and $n = 4$ are special cases, so we prove them both in the next result.

Proposition 2 *Alpha has a winning strategy for α-Zero on $\mathbb{Z}_2$, and Beta has a winning strategy for α-Zero on $\mathbb{Z}_4$.*

Table 1 Alpha's strategy for α-Zero when n is odd

If Beta defines e_{2i} to be	Then, Alpha defines e_{2i+1} to be
$e_{2i-1} + r = r$	$e_{2i} + (-r) = 0$
$e_{2i-1} \cdot r = 0$	$e_{2i} \cdot (-r) = 0$

Here, r is any nonzero element . We assume $e_{2i-1} = 0$ by structural induction

Table 2 Beta's strategy for α-Zero on $\mathbb{Z}_4$ with $\{a, b, c\} = \{\pm 1, 2\}$

If e_1 is	Then, e_2 should be	If e_3 is	Then e_4 should be
$e_0 \star a = 0 \star a$	$e_1 \cdot 0 = 0$	$e_2 + b = b$	$e_3 \cdot c = bc \neq 0$
		$e_2 \cdot b = 0$	$e_3 + c = c \neq 0$
$e_0 \star 0 = 0 \star 0 = 0$	$e_1 + 2 = 2$	$e_2 + (\pm 1) = \mp 1$	$e_3 + (\mp 1) = 2 \neq 0$
		$e_2 \cdot (\pm 1) = 2$	$e_3 \cdot (\mp 1) = 2 \neq 0$

Here, Alpha defines e_1 and e_3, while Beta defines e_2 and e_4. Without loss of generality, Alpha will first choose a and then b from $\{a, b, c\}$

Proof First, consider the game for α-Zero on $\mathbb{Z}_2$. Alpha defines e_1 to be $e_0 \cdot 1 = 0 \cdot 1 = 0$. Now, Beta must choose between defining e_2 to be $0 + 0 = 0$ and $0 \cdot 0 = 0$. In either case, Alpha wins.

Next, consider α-Zero on $\mathbb{Z}_4$. Define a, b, and c so that $\{a, b, c\} = \{\pm 1, 2\}$ and $\mathbb{Z}_4 = \{0, a, b, c\}$. We consider the case where Alpha plays 0 on the first move and where Alpha plays, without loss of generality, a on the first move separately. The strategy is summarized in Table 2. In all cases, Beta is able to define e_4 to be nonzero and win the game.

For α-Zero and β-Zero on $\mathbb{Z}_n$ with $n \geq 3$, it will turn out that the player with the last move will have a winning strategy. The rough idea of the strategy when $n \neq 4$ is to initially play all of the elements x in $\mathbb{Z}_n$ that square to 0. This can be ensured so long as a relatively small number of the elements of $\mathbb{Z}_n$ square to 0. The following lemma provides us with a sufficient bound on the number of elements that square to 0.

Lemma 1 *Let $n \geq 3$ such that $n \neq 4$. If $X = \{x \in \mathbb{Z}_n \mid x^2 = 0\}$, then $|X| \leq \lfloor \frac{n-1}{2} \rfloor$.*

Proof It is easy to verify that the result holds if $n \in \{3, 5, 6\}$, so assume that $n \geq 7$. Let $n = \prod_{i=1}^{k} p_i^{d_i}$ such the p_i are distinct primes and $d_i \geq 1$ for each i. Let $x \in \mathbb{Z}$ be such that $0 \leq x < n$ and $x^2 \equiv 0 \pmod{n}$. Since $x^2 \equiv 0 \pmod{n}$, we have that $x^2 \equiv 0 \pmod{p_i^{d_i}}$ for every i.

Note that there are at most $\sqrt{p_i^{d_i}}$ solutions y_i such that $y_i^2 \equiv 0 \pmod{p_i^{d_i}}$, since y_i must have the form $a_i \cdot p_i^{b_i}$, where $b_i = \lceil d_i/2 \rceil$ and $0 \leq a_i \leq p^{d_i - b_i} - 1$. Thus, there are at most $\sqrt{p_i^{d_i}}$ possible $y_i \in \mathbb{Z}_{p_i^{d_i}}$ such that $x \equiv y_i \pmod{p_i^{d_i}}$.

By the Chinese Remainder Theorem, we get that there are at most $\prod_{i=1}^{k}\sqrt{p_i^{d_i}}=\sqrt{n}$ possible values for x. Since $n \geq 7$, we have that there are at most $\sqrt{n} \leq \lfloor \frac{n-1}{2} \rfloor$ values x such that $x^2 \equiv 0 \pmod{n}$.

Proposition 2 shows us that Beta has a winning strategy for α-Zero on $\mathbb{Z}_4$. We now show that Beta has a winning strategy for α-Zero on $\mathbb{Z}_n$ when n is even and $n > 4$. To do so, we are going to generalize the argument so that we can also get a result for β-Zero simultaneously. We introduce the player *Last*, who is defined to be Alpha if n is odd and Beta otherwise. Last is so-named because Last makes the last move of the game.

Proposition 3 *Let $n \geq 3$ such that $n \neq 4$. Then,*

1. *Beta has a winning strategy for α-Zero if n is even, and*
2. *Alpha has a winning strategy for β-Zero if n is odd.*

Proof We will prove these two results simultaneously by showing that Last has a winning strategy. In both cases, Last wants $e_n \neq 0$.

First, note that Last makes $n/2$ total plays if n is even and $(n+1)/2$ plays if n is odd. Thus, Last plays $(n-2)/2$ plays *prior* to the last play if n is even and $(n-1)/2$ such plays if n is odd, both of which equal $\lfloor \frac{n-1}{2} \rfloor$.

Here is the strategy. Within the first $\lfloor \frac{n-1}{2} \rfloor$ plays, Last should choose all of the elements $x \in \mathbb{Z}_n$ such that $x^2 = 0$. By Lemma 1, this is possible. Once such x have all been chosen, Last can choose subsequent elements arbitrarily until the last play of the game.

At the end of the game, Last must define e_n to be either $e_{n-1}+x$ or $e_{n-1} \cdot x$ for some $e_{n-1}, x \in \mathbb{Z}_n$. If $e_{n-1} \neq -x$, then the former yields a victory for Last. Otherwise, Last will define e_n to be $e_{n-1} \cdot x = -x \cdot x = -x^2$. Since Last previously chose all of the elements that square to 0, we have $e_n = -x^2 \neq 0$ and Last wins.

We summarize the results for the α-Zero game in one statement.

Theorem 1 *Consider α-Zero played on $\mathbb{Z}_n$ for some $n \geq 1$. Then*

1. *Alpha has a winning strategy if $n = 2$ or n is odd, and*
2. *Beta has a winning strategy otherwise.*

Proof This follows immediately from Propositions 1, 2, and 3.

4 The β-Zero Game

In Proposition 3, we saw that Alpha has a winning strategy for β-Zero if n is odd and $n \geq 3$. It remains to determine what happens when $n = 1$ or n is even.

Theorem 2 *Consider β-Zero played on $\mathbb{Z}_n$ for some $n \geq 1$. Then,*

1. *Alpha has a winning strategy if n is odd and $n \geq 3$ and*
2. *Beta has a winning strategy otherwise.*

Table 3 Beta's strategy for β-Zero when n is even

If Alpha defines e_{2i+1} to be	Then, Beta defines e_{2i+2} to be
$e_{2i}\star 0 = 0$	$e_{2i+1} \cdot (n/2) = 0$
$e_{2i}\star(n/2)$	$e_{2i+1} \cdot 0 = 0$
$e_{2i} + r = r$	$e_{2i+1} + (-r) = 0$
$e_{2i} \cdot r = 0$	$e_{2i+1} \cdot (-r) = 0$

Here, $r \in \mathbb{Z}_n \backslash \{0, n/2\}$. We assume $e_{2i} = 0$ by structural induction

Proof Suppose that n is odd. If $n = 1$, we have $e_1 \in \{0+0, 0\cdot 0\} = \{0\}$, so Beta wins. The case where n is odd and $n \geq 3$ was done in Proposition 3.

Now assume that n is even. Beta's strategy is stated in Table 3. In all cases, $e_{2i+2} = 0$. Since n is even, $e_n = 0$ and Beta wins β-Zero.

5 Generalizations

We have completely determined which player has the winning strategy for the games α-Zero and β-Zero on $\mathbb{Z}_n$. However, there are many generalizations to consider. One easy variation is to have the criterion that Alpha wins if the game ends with $e_n \neq 0$. Call this game α-Nonzero on $\mathbb{Z}_n$, and define β-Nonzero to be the game on $\mathbb{Z}_n$ where Beta wins if the final element is nonzero. Since Alpha loses α-Nonzero exactly when Beta wins β-Zero and Beta loses β-Nonzero exactly when Alpha wins α-Zero, we can use results from the previous section to immediately get the result.

Theorem 3 *Let $n \geq 2$, and consider α-Nonzero and β-Nonzero played on $\mathbb{Z}_n$.*

1. *For games on $\mathbb{Z}_1$: Beta has a winning strategy for α-Nonzero, and Alpha has a winning strategy for β-Nonzero.*
2. *For α-Nonzero on $\mathbb{Z}_n$: Alpha has a winning strategy if n is odd, and Beta has a winning strategy if n is even.*
3. *For β-Nonzero on $\mathbb{Z}_n$: Alpha has a winning strategy if $n = 2$ or n is odd, and Beta has a winning strategy otherwise.*

Proof For the games on $\mathbb{Z}_1$, we have $e_1 \in \{0+0, 0\cdot 0\} = \{0\}$, so Beta wins α-Nonzero and Alpha wins β-Nonzero trivially.

Let $n = 2$. By Proposition 2 and Theorem 2, both Alpha and Beta can make the game end in 0, yielding the desired result. So assume that $n > 2$.

If n is odd, then Alpha wins α-Nonzero by Proposition 3 and wins β-Nonzero by Proposition 1. If n is even, then Beta wins β-Nonzero by Propositions 2 and 3 and wins α-Nonzero by Theorem 2.

A more interesting generalization is this: let α-Zero(a) on $\mathbb{Z}_n$ be just like α-Zero in that Alpha wins if $e_n = 0$, only the game starts with $e_0 = a$ for some possibly nonzero $a \in \mathbb{Z}_n$. In this notation, α-Zero is denoted by α-Zero(0). We can define β-Zero(a) similarly for all $a \in \mathbb{Z}_n$. We will mean α-Zero(a) and β-Zero(a) to imply that $a \in \mathbb{Z}_n$ for the appropriate n without explicitly stating that $a \in \mathbb{Z}_n$ below.

The results for α-Zero(a) for all n follow from the proofs of Propositions 1, 2, and 3.

Theorem 4 *Consider α-Zero(a) played on $\mathbb{Z}_n$ for some $n \geq 1$. Then*

1. *Alpha has a winning strategy for all a if $n = 2$ or n is odd, and*
2. *Beta has a winning strategy for all a otherwise.*

Proof First, consider the game for α-Zero(a) on $\mathbb{Z}_2$. The case where $a = 0$ was done in Proposition 2, so consider α-Zero(1). Alpha defines e_1 to be $e_0 + 1 = 1 + 1 = 0$. Now, Beta must choose between defining e_2 to be $0 + 0 = 0$ and $0 \cdot 0 = 0$. In either case, Alpha wins.

If n is odd, then the proof follows by the same argument as the proof of Proposition 1 since Alpha's first move is to define e_1 to be $e_0 \cdot 0 = a \cdot 0 = 0$; we did not need the fact that $e_0 = 0$ in the proof of Proposition 1.

So suppose that n is even and $n > 2$. The result follows from the proof of Proposition 3, since the proof did not require that e_0 be 0—Beta simply needs to play elements that square to zero to ensure the last element is nonzero.

Thus, the game α-Zero(a) is not much different for nonzero a as it is from α-Zero $=$ α-Zero(0), as the choice of a does not affect which player has a winning strategy. The game β-Zero(a) is more challenging, though. Indeed, note that the proof of Theorem 2 relies on the fact that $e_0 = 0$ to start its structural induction argument, so the proof does not apply directly if $a \neq 0$.

We start by considering β-Zero(a) on $\mathbb{Z}_2$, $\mathbb{Z}_4$, and $\mathbb{Z}_8$. The following proposition is the output of a computer program [2].

Proposition 4 *Consider β-Zero(a) on $\mathbb{Z}_n$.*

1. *If $n = 2$, then Beta has a winning strategy for all $a \in \mathbb{Z}_2$.*
2. *If $n = 4$, then Beta has a winning strategy if $a \in \{0, 2\}$ and Alpha has a winning strategy if $a \in \{1, 3\}$.*
3. *If $n = 8$, then Beta has a winning strategy if $a \in \mathbb{Z}_8 \backslash \{1, 5\}$ and Alpha has a winning strategy if $a \in \{1, 5\}$.*

The next lemma is a meta-strategy for Beta that we will use twice below.

Lemma 2 *Let n be even, and suppose that Beta can define $e_{2i+2} := e_{2i} \cdot s_i$ for some $s_i \in \mathbb{Z}_n$ for all i. If $as_k s_\ell = 0$ for some k and ℓ such that $k \neq \ell$, then Beta has a winning strategy for β-Zero(a) on $\mathbb{Z}_n$.*

Proof Since n is even and $e_{2i+2} = e_{2i} \cdot s_i$ for all i, we have

$$e_n = a \prod_{i=0}^{n/2-1} s_i = as_k s_\ell \cdot \prod_{\substack{0 \leq i < n/2 \\ i \notin \{k,\ell\}}} s_i = 0 \cdot \prod_{\substack{0 \leq i < n/2 \\ i \notin \{k,\ell\}}} s_i = 0.$$

Since n is even, Beta wins.

Table 4 Beta's strategy for β-Zero(a) when n and a are both even

If Alpha defines e_{2i+1} to be	Then Beta defines e_{2i+2} to be	t_k
$e_{2i} + 0 = e_{2i}$	$e_{2i+1} \cdot (n/2) = e_{2i} \cdot (n/2)$	$t_0 = n/2$
$e_{2i} \cdot 0 = 0$	$e_{2i+1} \cdot (n/2) = 0$	$t_0 = 0$
$e_{2i} \star (n/2)$	$e_{2i+1} \cdot 0 = 0$	$t_0 = 0$
$e_{2i} + r = r$	$e_{2i+1} + (-r) = e_{2i}$	$t_{\|r\|} = 1$
$e_{2i} \cdot r = 0$	$e_{2i+1} \cdot (-r) = e_{2i} \cdot (-r^2)$	$t_{\|r\|} = -r^2$

Here, $r \in \mathbb{Z}_n \backslash \{0, n/2\}$. We also define $|r|$ to be $-r$ if $n/2 < r < n$ and r if $0 < r < n/2$

While Lemma 2 describes the factors indexed according to the stage of the game—essentially i in the lemma—it will be more natural to describe the factors as indexed according to the most recently played elements. So, when Alpha defines $e_{2i+1} := e_{2i} \star k$ and Beta defines $e_{2i+2} = e_{2i} \cdot s_i$, we will refer to s_i as $t_{k'}$ for some element k' associated with k.

If n and a are both even, we see that Beta wins in the following result.

Proposition 5 *If n is even and $a \in \mathbb{Z}_n$ is even, then Beta has a winning strategy for β-Zero(a).*

Proof The case where $n = 2$ was done in Theorem 2, so assume that $n \geq 4$. Write $a = 2b$ for some $0 \leq b < a$. Beta can employ the strategy in Table 4, noting that $e_{2i+2} = e_{2i} \cdot t_k$ for the t_k specified in the table for each i. Then $at_0t_1 = 0$ if $t_0 = 0$ and $at_0t_1 = (2b)(n/2)t_1 = n(bt_1) = 0$ otherwise. Thus, Beta has a winning strategy by Lemma 2.

The following lemma will be used repeatedly to determine when Beta has a winning strategy for β-Zero(a) on $\mathbb{Z}_n$ when n is even. The proof outlines Beta's strategy.

Lemma 3 *Let n be even, and consider β-Zero(a) played on $\mathbb{Z}_n$. Suppose there exist $x, y \in \mathbb{Z}_n$ such that all of the following are true.*

1. $x, y \notin \{0, n/2\}$
2. $x \neq y$
3. $xy = 0$
4. $ax(n/2) = 0$
5. $ay(n/2) = 0$

Then, Beta has a winning strategy for β-Zero(a).

Proof Since $x, y \notin \{0, n/2\}$, we have that $-x \neq x$ and $-y \neq y$. Thus, $\{\pm x, \pm y\}$ contains two distinct elements if $y = -x$ and four distinct elements otherwise. Without loss of generality, assume that $0 < x < n/2$ and $0 < -y < n/2$.

Beta's strategy is summarized in Table 5. Note that by employing the strategy, Beta ensures that $e_{2i+2} = e_{2i} \cdot t_k$ for some t_k listed in Table 5 for each i. If $t_0 = 0$ or $t_x = 0$, then $at_0t_x = 0$ and Beta wins by Lemma 2. Thus, we may assume that $t_0 = n/2$ and $t_x \in \{\pm x, \pm y\}$. Then, $at_0t_x \in \{a(n/2)(\pm x), a(n/2)(\pm y)\} = \{0\}$ by assumption. Thus, Beta wins by Lemma 2.

Table 5 A possible strategy for Beta for β-Zero(a) on $\mathbb{Z}_n$ if n is even

If Alpha defines e_{2i+1} to be	Then Beta defines e_{2i+2} to be	t_k
$e_{2i}+0=e_{2i}$	$e_{2i+1}\cdot(n/2)=(n/2)e_{2i}$	$t_0=n/2$
$e_{2i}\cdot 0=0$	$e_{2i+1}\cdot(n/2)=0$	$t_0=0$
$e_{2i}\star(n/2)$	$e_{2i+1}\cdot 0=0$	$t_0=0$
$e_{2i}+z$	$e_{2i+1}\cdot w=e_{2i}w+zw=e_{2i}w$	$t_z=w$
$e_{2i}\cdot z$	$e_{2i+1}\cdot w=e_{2i}\cdot zw=0$	$t_z=0$
$e_{2i}+w$	$e_{2i+1}\cdot z=e_{2i}z+zw=e_{2i}z$	$t_z=z$
$e_{2i}\cdot w$	$e_{2i+1}\cdot z=e_{2i}\cdot zw=0$	$t_z=0$
$e_{2i}+r$	$e_{2i+1}+(-r)=e_{2i}$	$t_{\|r\|}=1$
$e_{2i}\cdot r$	$e_{2i+1}\cdot(-r)=-e_{2i}\cdot r^2$	$t_{\|r\|}=-r^2$

We let $(z, w)\in\{(x, y), (-y, -x)\}$ so that $zw=xy=0$, and we let $r\in\mathbb{Z}_n\backslash\{0, n/2, \pm x, \pm y\}$. We also define $|r|$ to be $-r$ if $n/2<r<n$ and r if $0<r<n/2$

Table 6 $n=2^k\prod_{i=1}^{\ell}p_i^{m_i}$ with x and y as in Lemma 3

If	(x, y)	xy	$ax(n/2)$ and $ay(n/2)$
$\ell>1$	$(\frac{n}{p_1}, \frac{n}{p_2})$	$(\frac{n}{p_1})(\frac{n}{p_2})=n(\frac{n}{p_1p_2})=0$	$a(\frac{n}{p_1})(\frac{n}{2})=an(\frac{n}{2p_1})=0$
			$a(\frac{n}{p_2})(\frac{n}{2})=an(\frac{n}{2p_2})=0$
$m_1>1$	$(\frac{n}{p_1}, -\frac{n}{p_1})$	$(\frac{n}{p_1})(-\frac{n}{p_1})=-n(\frac{n}{p_1^2})=0$	$a(\pm\frac{n}{p_1})(\frac{n}{2})=\pm an(\frac{n}{2p_1})=0$
$k>3$	$(\frac{n}{4}, -\frac{n}{4})$	$(\frac{n}{4})(-\frac{n}{4})=-n(\frac{n}{2^4})=0$	$a(\pm\frac{n}{4})(\frac{n}{2})=\pm an(\frac{n}{2^3})=0$
$n=8p_1$	$(2p_1, 4)$	$2p_1(4)=8p_1=n=0$	$a(2p_1)(4p_1)=ap_1(8p_1)=0$
			$a(4)(4p_1)=2a(8p_1)=0$

The following theorem is our main result for β-Zero(a) on $\mathbb{Z}_n$. Note that Beta repeatedly employs the strategy outlined in the proof of Lemma 3.

Theorem 5 *Consider β-Zero(a) on $\mathbb{Z}_n$.*

1. *If n is odd and $n\geq 3$, then Alpha has a winning strategy for β-Zero(a) for all $a\in\mathbb{Z}_n$.*
2. *If $n=1$ or n is even, then Beta has a winning strategy for β-Zero(a) for all a unless a is odd and there is an odd prime p such that $n\in\{4, 8, 2p, 4p\}$.*

Proof The cases where $a=0$ were done in Theorem 2, so we will assume that $a\neq 0$ and $n>1$. If n is odd, then Alpha has a winning strategy by the proof of Proposition 3, since the proof does not require that $e_0=0$. So, we may assume that n is even and write $n=2^k\prod_{i=1}^{\ell}p_i^{m_i}$ where the p_i are distinct odd primes, $m_i\geq 1$, and $k\geq 1$. Recall that if $\ell=0$, then $n=2^k$. Table 6 summarizes how Lemma 3 can be applied by identifying an appropriate x and y based on the prime decomposition of n.

Table 6 and Lemma 3 imply that Beta has winning strategy unless $\ell\leq 1$, $m_1\leq 1$, and $k\leq 3$, so the only remaining cases are when $n\in\{2, 4, 8, 2p, 4p, 8p\}$ for some odd prime p. Further, Beta has a winning strategy for $n=2$ by Proposition 4 and

Table 7 Summary of all results for α-Zero(a) and β-Zero(a) in $\mathbb{Z}_n$

Game	n	a	Winner	Reference
α-Zero(a)	2	All	Alpha	Theorem 4
	Odd	All	Alpha	Theorem 4
	Even ≥ 4	All	Beta	Theorem 4
β-Zero(a)	1	All	Beta	Theorem 2
	Odd ≥ 3	All	Alpha	Theorem 5
	2	All	Beta	Proposition 4
	4	$\{0, 2\}$	Beta	Proposition 4
	4	$\{\pm 1\}$	Alpha	Proposition 4
	8	$\mathbb{Z}_8 \backslash \{1, 5\}$	Beta	Proposition 4
	8	$\{1, 5\}$	Alpha	Proposition 4
	Even $\notin \{2, 4, 8, 2p, 4p\}$	all	Beta	Theorem 5
	$\{2p, 4p\}$	Even	Beta	Proposition 5
	$\{2p, 4p\}$	Odd	Unknown	

Here, a denotes the starting element of the game and p is an odd prime

for $n = 8p$ for some odd prime p by Table 6 and Lemma 3, so Beta has a winning strategy for all a if $n \notin \{4, 8, 2p, 4p\}$ for all odd primes p.

Finally, if $n \in \{4, 8, 2p, 4p\}$ and a is even, then Beta has a winning strategy by Proposition 5. So, Beta has a winning strategy for β-Zero(a) on $\mathbb{Z}_n$ for all even n unless perhaps if a is odd and $n \in \{4, 8, 2p, 4p\}$ for some odd prime p.

Note that the case where $n \in \{4, 8\}$ was done in Proposition 4, so the only outstanding cases for β-Zero(a) on $\mathbb{Z}_n$ are when $n \in \{2p, 4p\}$ for an odd prime p and a is odd.

6 Summary and Open Questions

We summarize the results for α-Zero(a) and β-Zero(a) on $\mathbb{Z}_n$ in Table 7 and conclude with several open questions.

1. Can we determine who wins β-Zero(a) for all n and all $a \in \mathbb{Z}_n$? That is, can we determine which player has a winning strategy when $n \in \{2p, 4p\}$ for an odd prime p when a is odd? Computer results on $\mathbb{Z}_n$ for $n \leq 14$ suggest that Alpha has the winning strategy exactly when a is a unit.
2. We can define the game α-Target(a, z) on $\mathbb{Z}_n$ to be the variation that $e_0 = a$ and Alpha wins exactly if $e_n = z$, where z might be nonzero. Similarly, Beta wins β-Target(a, z) exactly when $e_n = z$. For each n, a, and z, who has a winning strategy for α-Target(a, z) and β-Target(a, z) on $\mathbb{Z}_n$?
3. Note that all we need for this game are addition and multiplication, so this is really a game on rings. Let R be an arbitrary finite ring. We can define the game α-Target(R, a, z) for $a, z \in R$ to be the variation that $e_0 = a$ and Alpha wins exactly if $e_n = z$. Similarly, Beta wins β-Target(R, a, z) exactly when $e_n =$

z. For each R, a, and z, who has a winning strategy for α-Target(R, a, z) and β-Target(R, a, z)?

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Declarations

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Code availability We published our code at the webpage listed on [2].

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