Arnold Mathematical Journal
https://doi.org/10.1007/s40598-022-00201-6

RESEARCH CONTRIBUTION

Two-Sided Fundamental Theorem of Affine Geometry

Alexey Gorinov[1]

Received: 17 December 2020 / Revised: 29 October 2021 / Accepted: 4 March 2022

Abstract
The fundamental theorem of affine geometry says that if a self-bijection f of an affine space of dimenion n over a possibly skew field takes left affine subspaces to left affine subspaces of the same dimension, then f of the expected type, namely f is a composition of an affine map and an automorphism of the field. We prove a two-sided analogue of this: namely, we consider self-bijections as above which take affine subspaces to affine subspaces but which are allowed to take left subspaces to right ones and vice versa. We show that under some conditions these maps again are of the expected type.

Keywords Fundamental theorem of affine geometry · Division ring · Anti-automorphism · Semi-linear and semi-affine maps

1 Introduction

Let $\mathbf{k}$ be an associative division ring. The "fundamental theorem of affine geometry" (see e.g. [3] and references therein) is the statement that if a bijection $f : \mathbf{k}^n \to \mathbf{k}^n$ takes every left affine subspace to a left affine subspace of the same dimension, then f is an affine map composed with a map induced by an automorphism of $\mathbf{k}$, provided $\infty > n \geq 2$. The projective version of this result can be found e.g. in [2, Chapter 2]. Here we prove a two-sided analogue of the affine version. We will say that an affine subspace $\subset \mathbf{k}^n$ is *purely left* (respectively *purely right*) if it is left but not right (respectively right but not left). Affine subspaces which are both right and left will be called *two-sided* and affine subspaces which are purely left or purely right will be called *one-sided*.

In what follows affine subspaces of $\mathbf{k}^n$ of dimension 1 and 2 will be referred to as *lines* and *planes* respectively. By saying that a map $f : \mathbf{k}^n \to \mathbf{k}^n$ *takes* $A \subset \mathbf{k}^n$ *to* $B \subset \mathbf{k}^n$, or that A *goes to* B *under* f, we will mean that $f(A) = B$.

✉ Alexey Gorinov
agorinov@hse.ru

1 Faculty of Mathematics, National Research University Higher School of Economics, Moscow, Russia

A homomorphism $\mathbf{k}^n \to \mathbf{k}^n$ of left $\mathbf{k}$-modules is given by $x \mapsto xM$ where $x = (x_1, \dots, x_n)$ is a row vector, $M = (m_{ij})$ is an $n \times n$-matrix with entries in $\mathbf{k}$, and $(xM)_i = \sum_{l=1}^n x_l m_{li}$. Similarly, a homomorphism $\mathbf{k}^n \to \mathbf{k}^n$ of right $\mathbf{k}$-modules is given by $x \mapsto Mx$ where $x = \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}$ is a column vector, $M = (m_{ij})$ is a matrix as above, and $(Mx)_i = \sum_{l=1}^n m_{il} x_l$.

We will denote the centre of $\mathbf{k}$ by $Z(\mathbf{k})$. A map $f : \mathbf{k}^n \to \mathbf{k}^n$ is a homomorphism of $\mathbf{k}$-bimodules if and only if it is a homomorphism of left (or right) $\mathbf{k}$-modules and all entries of the corresponding matrix are in $Z(\mathbf{k})$, or equivalently, if and only if using the natural isomorphism $\mathbf{k}^n \cong \mathbf{k} \otimes_{Z(\mathbf{k})} Z(\mathbf{k})^n$ the map f can be written as $f = \mathrm{Id}_{\mathbf{k}} \otimes_{Z(\mathbf{k})} f'$ where $f' : Z(\mathbf{k})^n \to Z(\mathbf{k})^n$ is a homomorphism of $Z(\mathbf{k})$-vector spaces.

Our main result is the following version of the fundamental theorem of affine geometry for two-sided affine subspaces.

Theorem 1 *Let k be an integer $\in \{2, \dots, n-1\}$, and let $f : \mathbf{k}^n \to \mathbf{k}^n, n \geq 3$ be a bijective map that takes every left or right affine subspace of dimension k to a left or right affine subspace of dimension $\leq k$. The image of a left affine subspace may be right and vice versa. Then f is the composition of multiplication on the right by an $a \in \mathbf{k} \setminus \{0\}$, an isomorphism of $\mathbf{k}$-bimodules, a translation, and an automorphism or anti-automorphism of $\mathbf{k}$ applied component-wise.*

In particular, if $\mathbf{k} = \mathbb{H}$, the skew field of quaternions, then there exist an automorphism $g : \mathbb{H}^n \to \mathbb{H}^n$ of $\mathbb{H}$-bimodules, and elements $a, q \in \mathbb{H} \setminus \{0\}$ and $b \in \mathbb{H}^n$ such that f is the map

$$x \mapsto q(g(xa) + b)q^{-1},$$

possibly composed with the quaternion conjugation.

This theorem is a consequence of the following proposition.

Proposition 1 *Suppose n is an integer ≥ 3 and let $f : \mathbf{k}^n \to \mathbf{k}^n$ be a bijective map. Suppose there is a $k \in \{2, \dots, n-1\}$ such that f takes every left or right affine subspace of dimension k to a left or right affine subspace of dimension $\leq k$. The image of a left affine subspace may be right and vice versa. Then the following holds:*

(1) The map f takes every two-sided, respectively one-sided affine subspace of dimension ≤ 2 to a two-sided, respectively one-sided affine subspace of the same dimension.

(2) Suppose $\mathbf{k}$ is commutative, or $\mathbf{k}$ is non-commutative and there exists a two-sided plane P and a purely left line $L \subset P$ such that $f(L)$ is left. Then the image of every left affine subspace is a left affine subspace of the same dimension, and the map f can then be written as

$$f(x) = \sigma(g(xa) + b) \tag{1}$$

where $a \in \mathbf{k} \setminus \{0\}$, $b \in \mathbf{k}^n$, σ is an automorphism of $\mathbf{k}$ (applied component-wise) and $g : \mathbf{k}^n \to \mathbf{k}^n$ is an isomorphism of $\mathbf{k}$-bimodules.

(3) Suppose $\mathbf{k}$ *is non-commutative and for every two-sided plane* P *and purely left line* $L \subset P$ *the image* $f(L)$ *is right. Then* $\mathbf{k}$ *admits an anti-automorphism, which we will denote by* ε*. The composition* $\varepsilon \circ f$ *(where* ε *is applied component-wise) is then as in part (2) above.*

Let us deduce the statement of the first paragraph of Theorem 1 from Proposition 1. We first note that by part (1) of the proposition, either the hypothesis of part (2) or the hypothesis of part (3) is true. In the former case the statement follows from formula (1). In the latter case by part (3) of the proposition, $\mathbf{k}$ has an anti-automorphism ε, and we then apply part (2) to the map $\varepsilon \circ f$. To conclude the proof of Theorem 1 assuming Proposition 1 we use the Skolem-Noether theorem (see e.g. [6, Theorem 1.8]), which implies that every automorphism of $\mathbb{H}$ is inner.

Remark. There is an apparent asymmetry in formula (1): namely, this formula involves right multiplication by an $a \in \mathbf{k} \setminus \{0\}$, but not left multiplication. To see that this asymmetry is not really there, suppose a, b, g, σ are as in part (2) of Proposition 1. For $x \in \mathbf{k}^n$ set $\sigma_1(x) = axa^{-1}$. We then have $g(ax) + b = \sigma_1(g(xa) + \sigma_1^{-1}(b))$, so the map $\mathbf{k}^n \to \mathbf{k}^n$ given by $x \mapsto \sigma(g(ax) + b)$ can be written as

$$x \mapsto (\sigma \circ \sigma_1)(g(xa) + \sigma_1^{-1}(b)),$$

which is covered by formula (1).

Remark. In [1, Problem 2002–10] V. I. Arnold asked for a description of bijective maps $f : \mathbb{H}^n \to \mathbb{H}^n$ that take quaternionic affine subspaces to quaternionic affine subspaces but are allowed to interchange left and right subspaces. The author was able to provide an answer assuming f is a homeomorphism [4]. For $n \geq 3$, Theorem 1 gives a similar description over an arbitrary associative division ring.

Open questions. It would be interesting to know if there is an analogue of Theorem 1 for $n = 2$. A related question is whether the condition $k \in \{2, \dots, n-1\}$ in Theorem 1 can be replaced with $k \in \{1, \dots, n-1\}$. The answer to the latter question is no in general, but all examples the author is aware of are as follows: take $\mathbf{k} = \mathbb{Z}/2$; then affine lines $\subset \mathbf{k}^n$ are precisely two element subsets of $\mathbf{k}^n$, cf. [3, Remark 9]. So every self-bijection of $(\mathbb{Z}/2)^n$ takes lines to lines, but for $n \geq 3$ there are self-bijections of $(\mathbb{Z}/2)^n$ that are not affine. It follows from the main theorem of [3] that these are all examples that are possible when $\mathbf{k}$ is commutative, and a natural question is whether there are other examples when $\mathbf{k}$ is no longer assumed commutative.

It would also be interesting to know if other hypotheses of Theorem 1 can be weakened. For example, one way to weaken the hypotheses is to replace the bijectivity of f by injectivity or surjectivity, cf. [3] and [5], and another one is to require that f should take every left or right affine subspace of dimension k to a left or right affine subspace not necessarily of dimension $\leq k$.

Organisation of the paper. In Sect. 2 we prove a few preliminary results and deduce part (1) of Proposition 1. In Sect. 3 we reduce part (2) of Proposition 1 to the usual fundamental theorem of affine geometry. Finally, in Sect. 4 we prove part (3) of Proposition 1.

2 Linear Algebra Over a Division Ring

In order to prove the proposition we need to make a few observations first. The author was unable to find a reference for these, so proofs are provided for completeness. To begin with, let us summarise a few definitions and facts related to $\mathbf{k}$-modules. In the next paragraph we do this for left modules; the analogues for right modules are straightforward.

Every left $\mathbf{k}$-module is free. In view of this, in the sequel left $\mathbf{k}$-modules will often be referred to as *left* $\mathbf{k}$*-vector spaces*. A free generating set of a left $\mathbf{k}$-module V will be called a *basis*. All bases of V have the same number of elements, called the *(left) dimension* of V and denoted $\dim V$. If $V' \subset V$ is a left $\mathbf{k}$-submodule, then $\dim V' \leq \dim V$; moreover, if $\dim V < \infty$ and $V' \subsetneq V$, then $\dim V' < \dim V$.

For $i \in \{1, \ldots, n\}$ let $e_i \in \mathbf{k}^n$ be the element such that the i-th coordinate is 1 and all the rest are 0. For $I \subset \{1, \ldots, n\}$ we set E_1^I, respectively E_2^I to be the two-sided $\mathbf{k}$-vector subspace of $\mathbf{k}^n$ spanned by $e_i, i \in I$, respectively by $e_i, i \in \{1, \ldots, n\} \backslash I$.

Lemma 1 *Suppose $V \subset \mathbf{k}^n$ is a left vector subspace, and let m be the left dimension of V. There exists a subset $I \subset \{1, \ldots, n\}$ of cardinality $n - m$ such that $V \cap E_1^I = 0$ and the projection $p : \mathbf{k}^n = E_1^I \oplus E_2^I \to E_2^I$ restricted to V is an isomorphism $V \to E_2^I$. Moreover, $n - m$ is the maximum cardinality of $J \subset \{1, \ldots, n\}$ such that $V \cap E_1^J = 0$.*

Similar statements are true for right vector subspaces.

Proof We will consider the case when V is a left vector subspace. Choose a basis $x_1, \ldots x_m$ of V and form a matrix M with rows $x_1, \ldots x_m$. By using row operations we can transform M to get a matrix M' in row echelon form such that the i-th row of M' has a non-zero element at j_i-th place and zeroes before that, where $1 \leq j_1 < j_2 < \cdots < j_m \leq n$ is an increasing sequence of integers. (The row operations are as follows: given rows r_i, r_j such that $i, j \in \{1, \ldots, m\}, i \neq j$, we are allowed to 1. interchange r_i and r_j, and 2. replace r_i by $r_i + ar_j, a \in \mathbf{k}$.) The rows of M' will still be a basis of V.

Now set $I = \{1, \ldots, n\} \setminus \{j_1, \ldots, j_m\}$. The projection $p : \mathbf{k}^n = E_1^I \oplus E_2^I \to E_2^I$ restricted to V is an isomorphism $V \to E_2^I$, as it takes a basis to a basis. So $V \cap E_1^I = V \cap \ker p = 0$. Moreover, if $J \subset \{1, \ldots, n\}$ contains more than $n - m$ elements, then we cannot have $V \cap E_1^J = 0$ for dimension reasons. □

Lemma 2 *Suppose $V \subset \mathbf{k}^n$ is a left vector subspace which contains a right vector subspace W. Then $\dim V \geq \dim W$ where $\dim V$ is the left dimension of V and $\dim W$ is the right dimension of W. Moreover, the inequality is strict if and only if $W \subsetneq V$.*

The same is true with "left" and "right" interchanged. As a corollary, for two-sided affine or vector subspaces the left and right dimensions coincide.

Proof Let $m = \dim V$. The inequality $\dim V \geq \dim W$ follows straight from Lemma 1. To prove that $\dim V > \dim W$ if $W \subsetneq V$, let $I \subset \{1, \ldots, n\}$, E_1^I, E_2^I and $p : \mathbf{k}^n \to E_2^I$ be as in that lemma.

The projection p is a homomorphism of $\mathbf{k}$-bimodules, and its restriction to V is an isomorphism $V \to E_2^I$ of left $\mathbf{k}$-modules. When we restrict p to W we get a homomorphism of right $\mathbf{k}$-modules. So $W \subsetneq V$ if and only if $p(W)$ is a proper right subspace of $E_2^I \cong \mathbf{k}^m$, which is equivalent to $\dim W < m$. □

So from now on we do not distinguish between the left and right dimensions of an affine subspace of $\mathbf{k}^n$: when both make sense, they coincide.

Lemma 3 *The group of automorphisms of $\mathbf{k}^n$ as a $\mathbf{k}$-bimodule is transitive on the set of two-sided vector subspaces of $\mathbf{k}^n$ of any fixed dimension.*

Proof Suppose $V \subset \mathbf{k}^n$ is a two-sided vector subspace of dimension m. Let us consider V as a left vector subspace and $I, E_1^I, E_2^I, p : \mathbf{k}^n \to E_2^I$ be as in Lemma 1. We have $\mathbf{k}^n = E_1^I \oplus V = E_1^I \oplus E_2^I$. Moreover, p restricted to V is an injective homomorphism of $\mathbf{k}$-bimodules $V \to E_2^I$ of the same dimension, hence an isomorphism. So $\mathrm{id}_{E_1^I} \oplus p|_V : E_1^I \oplus V \to E_1^I \oplus E_2^I$ is an isomorphism of $\mathbf{k}$-bimodules that takes V to E_2^I, a coordinate subspace of dimension m. □

We will say that an affine subspace $A \subset \mathbf{k}^n$ of dimension > 0 and an affine line $L \subset \mathbf{k}^n$ have *the wrong intersection* if $A \cap L \neq \varnothing$, a point or L. Note that if this is the case, then both A and L are one sided and if one is left, then the other one is right; moreover, we have $L \not\subset A$, and by Lemma 2, $A \not\subset L$.

Lemma 4 *An affine subspace $A \subset \mathbf{k}^n$ of dimension > 0 is one sided if and only if there exists an affine line $L \subset \mathbf{k}^n$ that has the wrong intersection with A.*

Proof It suffices to consider the case when A is a vector subspace of $\mathbf{k}^n$, so let us assume that. Saying that e.g. A is left but not right is equivalent to saying that there is a non-zero $x \in A$ such that the right line L through the origin and x is not contained in A, which is equivalent to the intersection $L \cap A$ not being equal to either the empty set, a point or L. □

Lemma 5 *Suppose $\mathbf{k}$ is non-commutative, and let $P \subset \mathbf{k}^n$ be a two-sided plane. Then for every $X \in P$ there are lines L_1, L_2, L_3, L_4 through X such that*

- *L_1 and L_3 are purely right, and L_2 and L_4 are purely left;*
- *$L_1 \neq L_3$ and $L_2 \neq L_4$;*
- *L_2 has the wrong intersection with L_1 and L_3, and L_3 has the wrong intersection with L_2 and L_4;*

see Fig. 1.

Proof By Lemma 3 we may assume that $P = \mathbf{k}^2$. Let L_2 be any purely left line through X. (For example, one could take L_2 to be the left line through X and $X + e_1 + ae_2$ for $a \notin Z(\mathbf{k})$.) We then choose an $X' \neq X$ on L_2 and set L_3 to be the right line through X and X'. The line L_3 is then purely right and L_2 and L_3 have the wrong intersection. By Lemma 2, both sets $L_2 \backslash L_3$ and $L_3 \backslash L_2$ are non-empty. We now take L_1 to be the right line through X and an element of $L_2 \setminus L_3$ and similarly, we take L_4 to be the left line through X and an element of $L_3 \setminus L_2$. □

Lemma 6 *A purely left affine plane P cannot contain more than one right line through any of its points.*

Proof It suffices to consider the case when P contains two distinct right lines L_1, L_2 through the origin. Note that then P, L_1 and L_2 are abelian subgroups of $\mathbf{k}^n$, and P contains the right plane $L_1 + L_2$, which by Lemma 2 implies that P is two sided. □

Lemma 7 *Let f and k be as in Theorem 1. Then f takes every affine subspace $A \subset \mathbf{k}^n$ of dimension $\le k$ to an affine subspace of dimension* $\dim A$. *Moreover, if* $\dim A \le 2$ *and A is one sided, respectively two sided, then $f(A)$ is one sided, respectively two sided.*

Proof We first show by descending induction on $i \le k$ that every affine subspace of dimension i goes under f to an affine subspace of dimension $\le i$. To perform the induction step we note that every left affine subspace $A \subset \mathbf{k}^n$ can be written as the intersection $A' \cap A''$ where A' and A'' are left affine subspaces of dimension $\dim A + 1$ such that $f(A')$ and $f(A'')$ are both left or both right.

We now show that for every affine subspace $A \subset \mathbf{k}^n$ of dimension $\le k$ the image $f(A)$ has dimension $\dim A$ (and not less). To do this we note that A is an element of a flag

$$A_k \supsetneq A_{k-1} \supsetneq \cdots \supsetneq A_0$$

of left affine subspaces such that $\dim A_j = j$, cf. [5, proof of Lemma 10]. Recall that one the hypotheses of Proposition 1 is that $\dim f(A_k) \le k$. We conclude that $\dim f(A) = \dim A$ using Lemma 2 and the fact that f preserves strict inclusions.

Note that f takes every one-sided affine subspace A of dimension $\le k$ to a one-sided affine subspace, because A has the wrong intersection with some affine line by Lemma 4. So to prove Lemma 7 it remains to show that f takes two-sided planes, respectively lines to two-sided planes, respectively lines. Recall that we assume that $n \ge 3$, so it suffices to prove this statement for planes. Moreover, it suffices to consider the case when $\mathbf{k}$ is non-commutative.

Let $P \subset \mathbf{k}^n$ be a two-sided plane, and let $L_1, L_2, L_3, L_4 \subset P$ be lines as in Lemma 5. If $f(L_1)$ is a right line, then it is purely right, and so is $f(L_3)$, while $f(L_2)$ and $f(L_4)$ are purely left. Similarly, if $f(L_1)$ is a left line, then $f(L_1)$ and $f(L_3)$ are purely left and $f(L_2)$ and $f(L_4)$ are purely right. In any case the plane $f(P)$ contains

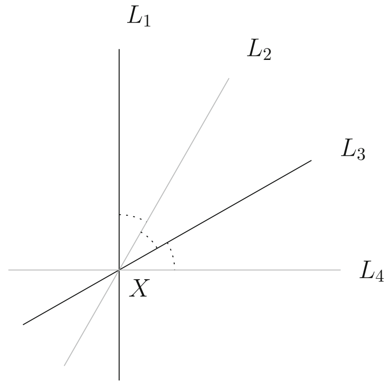
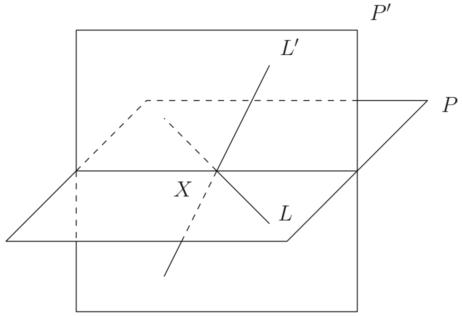


Fig. 1 Lemma 5; the dotted arcs indicate the wrong intersections

Fig. 2 Lemma 8



two purely left lines and two purely right lines that all pass through a single point, so we conclude using Lemma 6 that $f(P)$ is two sided. Lemma 7 is proved. □

Proof of Part (1) of Proposition 1. The result follows from Lemma 7. □

3 Proof of part (2) of Proposition 1

In this section $f : \mathbf{k}^n \to \mathbf{k}^n$ is a map that satisfies the hypotheses of part (2) of the proposition.

Lemma 8 *Let P be a two-sided affine plane and let $L \subset P$ be a purely left affine line such that $f(L)$ is purely left. Let $P' \neq P$ be a two-sided affine plane such that $P' \cap P$ is a line which contains some $X \in L$. Then f takes every purely left affine line $L' \subset P'$ through X to a purely left affine line, see Fig. 2; the same is true for every purely left affine line $L'' \subset P$ through X.*

Proof The intersection $P \cap P'$ is two sided, so $P \cap P' \neq L$. Let $L' \subset P'$ be a purely left line through X. Let us show that $f(L')$ is purely left.

Let R be the left plane spanned by L and L'; the plane R is purely left, as if it were two sided, so would be $L = P \cap R$ and $L' = P' \cap R$. So by Lemma 7 the plane $f(R)$ is purely left or purely right. If it were purely right, then $f(R) \cap f(P)$ would be right, as $f(P)$ is two sided by Lemma 7. But $f(R) \cap f(P) = f(R \cap P) = f(L)$, which is assumed purely left. So we conclude that $f(R)$ is purely left, which implies that $f(L') = f(P' \cap R) = f(P') \cap f(R)$ is left, hence purely left.

By symmetry we conclude that every purely left line $L'' \subset P$ through X goes to a purely left line. □

Here is a corollary of this lemma:

Lemma 9 *Let P and P' be affine two-sided planes such that $P \cap P'$ is a line. Then if some purely left affine line $L \subset P$ goes to a purely left affine line under f, so does every purely left affine line $\subset P \cup P'$.*

Proof By taking parallel copies of P' and using Lemma 8 we conclude that every purely left line $L_1 \subset P$ that intersects L is mapped to a purely left line. By replacing

L with one of these lines and applying the same argument we see that f takes every purely left line in P to a purely left line. Purely left lines $\subset P'$ can be handled in a similar way. □

Lemma 10 *The map f takes every purely left affine line which is contained in some two-sided plane P' to a purely left affine line.*

Proof Let P be as in part (2) of Proposition 1. By Lemma 9 it suffices to prove that there is a sequence

$$P = P_0, P_1, \dots, P_m = P'$$

of two-sided affine planes in $\mathbf{k}^n$ such that $P_i \cap P_{i-1}$ is a line for all $i = 1, \dots, m$. Moreover, by Lemma 3 it suffices to prove this in the case when P is spanned by e_1, e_2.

We proceed by induction on the least m such that the 2-dimensional vector subspace V' corresponding to P' is included in the 2-sided subspace $\mathbf{k}^m \subset \mathbf{k}^n$ spanned by $e_1, \dots, e_m$. Let us first consider the case $m = 2$. The plane P' is then

$$\{(x, y, a_3, \dots, a_n) \mid x, y \in \mathbf{k}\}$$

for some fixed $a_3, \dots, a_n \in \mathbf{k}$. The intersection of this subspace with

$$\{(x', 0, y', a_4, \dots, a_n) \mid x', y' \in \mathbf{k}\} \tag{2}$$

is a 2-sided line. So both P' and the plane P'' given by the same equations as P' but with a_3 replaced by 0 intersect (2) in a line. Repeating this for all a_i with $i > 3$ we construct a sequence of two-sided planes with the required properties.

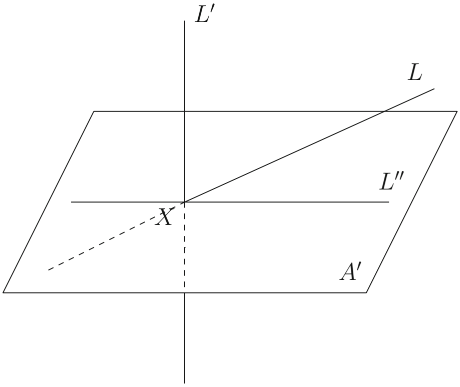
Suppose now $V' \subset \mathbf{k}^{m+1}$ but $V' \not\subset \mathbf{k}^m$. Then $W = V' \cap \mathbf{k}^m$ is a (two-sided) 1-dimensional vector subspace. Let $\bar{V} \subset \mathbf{k}^m$ be a two-sided 2-dimensional vector subspace that contains W. (Such a subspace exists by Lemma 3.) Set $\bar{P} = z + \bar{V}$ for some $z \in P'$. Note that $\bar{P} \cap P'$ is the two-sided line $z + W$. We now apply the induction hypothesis to $\bar{P}$. □

Lemma 11 *All left lines go to left lines under f.*

Proof Let L be a left line. The case when L is two sided is covered by Lemma 7, so in the rest of the proof we suppose that L is purely left. We use induction on the least m such that L is contained in an affine two-sided m-subspace. The case $m = 2$ is Lemma 10.

Suppose $m \geq 2$ and $L \subset A$, a two-sided affine $m+1$-subspace, but L is not contained in any affine two-sided m-subspace. Choose an $X \in L$ and let $A' \subset A$ and $L' \subset A$ be a two-sided affine m-subspace, respectively a 2-sided line through X such that $A' \cap L' = \{X\}$; these exist as $A \cong \mathbf{k}^{m+1}$ by Lemma 3. We also know that $L \cap L' = L \cap A' = X$, as L is purely left and is not included in any 2-sided affine subspace of dimension $m \geq 2$. Let P be the left plane through L and L'; note that P is purely left, as L can't be contained in a two-sided plane.

Fig. 3 Proof of Lemma 11



Since $P \subset A$ and $\dim P + \dim A' = \dim A + 1$, the intersection $P \cap A'$ is a left line L'', see Fig. 3. (Note that $L'' \neq L'$ as $L'' \subset A'$ but $L' \not\subset A'$.)

By Lemma 7, the image $f(P)$ is purely left or purely right, and it contains the left lines $f(L')$ (this is in fact a two-sided line since L' is two sided) and $f(L'')$ (here we use the induction hypothesis, L'' being a subspace of A'). So $f(P)$ is in fact purely left, again using Lemma 6, and so $f(L)$ is left: $f(P)$ contains the two-sided line $f(L')$ and so it cannot contain any more right lines. □

Now we apply the usual fundamental theorem of affine geometry to conclude that f is of the expected type.

Lemma 12 *The map f can be written as*

$$x \mapsto \sigma(G(x) + b)$$

where σ is an automorphism of $\mathbf{k}$ (applied component-wise), $b \in \mathbf{k}^n$ and $G : \mathbf{k}^n \to \mathbf{k}^n$ is a left linear map.

Proof For $\mathbf{k} \neq \mathbb{Z}/2$ this is an immediate consequence of Lemma 11 and the main theorem of [3]. If $\mathbf{k} = \mathbb{Z}/2$, then the lemma follows from [3, Corollary 3] and Lemma 7. □

Lemma 13 *The left and right lines spanned by a non-zero $x \in \mathbf{k}^n$ coincide iff $x = ax'$ with a and x' being non-zero elements of $\mathbf{k}$, respectively $\mathbf{k}^n$, and all components of x' being in the centre $Z(\mathbf{k})$ of $\mathbf{k}$.*

Proof Suppose the left and right lines through the origin and x coincide. Then for every $a \in \mathbf{k}$ there is an $a' \in \mathbf{k}$ such that $ax = xa'$. So if x_i, x_j are two non-zero coordinates of x, the inner automorphisms of $\mathbf{k}$ given by $a \mapsto x_i^{-1} a x_i$ and $a \mapsto x_j^{-1} a x_j$ coincide, which implies $x_i x_j^{-1} \in Z(\mathbf{k})$. □

Proof of part (2) of Proposition 1. We apply Lemma 12. Note that by Lemma 4, f takes two-sided affine lines to two-sided affine lines, and hence so must G. Let M be the matrix of G in the basis $e_1, \ldots, e_n$. The map G can then be written as $x \mapsto xM$ where

x is a row vector. To prove part (2) of Proposition 1 it remains to show that $M = aM'$, where $a \in \mathbf{k} \setminus \{0\}$ and all entries of M' are in $Z(\mathbf{k})$.

It follows from Lemma 13 applied to $G(e_i)$, $i = 1, \dots, n$ that the i-th row of M is an element $a_i \in \mathbf{k} \setminus \{0\}$ times a vector in $Z(\mathbf{k})^n$. We now show that $a_1, \dots, a_n$ are all equal up to multiplication by non-zero elements of $Z(\mathbf{k})$.

We have $M = \operatorname{diag}(a_1, \dots, a_n)N$ where N is an invertible matrix over $Z(\mathbf{k})$. The map $x \mapsto x \operatorname{diag}(a_1, \dots, a_n)$ must take two-sided affine lines to two-sided affine lines. Applying this map to $e_i + e_j$, which spans a two-sided line, and using Lemma 13 again we see that $a_i a_j^{-1} \in Z(\mathbf{k})$ for all $i, j = 1, \dots, n$. □

4 Proof of part (3) of Proposition 1

Proof assuming Lemma 14. Let $f : \mathbf{k}^n \to \mathbf{k}^n$ be a map that satisfies the hypotheses of part (3) of the proposition. By Lemma 7, f takes two-sided affine subspaces of dimension ≤ 2 to two-sided affine subspaces of the same dimension. So applying Lemma 3 we can construct a map $g : \mathbf{k}^n \to \mathbf{k}^n$ of $\mathbf{k}$-bimodules that takes $f(\mathbf{k}^2)$ to $\mathbf{k}^2$, where $\mathbf{k}^2 \subset \mathbf{k}^n$ is the two-sided plane spanned by e_1 and e_2. Part (3) of the proposition now follows from part (2) and the next lemma. □

Lemma 14 *Let $h : \mathbf{k}^2 \to \mathbf{k}^2$ be an bijective map which takes two-sided affine lines to two-sided affine lines. Suppose also that h takes every purely left affine line to a purely right one. Then $\mathbf{k}$ admits an anti-automorphism.*

In order to prove this lemma we will need the following result.

Lemma 15 *Let h be as in Lemma 14, and suppose moreover that $h(0) = 0$. Then h takes left linearly independent vectors to right linearly independent vectors, and is a homomorphism of abelian groups.*

Proof of Lemma 15. For $x, y \in \mathbf{k}^2$ such that $x \neq y$ we will denote the left (respectively right) line through x and y by $L(x, y)$ (respectively $R(x, y)$). Let us prove the first assertion of the lemma. Suppose $x, y \in \mathbf{k}^2$ are left linearly independent. Then $h(x), h(y)$ are right linearly independent: since h is a bijection, the lines

$$h(L(0, x)) = R(0, h(x)), h(L(0, y)) = R((0, h(y))$$

must be different.

We will now prove the second assertion. We need to check that for every $x, y \in \mathbf{k}^2$

$$h(x + y) = h(x) + h(y). \tag{3}$$

Clearly, it suffices to prove this assuming both x and y to be non-zero, which we do from now on.

We will say that two left lines $\subset \mathbf{k}^2$ are *parallel* iff they do not intersect, and similarly for right lines. If L_1 and L_2 are parallel lines, we write $L_1 \parallel L_2$. Note that if x, y are left linearly independent vectors in $\mathbf{k}^2$, then the following holds:

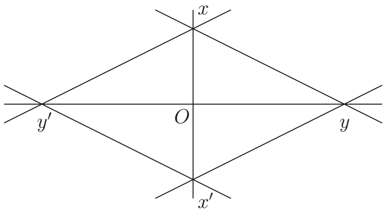


Fig. 4 Proof of Lemma 15

1. $x + y$ is the intersection point of the left line passing through x and parallel to $L(0, y)$ and the left line passing through y and parallel to $L(0, x)$.
2. Suppose the characteristic of $\mathbf{k}$ is $\neq 2$. If $x' \in L(0, x)$, $y' \in L(0, y)$, $x' \neq 0$, $y' \neq 0$, $L(x', y) \parallel L(y', x)$, $L(x', y') \parallel L(x, y)$, then $x' = -x$, $y' = -y$; see Fig. 4.

There are obvious analogues of these statements for right linearly independent vectors and right lines.

We will now check (3). Suppose first that x, y are left linearly independent. It follows from the first assertion of Lemma 15 that $h(x)$ and $h(y)$ are right linearly independent. So formula (3) follows from remark 1 above and from the fact that f takes parallel left lines to parallel right lines. Similarly, it follows from remark 2 above that if $x, y \in \mathbf{k}^2$ are left linearly independent, then $h(-x) = -h(x)$, $h(-y) = -h(y)$.

Suppose now that $x, y \in \mathbf{k}^2$ are non-zero but left linearly dependent, i.e., $y = ax$ for some $a \in \mathbf{k}$, $a \neq 0$. The case $a = -1$ has already been taken care of, so we may assume $a \neq -1$. Let z be a vector that is left linearly independent with x. Since $a \neq -1$, the vectors $x + z$, $y - z$ are left linearly independent, so we have

$$\begin{aligned} h(x+y) &= h(x+z+y-z) = h(x+z) + h(y-z) = h(x) + h(z) + h(y) + h(-z) \\ &= h(x) + h(y). \end{aligned}$$

So we have proved (3) for all $x, y \in \mathbf{k}^2$. □

Proof of Lemma 14. By composing h with a translation if necessary we may assume that $h(0) = 0$, so applying Lemma 15 we conclude that h is a homomorphism of abelian groups. Suppose $(a_1, a_2) \in \mathbf{k}^2$ is a non-zero vector. Define a map $\alpha_{a_1,a_2} : \mathbf{k} \to \mathbf{k}$ by the formula $h(ca_1, ca_2) = h(a_1, a_2)\alpha_{a_1,a_2}(c)$. All these maps are additive and bijective. If a_1, a_2 are both non-zero, we have

$$\begin{aligned} (h(a_1, 0) + h(0, a_2))\alpha_{a_1,a_2}(c) &= h(a_1, a_2)\alpha_{a_1,a_2}(c) = h(ca_1, ca_2) \\ &= h(a_1, 0)\alpha_{a_1,0}(c) + h(0, a_2)\alpha_{0,a_2}(c). \end{aligned}$$

Expressing both sides of this equation in terms of $h(1, 0)$ and $h(0, 1)$ and using the fact that these vectors are right linearly independent (see Lemma 15) we see that for all $a_1, a_2, c \in \mathbf{k}$ such that $a_1 \neq 0 \neq a_2$ we have

$$\alpha_{1,0}(a_1)\alpha_{a_1,a_2}(c) = \alpha_{1,0}(a_1)\alpha_{a_1,0}(c)$$

and

$$\alpha_{0,1}(a_2)\alpha_{a_1,a_2}(c) = \alpha_{0,1}(a_2)\alpha_{0,a_2}(c).$$

This implies that all α_{a_1,a_2} coincide. Indeed, if $a_1 \neq 0$ then for all $a_2 \in \mathbf{k}$ we have

$$\alpha_{a_1,a_2} = \alpha_{a_1,0} = \alpha_{a_1,1} = \alpha_{1,1}.$$

If $a_2 \neq 0$, then for all $a_1 \in \mathbf{k}$ we have

$$\alpha_{a_1,a_2} = \alpha_{0,a_2} = \alpha_{1,a_2} = \alpha_{1,1}.$$

Set $\alpha = \alpha_{1,1}$. We now check that α is an anti-automorphism of $\mathbf{k}$: if $a_1, a_2 \in \mathbf{k}$ are both non-zero, then

$$h(1,0)\alpha(a_1a_2) = h(a_1a_2, 0) = h(a_2, 0)\alpha(a_1) = h(1,0)\alpha(a_2)\alpha(a_1).$$

$\square$

Acknowledgements I am grateful to the referees for their careful reading of the paper, their comments and finding a mistake in the statement of the main theorem in the first version of the paper.

References

1. Arnold, V. I.: Arnold's Problems. Springer, Berlin (2005). Translated and revised edition of the Russian 2000 original
2. Artin, E.: Geometric Algebra. Interscience Publishers Inc., New York (1957)
3. Chubarev, A., Pinelis, I.: Fundamental theorem of geometry without the 1-to-1 assumption. Proc. AMS **127**(9):2735-2744 (1999)
4. Gorinov, A.G.: Pseudocomplex and Pseudo(bi)quaternion mappings. Funktsional. Anal. i Prilozhen. **38**(2), 84–85 (2004)
5. Gorinov, A.G., Howard, R., Johnson, V., McNulty, G.F.: Maps that must be affine or conjugate affine: a problem of Vladimir Arnold. Arnold Math. J. **6**, 213–229 (2020)
6. Lam, T.Y.: Algebraic Theory of Quadratic Forms. Revised Second Printing, Benjamin/Cummings Publishing Co. Inc., Reading (1980)