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RESEARCH CONTRIBUTION

Holomorphic Atiyah–Bott Formula for Correspondences

Grigory Kondyrev[1,2] · **Artem Prikhodko**[3,4]

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Abstract

We show how the formalism of 2-traces can be applied in the setting of derived algebraic geometry to obtain a generalization of the holomorphic Atiyah–Bott formula to the case when an endomorphism is replaced by a correspondence.

Contents

1 Introduction

Let M be a compact manifold, and let f be an endomorphism of M such that the fixed points of f are isolated. The famous Lefschetz fixed point theorem [9, Formula 71.1] expresses the super trace of the map $H^*(f)$ on cohomology induced by f in terms of some local data at the fixed points of f. In fact, in loc. cit. Lefschetz treated a more general problem: he considered a pair of maps $g, f : N \to M$ and gave a cohomological condition for f and g to have a coincidence point. The fixed point theorem above is a special case when $M = N$ and $g = \mathrm{Id}_M$.

✉ Grigory Kondyrev
gkond@math.northwestern.edu

Artem Prikhodko
artem.n.prihodko@gmail.com

1 HSE University, Moscow, Russian Federation

2 Northwestern University, Evanston, USA

3 Department of Mathematics, National Research University Higher School of Economics, Moscow, Russia

4 Center for Advanced Studies, Skoltech, Moscow, Russia

Variations of the Lefschetz fixed point theorem were also found to be very important in algebraic geometry. For example, in the setting of étale cohomology, the Lefschetz fixed point theorems were reestablished by the school of Grothendieck as a part of their program on the proof of Weil conjectures. The variant for coincidence is called Lefschetz–Verdier formula and appeared in [6, Corollaire 4.7]. However, this formula has the drawback that the local terms are in general quite implicit. Deligne made a conjecture that under favorable assumptions the local terms can be made precise, and this conjecture was subsequently proved in [4, 12], and in [13] with the view towards applications to the global functional Langlands correspondence.

Returning to the classical setting of the smooth manifolds, if all fixed points of the endomorphism f are simple, the local terms in the Lefschetz fixed point theorem can also be computed explicitly. Moreover, in this case, the theorem itself admits a vast generalization, which is due to Atiyah and Bott [1, 2]. Specifically, given an elliptic complex E on M and a bundle map $b\colon f^{-1}E \to E$, there is an equality

$$L(E, b) := \sum_i (-1)^i \mathsf{Tr}(H^i(b)_{|H^i(M,E)}) = \sum_{x=f(x)} \frac{\mathsf{Tr}_k(E_x \simeq E_{f(x)} \xrightarrow{b_x} E_x)}{\det(1 - d_x f)}. \quad (1)$$

In [8], we showed how one can use a simple traces machinery in $(\infty, 2)$-categories to prove an algebro-geometric analogue of the Atiyah–Bott formula (1). In this note, we explain how to adapt our arguments to deduce a version of this theorem for a pair of morphisms, that is, a version of the Lefschetz–Verdier fixed point formula for vector bundles on algebraic varieties (Theorem 1.6), which we will further call the **holomorphic Atiyah–Bott formula for correspondence**.

Convention For the rest of this work, we fix an algebraically closed base field k.

To state the holomorphic Atiyah–Bott formula for correspondences we first introduce necessary notations.

Definition 1.1 Let X, Y be a pair of k-schemes. A **correspondence** is a pair of morphisms $g, f\colon Y \to X$.

Let us now denote by Vect_k the unbounded derived category of cochain complexes over k, and by $\mathrm{QCoh}(X)$ the unbounded derived category of quasi-coherent sheaves on X (see conventions below). Recall that for an endomorphism $f\colon X \to X$ a lax-equivariant structure on a sheaf $E \in \mathrm{QCoh}(X)$ is a map $b\colon f^*E \to E$. We first generalize this notion to the case of correspondences.

To do this, let us assume that both X and Y are smooth and proper. Under these assumptions, the pushforward functor g_* on quasi-coherent sheaves admits a right adjoint $g^!$, and there is a natural equivalence $g^!(-) \simeq g^*(-) \otimes \omega_g$, where $\omega_g := g^!(\mathcal{O}_X)$ is the relative dualizing complex. This motivates the following

Definition 1.2 Let $g, f\colon Y \to X$ be a correspondence. A **lax** (g, f)**-equivariant structure** on $E \in \mathrm{QCoh}(X)$ is a map $b\colon f^*E \to g^!E$ in $\mathrm{QCoh}(Y)$.

Recall now the notion of a dualizable object (see Definition 2.1). Similar to the classical Lefschetz number, in the case E is coherent, any lax (g, f)-equivariant structure on E produces an element of k:

Definition 1.3 For a lax (g, f)-equivariant coherent sheaf $(E \in \mathrm{QCoh}(X), b : f^*E \to g^!E)$ on X we define its **Lefschetz number** $\mathrm{L}(E, b) \in k$ of b as the trace (Definition2.1) in Vect_k of the corresponding endomorphism

$$\Gamma(X, E) \longrightarrow \Gamma(X, f_* f^* E) \simeq \Gamma(Y, f^* E) \xrightarrow{\Gamma(Y,b)} \Gamma(Y, g^! E) \simeq \Gamma(X, g_* g^! E) \longrightarrow \Gamma(X, E)$$

on global sections, where the last morphism is obtained from the counit of the adjunction $g_* \dashv g^!$. Note that the trace makes sense since $\Gamma(X, E)$ is a dualizable object in Vect_k due to the properness of X.

To state how the Lefschetz number can be described in local terms, for a correspondence as above let us denote by $Y^{g=f,\mathrm{cl}}$ the **classical fixed point scheme**, that is, the fiber product

$$\begin{array}{ccc} Y^{g=f,\mathrm{cl}} & \longrightarrow & Y \\ \downarrow & & \downarrow {\scriptstyle (g,f)} \\ X & \xrightarrow[\Delta]{} & X \times X \end{array}$$

in the category of reduced schemes.

Definition 1.4 Given a correspondence $g, f : Y \to X$, we say that g **is étale at the fixed points of** (g, f) if the morphism g is étale at each $y \in Y^{g=f,\mathrm{cl}}$.

Remark 1.5 Note that if g is étale at the fixed points, then there is a canonical trivialization $\omega_{g|Y^{g=f}} \simeq \mathcal{O}_{Y^{g=f,\mathrm{cl}}}$ in $\mathrm{QCoh}(Y^{g=f,\mathrm{cl}})$.

We can now give a description of the local terms of the Atiyah–Bott formula. Let $g, f : Y \to X$ be a correspondence such that g is étale at the fixed points, and let $y \in Y^{g=f,\mathrm{cl}}$ be a fixed point. Note that then:

- Given a lax (g, f)-equivariant sheaf $(E \in \mathrm{QCoh}(X), b : f^*E \to g^!E)$ on X, we can restrict b to the fiber over y, obtaining an endomorphism

$$E_{f(y)} \simeq (f^*E)_y \xrightarrow{b_y} (g^!E)_y \simeq E_{g(y)} \otimes \omega_{g,y} \simeq E_{g(y)}$$

  in $\mathrm{QCoh}(\{x\}) \simeq \mathrm{Vect}_k$, where $x := f(y) = g(y)$ and in the last equality we have used the trivialization of $\omega_{g|Y^{g=f}}$ from Remark 1.5.
- Since by our assumptions $d_y g$ is invertible, we obtain an endomorphism

$$\mathbb{T}_x X \xrightarrow[\sim]{(d_y g)^{-1}} \mathbb{T}_y Y \xrightarrow{d_y f} \mathbb{T}_x X$$

  of the tangent space of X at our fixed point.

Finally, we can formulate the main result of this work.

Theorem 1.6 (Holomorphic Atiyah–Bott formula for correspondences) *Let X, Y be a pair of smooth proper k-schemes and let $(g, f)\colon Y \to X \times X$ be a correspondence such that the graph of Y intersects the diagonal in $X \times X$ transversally* (see Definition 3.1) *and g is étale at the fixed points* (Definition 1.4). *Then for any lax (g, f)-equivariant perfect sheaf $(E \in \mathrm{QCoh}(X), b\colon f^*E \to g^!E)$ on X there is an equality*

$$\mathrm{L}(E, b) = \sum_{f(y)=g(y)} \frac{Tr_{\mathrm{Vect}_k}(E_{f(y)} \xrightarrow{b_y} E_{g(y)})}{\det(1 - d_y f \circ (d_y g)^{-1})}. \tag{2}$$

Example 1.7 In the case $Y = X$ and $g = \mathrm{Id}_X$, we recover the usual Holomorphic Atiyah–Bott formula [8, Theorem 3.1.2].

One interesting application of Theorem 1.6 is in the context of birational geometry. Namely, given a rational endomorphism $\phi\colon X \dashrightarrow X$ of a smooth proper variety X one can take Y to be the closure of the graph of ϕ in $X \times X$ and $g, f\colon Y \to X$ to be the projections to the first and second factors. Let us illustrate how this works in the simplest nontrivial example.

Example 1.8 Assume that char $k \neq 2$. Let $X = \mathbb{P}^2$ and consider the Cremona transformation

$$\phi(x : y : z) := (\frac{1}{x} : \frac{1}{y} : \frac{1}{z}).$$

By taking the closure of the graph of ϕ, we obtain a correspondenc $\mathbb{P}^2 \xleftarrow{g} Y \xrightarrow{f} \mathbb{P}^2$, where Y is given by

$$\{xu = yv = zw\} \subseteq \mathbb{P}^2(x : y : z) \times \mathbb{P}^2(u : v : w).$$

The intersection with the diagonal $x = u, y = v, z = w$ consists of 4 points $(\pm 1 : \pm 1 : \pm 1)$. Consider $E = \omega_{\mathbb{P}^2}$. There is a canonical lax-equivariant structure on E given by the pullback on forms $t\colon f^*\omega_{\mathbb{P}^2} \to \omega_Y \simeq g^!\omega_{\mathbb{P}^2}$.

Let us first compute the Lefschetz number of t. Since

$$\Gamma(\mathbb{P}^2, \omega_{\mathbb{P}^2}) \simeq H^{2,2}(\mathbb{P}^2)[2] \simeq k[2],$$

It is enough to understand the map on $H^{2,2}(\mathbb{P}^2)$, which is the degree of ϕ since $\mathbb{P}^2$ is 2-dimensional. Since the Cremona transform is birational, it follows that $L(t) = 1$.

Now let us compute the right-hand side of the Atiyah–Bott formula. All fixed points lie in the affine chart $z = 1$ with inhomogeneous coordinates (X, Y). The differential of ϕ at a point $x = (X_0, Y_0)$ is given by the matrix

$$\begin{pmatrix} -1/X_0^2 & 0 \\ 0 & -1/Y_0^2 \end{pmatrix}.$$

Since for all fixed points $X_0^2 = Y_0^2 = 1$, we have

$$\det(d_x g - d_x f) = \det\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right) = \det\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} = 4.$$

Also in this case the induced map $f^*\omega_{\mathbb{P}^2,x} \to g^!\omega_{\mathbb{P}^2,x}$ is just the identity; hence, we find that the right-hand side of the Atiyah–Bott formula is

$$4 \cdot \frac{1}{4} = 1.$$

Conventions.

(1) All categories we work with are assumed to be $(\infty, 1)$-categories (we refer the reader to [10] for the theory of $(\infty, 1)$-categories). For an $(\infty, 1)$-category $\mathscr{C}$ we will denote by $\mathscr{C}^{\simeq}$ the underlying ∞-groupoid of $\mathscr{C}$ obtained by discarding all the non-invertible morphisms from $\mathscr{C}$.

(2) For a field k we denote by Vect_k the stable symmetric monoidal $(\infty, 1)$-category of unbounded cochain complexes over k up to quasi-isomorphism with the canonical $(\infty, 1)$-enhancement.

(3) We will denote by $\mathrm{Pr}^{\mathrm{L}}_{\infty}$ the $(\infty, 1)$-category of presentable $(\infty, 1)$-categories and continuous functors with a symmetric monoidal structure from [11, Proposition 4.8.1.14.]. Note that Vect_k is a commutative algebra object in $\mathrm{Pr}^{\mathrm{L}}_{\infty}$. By [11, Corollary 5.1.2.6.] it follows that the presentable stable $(\infty, 1)$-category of k-linear presentable $(\infty, 1)$-categories and k-linear functors $\mathrm{Cat}_k := \mathsf{Mod}_{\mathrm{Vect}_k}(\mathrm{Pr}^{\mathrm{L}}_{\infty})$ admits a natural symmetric monoidal structure. We will also denote by $2\,\mathrm{Cat}_k$ the symmetric monoidal $(\infty, 2)$-category of k-linear presentable $(\infty, 1)$-categories and continuous k-linear functors, so that the underlying $(\infty, 1)$-category of $2\,\mathrm{Cat}_k$ is precisely Cat_k.

(4) We will further work in the setting of derived algebraic geometry over some fixed algebraically closed field k, where derived affine schemes are modeled by commutative simplicial k-algebras.[1] For a derived prestack X we will denote the k-linear symmetric monoidal $(\infty, 1)$-category of unbounded complexes of quasi-coherent sheaves on X by $\mathrm{QCoh}(X) \in \mathrm{CAlg}(\mathrm{Cat}_k)$. Similarly, all the functors between quasi-coherent sheaves are assumed to be derived in an appropriate sense. We refer the reader to [5] for an introduction to the basic concepts of derived algebraic geometry using the functor of points approach.

2 Traces and Morphisms of Traces

We start with the following classical definition.

Definition 2.1 Let $(\mathscr{C}, \otimes, I)$ be a symmetric monoidal $(\infty, 1)$-category. Then:

[1] If characteristic of k is zero this category is equivalent to the ∞-category of connective commutative dg-algebras.

1. An object $X \in \mathscr{C}$ is called **dualizable**, if there exist an object $X^\vee \in \mathscr{C}$ together with **unit** $I \longrightarrow X \otimes X^\vee$ and **counit** $X \otimes X^\vee \longrightarrow I$ maps satisfying triangle identities.
2. Let $X \xrightarrow{f} X$ be a morphism in $\mathscr{C}$ where the object X is dualizable. Then the **trace of** f denoted by $\mathsf{Tr}_{\mathscr{C}}(f) \in \mathrm{Hom}_{\mathscr{C}}(I, I)$ of f is defined as the composite

$$I \xrightarrow{\mathsf{coev}} X \otimes X^\vee \xrightarrow{f \otimes \mathrm{Id}_{X^\vee}} X \otimes X^\vee \underset{\sim}{\xrightarrow{\mathsf{Twist}}} X^\vee \otimes X \xrightarrow{\mathsf{ev}} I$$

in $\mathscr{C}$.

Example 2.2 Notice that an object $V \in \mathrm{Vect}_k$ is dualizable if an only if it has finite-dimensional cohomology spaces nonzero only in finitely many degrees. If $V \in \mathrm{Vect}_k$ is dualizable and $f \in \mathrm{Hom}_{\mathsf{Vect}_k}(V, V)$ is some morphism, then the trace $\mathsf{Tr}_{\mathsf{Vect}_k}(f) \in \mathrm{Hom}_{\mathsf{Vect}_k}(k, k) \simeq k$ is the alternating sum of the ranks of the map on the cohomology spaces of V induced by f. In particular, in case $f = \mathrm{Id}_V$ we see that $\mathsf{Tr}_{\mathsf{Vect}_k}(\mathrm{Id}_V) = \mathcal{X}_{\mathsf{Vect}_k}(V) \in k$ is simply the Euler characteristic of V.

Remark 2.3 Suppose we are given a dualizable object $X \in \mathscr{C}$ together with a map $X \xrightarrow{f} Y \otimes X$, where $Y \in \mathscr{C}$ is some object. Similar to the second part of Definition 2.1, one can then consider the composite

$$I \xrightarrow{\mathsf{coev}} X \otimes X^\vee \xrightarrow{f \otimes \mathrm{Id}_{X^\vee}} Y \otimes X \otimes X^\vee \underset{\sim}{\xrightarrow{\mathrm{Id}_Y \otimes \mathsf{Twist}}} Y \otimes X^\vee \otimes X \xrightarrow{\mathrm{Id}_Y \otimes \mathsf{ev}} Y$$

which we will further call a **twisted trace** of f.

The notion of trace is extremely useful in the setting of derived algebraic geometry. Namely, note that by [3, Theorem 1.2] for any perfect derived stacks X, Y (see [3, Definition 3.2]) there is a canonical equivalence $\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(Y) \simeq \mathrm{QCoh}(X \times Y)$ obtained from the bicontinuous functor

$$\begin{array}{ccc} \mathrm{QCoh}(X) \times \mathrm{QCoh}(Y) & \underset{\sim}{\longrightarrow} & \mathrm{QCoh}(X \times Y) \\ (\mathcal{F}, \mathcal{G}) & \longmapsto & (q_1^*\mathcal{F}) \otimes (q_2^*\mathcal{G}), \end{array}$$

where

$$X \xleftarrow{q_1} X \times Y \xrightarrow{q_2} Y$$

are the projection maps. In particular, the object $\mathrm{QCoh}(X) \in \mathrm{Cat}_k$ is self-dual, with the unit and counit maps given by

$$\mathrm{Vect}_k \xrightarrow{\Delta_* \mathcal{O}_X} \mathrm{QCoh}(X \times X) \simeq \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X)$$

$$\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \simeq \mathrm{QCoh}(X \times X), \xrightarrow{\Gamma(\Delta^* -)} \mathrm{Vect}_k$$

where $X \xrightarrow{\Delta} X \times X$ is the diagonal map and $\mathrm{QCoh}(X) \xrightarrow{\Gamma(-)} \mathrm{Vect}_k$ is the (derived) global sections functor.

A convenient way to calculate traces of various endomorphisms of the dualizable object $\mathrm{QCoh}(X) \in \mathrm{Cat}_k$ is provided by the formalism of kernels. Namely, by [3, Theorem 1.2] there is an equivalence

$$\begin{aligned} \mathrm{QCoh}(X \times X) &\xrightarrow[\sim]{} \mathrm{Fun}_{\mathrm{Cat}_k}\big(\mathrm{QCoh}(X), \mathrm{QCoh}(Y)\big) \\ \mathcal{K} &\longmapsto q_{2*}(\mathcal{K} \otimes (q_1^* -)) \end{aligned}$$

of $(\infty, 1)$-categories. The sheaf $\mathcal{K}$ is frequently called the **kernel** of the corresponding functor. Moreover, by [8, Proposition 2.1.6] we have an equivalence

$$\mathrm{Tr}_{\mathrm{Cat}_k}(q_{2*}(\mathcal{K} \otimes (q_1^* -))) \simeq \Gamma(X, \Delta^* \mathcal{K}) \in \mathrm{Vect}_k \tag{3}$$

allowing us to instantly calculate trace of an endomorphism of $\mathrm{QCoh}(X) \in \mathrm{Cat}_k$ in terms of its kernel.

It is now straightforward to see that notion of trace allows us to recover derived fixed points schemes in the setting of derived algebraic geometry:

Definition 2.4 Let $X \xleftarrow{g} Y \xrightarrow{f} X$ be a correspondence of derived stacks. We then define the **derived fixed points stack** denoted $Y^{g=f}$ as the pullback

$$\begin{array}{ccc} Y^{g=f} & \xrightarrow{j} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle (g,f)} \\ X & \xrightarrow[\Delta]{} & X \times X \end{array}$$

of derived stacks.

Proposition 2.5 *In the setting of Definition 2.4 for a sheaf* $\mathcal{G} \in \mathrm{QCoh}(Y)$ *there is a canonical equivalence*

$$\mathit{Tr}_{\mathrm{Cat}_k}\big(f_*(\mathcal{G} \otimes g^* -)\big) \simeq \Gamma(Y^{g=f}, j^* \mathcal{G})$$

in Vect_k.

Proof First, note that the kernel of the functor

$$\mathrm{QCoh}(X) \xrightarrow{f_*(\mathcal{G} \otimes g^* -)} \mathrm{QCoh}(X)$$

is given by the sheaf $(g, f)_* \mathcal{G} \in \mathrm{QCoh}(X \times X)$. Indeed, due to projection formula (see [3, Proposition 3.10]) we have

$$q_{2*}\big((g, f)_* \mathcal{G} \otimes (q_1^* -)\big) \simeq q_{2*}(g, f)_*(\mathcal{G} \otimes (g, f)^* q_1^* -) \simeq f_*(\mathcal{G} \otimes g^* -).$$

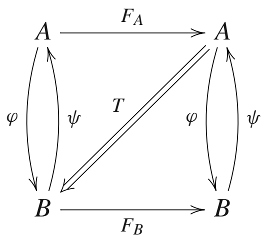
Consequently, using Eq. (3) and projection formula we obtain

$$\mathsf{Tr}_{\mathrm{Cat}_k}\big(f_*(\mathcal{G}\otimes g^*-)\big)\simeq\Gamma(X,\Delta^*(g,f)_*\mathcal{G})\simeq\Gamma(X,i_*j^*\mathcal{G})\simeq\Gamma(Y^{g=f},j^*\mathcal{G})$$

as claimed.

Now note that since Cat_k is the underlying $(\infty,1)$-category of the $(\infty,2)$-category $2\,\mathrm{Cat}_k$, it is natural to ask whether the 2-morphisms in $2\,\mathrm{Cat}_k$ provide some additional functoriality on the level of traces. Motivated by this, let us discuss the notion of trace in $(\infty,2)$-categories. Let $\mathscr{E}$ be a symmetric monoidal $(\infty,2)$-category (that is, a commutative algebra object in the $(\infty,1)$-category of $(\infty,2)$-categories, see [5, Chapter V.3, 1.4.1.]). As was shown in [8], in this case, the trace constructions admits additional functoriality:

Proposition 2.6 [8, Proposition 1.2.3] *Let $\mathscr{E}$ be a symmetric monoidal $(\infty,2)$-category and suppose we are given a* (not necessary commutative) *diagram*



in $\mathscr{E}$, where φ is left adjoint to ψ and

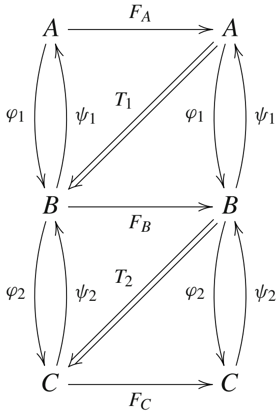
$$\varphi\circ F_A\xrightarrow{\;T\;}F_B\circ\varphi$$

is a 2-morphism in $\mathscr{E}$. Then there exist a natural morphism

$$\mathsf{Tr}_{\mathscr{E}}(F_A)\xrightarrow{\mathsf{Tr}(\varphi,T)}\mathsf{Tr}_{\mathscr{E}}(F_B)$$

in the $(\infty,1)$-category $\mathrm{Hom}_{\mathscr{E}}(I,I)$ *called a* **morphism of traces induced by** T.

Moreover, given a diagram



in $\mathscr{E}$, where φ_1 is left adjoint to ψ_1, φ_2 is left adjoint to ψ_2 and

$$\varphi_1 \circ F_A \xrightarrow{T_1} F_B \circ \varphi_1$$

$$\varphi_2 \circ F_B \xrightarrow{T_2} F_C \circ \varphi_2$$

are 2-morphisms there is an equivalence

$$\mathit{Tr}(\varphi_2 \circ \varphi_1, T_2 \circ_{\mathit{vert}} T_1) \simeq \mathit{Tr}(\varphi_2, T_2) \circ \mathit{Tr}(\varphi_1, T_1),$$

where $\circ_{\mathit{vert}}$ is the vertical composition of 2-morphisms.

We refer the reader to [8, Example 1.2.5] for an explicit description of the $(\infty, 2)$-categorical trace $\mathsf{Tr}(\varphi, T)$.

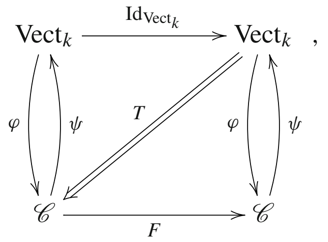
We now provide several examples in the case $\mathscr{E} = 2\,\mathrm{Cat}_k$:

Example 2.7 [8, Definition 1.2.9] In the case $\mathscr{E} = 2\,\mathrm{Cat}_k$, suppose we are given some dualizable object $\mathscr{C} \in 2\,\mathrm{Cat}_k$ together with an endofunctor $\mathscr{C} \xrightarrow{F} \mathscr{C}$. Note that there is a canonical equivalence

$$\mathrm{Fun}_{2\,\mathrm{Cat}_k}(\mathrm{Vect}_k, \mathscr{C})^{\mathsf{ladj}} \xrightarrow[\sim]{\mathsf{ev}_k} \mathscr{C}^{\mathsf{comp}} ,$$

where $\mathrm{Fun}_{2\,\mathrm{Cat}_k}(\mathrm{Vect}_k, \mathscr{C})^{\mathsf{ladj}} \subseteq \mathrm{Fun}_{2\,\mathrm{Cat}_k}(\mathrm{Vect}_k, \mathscr{C})$ is the full $(\infty, 1)$-subcategory spanned by those morphisms in $2\,\mathrm{Cat}_k$ which admit a right adjoint, and $\mathscr{C}^{\mathsf{comp}} \subseteq \mathscr{C}$ is the full $(\infty, 1)$-subcategory of compact objects (see [5, 7.1.1]).

In particular, given a compact object $E \in \mathscr{C}^{\mathsf{comp}}$ together with a morphism $E \xrightarrow{t} F(E)$ in $\mathscr{C}$, we can apply the $(\infty, 2)$-categorical trace construction to the diagram



where ϕ is the functor obtained from the compact object $E \in \mathscr{C}^{\mathsf{comp}}$ and T is the 2-morphism obtained from t.

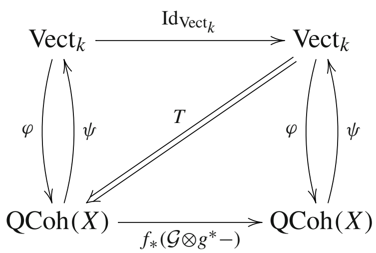
The corresponding element

$$k \simeq \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{Vect}_k}) \xrightarrow{\mathsf{Tr}(\varphi, T)} \mathsf{Tr}_{2\,\mathrm{Cat}_k}(F) \in \mathrm{Hom}_{2\,\mathrm{Cat}_k}(\mathrm{Vect}_k, \mathrm{Vect}_k) \simeq \mathrm{Vect}_k$$

is called the **Chern character of** E and is denoted by $\mathrm{ch}(E, t) \in \mathsf{Tr}_{2\,\mathrm{Cat}_k}(F)$.

Example 2.8 (Chern character for vector spaces) In the case $\mathscr{C} = \mathrm{Vect}_k$, $F = \mathrm{Id}_{\mathrm{Vect}_k}$ in the setting of Example 2.7 (in particular, we have $t \in \mathrm{Hom}_{\mathrm{Vect}_k}(E, E)$ and $\mathsf{Tr}_{2\,\mathrm{Cat}_k}(F) \simeq k$) directly by definition we have an equality $\mathrm{ch}(V, t) = \mathsf{Tr}_{\mathrm{Vect}_k}(t)$ of two numbers.

Example 2.9 (Chern character for lax-equivariant sheaf) Let $X \xleftarrow{g} Y \xrightarrow{f} X$ be a correspondence of perfect derived stacks (see [3, Definition 3.2]), and $E \in \mathrm{Perf}(X)$ be a perfect sheaf (by [3, 3.1] equivalently compact/dualizable object of $\mathrm{QCoh}(X)$) equipped with a map $t\colon E \to f_*(\mathcal{G} \otimes g^*E)$ for some $\mathcal{G} \in \mathrm{QCoh}(Y)$. Then similar to [8, Remark 2.2.4] one checks that the Chern character $\mathrm{ch}(E, t)$ of E obtained from the diagram



is equivalent to the twisted trace (see Remark 2.3) of the induced map

$$i^*E \simeq j^*f^*E \xrightarrow{j^*(b)} j^*(\mathcal{G} \otimes g^*E) \simeq j^*\mathcal{G} \otimes j^*g^*E \simeq j^*\mathcal{G} \otimes i^*E$$

in $\mathrm{QCoh}(Y^{g=f})$, where $b\colon f^*E \to \mathcal{G} \otimes g^*E$ is the morphism which corresponds to $t \in \mathrm{Hom}_{\mathrm{QCoh}(X)}\big(E, f_*(\mathcal{G} \otimes g^*E)\big)$ via the adjunction $f^* \dashv f_*$.

Example 2.10 The case we are most interested in is when both X and Y are smooth and $\mathcal{G} = \omega_g := g^!(\mathcal{O}_X)$. In this case $g^*(-) \otimes \omega_g \simeq g^!(-)$, so a morphism of traces corresponding to the lax-equivariant structure from Definition 1.2 under the identification from the previous example corresponds to the twisted trace with the value in $\Gamma(Y^{g=f}, j^*\omega_g)$.

Remark 2.11 In the setting of Example 2.9, suppose we are given a map

$$\begin{array}{ccccc} Z & \xleftarrow{g'} & W & \xrightarrow{f'} & Z \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} \\ X & \xleftarrow[g]{} & Y & \xrightarrow[f]{} & X \end{array}$$

of correspondences (in particular, we automatically get a map $l\colon W^{g'=f'} \to Y^{g=f}$ on derived fixed points). The map $b\colon f^*E \to \mathcal{G} \otimes g^*E$ in $\mathrm{QCoh}(Y)$ then gives a map

$$b' : (f')^*u^*E \simeq v^*f^*E \xrightarrow{v^*(b)} v^*\mathcal{G} \otimes v^*g^*E \simeq v^*\mathcal{G} \otimes (g')^*u^*E$$

and hence by adjunction a map $t'\colon (u^*E) \to f'_*(v^*\mathcal{G} \otimes (g')^*(u^*E))$. Moreover, using Example 2.9 we obtain a canonical equivalence

$$l^* \operatorname{ch}(E, t) \simeq \operatorname{ch}(u^*E, t')$$

of Chern characters.

Example 2.12 (Todd class) One of the main theorems of [7] is that for a classical smooth scheme X the morphism of traces

$$\bigoplus_{p,q} H^{q,p}(X) \overset{\mathrm{HKR}}{\simeq} \pi_*\mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(X)}) \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(-\otimes\mathcal{O}_X)} \pi_*\mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(X)}) \overset{\mathrm{HKR}\circ t_S^{-1}}{\simeq} \bigoplus_{p,q} H^{q,p}(X)$$

induced by the commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(X) & \xrightarrow{\mathrm{Id}_{\mathrm{QCoh}(X)}} & \mathrm{QCoh}(X) \\ \mathcal{O}_X \big\downarrow\big\uparrow & & -\otimes\mathcal{O}_X \big\downarrow\big\uparrow \\ \mathrm{ICoh}(X) & \xrightarrow[\mathrm{Id}_{\mathrm{ICoh}(X)}]{} & \mathrm{ICoh}(X) \end{array}$$

is given by the multiplication by the classical Todd class, where $\mathrm{ICoh}(X)$ here is the $(\infty, 1)$-category of ind-coherent sheaves and HKR is the Hochschild–Kostant–Rosenberg isomorphism. We refer the reader to [7] for more details.

3 Proof of the Atiyah–Bott Formula

In this section, we show how one can adapt the proof of [8] to the case of correspondences. First we make precise the transversality condition from Theorem 1.6.

Definition 3.1 Let $X \xleftarrow{g} Y \xrightarrow{f} X$ be a correspondence of algebraic k-varieties. We say that $Y \to X \times X$ **intersects with the diagonal** $X \to X \times X$ **transversely** if the derived fixed point scheme $Y^{g=f}$ (see Definition 2.4) is discrete. In more classical terms this is equivalent to the following pair of conditions:

- The underlying classical scheme $Y^{g=f,\mathrm{cl}}$ is discrete.
- The induced map on tangent spaces $1 - d_y f \circ (d_y g)^{-1}$ is invertible for all $y \in Y^{g=f,\mathrm{cl}}$.

Proof of Theorem 1.6 The strategy of the proof is similar to [8, Proposition 3.2]. We have a diagram

$$\begin{array}{ccc} \mathrm{Vect}_k & \xrightarrow{\mathrm{Id}_{\mathrm{Vect}_k}} & \mathrm{Vect}_k \\ E \big\downarrow & \overset{T_1}{\swarrow\!\!\swarrow} & \big\downarrow E \\ \mathrm{QCoh}(X) & \xrightarrow{f_* g^!} & \mathrm{QCoh}(X) \\ \Gamma \big\downarrow & \underset{T_2}{\swarrow\!\!\swarrow} & \big\downarrow \Gamma \\ \mathrm{Vect}_k & \xrightarrow[\mathrm{Id}_{\mathrm{Vect}_k}]{} & \mathrm{Vect}_k \end{array}$$

in $2\,\mathrm{Cat}_k$, where the 2-morphism T_1 is obtained from the lax (g, f)-equivariant structure on E and the morphism T_2 is given by

$$\Gamma(X, f_* g^! -) \simeq \Gamma(X, g^! -) \simeq \Gamma(X, g_* g^! -) \longrightarrow \Gamma(X, -)\ .$$

By applying the 2-trace formalism Proposition 2.6 to the commutative diagram, we obtain a commutative triangle

$$\begin{array}{ccc} k & \xrightarrow{\mathrm{ch}(E,t)} & \mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_* g^!) \\ & \searrow^{\mathsf{Tr}(\Gamma(X,E),\, T_2 \circ_{\mathsf{vert}} T_1)} & \big\downarrow {\scriptstyle \mathsf{Tr}(\Gamma, T_2)} \\ & & k \end{array}$$

in Vect_k, that is, an equality

$$\mathsf{Tr}(\Gamma, T_2) \circ \mathrm{ch}(E, t) \simeq \mathsf{Tr}(\Gamma(X, E), T_2 \circ_{\mathsf{vert}} T_1) \tag{4}$$

of two numbers. First, we note that using Example 2.8, we have

$$\mathsf{Tr}\big(\Gamma(X, E), T_2 \circ_{\mathsf{vert}} T_1\big) \simeq \text{Ł}(E, b).$$

Second, by Proposition 2.5, we have an equivalence

$$\mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_* g^!(-)) \simeq \mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_* g^*(-) \otimes \omega_g) \simeq \Gamma(Y^{g=f}, j^*\omega_g),$$

where $j\colon Y^{g=f} \to Y$ is the natural map. Moreover, by Remark 1.5 we have $j^*\omega_g \simeq \mathcal{O}_{Y^{g=f}}$, and hence

$$\mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_* g^!(-)) \simeq \bigoplus_{f(y)=g(y)} k e_y,$$

where $e_y := 1 \in \Gamma(\{y\}, \mathcal{O}_y)$. In particular, we can write

$$\mathrm{ch}(E, t) = \sum_{f(y)=g(y)} \mathrm{ch}(E, t)_y e_y.$$

Moreover, for $y \in Y^{g=f}$ by Example 2.9, we obtain

$$\mathrm{ch}(E, t)_y \simeq \mathsf{Tr}_{\mathrm{Vect}_k}(E_{f(y)} \xrightarrow{b_y} E_{g(y)}).$$

It suffices now to show that the map

$$\int_{Y^{g=f}} : \bigoplus_{f(y)=g(y)} k e_y \simeq \mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_* g^!) \xrightarrow{\mathsf{Tr}(\Gamma, T_2)} k$$

sends e_y to $1/\det(1 - d_y f \circ (d_y g)^{-1})$. Note that $\int_{Y^{g=f}}$ is independent of E; hence, to compute $\lambda_y := \int_{Y^{g=f}}(e_y)$, we can apply (4) to the special case $E := x_* k$, a skyscraper sheaf at a fixed point $x = f(y) = g(y)$, with the natural lax (g, f)-equivariant structure given by

$$f^*E \simeq f^*f_*y_*k \longrightarrow y_*k \longrightarrow g^!g_*y_*k \simeq g^!E.$$

In this case, (4) produces an equality

$$1 = Ł(E, b) = \lambda_y\,\mathrm{ch}(E, t)_y = \lambda_y \mathrm{Tr}_{\mathrm{Vect}_k}(E_{f(y)} \xrightarrow{b_y} E_{g(y)})$$

and so we want to see that the trace $\mathrm{Tr}_{\mathrm{Vect}}\big((f_*y_*k)_{f(y)} \xrightarrow{b_y} (f_*y_*k)_{g(y)}\big)$ is equal to $\det(1 - d_y f \circ (d_y g)^{-1})$. But by smoothness, the cohomology of the derived fiber E_x can be computed as

$$H^*(E_x) \simeq \Lambda^*(\mathbb{T}^*_{X,x})$$

and the induced map $H^*(b_y)$ is precisely $\Lambda^*((d_y f \circ (d_y g)^{-1})^\vee)$. The result then follows from the well-known relation between traces and determinants, see [8, Lemma 3.2.5] for more details. □

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