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RESEARCH CONTRIBUTION

Counting Tripods on the Torus

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Abstract
Motivated by the problem of counting finite BPS webs, we count certain immersed metric graphs, *tripods*, on the flat torus. Classical Euclidean geometry turns this into a lattice point counting problem in $\mathbb{C}^2$, and we give an asymptotic counting result using lattice point counting techniques.

Keywords Tripods · Lattice point counting · Trivalent graphs

1 Introduction

Given two points z, w in the plane $\mathbb{C}$, the set of points p that minimize the sum of distances $|z - p| + |w - p|$ is exactly the line segment connecting z and w. At any point p on this segment, the line segments between z and p and w and p have angle $2\pi/2 = \pi$. Given *three points* z, w, u in $\mathbb{C}$ such that the triangle they form has largest angle at most $2\pi/3$, the *Fermat point* p minimizes the sum of the lengths $|z-p|+|w-p|+|u-p|$. A classical result in Euclidean geometry says that the angles

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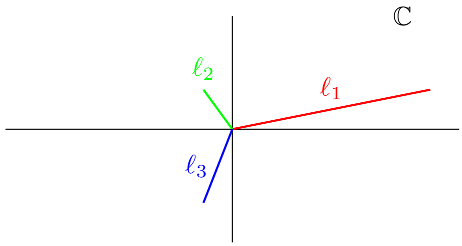
Fig. 1 The graph $\mathcal{G}(\ell_1, \ell_2, \ell_3) \subset \mathbb{C}$



between the line segments connecting p to z, w, u are $2\pi/3$. We call the configuration of line segments a *tripod*.

An integral lattice point $m + in \in \mathbb{Z}[i]$ is called *primitive* if m and n are coprime. A classical result states that as R tends to infinity, the number of primitive points in the circle of radius R centered at the origin grows like $\frac{R^2}{\zeta(2)}$. These points correspond to embedded closed geodesics in the flat torus $\mathbb{C}/\mathbb{Z}[i]$. Inspired by the work of [4], we ask for a similar asymptotic for the number of *immersed* graphs in flat tori that are the projections of tripods whose vertices are at points of a unimodular lattice.

Let Λ be a lattice in $\mathbb{C}$. Throughout, we will assume that Λ has unit covolume. A primitive vector in the complex plane descends to a primitive closed trajectory on $\mathbb{T} = \mathbb{C}/\Lambda$ with the origin marked. Analogously, a tripod on the torus will be an immersed tripod from the plane such that all three endpoints of the tripod descend to 0. More formally, $\mathcal{I}$ is an isometrically *immersed* copy in $\mathbb{T}$ of a metric graph $\mathcal{G} = \mathcal{G}(\ell_1, \ell_2, \ell_3) \subset \mathbb{C}$ (see Fig. 1) given by positive parameters ℓ_1, ℓ_2, ℓ_3,

$$\mathcal{G} = \{t\colon 0 \le t \le \ell_1\} \bigcup \{te^{2\pi i/3}\colon 0 \le t \le \ell_2\} \bigcup \{te^{4\pi i/3}\colon 0 \le t \le \ell_3\}.$$

The image of the tripod is in fact an immersed copy of an equiangular Θ-graph; that is, a graph with two vertices with three edges between them. The vertices are the point 0 and the tripod point p, and it is not difficult to check that the line segments must meet with angle $2\pi/3$ at 0. We will later discuss (see Sect. 3) that associated to each tripod is a (minimal) cover of the torus where the tripod becomes embedded. The degree of the cover is one more than the number of transverse self-intersections of the original tripod, and the embedded tripod gives a representation of the covering torus as an equiangular hexagon with parallel sides identified by translation.

Definition 1.1 A *tripod* $\curlyvee$ consists of a pair $(\mathcal{G}(\ell_1, \ell_2, \ell_3), \mathcal{I})$, where $\mathcal{I}\colon \mathcal{G} \to \mathbb{T}$ is an isometric immersion, with $\mathcal{I}(0) = p$, and

$$\mathcal{I}(\ell_1) = \mathcal{I}(\ell_2 e^{2\pi i/3}) = \mathcal{I}(\ell_3 e^{4\pi i/3}) = 0.$$

The *length* of the tripod is denoted $\ell(\curlyvee)$, and is given by

$$\ell(\curlyvee) = \ell_1 + \ell_2 + \ell_3.$$

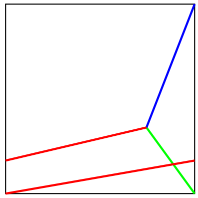


Fig. 2 A (primitive) tripod Υ drawn in a fundamental domain for $\mathbb{T}$

We say a tripod Υ is *primitive* if it is not a scaled copy of another tripod. A primitive tripod is called *reduced* if the only lattice points lying on its legs are at its endpoints and the tripod point is not at a lattice point (Fig. 2).

The distinction between primitive and reduced lattices is important for future work. In the case of lattice points, the concepts are identical: a line segment from the origin to a lattice point passes through another lattice point if and only if it is a scaled copy of the vector from the origin to that other lattice point. However, this is not the case for tripods. In future work, we plan to analyze tripods on higher genus translation surfaces, which admit cone points. In the presence of cone points, it is natural to define a tripod as having endpoints at the cone points and no cone points on either of the legs or at the tripod point. This is essential because there is no unique continuation of the leg of a tripod through a cone point on a translation surface because the angle at the cone point is greater than 2π. Thus, the tripod would not be well-defined.

Given a lattice Λ, let

$$N_{\Upsilon}(R, \Lambda) := \#\{\Upsilon\colon \Upsilon \text{ primitive with endpoints in } \Lambda \text{ and } \ell(\Upsilon) \leq R\}.$$

Furthermore, let

$$N_{red,\Upsilon}(R, \Lambda) := \#\{\Upsilon\colon \Upsilon \text{ reduced with endpoints in } \Lambda \text{ and } \ell(\Upsilon) \leq R\}.$$

What is the asymptotic behavior of $N_{\Upsilon}(R, \Lambda)$ Our main result is:

Theorem 1.1 *For all* (*unit covolume*) *lattices* Λ *in* $\mathbb{C}$,

$$\lim_{R\to\infty} \frac{N_{\Upsilon}(R, \Lambda)}{R^4} = \frac{1}{\zeta(4)} \cdot \frac{\sqrt{3}\pi}{24} = \frac{15\sqrt{3}}{4\pi^3}.$$

We will prove this claim by turning our problem into a problem of counting pairs $(z, w) \subset \Lambda = \mathbb{Z} + \mathbb{Z}\tau$ satisfying certain conditions. The $1/\zeta(4)$ term in our theorem arises from the fact that we will be counting pairs

$$z = a + b\tau, \quad w = c + d\tau, \qquad a, b, c, d \in \mathbb{Z}$$

with $\gcd(a, b, c, d) = 1$. The term $\frac{\sqrt{3}\pi}{24}$ represents the volume of a region in $\mathbb{C}^2$ in which we will be counting dilations of sets of points.

To clarify how the concepts of primitive versus reduced tripods affect their asymptotics, we prove the following two theorems. Let $N_{\text{nonred},\Upsilon}(R, \Lambda)$ be the count of the nonreduced tripods up to length R.

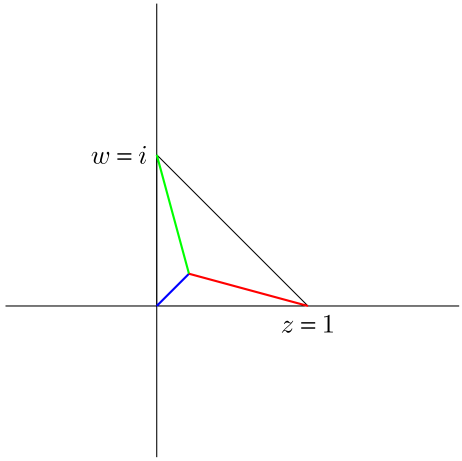


Fig. 3 Tripod $\curlyvee$ inscribed in triangle $\Delta(0, 1, i)$. The tripod point is $\left(\frac{1}{3+\sqrt{3}}, \frac{1}{3+\sqrt{3}}\right)$

Theorem 1.2 *For almost every lattice* Λ, $N_{\text{nonred},\curlyvee}(R, \Lambda) = 0$.

On the other hand, we show that Theorem 1.2 does not hold for all lattices.

Theorem 1.3 *Let* $\zeta = e^{2\pi i/6}$. *Then, for* $\Lambda = \mathbb{Z} + \mathbb{Z}\zeta$, *there is a positive constant* C *such that*

$$N_{\text{nonred},\curlyvee}(R, \Lambda) \geq CR^4 \qquad \textit{for sufficiently large } R.$$

1.1 Lifting

Note that if we lift a (primitive) tripod $\curlyvee$ from $\mathbb{T}$ to $\mathbb{C}$, we obtain a center point $\tilde{p}$ and segments emanating from $\tilde{p}$ to points in Λ. We can always choose our lift so that one of these points is 0, and we call the other two z, w with, say $\arg(z) < \arg(w)$. To remove ambiguity, we insist that the point $\tilde{p}$ lies in the sector of $\mathbb{C}$ specified by

$$0 \leq \arg(\tilde{p}) < 2\pi/3.$$

Only certain pairs (z, w) will yield triangles which have inscribed tripods. The length of the tripod can be computed explicitly in terms of z and w, thus turning our problem into a lattice point counting problem in $\mathbb{C}^2$ (Fig. 3).

1.2 Differentials, Saddle Connections, and Tripods

Our problem is also inspired by the problem of counting *saddle connections* for *quadratic differentials*. Given a holomorphic quadratic differential q on a compact Riemann surface X, there is a singular flat metric associated to q with conical singularities of angle $(k+2)\pi$ at zeros of order k of the differential. A *saddle connection* is a geodesic trajectory connecting two singular points with no singularities in its interior. Alternatively, one can think of it as a regular point p on (X, q) with two geodesic rays

γ_1, γ_2 emanating from p with an angle of π, each terminating in a singular point. Note that angles of π correspond to different local choices of a square root of q. Of course there are (infinitely) many choices of p in this setting. The problem of counting saddle connections is very well studied, see [3], for example. Our problem can be thought of as a problem naturally adapted to the setting of holomorphic *cubic* differentials c, where it is natural to consider trajectories emanating from a point at angles of $2\pi/3$, each corresponding to a different choice of cube root of the differential c. More generally, finite BPS webs arise in the work of [4] and are immersed graphs on Riemann surfaces with trivalent internal vertices and specified leaves, associated to lists of holomorphic k-differentials of varying orders k, which generalize the idea of saddle connections to the setting of higher Teichmüller theory. This work is motivated by discussions with Andy Neitzke at MSRI in Fall 2019, where we discussed the general problem of counting finite BPS webs of bounded length on higher genus surfaces. Our work can be viewed as a model case for counting these kinds of webs. See also the work of Douglas and Sun [2, Figure 8] for pictures of graphs on the once-punctured torus which appear to be closely related to our problem.

Remark 1 We remark that recent work of Koziarz and Nguyen [7] shows that the leading term for the normalized asymptotics for counting certain types of triangulations on surfaces is in $\mathbb{Q} \cdot (\sqrt{3}\pi)^N$ for an appropriate power N. Nevertheless, we do not see an obvious relation between the results. There is no primitivity assumption in the work of [7], which accounts for the $\zeta(4)$ factor in our work. However, if the primitivity is removed, then the R^4 growth rate of $N_\Upsilon(R, \Lambda)$ behaves as $\sqrt{3}\pi/24 \notin \mathbb{Q} \cdot (\sqrt{3}\pi)^4$. So there does appear to be a fundamental difference in the objects being counted.

Organization

In Sect. 2, we explain the Euclidean geometry which allows us to translate the problem to a lattice point counting problem. In Sect. 3, we summarize some nice properties of tripods, lengths, intersections, and covers. In Sect. 4, we state precisely the lattice point counting problem in $\mathbb{C}^2$ and the lattice point counting results which we use to solve the problem. In Sect. 5, we compute the volume of a region in $\mathbb{C}^2$ which gives us the main term in the asymptotic formula. In Sect. 6, we prove Theorems 1.2 and 1.3 concerning nonreduced tripods.

2 Fermat Points and Steiner Trees

2.1 Inscribed Tripods

We recall some beautiful facts from classical Euclidean geometry which are crucial for our translation of our counting problem to a lattice point counting problem. The following is due to Torricelli. The problem was posed to him by Fermat, and published by Torricelli's student Viviani. An excellent history of this problem (and the more general Steiner tree problem) can be found in [1].

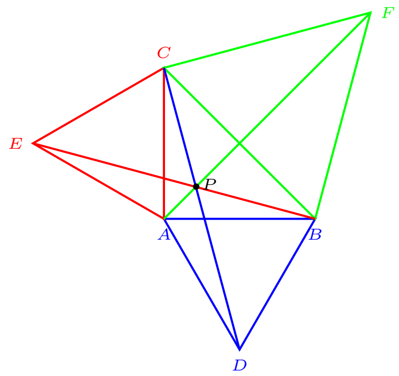


Fig. 4 Constructing a tripod (aka Steiner tree) $\curlyvee$ inscribed in a triangle $\Delta(ABC)$

Theorem 2.1 *A triangle $\Delta(ABC)$ contains an inscribed tripod if and only if the largest angle is at most $2\pi/3$. In this case, the tripod $\curlyvee(ABC)$ is constructed by constructing equilateral triangles on each side on the original triangle (which do not intersect the interior), and drawing lines connecting the opposite vertex of each equilateral triangle to the opposite vertex of the original triangle. That is,*

- *build an equilateral triangle $\Delta(ABD)$ on the side AB,*
- *an equilateral triangle $\Delta(ACE)$ on the side AC,*
- *an equilateral triangle $\Delta(BCF)$ on the side BC,*
- *build the segments DC, BE, and AF,*
- *the three line segments (DC, BE, AF) intersect at a common point, P, and the line segments AP, BP, CP, form the tripod. See Fig. 4.*

Furthermore, the point P is the unique point which minimizes the sum of distances

$$|AQ| + |BQ| + |CQ|$$

over all points Q in the plane. The tripod is known as the Steiner tree *associated to the points A, B, C [5]. Moreover, the length of the tripod $\ell(\curlyvee(ABC)) = |AP| + |BP| + |CP|$ is equal to the length of each of the auxiliary segments,*

$$\ell(\curlyvee(ABC)) = |DC| = |BE| = |AF|.$$

The above result can be rephrased using numbers in the complex plane. Suppose

$$z, w \in \mathbb{C} \text{ such that } \arg(z) < \arg(w) \text{ and } \Delta(0, z, w) \text{ has all angles at most } 2\pi/3,$$

Applying the above result to the triangle $\Delta(0, z, w)$, we obtain the following result.

Lemma 2.2 *Let $\curlyvee(z, w)$ denote the tripod inscribed in the triangle $\Delta(0, z, w)$, and let p denote the tripod point of $\curlyvee(z, w)$. Then,*

$$\ell(\curlyvee(z, w)) = |e^{i\pi/3} z + e^{-i\pi/3} w| \tag{2.1}$$

and

$$\arg(p) = \arg(e^{i\pi/3}z + e^{-i\pi/3}w). \tag{2.2}$$

Proof We write $\curlyvee$ for $\curlyvee(z, w)$. Let p denote the tripod point. Then,

$$\ell(\curlyvee) = |p| + |w - p| + |z - p|.$$

On the other hand, we assume that $\arg(z) < \arg(w)$ and by the fact that the line segments $p0$, pz, and pw are at angles of $2\pi/3$, we have that the three complex numbers

$$e^{i\pi/3}(z - p), \quad p, \quad e^{-i\pi/3}(w - p)$$

are parallel (that is, their ratios are real and positive). Therefore, the magnitude of their sum is the sum of their magnitudes. Since

$$1 = e^{i\pi/3} + e^{-i\pi/3},$$

it follows that

$$\begin{aligned}\ell(\curlyvee) &= |p| + |w - p| + |z - p| \\ &= |p| + |e^{-i\pi/3}(w - p)| + |e^{i\pi/3}(z - p)| \\ &= |p + e^{-i\pi/3}(w - p) + e^{i\pi/3}(z - p)| \\ &= |e^{-i\pi/3}w + e^{i\pi/3}z|. \end{aligned} \tag{2.3}$$

Our earlier observation that the complex numbers p, $e^{-i\pi/3}(w - p)$, $e^{i\pi/3}(z - p)$ all have the same argument implies that

$$\begin{aligned}\arg(p) &= \arg(p + e^{-i\pi/3}(w - p) + e^{i\pi/3}(z - p)) \\ &= \arg(e^{-i\pi/3}w + e^{i\pi/3}z)\end{aligned}$$

as claimed. □

If $\curlyvee$ is a tripod with endpoints 0, z, and w, and tripod point p, then $u = e^{i\pi/3}z + e^{-i\pi/3}w$ is the third vertex of the equilateral triangle with vertices at z and w. We call u the *Torricelli point* of the tripod; see Fig. 5. By Lemma 2.2, the point u satisfies $\arg(u) = \arg(p)$ and $\mathbb{P}|u| = \ell(\curlyvee) = |p| + |z - p| + |w - p|$.

Remark 2 There are also other ways of defining lengths of tripods. For example, given a tripod $\curlyvee$ inscribed in the triangle $\Delta(ABC)$, we could define its *triangle length* ℓ_Δ to be

$$\ell_\Delta(\curlyvee) = |AB| + |AC| + |BC|.$$

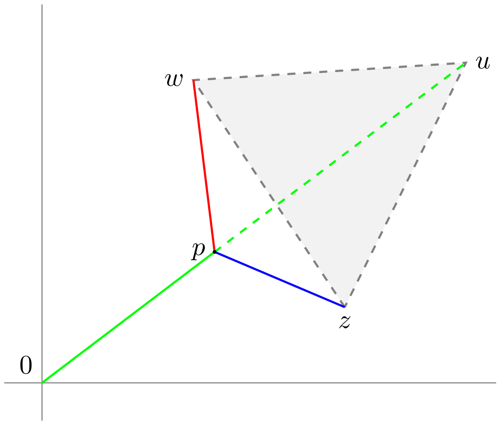


Fig. 5 Tripod with endpoints $0, z, w$, and Torricelli point u

2.2 Steiner Trees

More generally, it could be interesting to consider projections to the torus of solutions to the *Euclidean Steiner tree problem* with integer vertices: given N points in the plane, find the connected embedded graph with minimal total length with vertices at these points. For two points, this is of course the straight line, and more generally, it is not hard to see that the minimizer must be a tree.

3 Tripod Properties

In this section, we consider the number of self-intersections of a tripod on a torus and the number of subregions that a tripod divides a torus into. We relate these tripod properties to its lengths, defined in the previous section.

3.1 Lattice Index and Tripod Lengths

Given a tripod $\curlyvee$ in a lattice Λ, let $\Lambda(\curlyvee)$ be the minimal lattice in $\mathbb{R}^2$ which contains $\curlyvee$ as a tripod. We call $\Lambda(\curlyvee)$ the *spanning lattice* of $\curlyvee$; it is a sublattice of Λ. We define the *lattice index* of $\curlyvee$ in Λ as the index $[\Lambda\colon \Lambda(\curlyvee)]$. By our running assumption that Λ has unit covolume, we have $\operatorname{covol}(\Lambda(\curlyvee)) = [\Lambda\colon \Lambda(\curlyvee)]$. Recall that the tripod length is given by $\ell(\curlyvee) = \ell_1 + \ell_2 + \ell_3$. We define the L_2-length of a tripod by

$$L_2(\curlyvee) = (\ell_1^2 + \ell_2^2 + \ell_3^2)^{1/2}.$$

Proposition 3.1 *Let Λ be a lattice in $\mathbb{R}^2$ with unit covolume, and let $\curlyvee$ be a tripod in Λ. The lattice index $[\Lambda\colon \Lambda(\curlyvee)]$ is related to the tripod lengths $\ell(\curlyvee)$ and $L_2(\curlyvee)$ by*

$$[\Lambda\colon \Lambda(\curlyvee)] = \operatorname{covol}(\Lambda(\curlyvee)) = \frac{\sqrt{3}}{4}\left(\ell(\curlyvee)^2 - L_2(\curlyvee)^2\right).$$

Proof Suppose $\curlyvee$ spans the triangle with vertices $0, z, w$. Then, the area of a fundamental domain of $\Lambda(\curlyvee)$ is twice the area of the triangle, $\mathrm{covol}(\Lambda(\curlyvee)) = 2Area(\Delta(0, z, w))$. The tripod dissects the triangle $\Delta(0, z, w)$ into three subtriangles, each with internal angle $2\pi/3$. By adding up these areas, we have

$$Area(\Delta(0, z, w)) = \frac{\sqrt{3}}{4}\left(\ell_1\ell_2 + \ell_1\ell_3 + \ell_2\ell_3\right).$$

Therefore,

$$\begin{aligned}\mathrm{covol}(\Lambda(\curlyvee)) &= \frac{\sqrt{3}}{2}\left(\ell_1\ell_2 + \ell_1\ell_3 + \ell_2\ell_3\right)\\ &= \frac{\sqrt{3}}{4}\Big((\ell_1 + \ell_2 + \ell_3)^2 - (\ell_1^2 + \ell_2^2 + \ell_3^2)\Big).\end{aligned}$$

This proves the second part of the proposition. The first part follows from the assumption that Λ has unit covolume, so $[\Lambda\colon \Lambda(\curlyvee)] = \mathrm{covol}(\Lambda(\curlyvee))$. ◻

3.2 Counting Intersections and Regions

Proposition 3.2 *Suppose $\curlyvee$ is a tripod in Λ with index $n = [\Lambda\colon \Lambda(\curlyvee)]$, with only transverse self-intersections on the torus $\mathbb{T} = \mathbb{C}/\Lambda$. Then, the number of self-intersections of $\curlyvee$ on $\mathbb{T}$ is $n - 1$.*

Proof Suppose the tripod $\curlyvee$ spans the triangle with vertices $0, z, w$. Let $\Lambda(\curlyvee)$ denote the lattice in $\mathbb{C}$ spanned by z and w; we have $\Lambda(\curlyvee) \subset \Lambda$ by assumption that $\curlyvee$ is a tripod in Λ.

Let $n = [\Lambda\colon \Lambda(\curlyvee)]$. If we lift the tripod $\curlyvee$ from $\mathbb{C}/\Lambda$ to $\mathbb{C}/\Lambda(\curlyvee)$, then the preimage of $\curlyvee$ is a union of n tripods $\curlyvee_1, \curlyvee_2, \ldots, \curlyvee_n$ which are translates of each other. Each $\curlyvee_i$ has no self-intersections on $\mathbb{C}/\Lambda(\curlyvee)$. We claim that for each $i \neq j$, the intersection $\curlyvee_i \cap \curlyvee_j$ consists of exactly two points in the covering torus $\mathbb{C}/\Lambda(\curlyvee)$. To verify this claim, first observe that $|\curlyvee_i \cap \curlyvee_j| = |\curlyvee_i \cap (\curlyvee_j + \epsilon)|$ as ϵ varies over a small neighborhood U of zero in $\mathbb{R}^2$, with the neighborhood chosen such that the intersection remains transverse. By moving the copy $\curlyvee_j$ toward $\curlyvee_i$, along a path that keeps the intersection transverse, we have $|\curlyvee_i \cap \curlyvee_j| = |\curlyvee_i \cap (\curlyvee_i + \epsilon)|$. Finally, we verify that $|\curlyvee_i \cap (\curlyvee_i + \epsilon)| = 2$ when intersecting a tripod with a small translate of itself, as demonstrated in Fig. 6. It follows that $|\curlyvee_i \cap \curlyvee_j| = 2$ as claimed.

From this claim, it follows that $\curlyvee = \cup_i \curlyvee_i$ has $2\binom{n}{2} = n(n-1)$ self-intersections on the torus $\mathbb{C}/\Lambda(\curlyvee)$. The quotient map $\mathbb{C}/\Lambda(\curlyvee) \to \mathbb{C}/\Lambda$ to the original torus has degree n, so this implies that the tripod on $\mathbb{C}/\Lambda$ has $n - 1$ self-intersections. ◻

Proposition 3.3 *Suppose $\curlyvee$ is a tripod in Λ with index $n = [\Lambda\colon \Lambda(\curlyvee)]$, with only transverse self-intersections on the torus $\mathbb{T} = \mathbb{C}/\Lambda$. Then, the complement $\mathbb{T} \setminus \curlyvee$ consists of n connected regions.*

Proof The tripod $\curlyvee$ induces a cell structure on $\mathbb{T}$ as follows. The vertices (0-cells) are the two tripod points of $\curlyvee$ and all self-intersection points. The edges (1-cells) are the

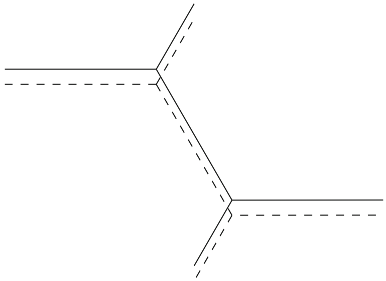


Fig. 6 Intersection of $\curlyvee_i$ and $\curlyvee_i + \epsilon$

segments of $\curlyvee$ after subdividing along vertices, and the faces (2-cells) are the connected components of $\mathbb{T} \setminus \curlyvee$. Note that the components of $\mathbb{T} \setminus \curlyvee$ are simply connected because their lifts to the cover $\mathbb{C}/\Lambda(\curlyvee)$ are homeomorphic and simply connected. Let c_i denote the number of i-cells, for $i = 0, 1, 2$. The Euler characteristic of $\mathbb{T}$ is zero, so

$$c_0 - c_1 + c_2 = 0.$$

By Proposition 3.2, the number of vertices in this cell decomposition is $c_0 = 2 + (n-1) = n+1$. To compute the number of edges, we observe that two vertices have degree 3 while the other $n-1$ vertices have degree 4. Thus,

$$c_1 = \frac{1}{2}\sum_v \deg(v) = \frac{1}{2}(3 \cdot 2 + 4(n-1)) = 2n+1.$$

Finally, using the Euler characteristic relation, we have $c_2 = c_1 - c_0 = 2n+1-(n+1) = n$, as claimed. $\square$

3.3 Counting Tripods by Spanning Lattice

Given a tripod $\curlyvee$, recall that $\Lambda(\curlyvee)$ denotes the minimal lattice which contains the endpoints of $\curlyvee$; we call $\Lambda(\curlyvee)$ the *spanning lattice* of $\curlyvee$.

The association of $\curlyvee$ to $\Lambda(\curlyvee)$ defines a map

$$(\text{ tripods in } \Lambda) \to (\text{sublattices of } \Lambda).$$

This map is surjective, but not injective. For a fixed sublattice $\Lambda_0 \subset \Lambda$, the set of tripods

$$\{\curlyvee \colon \Lambda(\curlyvee) = \Lambda_0\}$$

is finite, but as Λ_0 varies the size of this preimage is unbounded. In particular, the size of this preimage grows arbitrarily large asymptotically in proportion to the ratio

$$(\text{length of second-shortest vector in } \Lambda_0)/(\text{length of shortest vector in } \Lambda_0).$$

Proposition 3.4 *Given a lattice Λ in $\mathbb{C}$, there are finitely many tripods $\curlyvee$ such that* $\Lambda(\curlyvee) = \Lambda$.

Proof Without loss of generality, assume that Λ has unit covolume. Say a triangle with endpoints in Λ is a *unit triangle* if it has area $1/2$. By Theorem 2.1, it suffices to show that Λ contains finitely many unit triangles with angles at most $2\pi/3$, up to translation so that one triangle vertex is at the origin.

Suppose $\Delta(0, z, w)$ is a unit triangle with angles at most $2\pi/3$. If θ is the angle of the triangle at 0, then the triangle area satisfies

$$\frac{1}{2}|z||w| \sin\theta = \text{area}(\Delta(0, z, w)) = \frac{1}{2}.$$

Up to translation, we may assume that the largest angle of $\Delta(0, z, w)$ is at 0, so that $\theta \geq \pi/3$, and our initial hypothesis is that $\theta \leq 2\pi/3$. Therefore, $\sin\theta \geq \sqrt{3}/2$, which implies

$$|z||w| = \frac{1}{\sin\theta} \leq \frac{2}{\sqrt{3}}.$$

Let $L = \min\{|z|\colon z \in \Lambda,\ z \neq 0\}$. The above bound implies that z and w lie in the set $\{z \in \Lambda\colon |z| \leq \frac{2}{\sqrt{3}L}\}$ which is finite. This verifies that there are finitely many unit triangles in Λ up to translation with angles at most $2\pi/3$. $\square$

4 Lattice Point Counting

4.1 Lifting

We now describe how to turn our counting problem for tripods on the torus into a lattice point counting problem in $\mathbb{C}^2$. Given a tripod $\curlyvee$ on $\mathbb{T} = \mathbb{C}/\Lambda$, we fix a lift to $\mathbb{C}$ by choosing the center point $\tilde{p}$ to lie in the sector $0 \leq \arg(\tilde{p}) < 2\pi/3$. The lifted tripod will have one endpoint at 0. Denote the other endpoints by $z, w \in \Lambda$, with $\arg(z) < \arg(w)$. (By $\arg(z) < \arg(w)$, we mean $\arg(z) < \arg(w) < \pi + \arg(z)$.) See Fig. 7 for an example of determining a lift.

4.2 Angle Bound

By Theorem 2.1, a necessary and sufficient condition for $0, z, w$ to be endpoints of a tripod is that

$$\text{the angles of the triangle } \Delta(0, z, w) \text{ are at most } 2\pi/3. \tag{4.1}$$

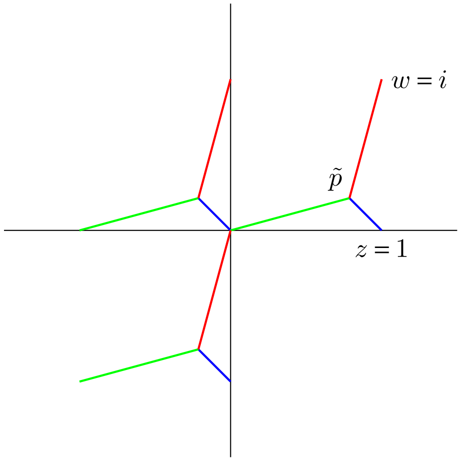


Fig. 7 Tripod $\curlyvee$ lifted to $\mathbb{C}$, with endpoints $0, z, w$. The lift with $\arg(\tilde{p}) \in [0, 2\pi/3)$ is labeled

4.3 Length Bound

Finally, if we want $\ell(\curlyvee(z, w)) \leq R$, we need, by Lemma 2.2, that the Torricelli point $u = ze^{i\pi/3} + we^{-i\pi/3}$ is distance at most R from the origin,

$$|ze^{i\pi/3} + we^{-i\pi/3}| \leq R. \tag{4.2}$$

Putting (4.1) and (4.2) together, we obtain:

Lemma 4.1 *Suppose $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ is a lattice in $\mathbb{C} \cong \mathbb{R}^2$ with $Im(\tau) > 0$. The number of primitive tripods $N_\curlyvee(R, \Lambda)$ is given by the number of pairs $(z, w) \in \Lambda^2$ satisfying the following conditions:*

$$\begin{aligned} z = a + b\tau, \quad w = c + d\tau, \quad \gcd(a, b, c, d) = 1, \\ \arg(z) < \arg(w), \quad \Delta(0, z, w) \textit{ has all angles } \leq 2\pi/3, \\ |ze^{i\pi/3} + we^{-i\pi/3}| \leq R \textit{ and } 0 \leq \arg(e^{i\pi/3}z + e^{-i\pi/3}w) < 2\pi/3. \end{aligned} \tag{4.3}$$

The following corollary follows from standard lattice point counting results [6, §24.10].

Corollary 4.2

$$\lim_{R\to\infty} \frac{N_\curlyvee(R, \Lambda)}{R^4} = \frac{1}{\zeta(4)} \operatorname{vol}(\Omega^\curlyvee),$$

where

$$\Omega^\curlyvee = \left\{ (z, w) \in \mathbb{C}^2 : \begin{array}{c} \arg(z) < \arg(w) < \pi + \arg(z), \\ \Delta(0, z, w) \textit{ has all angles } \leq 2\pi/3, \\ |e^{i\pi/3}z + e^{-i\pi/3}w| \leq 1 \textit{ and } 0 \leq \arg(e^{i\pi/3}z + e^{-i\pi/3}w) < 2\pi/3 \end{array} \right\}.$$

Proof $N_{\curlyvee}(R, \Lambda)$ counts primitive points in $\Lambda^2 = (\mathbb{Z} + \mathbb{Z}\tau)^2$ in the dilated set $R\Omega^{\curlyvee}$. $\Omega^{\curlyvee}$ is a compact region with smooth boundary (in $\mathbb{C}^2 \cong \mathbb{R}^4$), and so by [6, §24.10], $N_{\curlyvee}(R, \Lambda)$ is asymptotic to

$$\frac{1}{\zeta(4)} \operatorname{vol}(\Omega^{\curlyvee}) R^4,$$

where the factor of $1/\zeta(4)$ is the probability that a random integer vector $(a, b, c, d) \in \mathbb{Z}^4$ is primitive, that is, $\gcd(a, b, c, d) = 1$. □

5 Volumes

To finish the proof of Theorem 1.1, we need to compute the volume of $\Omega^{\curlyvee}$. This is given by

Lemma 5.1

$$\operatorname{vol}(\Omega^{\curlyvee}) = \frac{\sqrt{3}}{24}\pi = \frac{1}{4} \cdot \frac{2\pi}{3} \cdot \frac{\sqrt{3}}{4} \tag{5.1}$$

Proof of Theorem 1.1 Combining Corollary 4.2 and Lemma 5.1, we obtain Theorem 1.1.

5.1 Proof of Lemma 5.1

To prove Lemma 5.1, we will apply the following volume-preserving change of coordinates. Recall that $u = e^{i\pi/3}z + e^{-i\pi/3}w$ is the Torricelli point of the tripod (see Section 2). Let $\phi\colon \mathbb{C}^2 \to \mathbb{C}^2$ be defined by

$$\phi\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{i\pi/3} & e^{-i\pi/3} \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z \\ e^{i\pi/3}z + e^{-i\pi/3}w \end{pmatrix}.$$

The inverse map ϕ^{-1} with $u = e^{i\pi/3}z + e^{-i\pi/3}w$ is

$$\phi^{-1}\begin{pmatrix} z \\ u \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{-i\pi/3} & e^{i\pi/3} \end{pmatrix}\begin{pmatrix} z \\ u \end{pmatrix} = \begin{pmatrix} z \\ e^{-i\pi/3}z + e^{i\pi/3}u \end{pmatrix}.$$

By Lemma 2.2, the point u satisfies $\arg(u) = \arg(p)$ and $|u| = \ell(\curlyvee) = |p| + |z - p| + |w - p|$.

Recall that $\Omega^{\curlyvee}$ is defined as

$$\Omega^{\curlyvee} = \left\{ \begin{pmatrix} z \\ w \end{pmatrix} \in \mathbb{C}^2 \colon \begin{array}{c} \arg(z) < \arg(w) < \pi + \arg(z), \\ \text{triangle } \Delta(0, z, w) \text{ has all angles } \le 2\pi/3, \\ |e^{i\pi/3}z + e^{-i\pi/3}w| \le 1 \text{ and } 0 \le \arg(e^{i\pi/3}z + e^{-i\pi/3}w) < 2\pi/3 \end{array} \right\}.$$

Its image under ϕ is

$$\phi(\Omega^{\curlyvee}) = \left\{ \begin{pmatrix} z \\ u \end{pmatrix} \in \mathbb{C}^2 \colon \begin{array}{c} \arg(z) < \arg(e^{-i\pi/3}z + e^{i\pi/3}u) < \pi + \arg(z) \\ \text{triangle } \Delta(0, z, e^{-i\pi/3}z + e^{i\pi/3}u) \text{ has all angles } \leq 2\pi/3, \\ |u| \leq 1, \quad 0 \leq \arg(u) < 2\pi/3 \end{array} \right\}. \tag{5.2}$$

For $u \in \mathbb{C}$ satisfying $|u| \leq 1$ and $0 \leq \arg(u) < 2\pi/3$, let

$$\begin{aligned} \phi(\Omega^{\curlyvee})_u &= \{z \in \mathbb{C} \colon (z, u) \in \phi(\Omega^{\curlyvee})\} \\ &= \left\{ z \in \mathbb{C} \colon \begin{array}{r} 0,\ z,\ \text{and}\, e^{-i\pi/3}z + e^{i\pi/3}u \text{ are endpoints of a tripod} \\ \text{with } \arg(z) < \arg(e^{-i\pi/3}z + e^{i\pi/3}u) < \pi + \arg(z) \end{array} \right\}. \end{aligned}$$

and as a special case

$$\phi(\Omega^{\curlyvee})_1 = \left\{ z \in \mathbb{C} \colon \begin{array}{r} 0,\ z,\ \text{and}\, e^{-i\pi/3}z + e^{i\pi/3} \text{ are endpoints of a tripod} \\ \text{with } \arg(z) < \arg(e^{-i\pi/3}z + e^{i\pi/3}) \end{array} \right\}. \tag{5.3}$$

Lemma 5.2 *The region $\phi(\Omega^{\curlyvee})_1$ is equal to the equilateral triangle $\Delta(0, 1, e^{-i\pi/3}) \subset \mathbb{C}$ (Fig. 9).*

Proof By definition $\phi(\Omega^{\curlyvee})_1$ consists of points z such that the triangle $\Delta(0, z, e^{-i\pi/3}z + e^{i\pi/3})$ has all angles at most $2\pi/3$. By Theorem 2.1, such a triangle has an inscribed tripod. Using Lemma 2.2 with $w = e^{-i\pi/3}z + e^{i\pi/3}$ and the computation

$$e^{i\pi/3}z + e^{-i\pi/3}w = e^{i\pi/3}z + (e^{-2i\pi/3}z + 1) = 1,$$

such a tripod has length 1 and has its tripod point on the positive real axis.

Such a tripod has the following description, illustrated in Fig. 8. The center tripod point p lies on the positive real axis, and let a denote its distance from the origin. The lower endpoint is z, and let b denote its distance from p, while the upper endpoint w is at distance $1 - a - b$ from p. Each tripod leg has nonnegative length, so we assume $0 \leq a, b \leq 1$ and $0 \leq a + b \leq 1$.

In such a tripod, $z = a + be^{-i\pi/3}$. The points

$$\{z = a + be^{-i\pi/3} \colon a \geq 0,\ b \geq 0,\ a + b \leq 1\}$$

are exactly those inside the equilateral triangle with endpoints 0, 1, and $e^{-i\pi/3}$.

It follows that $\phi(\Omega^{\curlyvee})_1 = \Delta(0, 1, e^{-i\pi/3})$ as claimed. □

Lemma 5.3 *The region $\phi(\Omega^{\curlyvee})_u$ has area $\frac{\sqrt{3}}{4}|u|^2$.*

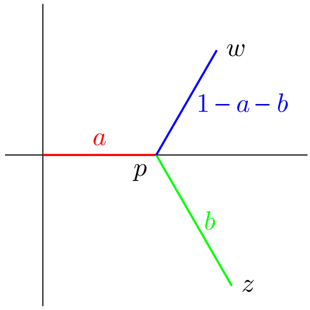


Fig. 8 A tripod in $\mathbb{C}$

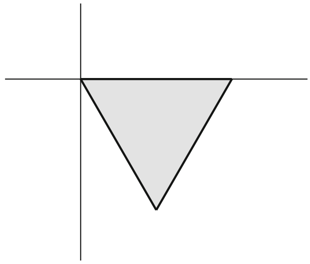


Fig. 9 The unit equilateral triangle $\Delta(0, 1, e^{-i\pi/3})$ in $\mathbb{C}$

Proof When $u = 1$, the claim follows from Lemma 5.2. For arbitrary u, the association $u \mapsto \phi(\Omega^{\curlyvee})_u$ is equivariant under multiplication, i.e.,

$$\phi(\Omega^{\curlyvee})_u = \{uz : z \in \phi(\Omega^{\curlyvee})_1\}.$$

The region $\phi(\Omega^{\curlyvee})_u$ has real dimension 2, so it follows that

$$\mathrm{vol}(\phi(\Omega^{\curlyvee})_u) = |u|^2\,\mathrm{vol}(\phi(\Omega^{\curlyvee})_1) = \frac{\sqrt{3}}{4}|u|^2$$

as claimed. □

Proof of Lemma 5.1 First, note that $\mathrm{vol}(\Omega^{\curlyvee}) = \mathrm{vol}(\phi(\Omega^{\curlyvee}))$ since ϕ has unit-magnitude Jacobian:

$$|Jac(\phi)| = \left|\det\begin{pmatrix} 1 & 0 \\ e^{i\pi/3} & e^{-i\pi/3}\end{pmatrix}\right| = 1.$$

Then, to compute $\phi(\Omega^{\curlyvee})$: we slice the region according to the u-coordinate.

$$\mathrm{vol}(\phi(\Omega^{\curlyvee})) = \int\int_{(z,u)\in\phi(\Omega^{\curlyvee})} dz\,du = \int_u \mathrm{vol}(\phi(\Omega^{\curlyvee})_u)du.$$

From the definition of $\phi(\Omega^{\curlyvee})$, it is clear that the region $\phi(\Omega^{\curlyvee})_u$ is nonempty only if

$$u = re^{i\theta} \quad \text{where } 0 \le r \le 1 \text{ and } 0 \le \theta \le 2\pi/3.$$

Using this substitution $u = re^{i\theta}$ and the identity $\mathrm{vol}(\phi(\Omega^{\curlyvee})_u) = \frac{\sqrt{3}}{4}|u|^2 = \frac{\sqrt{3}}{4}r^2$ from Lemma 5.3, we have

$$\begin{aligned}\mathrm{vol}(\phi(\Omega^{\curlyvee})) &= \int_{\theta=0}^{2\pi/3}\int_{r=0}^{1}\frac{\sqrt{3}}{4}r^2\cdot r\,dr\,d\theta\\ &= \frac{\sqrt{3}}{4}\int_{\theta=0}^{2\pi/3}d\theta\int_{r=0}^{1}r^3\,dr = \frac{\sqrt{3}}{4}\cdot\frac{2\pi}{3}\cdot\frac{1}{4} = \frac{\sqrt{3}\pi}{24}\end{aligned}$$

as claimed.

6 Nonreduced Tripods

Proof of Theorem 1.2 We will consider a lattice with a nonreduced tripod and show that the real and imaginary parts of the complex number determining the lattice are related by an equation with rational coefficients. From this, we conclude that a lattice admitting a nonreduced tripod must lie in a countable union of positive codimension subsets of the space of lattices.

Consider a lattice Λ in $\mathbb{C}$. Consider a tripod in this lattice with the property that at least one of its legs has a lattice point on its interior. We consider the following transformations. Let ℓ_1 be a leg that has a lattice point in its interior. Without loss of generality, translate the lattice so that the endpoint of ℓ_1 is the origin. Next observe that rotating a tripod preserves the tripod property. Therefore, we rotate the lattice Λ about the origin to a new lattice such that ℓ_1 lies in the positive real axis. Next, we scale the entire lattice so that one of its basis vectors is 1 and we write $\Lambda' = \mathbb{Z} \oplus \tau\mathbb{Z}$, where $\tau = s + it$. We make no claims that Λ' has unit covolume. From now on, we work entirely with respect to this tripod in the lattice Λ'.

Let z and w be the other endpoints of the tripod. Then, there exist integers $a_z, b_z, a_w, b_w \in \mathbb{Z}$ such that $z = a_z + b_z\tau$ and $w = a_w + b_w\tau$. Furthermore, the tripod point is simply given by a real number r_p and the assumption that ℓ_1 has a lattice point on it implies $r_p \geq 1$. Our assumption that the tripod point lies on the positive real axis, along with our convention that $\arg(z) < \arg(w)$, implies that $\arg(z - r_p) = -\pi/3$ and $\arg(w - r_p) = \pi/3$. See Fig. 8, where $p = r_p$. The angle conditions on the tripod points imply that

$$\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(a_z + b_z\tau - r_p\right) \in \mathbb{R} \quad \text{and} \quad \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(a_w + b_w\tau - r_p\right) \in \mathbb{R}.$$

In particular, the imaginary parts of both quantities are 0. Recall that $\tau = s + it$. After multiplying by 2, this yields the equations

$$b_z t + \sqrt{3}\left(a_z + b_z s - r_p\right) = 0 \quad \text{and} \quad b_w t - \sqrt{3}\left(a_w + b_w s - r_p\right) = 0.$$

It is convenient to define $t = t'\sqrt{3}$ to get

$$b_z t' + a_z + b_z s - r_p = 0 \qquad \text{and} \qquad b_w t' - a_w - b_w s + r_p = 0.$$

Adding these equations yields

$$(b_z + b_w)t' + (a_z - a_w) + (b_z - b_w)s = 0.$$

First we consider the case where $b_z + b_w \neq 0$. In this case, $t' = t/\sqrt{3} = q_1 + q_2 s$ for some $q_1, q_2 \in \mathbb{Q}$. However, this implies that $\tau = s + it$, which defines the lattice Λ, is not free to be any point in the complex plane because its real and complex parts are related by an equation. This completes the proof in this case.

Next, we consider the case where $b_z + b_w = 0$. We claim that $b_z - b_w \neq 0$ because otherwise, we would have $b_z = b_w = 0$ and this would contradict the assumption that 0, z and w form a tripod in Λ. Therefore, we conclude that $s \in \mathbb{Q}$ in which case we have again reduced to a measure zero subset of the space of lattices.

6.1 Many Nonreduced Tripods

In this section, fix $\Lambda = \mathbb{Z} + \mathbb{Z}\zeta$ as the triangular lattice, where $\zeta = e^{2\pi i/6}$ denotes a sixth root of unity.

Proof of Theorem 1.3 Consider a tripod $\curlyvee$ in Λ with endpoints at 0, $z = a + b\zeta$, and $w = c + d\zeta$, where a, b, c, d are integers. Assume that $\arg(z) < \arg(w)$. Let $u = e^{i\pi/3}z + e^{-i\pi/3}w$ be the Torricelli point of the tripod $\curlyvee(0, z, w)$ (see Sect. 2); u also appears in the map $\phi(z, w) = (z, u)$ in the proof of Lemma 5.1. The point u satisfies

$$\begin{aligned} u = z\zeta + w\zeta^{-1} &= a\zeta + b\zeta^2 + c\zeta^{-1} + d \\ &= a\zeta + b(\zeta - 1) + c(1 - \zeta) + d \\ &= (-b + c + d) + (a + b - c)\zeta. \end{aligned}$$

This shows that $u \in \Lambda$. The tripod point p of $\curlyvee$ is a positive real multiple of u. Conversely, if $z \in \Lambda$ and $u = z\zeta + w\zeta^{-1} \in \Lambda$, then it is straightforward to check that also $w \in \Lambda$.

Recall that a tripod $\curlyvee$ is *nonreduced* if its interior contains a lattice point. If the leg $0p$ contains a lattice point in its interior, then $0u \supset 0p$ also contains a lattice point so u must be a nonprimitive lattice point of Λ. Conversely, if u is a nonprimitive lattice point so that $0u$ contains lattice points in its interior, a sufficient condition for $0p$ to contain a lattice point in its interior is that

$$\ell(0p) > \frac{1}{2}\ell(0u).$$

Therefore, as a lower bound for the number of nonreduced tripods, we have

$$\#\{\text{nonreduced } \curlyvee \text{ in } \Lambda\} \geq \sum_{\substack{\text{nonprimitive} \\ u\in\Lambda}} \#\left\{\text{tripods } \curlyvee : \begin{array}{c}\text{tripod point } p \in 0u,\ \ell(0p) > \frac{1}{2}\ell(0u), \\ 0p \text{ is longest leg of } \curlyvee\end{array}\right\}.$$

The condition that $0p$ is the longest tripod leg is needed to avoid overcounting on the torus $\mathbb{R}^2/\Lambda$. Note that the length of the tripod $\ell(\curlyvee)$ is equal to the magnitude $|u| = \ell(0u)$ of the Torricelli point.

Suppose we fix some u with $|u| = R$. Then, the set

$$\{\text{tripods } \curlyvee : \text{tripod point } p \in 0u\}$$

is parametrized by choosing w inside the equilateral triangle $\Delta(0, u, \zeta u)$ with side length R, while the set

$$\left\{\text{tripods } \curlyvee : \text{tripod point } p \in 0u,\ 0p \text{ is longest leg of } \curlyvee\right\}$$

is parametrized by choosing w inside a subregion of the previous triangle of one-third size, see Fig. 10 (middle). Finally, the set

$$\left\{\text{tripods } \curlyvee : \begin{array}{c}\text{tripod point } p \in 0u,\ 0p \text{ is longest leg of } \curlyvee, \\ \ell(0p) > \frac{1}{2}\ell(0u)\end{array}\right\}$$

is parametrized by choosing w inside the subregion shown in Fig. 10 (right), which has one-fourth the size of the original triangle.

Therefore, for fixed lattice point u with $|u| = R$,

$$\begin{aligned}
&\#\left\{\text{tripods } \curlyvee : \begin{array}{c}\text{tripod point } p \in 0u,\ \ell(0p) > \frac{1}{2}\ell(0u), \\ 0p \text{ is longest leg of } \curlyvee\end{array}\right\} \\
&\quad = \#\{\text{lattice points in subregion of } \Delta(0, u, \zeta u)\} \\
&\quad = \frac{\text{vol(triangle subregion)}}{\text{covol}(\Lambda)} + O(R) \\
&\quad = \frac{\sqrt{3}R^2/16}{\sqrt{3}/2} + O(R) \\
&\quad = \frac{1}{8}R^2 + O(R).
\end{aligned}$$

Now, it remains to sum over the possible choices of u. Summing over all lattice points u, we would have

$$\begin{aligned}
\sum_{\substack{u\in\Lambda \\ |u|\leq R}} \#\left\{\text{tripods } \curlyvee : \begin{array}{c}\text{tripod point } p \in 0u,\ \ell(0p) > \frac{1}{2}\ell(0u), \\ 0p \text{ is longest leg of } \curlyvee\end{array}\right\} &= \sum_{\substack{u\in\Lambda \\ |u|=r\leq R}} \frac{1}{8}r^2 + O(r) \\
&= \int_{r=0}^{R} \left(\frac{\pi}{4}r^3 + O(r^2)\right) dr \\
&= \frac{\pi}{16}R^4 + O(R^3).
\end{aligned}$$

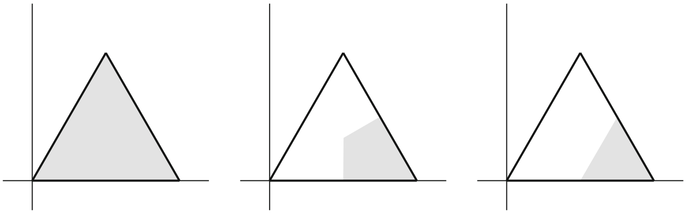


Fig. 10 An equilateral triangle and relevant subregions

If we instead sum over only those u which are nonprimitive lattice points, our answer changes asymptotically by a constant factor

$$1-\zeta(2)^{-1}=\lim_{R\to\infty}\frac{\#\{u\in\Lambda\colon |u|\leq R,\ u\text{ nonprimitive}\}}{\#\{u\in\Lambda,\ |u|\leq R\}}=1-\frac{6}{\pi^2}\approx 0.392.$$

Therefore,

$$\#\{\text{nonreduced }\curlyvee\text{ in }\Lambda\colon \ell(\curlyvee)\leq R\}=\left(1-\frac{6}{\pi^2}\right)\frac{\pi}{16}R^4+O(R^3),$$

so we may take any positive constant $C<(1-\frac{6}{\pi^2})\frac{\pi}{16}\approx 0.0770$.

Note that the total number of tripods in Λ satisfies

$$\#\{\curlyvee\text{ in }\Lambda\colon \ell(\curlyvee)\leq R\}\sim\frac{\pi}{12}R^4.$$

Corollary 6.1 *The number of nonreduced, primitive tripods in Λ satisfies*

$$\#\{\curlyvee\textit{ nonreduced and primitive}\colon \ell(\curlyvee)\leq R\}\geq CR^4$$

for some positive constant C, for sufficiently large R.

Proof The previous theorem showed that

$$\begin{aligned}\#\{\text{nonreduced }\curlyvee\text{ in }\Lambda\colon \ell(\curlyvee)\leq R\}&\gtrsim\left(1-\frac{6}{\pi^2}\right)\frac{\pi}{16}R^4\\&\sim\left(1-\frac{6}{\pi^2}\right)\frac{3}{4}\#\{\text{total }\curlyvee:\ell(\curlyvee)\leq R\},\end{aligned}$$

where the constant

$$C_1=\left(1-\frac{6}{\pi^2}\right)\frac{3}{4}\approx 0.294.$$

We also know that the number of primitive tripods satisfies

$$\#\{\text{primitive } \curlyvee : \ell(\curlyvee) \leq R\} \sim \zeta(4)^{-1}\#\{\text{total } \curlyvee : \ell(\curlyvee) \leq R\},$$

where

$$C_2 = \zeta(4)^{-1} \approx 0.924.$$

Together these bounds imply that

$$\begin{aligned}\#\{\text{nonreduced, primitive } \curlyvee : \ell(\curlyvee) \leq R\} &\gtrsim (C_1 + C_2 - 1)\#\{\text{total } \curlyvee,\ \ell(\curlyvee) \leq R\} \\ &\sim (C_1 + C_2 - 1)\frac{\pi}{12}R^4.\end{aligned}$$

We may take C to be any positive constant less than $(C_1 + C_2 - 1)\frac{\pi}{12}$. □

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Appendix: Numerics

We obtained experimental evidence for Theorem 1.1 using the following elementary Sage code, which computes the number of tripods in $\mathbb{Z}[i]$ of length at most R.

```
from itertools import product
from time import time
def is_prim(a,b,c,d):
    '''checks that a+bi and c+di are primitive'''
    return gcd(gcd(a,b), gcd(c,d))==1
def is_longest(a,b,c,d):
    '''checks that the largest angle of the triangle with vertices at 0, a+bi,
      c+di
    is at 0, by checking that the length of the side opposite 0,
    whose length is |(a-c) + (b-d)i|, is longest.'''
    return min(a^2+b^2, c^2+d^2) > 2*a*c+2*b*d
def is_positively_oriented(a, b, c, d):
    '''checks that the triangle with vertices at 0, a+bi, c+di is positively
     oriented'''
    return a*d-b*c > 0
def is_tripod(a,b,c,d):
    '''checks that the triangle with vertices at 0, a+bi, c+di admits a tripod'''
    return is_longest(a, b, c, d) and RR(2*a*c+2*b*d+ sqrt((a^2+b^2)*(c^2+d^2)))>0
def tripod_length_squared(a,b,c, d, R):
    '''computes the length-squared of the tripod'''
    return RR((a-c)^2 + (b-d)^2 + a*c + b*d + sqrt(3)*(a*d-b*c)) < RR(R^2)
def tripod_counts(R): #guess is 0.20947986097*R^4 = RR(15*sqrt(3)/(4*pi^3)) R^4
    '''returns the length of the list of tripods of length at most $R$'''
    L=[(a,b, c, d) for (a,b,c,d) in product(range(-1.5*R, 1.5*R), repeat=4)
       if is_prim(a,b,c,d) and is_longest(a,b,c,d) and is_positively_oriented
       (a,b,c,d)]
    return len(L)
```

For $R = 35$, this yields

```
tripod_counts(35) = 312488
```

For comparison,

$$\left| \frac{312488}{(35)^4} - \frac{15\sqrt{3}}{4\pi^3} \right| = 0.00124129370635984\ldots.$$

We make no claims that this Sage code is particularly efficient.

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