



On Affine Real Cubic Surfaces

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Abstract

We prove that the space of affine, transversal at infinity, nonsingular real cubic surfaces has 15 connected components. We give a topological criterion to distinguish them and show also how these 15 components are adjacent to each other via wall-crossing.

Keywords Real affine cubic surfaces · Deformation classification · Wall-crossing

Mathematics Subject Classification Primary 14R05; Secondary 14P25 · 14J10 · 14J26

On résout les problèmes qu'on se pose et non les problèmes qui se posent.
Henri Poincaré

1 Introduction

1.1 Main Task

We consider an affine three-space as a chart $\mathbb{P}^3 \setminus \mathbb{P}^2$ of a projective space \mathbb{P}^3 with a fixed hyperplane \mathbb{P}^2 . Accordingly, by an *affine cubic surface transversal at infinity*, we mean the complement $X \setminus A$, where $X \subset \mathbb{P}^3$ is a projective cubic surface transversal to \mathbb{P}^2 and $A = X \cap \mathbb{P}^2$. Occasionally, we refer to affine cubics as to pairs (X, A) .

The space of nonsingular affine cubic surfaces transversal at infinity is $\mathbb{P}^{19} \setminus (\Delta \cup \Delta')$, where \mathbb{P}^{19} is the space of projective cubic surfaces, $\Delta \subset \mathbb{P}^{19}$ is the hypersurface formed by singular surfaces, and Δ' is the hypersurface formed by surfaces which are not transversal to \mathbb{P}^2 .

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Our main objective is to classify up to deformation the real nonsingular affine cubic surfaces transversal at infinity. These surfaces form the real part $\mathbb{P}_{\mathbb{R}}^{19} \setminus (\Delta_{\mathbb{R}} \cup \Delta'_{\mathbb{R}})$ of $\mathbb{P}^{19} \setminus (\Delta \cup \Delta')$. We declare two such surfaces *deformation equivalent* if they belong to the same connected component of $\mathbb{P}_{\mathbb{R}}^{19} \setminus (\Delta_{\mathbb{R}} \cup \Delta'_{\mathbb{R}})$.

Often, it is convenient to work with a larger space of surfaces, $\mathbb{P}_{\mathbb{R}}^{19} \setminus (\dot{\Delta}_{\mathbb{R}} \cup \dot{\Delta}'_{\mathbb{R}})$, where the semi-algebraic hypersurface $\dot{\Delta}_{\mathbb{R}} \subset \mathbb{P}_{\mathbb{R}}^{19}$ (resp. $\dot{\Delta}'_{\mathbb{R}} \subset \mathbb{P}_{\mathbb{R}}^{19}$) is formed by real affine cubic surfaces with a real singular point (resp. not transversal to \mathbb{P}^2 at some real point). Since both the spaces, $\mathbb{P}_{\mathbb{R}}^{19} \setminus (\dot{\Delta}_{\mathbb{R}} \cup \dot{\Delta}'_{\mathbb{R}})$ and $\mathbb{P}_{\mathbb{R}}^{19} \setminus (\Delta_{\mathbb{R}} \cup \Delta'_{\mathbb{R}})$, are open in Euclidean topology and differ by a codimension 2 semi-algebraic set, this does not change the equivalence relation.

1.2 Deformation Classification

Recall that, for every real algebraic M -surface X , there exists a quadratic $\mathbb{Z}/4$ -valued function (called *Rokhlin–Guillou–Marin quadratic function*), which is defined on the kernel of the inclusion homomorphism $H_1(X_{\mathbb{R}}; \mathbb{Z}/2) \rightarrow H_1(X_{\mathbb{C}}; \mathbb{Z}/2)$ and takes value $2 \in \mathbb{Z}/4$ on each real vanishing cycle (see, e.g., [1]).

Theorem 1.2.1¹ *There are 15 deformation classes of real affine nonsingular and transversal at infinity cubic surfaces $X \setminus A$. For all but two exceptional classes, such surfaces are deformation equivalent if and only if their real parts $X_{\mathbb{R}} \setminus A_{\mathbb{R}}$ are homeomorphic. The two exceptional classes are those for which:*

- X is an M -surface, that is, $\chi(X_{\mathbb{R}}) = -5$,
- A is an M -curve, that is, $A_{\mathbb{R}}$ has two components,
- both components of $A_{\mathbb{R}}$ give non-zero classes in $H_1(X_{\mathbb{R}})$.

The number of real lines intersecting an oval-component $O \subset A_{\mathbb{R}}$ is 12 for one of these exceptional classes and 16 for another. The Rokhlin–Guillou–Marin quadratic function $q : H_1(X_{\mathbb{R}}) \rightarrow \mathbb{Z}/4$ takes value $q(O) = 2$ in the first case and $q(O) = 0$ in the second.

The topological types of $X_{\mathbb{R}} \setminus A_{\mathbb{R}}$ used as a classification invariant in Theorem 1.2.1 are determined by $X_{\mathbb{R}}$ (five columns in Table 1), the number of components of $A_{\mathbb{R}}$ (one or two circles) and the number of components of $X_{\mathbb{R}} \setminus A_{\mathbb{R}}$ (one, two, or three). In Table 1, we list possible combinations and give the corresponding number of deformation classes of (X, A) in each case.

More concretely:

- Each of the five classes of X contains precisely one deformation class of affine cubics $X \setminus A$ with connected real locus $A_{\mathbb{R}}$.
- The class of X with $X_{\mathbb{R}} = \mathbb{R}P^2$ contains precisely one deformation class of $X \setminus A$ with 2-component $A_{\mathbb{R}}$.

¹ **Added in proofreading.** As was pointed out to us by J. Capco, a somewhat different description for 10 of these 15 deformation classes and some hints how to complete the classification were given by B. Segre [6], pp. 114–124.

Table 1 The number of real deformation classes of affine cubics $X \setminus A$ depending on the topology of $X_{\mathbb{R}}$ (columns) and $A_{\mathbb{R}}$ (rows). Here, \mathbb{T}^2 stands for $S^1 \times S^1$

$X_{\mathbb{R}}$	$\mathbb{R}\mathbb{P}^2 \# 3\mathbb{T}^2$	$\mathbb{R}\mathbb{P}^2 \# 2\mathbb{T}^2$	$\mathbb{R}\mathbb{P}^2 \# \mathbb{T}^2$	$\mathbb{R}\mathbb{P}^2$	$\mathbb{R}\mathbb{P}^2 \amalg S^2$
$A_{\mathbb{R}} = S^1$	1	1	1	1	1
$A_{\mathbb{R}} = S^1 \amalg S^1, b_0(X_{\mathbb{R}} \setminus A_{\mathbb{R}}) = \begin{cases} 1 \\ 2 \\ 3 \end{cases}$	2	1	1	0	0
	1	1	1	1	1
	0	0	0	0	1

- Each of the two classes of X with $X_{\mathbb{R}} = \mathbb{R}\mathbb{P}^2 \# k(S^1 \times S^1), k = 1, 2$, contains precisely two classes of $X \setminus A$ with two-component $A_{\mathbb{R}}$: for one class, the oval of $A_{\mathbb{R}}$ is null-homologous in $X_{\mathbb{R}}$ and for another is not.
- The class of X with $X_{\mathbb{R}} = \mathbb{R}\mathbb{P}^2 \# 3(S^1 \times S^1)$ contains precisely three classes of $X \setminus A$ with two-component $A_{\mathbb{R}}$; for one class, the oval of $A_{\mathbb{R}}$ is null-homologous in $X_{\mathbb{R}}$ and for the other two is not. For one of the latter two classes, the oval is homologous to a real vanishing class in $H_1(X_{\mathbb{R}})$ and for another is not.
- The class of X with $X_{\mathbb{R}} = \mathbb{R}\mathbb{P}^2 \amalg S^2$ contains precisely two classes of $X \setminus A$ with two-component $A_{\mathbb{R}}$: for one class, the components of $A_{\mathbb{R}}$ lie in different components of $X_{\mathbb{R}}$ and for another in the same component $\mathbb{R}\mathbb{P}^2$.

1.3 Adjacency of Deformation Classes

Two deformation classes are said to be *adjacent* if they meet along a maximal dimension stratum of $\Delta_{\mathbb{R}} \setminus \Delta'_{\mathbb{R}}$. These strata are formed by those transversal at infinity hypersurfaces which have a node and no other singular points, and they constitute the connected components of the smooth part of $\Delta_{\mathbb{R}} \setminus \Delta'_{\mathbb{R}}$. We call these strata *walls* and name the graph representing the above adjacency relation the *wall-crossing graph*.

We depict the vertices of this graph by circles labeled inside with the number of real lines intersecting $A_{\mathbb{R}}$. If $A_{\mathbb{R}}$ is connected, it is just the total number of real lines on X . If not, then it is a pair of numbers indicating the number of lines intersecting the one-sided in \mathbb{P}^2 component of $A_{\mathbb{R}}$ and the two-sided component called *the oval*. The edges are decorated with the number of real lines intersecting the node that appears at the instance of wall-crossing. Following the same convention as for vertices, when $A_{\mathbb{R}}$ has two connected components, this number is split into a pair. In addition, we color in black the vertices representing (X, A) with two-component $A_{\mathbb{R}}$ whose oval bounds a disc in the non-orientable component of $X_{\mathbb{R}}$.

Theorem 1.3.1 *The wall-crossing graph for affine cubics is as shown in Fig. 1: its left-hand-side corresponds to (X, A) with one-component curves $A_{\mathbb{R}}$, and the right-hand-side to (X, A) with two-component $A_{\mathbb{R}}$.*

Note that there may be several walls that separate the same pair of deformation classes and, thus, represent the same edge on the graph of Fig. 1 (see details in Sect. 5.2). For instance, it is so for a pair of walls which are adjacent to the same codimension 1 cuspidal stratum of $\Delta_{\mathbb{R}} \setminus \Delta'_{\mathbb{R}}$ (representing A_2 -singularity on a cubic). This motivated

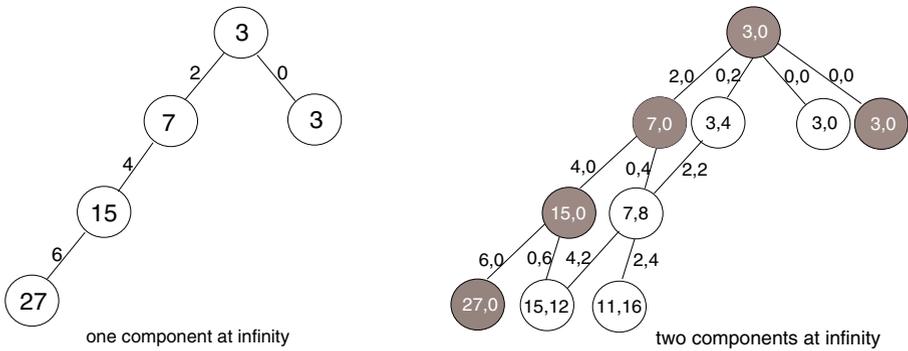


Fig. 1 The wall-crossing graph for transversal at infinity affine cubic surfaces

us to consider *extended walls* that we define as the connected components of $(\Delta_{\mathbb{R}} \setminus \Delta'_{\mathbb{R}}) \cup (\Delta^c_{\mathbb{R}} \setminus \Delta'_{\mathbb{R}})$, where Δ^c is the union of the cuspidal strata of Δ . We establish that the correspondence between the edges of the graph in Fig. 1 and extended walls is in fact bijective by combining Theorem 1.3.1 with the following one.

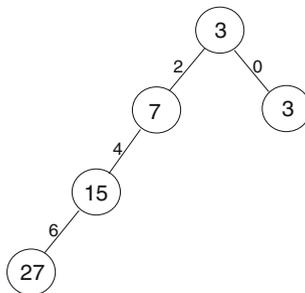
Theorem 1.3.2 *Each edge of the graph in Fig. 1 represents just one extended wall.*

2 Preliminaries

2.1 On the Projective Real Nonsingular Cubic Surfaces (cf., [4, 6])

Theorem 2.1.1 *There are five deformation classes of real nonsingular projective cubic surfaces. Two real nonsingular projective cubic surfaces are deformation equivalent if and only if their real parts $X_{\mathbb{R}}$ are homeomorphic. For four of them, $X_{\mathbb{R}} = \#_{2\mu+1} \mathbb{R}P^2$, $\mu = 0, 1, 2, 3$. For the fifth one, $X_{\mathbb{R}} = \mathbb{R}P^2 \# S^2$.*

The wall-crossing graph of real nonsingular projective cubic surfaces coincides with the left-hand-side graph of Fig. 1 and is reproduced below. The number of real lines for each class is encircled and the number of lines passing through the node corresponding to a wall-crossing decorates the edges.



Note also that representatives of all five deformation classes of real projective nonsingular cubic surfaces can be obtained by small perturbations of Cayley’s four-

nodal cubic surface, $\sum_{i=0}^4 x_i^3 = \frac{1}{4}(\sum_{i=0}^4 x_i)^3$ (see <https://mathworld.wolfram.com/CayleyCubic.html>).

2.2 Blowup Models of Cubic Surfaces (cf., [6])

A set of real points or real lines on a real algebraic surface is said to be *real* if it is invariant under the complex conjugation on the surface.

A set of six points $S \subset \mathbb{P}^2$ will be called a *typical 6-tuple* if S does not contain collinear triples and all six points are not coconic.

Theorem 2.2.1 *For any real projective nonsingular cubic surface X with connected real part, there exists a real set of six pairwise disjoint lines on X . Blowing down these lines yields a real plane with a typical real 6-tuple $S \subset \mathbb{P}^2$. Conversely, blowing up \mathbb{P}^2 at any real typical 6-tuple S gives a unique (up to a real projective transformation) real projective nonsingular cubic surface X with connected real part equipped with a real set of six skew lines.*

This blow-down transforms isomorphically any real nonsingular hyperplane sections $A \subset X$ into a real nonsingular plane cubic curves $C \subset \mathbb{P}^2$, $S \subset C$. In this way, it establishes a bijection between the set of the former sections A with the set of the latter curves C .

Recall that for a typical 6-tuple $S = \{x_1, \dots, x_6\} \subset \mathbb{P}^2$, the 27 lines in the corresponding cubic surface X include:

- six exceptional curves E_i corresponding to blowing up at x_i ;
- $\binom{6}{2} = 15$ proper images of straight lines $x_i x_j \subset \mathbb{P}^2$, $1 \leq i < j \leq 6$;
- six proper images Q_i of plane conics passing through the five points x_j , $j \neq i$.

Theorem 2.2.1 allows us to represent any real affine nonsingular and transversal at infinity cubic surface (X, A) with connected $X_{\mathbb{R}}$ via a real nonsingular plane cubic curve C equipped with a real typical 6-tuple $S \subset C$. Since such a representation determines (X, A) only up to a real affine transformation, which may preserve or reverse orientation, such approach ignores a possibility that some affine equivalent cubics may be not deformation equivalent. For this reason, in our first step of proving Theorem 1.2.1, we classify pairs (X, A) only up to a weaker equivalence in which we call two pairs *coarse deformation equivalent* if one is deformation equivalent to the image of another under an affine transformation.

As is well known and easy to show, Theorem 2.2.1 extends to a wider class of 6-tuples, as well as to families of them. In particular, the following statement holds.

Theorem 2.2.2 *Given a continuous family (C_t, S_t) formed by nonsingular real plane cubics C_t and real 6-tuples $S_t \subset C_t$ such that the linear system of cubic curves passing through S_t has projective dimension 2, there is a unique (up to a family of real projective transformations) family of real projective cubic surfaces X_t giving rise to the given data including the proper image A_t of C_t as a real nonsingular hyperplane section of X_t .*

Note that the dimension condition imposed on S in Theorem 2.2.2 is fulfilled if, for example, S is in a *one-nodal* position, that is, if S lies on a nonsingular conic, or if S contains one and only one triple of points aligned.

2.3 Counting Real Lines Intersecting a Given Component of $A_{\mathbb{R}}$

Here, we treat a particular case of blowup models, namely, the case where $C_{\mathbb{R}}$ has two connected components: an oval and a one-sided component. We denote by $S_{a,b}$, $a, b \geq 0$, $a + b \in \{0, 2, 4, 6\}$, the set of pairs (C, S) , where $C \subset \mathbb{P}^2$ is a nonsingular cubic curve, $C_{\mathbb{R}}$ has two connected components, and $S \subset C$ is a real typical 6-tuple which includes $\mu \leq 3$ pairs of complex-conjugate imaginary points and $a + b = 6 - 2\mu$ real ones, among which a points lie on the one-sided component of $C_{\mathbb{R}}$ and the remaining b real points lie therefore on the oval (two-sided component). By $\Sigma_{a,b}$, we denote the set of real affine cubic surfaces $X \setminus A$, represented by pairs $(C, S) \in S_{a,b}$, and put $\Sigma_{\mu} = \cup_{a+b=6-2\mu} \Sigma_{a,b}$.

Proposition 2.3.1 *The set Σ_{μ} is partitioned into non-empty subsets as follows:*

- for $\mu = 0$ into 3 subsets: $\Sigma_{6,0}$, $\Sigma_{0,6} = \Sigma_{3,3} = \Sigma_{4,2}$ and $\Sigma_{1,5} = \Sigma_{2,4} = \Sigma_{5,1}$,
- for $\mu = 1$ into 2 subsets: $\Sigma_{4,0}$ and $\Sigma_{0,4} = \Sigma_{1,3} = \Sigma_{2,2} = \Sigma_{3,1}$,
- for $\mu = 2$ into 2 subsets $\Sigma_{2,0}$ and $\Sigma_{0,2} = \Sigma_{1,1}$.

The case $\Sigma_3 = \Sigma_{0,0}$ is trivial.

Proof First, note that the indicated subsets are non-empty because one can always displace the required number of real points on the two-components of a cubic $C_{\mathbb{R}}$. Next, these sets either coincide or are disjoint and, thus, give a partition.

If $b \geq 3$, we can apply a standard quadratic Cremona transformation which changes exceptional divisors over points x_1, x_2, x_3 on the oval $O \subset C_{\mathbb{R}}$ by the proper transform of lines x_1x_2, x_2x_3 and x_3x_1 . This Cremona transformation takes the oval with b points into a one-sided component with $b - 3$ points and, thus, identifies $\Sigma_{a,b}$ with $\Sigma_{b-3, a+3}$. Hence, $\Sigma_{3,3} = \Sigma_{0,6}$, $\Sigma_{2,4} = \Sigma_{1,5}$ for $\mu = 0$ and $\Sigma_{1,3} = \Sigma_{0,4}$ for $\mu = 1$.

If $a \geq 2$ and $b \geq 1$, we apply the same Cremona transformation, but with x_1 chosen on the oval, and x_2 with x_3 on the one-sided component. This also takes the oval into one-sided component and, thus, identifies $\Sigma_{a,b}$ with $\Sigma_{b+1, a-1}$. Hence, $\Sigma_{5,1} = \Sigma_{2,4}$, $\Sigma_{4,2} = \Sigma_{3,3}$ for $\mu = 0$, and $\Sigma_{3,1} = \Sigma_{2,2}$ for $\mu = 1$.

If $b \geq 1$ and $\mu \geq 1$, we apply again the same transformation, but with x_2, x_3 to be chosen imaginary complex conjugate and x_1 placed on the oval. This also takes the oval into one-sided component and, thus, identifies $\Sigma_{a,b}$ with $\Sigma_{b-1, a+1}$. Therefore, $\Sigma_{3,1} = \Sigma_{0,4}$, $\Sigma_{2,2} = \Sigma_{1,3}$ for $\mu = 1$, and $\Sigma_{1,1} = \Sigma_{0,2}$ for $\mu = 2$. \square

To distinguish affine cubics (X, A) by counting real lines intersecting a given component of $A_{\mathbb{R}}$, we make first a count in terms of (\mathbb{P}^2, C, S) and then translate it in terms of (X, A) .

Lemma 2.3.2 *If $(C, S) \in S_{a,b}$, then the proper image $\tilde{O} \subset A_{\mathbb{R}}$ of the oval $O \subset C_{\mathbb{R}}$ is intersected by $2b + ab$ real lines of X if b is even, and by $6 - 2\mu + ab$ if b is odd.*

Proof For each of the b points $x_i \in O$, the component \tilde{O} is intersected by the line E_i and by the proper images of lines $\tilde{x}_j x_j$ for each of the a points $x_j \in S$ on the one-sided component $A_{\mathbb{R}} \setminus O$. In addition, \tilde{O} intersects b lines Q_i if b is even, or $a = 6 - 2\mu - b$ lines Q_j if b is odd. \square

Lemma 2.3.3 *Assume that $(C, S) \in \mathcal{S}_{a,b}$. Then, the proper image $\tilde{O} \subset A_{\mathbb{R}}$ of the oval $O \subset C_{\mathbb{R}}$ is an oval of $A_{\mathbb{R}}$ if and only if b is even.*

Proof This is because after blowing up at a point of a curve on a surface, the tubular neighborhood of the proper image of the curve alternates its orientability. \square

Proposition 2.3.4 *Assume $(X, A) \in \Sigma_{a,b}$. Then the oval of A intersects 0 lines if $b = 0$. If $b > 0$, then this oval intersects:*

- 12 lines for $\Sigma_{0,6} = \Sigma_{3,3} = \Sigma_{4,2}$ and 16 lines for $\Sigma_{1,5} = \Sigma_{2,4} = \Sigma_{5,1}$;
- 8 lines if $a + b = 4$;
- 4 lines if $a + b = 2$.

Proof Lemma 2.3.2 gives answer $2b + ab$ for even b , which is the required number by Lemma 2.3.3. For odd b , these Lemmas imply that the one-sided component \tilde{O} intersects $6 - 2\mu + ab$ lines and, thus, the oval of $A_{\mathbb{R}}$ intersects the remaining real lines. \square

3 Proof of the Main Theorem

3.1 Coarse Deformation Classes via Blow-up Models

Lemma 3.1.1 *If the triples $(\mathbb{P}_{\mathbb{R}}^2, C_{\mathbb{R}}^1, S_{\mathbb{R}}^1)$, $(\mathbb{P}_{\mathbb{R}}^2, C_{\mathbb{R}}^2, S_{\mathbb{R}}^2)$ are homeomorphic, then the associated real affine cubic surfaces (X^1, A^1) , (X^2, A^2) are coarse deformation equivalent.*

Proof According to Theorem 2.2.2 to prove coarse deformation equivalence of (X^1, A^1) and (X^2, A^2) , it is sufficient to build a real deformation between $(C_{\mathbb{R}}^1, S_{\mathbb{R}}^1)$ and $(C_{\mathbb{R}}^2, S_{\mathbb{R}}^2)$ using at worst one-nodal 6-tuples. Such a real deformation equivalence can be built in three steps: deforming C^1 to C^2 , transporting S^1 from C^1 to C^2 along a chosen deformation by means of a family of typical 6-tuples, and finally moving the transported S^1 to S^2 . A real deformation between C^1 and C^2 exists due to the deformation classification of real plane cubic curves. Given two typical conj-invariant 6-tuples S^1, S^2 on the same real nonsingular plane cubic curve C , we may join them by a generic path $S^t, t \in [1, 2]$, if the pairs $(C_{\mathbb{R}}, S_{\mathbb{R}}^1), (C_{\mathbb{R}}, S_{\mathbb{R}}^2)$ are homeomorphic. It may happen that at a finite number of times, the 6-tuples S^t go through a one-nodal position. But since in the associated family of real surfaces (X, A) , the topology of the real part of the cubic surfaces is not changing, and since when X acquires a node the curve A does not pass through the node (and remains nonsingular), we may conclude that at each nodal instance we touch the discriminant Δ^a without crossing it, and thus, due to the deformation classification of real nonsingular projective cubic surfaces, may bypass the family of surfaces by a small real deformation. \square

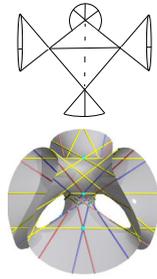


Fig. 2 Cayley's four-nodal cubic surface with a tetrahedron-like piece T (top figure) gives after a small perturbation a cubic surface X (at the bottom), with nodes replaced by one-handles. A plane truncating T near its vertex traces on $X_{\mathbb{R}}$ a cubic curve $A_{\mathbb{R}}$, whose oval is a vanishing cycle. Another plane separating one pair of vertices of T from another traces on $X_{\mathbb{R}}$ a cubic curve $A_{\mathbb{R}}$, whose oval is homologous to the sum of two disjoint vanishing cycles. In each case, the oval intersects $4n$ lines in $X_{\mathbb{R}}$, where n is the number of edges of T intersected by the plane (resp. 3 and 4)

Proposition 3.1.2 *The real transversal at infinity affine cubics (X, A) for which both $X_{\mathbb{R}}$ and $A_{\mathbb{R}}$ are connected form 4 coarse deformation classes: each class is determined by the deformation class of X .*

Proof Using Theorem 2.2.1, we interpret the claim in the language of blowup models of (X, A) , that is, triples (\mathbb{P}^2, C, S) . Since, as it follows from Theorem 2.1.1, the class of X is determined by $\mu \in \{0, 1, 2, 3\}$, for construction of A for a fixed class of X , it is enough to pick a connected cubic $A_{\mathbb{R}}$ and place on it $6 - 2\mu$ real points (and μ pairs of imaginary points on $A \setminus A_{\mathbb{R}}$). For uniqueness of the coarse deformation class of such pairs (X, A) , it is enough to apply Lemma 3.1.1. \square

Proposition 3.1.3 *The real transversal at infinity affine cubics (X, A) for which $X_{\mathbb{R}}$ is connected, while $A_{\mathbb{R}}$ is not, form eight coarse deformation classes. For all but two exceptional cases, indicated in Theorem 1.2.1, the coarse deformation class is determined by the topology of $X_{\mathbb{R}} \setminus A_{\mathbb{R}}$, while the exceptional cases are distinguished as stated in Theorem 1.2.1.*

Proof We use Theorem 2.2.1 and Lemma 3.1.1, like in the case of connected $A_{\mathbb{R}}$, and then apply Proposition 2.3.1 giving eight coarse deformation classes of (X, A) : three for $\mu = 0$, two for each of $\mu \in \{1, 2\}$ and one for $\mu = 3$. Proposition 2.3.4 gives the numbers of lines intersecting each component of $A_{\mathbb{R}}$ and, in particular, describes the two exceptional classes for $\mu = 0$ in terms of real lines, as stated in Theorem 1.2.1. To obtain the other description, in terms of Rokhlin–Guillou–Marin quadratic function, it is sufficient to observe on examples (see Fig. 2) that in the case $\Sigma_{0,6} = \Sigma_{3,3} = \Sigma_{4,2}$ (where the number of real lines intersecting the oval is 12), the oval is a vanishing cycle, and in the other case, $\Sigma_{5,1} = \Sigma_{2,4} = \Sigma_{1,5}$ (where the number of real lines intersecting the oval is 16), the oval is homologous to the sum of two disjoint vanishing cycles. Therefore, in the first case, the quadratic function takes value $2 \in \mathbb{Z}/4$ on the oval, while in the other case its value is $0 \in \mathbb{Z}/4$. \square

3.2 Affine Cubics with Disconnected $X_{\mathbb{R}}$

Lemma 3.2.1 *If a real cubic hypersurface $X_{\mathbb{R}} \subset \mathbb{P}_{\mathbb{R}}^{n+1}$ has at least one contractible n -dimensional component in $\mathbb{P}_{\mathbb{R}}^{n+1}$, then $X_{\mathbb{R}}$ has no singular points.*

Proof Let $F \subset X_{\mathbb{R}}$ be a contractible n -dimensional component. Then it decomposes $\mathbb{P}_{\mathbb{R}}^{n+1}$ into ≥ 2 connected components so that at least one of them is contractible in $\mathbb{P}_{\mathbb{R}}^{n+1}$. Pick a point inside this contractible component. Each real line passing through this point intersects F at an even number ≥ 2 of real points, and in addition the other component of $X_{\mathbb{R}}$ at ≥ 1 points. So, existence of a singular point contradicts the Bézout theorem. \square

Proposition 3.2.2 *The real transversal at infinity affine cubic surfaces (X, A) with disconnected $X_{\mathbb{R}}$ form three deformation classes. One is formed by cubics with connected $A_{\mathbb{R}}$ and two with disconnected. For cubics of one of the latter two classes, the oval of $A_{\mathbb{R}}$ is contained in the spherical component of $X_{\mathbb{R}}$, while for the other class it is contained in the non-spherical component.*

Proof We start with the case in which the spherical component $F \subset X_{\mathbb{R}}$ contains an oval $\Gamma = F \cap \mathbb{P}_{\mathbb{R}}^2$ of $A_{\mathbb{R}} = X_{\mathbb{R}} \cap \mathbb{P}_{\mathbb{R}}^2$. Then, we select as a standard model a pair (X', A') obtained by a small perturbation of the product of a sphere with a plane so that the spherical component F' of $X'_{\mathbb{R}}$ and the oval component of the curve $A'_{\mathbb{R}}$ are contained inside F and, respectively, Γ , while the non-spherical component of $X'_{\mathbb{R}}$ lies outside F . Next, we pick cubic equations f for $X_{\mathbb{R}}$ and g for $X'_{\mathbb{R}}$ so that $fg > 0$ is inside F' . Then, by Bolzano intermediate value theorem, the cubic surface $X_{\mathbb{R}}(t)$ defined by $tf + (1 - t)g$ has for any $t \in [0, 1]$ at least one two-dimensional component contained inside F . Similarly, $A_{\mathbb{R}}(t) = X_{\mathbb{R}}(t) \cap \mathbb{P}_{\mathbb{R}}^2$ has at least one one-dimensional component inside Γ . These components have to be contractible and so, by Lemma 3.2.1, the pairs $(X_{\mathbb{R}}(t), A_{\mathbb{R}}(t))$ are nonsingular for all $t \in [0, 1]$. Thus, $(X_{\mathbb{R}}, A_{\mathbb{R}})$ and $(X'_{\mathbb{R}}, A'_{\mathbb{R}})$ are deformation equivalent, and it remains to be noticed that all the standard models (X', A') are deformation equivalent to each other.

In the other two cases, we pick as a standard model a pair (X', A') so that X' is defined by $g = wq + \epsilon f_{\infty}$, $0 < |\epsilon| \ll 1$, where $\mathbb{P}^2 = \{w = 0\}$, $f_{\infty} = f(x, y, z, 0)$ (f still defines X), and $q = 0$ defines a small sphere contained inside F and chosen so that $fg > 0$ inside F' . By the same arguments as above, the cubic surfaces $X(t)$ defined by $tf + (1 - t)g = 0$ are all nonsingular. In addition, $A(t) = A = A'$ for each t . So, it remains to be noticed that all the standard models $(X'_{\mathbb{R}}, A'_{\mathbb{R}})$ with homeomorphic $A'_{\mathbb{R}}$ are deformation equivalent, since all real nonsingular cubic curves with the same topology are deformation equivalent. \square

3.3 Achirality and End of Proof of Theorem 1.2.1

To finish the proof of Theorem 1.2.1 there remains to check achirality in each of coarse deformation classes.

Proposition 3.3.1 *Each coarse deformation class of real affine surface (X, A) contains a surface invariant under some real affine reflection.*

Proof The Cayley four-nodal real cubic surface $\sum_{i=0}^4 x_i^3 = \frac{1}{4}(\sum_{i=0}^3 x_i)^3$ is invariant under the projective transformations induced by permutations of the coordinates x_0, \dots, x_3 . In particular, it is invariant under six reflections induced by the $6 = \binom{4}{2}$ transpositions. So, we may obtain examples of achiral real affine surfaces just by choosing one of these reflections and considering a reflection invariant real perturbation of the Cayley surface and a reflection invariant real projective plane as the plane at infinity. It is then straightforward to construct in such a way a representative for each of our 15 coarse deformation classes. (In particular, the both plane sections indicated in caption to Fig. 2 can be chosen invariant with respect to a reflection fixing two of four vertices of the tetrahedron and permuting the two others.)

Recall that for all but two exceptional cases, a coarse deformation class is determined by simple topological properties of the pair $(X_{\mathbb{R}}, A_{\mathbb{R}})$. As for the two exceptional cases, already the construction indicated in the proof of Proposition 3.1.3 provides reflection invariant examples. \square

3.4 Proof of Theorem 1.3.1

The forgetful map $(X, A) \mapsto X$ induces a projection $\Gamma^{\text{af}} \rightarrow \Gamma$ of the wall-crossing graph Γ^{af} of real transversal at infinity affine cubic surfaces to the wall-crossing graph Γ of projective real nonsingular cubic surfaces that is shown in Theorem 2.1.1. For the connected component of Γ^{af} that describes adjacency for deformation classes of affine cubics (X, A) with connected $A_{\mathbb{R}}$, this projection is an isomorphism, as it follows from its bijectivity at the level of vertices established in Proposition 3.1.2.

For the remaining part of Γ^{af} (that describes adjacency in the cases with two-component $A_{\mathbb{R}}$), the set of vertices is described in Propositions 3.1.3 and 3.2.2. Labels of the vertices are determined by Proposition 2.3.4. To determine the edges and their labels, we use the following observation: if under wall-crossing the Euler characteristic of $X_{\mathbb{R}}$ increases, then the number of real lines decreases by $0 \leq 2k \leq 12$ (where k is the number of real lines passing through the node at the instance of wall-crossing) and, in particular, each of the labels a and b decreases. Existence of the edges with labels (a, b) such that $a, b \in \{0, 2, 4, 6\}$, $a + b \leq 6$ follows from Theorem 2.2.2 and known examples of transversal pairs of conic and cubic, see [3, Section 9.2].

4 Edges of the Wall-Crossing Graph

4.1 Extended Walls

Let us assume that an affine cubic $X \setminus A$, $A = X \cap \mathbb{P}^3$, has a double point $x \in X \setminus A$ and consider affine coordinates centered at x . Then, such an affine cubic is defined by equation $f_2 + f_3 = 0$, where f_2 and f_3 are nonzero ternary quadratic and cubic forms: f_3 defines the (cubic) curve $A \subset \mathbb{P}^2$ and f_2 a conic.

The following criterion is straightforward from definitions (*c.f.*, for example, [2, Lemma 2.2]).

Lemma 4.1.1 *An affine cubic surface $\{f_2 + f_3 = 0\}$ is transversal at infinity and has no singular point except the node or cusp at the coordinate origin, if and only if $\{f_3 = 0\} \subset \mathbb{P}^2$ is a nonsingular cubic transversal to the conic $\{f_2 = 0\} \subset \mathbb{P}^2$.*

Proposition 4.1.2 *Pick any real nonsingular plane cubic curve C and any label: “ a ” if $C_{\mathbb{R}}$ is connected, or “ a, b ” if it has two components. Then the above construction gives a bijection between the set of extended walls representing the edges of graphs on Fig. 1 with the chosen label and the connected components of the space of real conics B intersecting transversely cubic C , with the number of intersection points on each component of $C_{\mathbb{R}}$ prescribed by the label.*

Proof It is a straightforward consequence of Lemma 4.1.1, Theorem 1.3.1, and connectedness of the group of real affine transformations preserving orientation. \square

Remark 4.1.3 The only label marking more than one edge on Fig. 1 is “0, 0”.

4.2 Auxiliary Covering

Let C be a nonsingular plane cubic curve. Denote by $C^{(n)}$ its n -fold symmetric power and by $\dot{C}^{(n)}$ its Zariski open subset formed by collections of n distinct points. For any effective divisor of degree 5 on C , which we consider as a point $D \in C^{(5)}$, there is a unique conic B that cuts on C a divisor $B \cdot C \geq D$. This correspondence, $D \mapsto B \cdot C \in C^{(6)}$, defines a regular map $f : C^{(5)} \rightarrow C^{(6)}$ whose image $V = f(C^{(5)}) \subset C^{(6)}$ is a codimension 1 smooth subvariety of $C^{(6)}$ (in fact, this image is isomorphic to \mathbb{P}^5 due to its identification with the space of plane conics).

In accordance with Proposition 4.1.2, we are especially interested in a Zariski open subset of V defined as $\dot{V} = f(C^{(5)}) \cap \dot{C}^{(6)} = f(\dot{C}^{(5)}) \cap \dot{C}^{(6)}$.

If the cubic curve C is real, the (open) manifold $\dot{C}_{\mathbb{R}}^{(n)} = \{D \in \dot{C}^{(n)} : \text{conj } D = D\}$ splits into several connected components enumerated by the number of points in $D \in \dot{C}_{\mathbb{R}}^{(n)}$ on each of the connected components of $C_{\mathbb{R}}$. Namely, we have a partition

$$\dot{C}_{\mathbb{R}}^{(n)} = \begin{cases} \bigsqcup_{r \geq 0, r = n \bmod 2} \dot{C}_{\mathbb{R}}^r, & \text{if } C_{\mathbb{R}} \text{ is connected,} \\ \bigsqcup_{r, s \geq 0, r + s = n \bmod 2} \dot{C}_{\mathbb{R}}^{r, s} & \text{if } C_{\mathbb{R}} \text{ has two components,} \end{cases}$$

where r stands for the number of points of $D \in \dot{C}_{\mathbb{R}}^{(n)}$ on the one-sided component of $C_{\mathbb{R}}$ and s for that on the oval.

In the case $n = 6$, for each $a = 0 \pmod 2, a \geq 0$ and, respectively, each $a + b = 0 \pmod 2$ with $a, b \geq 0$, we put

$$\dot{V}_{\mathbb{R}}^a = V \cap \dot{C}_{\mathbb{R}}^a, \quad \dot{V}_{\mathbb{R}}^{a,b} = V \cap \dot{C}_{\mathbb{R}}^{a,b}.$$

The above map f respects this partition. It restricts to a collection of maps

$$\begin{cases} \dot{f} : \dot{C}_{\mathbb{R}}^r \rightarrow V_{\mathbb{R}}, & \text{if } C_{\mathbb{R}} \text{ is connected,} \\ \dot{f} : \dot{C}_{\mathbb{R}}^{r,s} \rightarrow V_{\mathbb{R}}, & \text{if } C_{\mathbb{R}} \text{ has two components.} \end{cases}$$

Note that $\dot{f}(\dot{C}_{\mathbb{R}}^r)$ and $\dot{f}(\dot{C}_{\mathbb{R}}^{r,s})$ are contained in the closure of $\dot{V}_{\mathbb{R}}^{r+1}$ and, respectively, $\dot{V}_{\mathbb{R}}^{\bar{r},\bar{s}}$, where for $x = r, s$ we let $\bar{x} := x + 1$ if x is odd and $\bar{x} = x$ if even.

Lemma 4.2.1 *Each restriction map $\dot{f} : \dot{C}_{\mathbb{R}}^r \rightarrow V_{\mathbb{R}}$ or $\dot{f} : \dot{C}_{\mathbb{R}}^{r,s} \rightarrow V_{\mathbb{R}}$ is smooth with only fold singular points. Namely, \dot{f} is a local diffeomorphism at a point D if $\dot{f}(D) \in \dot{V}_{\mathbb{R}}$, while for $\dot{f}(D) \in V_{\mathbb{R}} \setminus \dot{V}_{\mathbb{R}}$ the map \dot{f} is given by $\dot{f}(x_1, \dots, x_4, y) = (x_1, \dots, x_4, y^2)$ in appropriate local coordinates of the domain and codomain. Moreover, $\dot{f}^{-1}(\dot{f}(D)) = D$ for every fold point D (a point with $\dot{f}(D) \in V_{\mathbb{R}} \setminus \dot{V}_{\mathbb{R}}$).*

Proof Straightforward from the Abel–Jacobi theorem, implying that for each 5-tuple $D = \{p_1, \dots, p_5\}$ in the domain of \dot{f} the 6th point in the 6-tuple $\dot{f}(D) = \{p_1, \dots, p_6\}$ is determined by the condition $p_1 + \dots + p_6 = 0$ with respect to the group structure in C with a flex-point of C taken for zero. \square

The following result is an immediate consequence of Lemma 4.2.1.

Proposition 4.2.2 *Each image $\dot{f}(\dot{C}_{\mathbb{R}}^r)$, $\dot{f}(\dot{C}_{\mathbb{R}}^{r,s})$ is a manifold with the boundary formed by fold-points of \dot{f} and the interior part being $\dot{V}_{\mathbb{R}}^{r+1}$, $\dot{V}_{\mathbb{R}}^{\bar{r},\bar{s}}$, respectively.*

4.3 Proof of Theorem 1.3.2

Let us treat, first, the case of edges with labels a and a, b different from 0 and 0, 0. In such a case, according to Proposition 4.1.2, it is sufficient to show that each of the manifolds $\dot{V}_{\mathbb{R}}^a$ and $\dot{V}_{\mathbb{R}}^{a,b}$ is connected.

To prove their connectedness, we put $a = r + 1$ or $a, b = \bar{r}, \bar{s}$, respectively, and deduce the connectedness of $\dot{f}(\dot{C}_{\mathbb{R}}^r)$ and $\dot{f}(\dot{C}_{\mathbb{R}}^{r,s})$ from the connectedness of $\dot{C}_{\mathbb{R}}^r$ and $\dot{C}_{\mathbb{R}}^{r,s}$ (already observed in Sect. 4.2). By virtue of Proposition 4.2.2, this implies the connectedness of $\dot{V}_{\mathbb{R}}^{r+1} = \dot{V}_{\mathbb{R}}^a$ and $\dot{V}_{\mathbb{R}}^{\bar{r},\bar{s}} = \dot{V}_{\mathbb{R}}^{a,b}$.

For the case of labels 0 or 0, 0, we observe that the set of real quadratic forms taking a fixed sign on each component of $C_{\mathbb{R}}$ is convex and that non-transversality with C is a codimension 2 condition (since it happens simultaneously at pairs of complex-conjugate points). Due to Proposition 4.1.2, this gives one extended wall for connected $C_{\mathbb{R}}$ and two walls for disconnected. This corresponds to one edge labeled with 0 and, respectively, two edges labeled 0, 0 on Fig. 1.

5 Concluding Remarks

5.1 Intersecting a Cubic by Curves of Arbitrary Degree

Our proof of Theorem 1.3.2 given in Sect. 4 is based on enumeration of connected components of the space of real conics intersecting transversally a fixed nonsingular real cubic $C \subset \mathbb{P}^2$. If we replace conics by curves of arbitrary degree $d \geq 1$, we may similarly define a map $f : C^{(3d-1)} \rightarrow C^{(3d)}$, let $V = f(C^{(3d-1)})$, consider a similar partition of $\dot{V}_{\mathbb{R}}$ into \dot{V}^a or $\dot{V}^{a,b}$, and prove the connectedness of $\dot{V}^{a,b}$ for $(a, b) \neq (0, 0)$ and of \dot{V}^a for any a .

The case $(a, b) = (0, 0)$ is possible only if d is even. Then the same sign and convexity arguments as for $d = 2$ apply and show that for any even d the space $\dot{V}^{0,0}$, that is the space of real nonsingular plane curves of degree d transversal to C and having no real points common with C , consists of two connected components. The latter ones are distinguished by comparison of the signs that an underlying real form of degree d takes on the real components of C .

5.2 Number of Ordinary Walls in an Extended Wall

Recall that by definition (see Sect. 1) the “ordinary” walls are the connected components of $\Delta_{\mathbb{R}} \setminus \Delta'_{\mathbb{R}}$, while the extended walls are the connected components of a bigger space obtained by adding the cuspidal strata. As it is natural to expect, there exist extended walls that contain more than one ordinary wall. A complete answer is given in Table 2 that indicates the precise number of ordinary walls in each of the extended walls. The latter ones are marked with the labels as in the previous section (like the corresponding edges on Fig. 1). Note that label 0, 0 is attributed to two different extended walls, so, “1” under “0, 0” says that each of them contains just one ordinary wall.

In fact, the set of ordinary walls is in a natural bijective correspondence with the set of deformation classes of real plane quintics that split into a nonsingular cubic C and nonsingular conic B meeting each other transversally (cf. Lemma 4.1.1). Therefore, the results shown in the table can be obtained from [3, Section 9.2], where the latter classification was presented as follows.

Table 2 Number of ordinary walls in each extended wall

Extended wall label	0	2	4	6	0, 0	0, 2	2, 0	0, 4	2, 2	4, 0	0, 6	2, 4	4, 2	6, 0
Number of walls	1	1	1	3	1	1	2	1	1	2	1	2	1	3

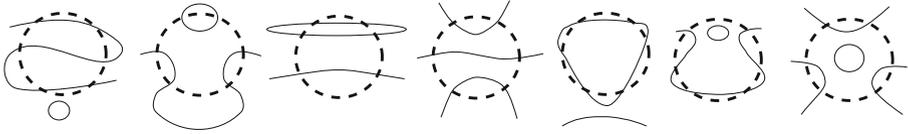


Fig. 3 The extremal mutual positions of a cubic and a conic

On one hand, each deformation class is determined by the topology of the mutual arrangement of $C_{\mathbb{R}}$ and $B_{\mathbb{R}}$ in the real projective plane. On the other hand, as it was shown by G. Polotovskiy [5], there are 25 such arrangements: 7 extremal classes are shown in Fig. 3, while the other, non-extremal, cases are obtained from the extremal ones by repeating the following moves. One move consists in erasing an oval (of the cubic or of the conic) containing no intersection points, and the other in shifting a piece of curve containing a pair of consecutive (both on the conic and the cubic) intersection points, so that these two intersections disappear.

5.3 Vasilliev's Conjectures

The deformation classification in this paper is naturally related to enumeration of connected components of non-discriminant perturbations of parabolic singularities of type P_8 . In a recent preprint [7], Vasilliev provided lower bounds on the number of such components for each of the eight classes of parabolic singularities and conjectured that his bounds are sharp. Our results confirm partially this conjecture.

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