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RESEARCH CONTRIBUTION

Supertraces on Queerified Algebras

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Abstract
We describe supertraces on "queerifications" (see arXiv:2203.06917) of the algebras of matrices of "complex size", algebras of observables of Calogero–Moser model, Vasiliev higher spin algebras, and (super)algebras of pseudo-differential operators. In the latter case, the supertraces establish complete integrability of the analogs of Euler equations to be written (this is one of several open problems and conjectures offered).

Keywords Simple Lie superalgebra · Queerification · Trace · Supertrace

Mathematics Subject Classification Primary 17B20 · 16W55; Secondary 81Q60 · 17B70

1 Introduction

The goal of this note is to give a list of supertraces in a quite general new situation (Sects. 2–5); to make the exposition self-contained we remind certain known results (Sect. 6 and partly Sects. 4, 5).

The traces on Lie algebras, and even (here: not odd) supertraces on Lie superalgebras, are known to be very useful, for example, in representation theory, see, e.g., [6] and references therein. The *odd* supertraces are less known, and hence less popular. In Sect. 5, we show one of their usages not previously explored: application to the study of integrability of certain dynamical systems.

Inspired by [4], where Lie algebras are queerified over an algebraically closed ground field of characteristic $p = 2$ to produce the complete list of simple finite-

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dimensional Lie superalgebras in characteristic $p = 2$, this new method—Lie queerification—producing many new simple Lie superalgebras from associative algebras and superalgebras over $\mathbb{C}$ is applied in [15] to several infinite-dimensional algebras of interest in theoretical physics. A number of papers were devoted to the description of traces on these algebras, and supertraces on these algebras considered as superalgebras, see [10–13]. In this note, we describe the supertraces on the Lie queerifications of these algebras and superalgebras, having added one more type of example.

2 Preliminaries

2.1 From Associative to Lie

Let $\mathbb{K}$ be an algebraically closed ground field of characteristic $p \neq 2$; unless otherwise stated, we consider $\mathbb{K} = \mathbb{C}$.

Let A be any associative algebra, and let A^L be the Lie algebra whose space is A but multiplication is given by the commutator $[a, b] := ab - ba$ for any $a, b \in A$.

Let $A := A_{\bar{0}} \oplus A_{\bar{1}}$ be a $\mathbb{Z}/2$-graded algebra; let p denote the parity function: $p(a) = i \in \mathbb{Z}/2$ for any non-zero $a \in A_i$. If A a $\mathbb{Z}/2$-graded associative algebra, let A^S be the Lie superalgebra whose space is A but the multiplication is given by the supercommutator, which by the modern habitual abuse of notation is also denoted $[\cdot, \cdot]$, although defined differently, namely as

$$[a, b] := ab - (-1)^{p(a)p(b)} ba \text{ for any homogeneous } a, b \in A$$

and extended to inhomogeneous elements via linearity. Let $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ be the first derived Lie algebra (resp. Lie superalgebra), a.k.a. commutant (resp. supercommutant), of the Lie algebra (resp. Lie superalgebra) $\mathfrak{g}$.

Not every $\mathbb{Z}/2$-graded algebra A is called *superalgebra*: only if multiplication in A or—if A is associative—in A^S depends on the parity. Thus, a liefication A^L of a $\mathbb{Z}/2$-graded algebra A is a Lie algebra, whereas a liefication A^S of a $\mathbb{Z}/2$-graded superalgebra A is a Lie superalgebra (satisfying axioms of anti-commutativity and Jacobi identity with signs depending on parity).

Recall that a *trace* (called *supertrace* in the super setting, for emphasis) on a given Lie algebra (resp. Lie superalgebra) $\mathfrak{g}$ is a linear function that vanishes on its commutant (resp. supercommutant), so there are $\dim(\mathfrak{g}/\mathfrak{g}')$ linearly independent traces (resp. supertraces) on $\mathfrak{g}$; some of the supertraces can be even and some of them odd.

2.2 Queerifications in Characteristic $p \neq 2$ (from [4])

Let A be an associative algebra. The space of the associative algebra $\mathrm{Q}(A)$—the ***associative*** *queerification* of A—is $A \oplus \Pi(A)$, where Π is the change of parity functor, with the same multiplication in A; let the left action of A on $\Pi(A)$, considered as a copy of A, and multiplication in $\Pi(A)$ be

$$x\Pi(y) := \Pi(xy), \quad \Pi(x)y := \Pi(xy),$$
$$\Pi(x)\Pi(y) := xy \text{ for any } x, y \in A.$$

Set $\mathrm{Q}(n) := \mathrm{Q}(\mathrm{Mat}(n))$.

We will be mostly interested in the following: ***Lie*** *queerifications* $\mathfrak{q}(A)$:

1. when A is an associative algebra, “Liefication" yields Lie algebra A^L;
2. when A is a $\mathbb{Z}/2$-graded associative algebra, “Liefication” yields $\mathbb{Z}/2$-graded Lie algebra A^L;
3. when A is an associative superalgebra (this case differs from case 2) because passing to A^S the supercommutators instead of commutators are considered), “super Liefication" yields Lie superalgebra A^S which is $\mathbb{Z}/2$-graded by parity.

In Case (1), as spaces, $\mathfrak{q}(A) := A^L \oplus \Pi(A)$, so $\mathfrak{q}(A)_{\bar 0} = A^L$ and $\mathfrak{q}(A)_{\bar 1} = \Pi(A)$, with the bracket given by the following expressions and super anti-symmetry, i.e., anti-symmetry amended by the Sign Rule:

$$\begin{aligned}[][x, y] := xy - yx; \quad [x, \Pi(y)] := \Pi(xy - yx); \quad [\Pi(x), \Pi(y)]\\ := xy + yx \ \text{ for any } x, y \in A.\end{aligned} \tag{1}$$

The term “queer”, now conventional, is taken after the Lie superalgebra $\mathfrak{q}(n) := \mathfrak{q}(\mathrm{Mat}(n))$. (The associative superalgebra $\mathrm{Q}(n)$ is an analog of $\mathrm{Mat}(n)$; likewise, the Lie superalgebra $\mathfrak{q}(n)$ is an analog of $\mathfrak{gl}(n)$ for several reasons, for example, due the role of these two types of analogs in Schur’s Lemma and in the classification of central simple superalgebras, see [14, Ch.7].) We express the elements of the Lie superalgebra $\mathfrak{q}(n)$ by means of a pair of matrices:

$$(X, Y) \longleftrightarrow \begin{pmatrix} X & Y \\ Y & X \end{pmatrix} \in \mathfrak{gl}(n|n), \text{ where } X, Y \in \mathrm{Mat}(n). \tag{2}$$

For any associative A, we will similarly denote the elements of $\mathfrak{q}(A)$ by pairs (X, Y), where $X, Y \in A$. The brackets between these elements are as follows:

$$\begin{aligned}&[(X_1, 0), (X_2, 0)] := ([X_1, X_2], 0), \quad [(X, 0), (0, Y)] := (0, [X, Y]),\\ &[(0, Y_1), (0, Y_2)] := (Y_1Y_2 + Y_2Y_1, 0).\end{aligned} \tag{3}$$

We define Lie queerifications in cases (2) and (3) in the next section.

3 Traces and Supertraces: Generalities

3.1 Theorem *Let A be an associative algebra. Then, the estimate of the number of traces in the three cases of its Lie queerification are as follows:*

Case 1. *Let $Q(A) := A \oplus \Pi(A)$ be the associative queerification of A, let $\mathfrak{g} := A^L$ be the corresponding Lie algebra and let the Lie superalgebra $\mathfrak{qg} := (Q(A))^S = \mathfrak{q}(A)$ be the Lie queerification of A. Then, there are as many odd supertraces on $\mathfrak{qg}$ as there are traces on $\mathfrak{g}$; there are fewer even supertraces on $\mathfrak{qg}$ than there are traces on $\mathfrak{g}$. In particular, if A has unit, then there are no even supertraces on $\mathfrak{qg}$.*
Case 2. *Let A be a $\mathbb{Z}/2$-graded associative algebra, $\mathfrak{g} := A^L$. Let $i, j = \bar{0}, \bar{1}$, let n_i be the number of traces on $\mathfrak{g}$ of grade i, and let $N_{i,j}$ be the number of supertraces on $\mathfrak{qg}$ of grade (i, j). Then,*

$$N_{\bar{1},\bar{0}}=n_{\bar{0}}, \quad N_{\bar{1},\bar{1}} = n_{\bar{1}}, \quad N_{\bar{0},\bar{0}}=\mathrm{codim}_{A_{\bar{0}}}((A_{\bar{0}})^2+(A_{\bar{1}})^2), \quad N_{\bar{0},\bar{1}}=\mathrm{codim}_{A_{\bar{1}}}(A_{\bar{0}}A_{\bar{1}}). \tag{4}$$

In particular, if A has unit, then there are no supertraces on $\mathfrak{qg}$ of grades $(\bar{0}, \bar{0})$ and $(\bar{0}, \bar{1})$.
Case 3. *Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be an associative superalgebra, $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} := A^S$. Let $Q(A) := A \oplus \Pi(A)$ and $\mathfrak{qg} := (Q(A))^S$. In notation of Case* 2*),*

$$N_{\bar{0},\bar{1}} = n_{\bar{1}}, \quad N_{\bar{0},\bar{0}} = \mathrm{codim}_{A_{\bar{0}}}((A_{\bar{0}})^2 + (A_{\bar{1}})^2), \quad N_{\bar{1},\bar{1}} = \mathrm{codim}_{A_{\bar{1}}}(A_{\bar{0}}A_{\bar{1}}). \tag{5}$$

In particular, if A has unit, then there are no even supertraces on $\mathfrak{qg}$, i.e., supertraces of grades $(\bar{0}, \bar{0})$ and $(\bar{1}, \bar{1})$.

Proof **Case 1**. Clearly, $\mathfrak{qg} = \mathfrak{g} \oplus \Pi(\mathfrak{g})$ as spaces.

Denote, for brevity, $\mathfrak{u} := (\mathfrak{qg})'$. By definition, the supercommutant $\mathfrak{u} := \mathfrak{u}_{\bar{0}} \oplus \mathfrak{u}_{\bar{1}}$ is spanned by elements $[a, b]$ for any $a, b \in \mathfrak{qg}$. In particular,

$$\mathfrak{u}_{\bar{1}} := \mathrm{Span}([x, \Pi(y)] = \Pi([x, y]) \mid x, y \in (\mathfrak{qg})_{\bar{0}}),$$

and hence $\mathfrak{u}_{\bar{1}} = \Pi([\mathfrak{g}, \mathfrak{g}])$. Therefore, there are as many odd supertraces on $\mathfrak{qg}$ as there are traces on $\mathfrak{g}$.

Clearly, $\mathfrak{u}_{\bar{0}}$ is the sum of two ideals of $\mathfrak{g}$:

$$\mathfrak{u}_{\bar{0}} = [\mathfrak{g}, \mathfrak{g}] + [\Pi(\mathfrak{g}), \Pi(\mathfrak{g})].$$

Observe that the second summand does not have to be contained in the first one. Therefore, there are fewer even supertraces on $\mathfrak{qg}$ than there are traces on $\mathfrak{g}$.

In particular, if A has unit $\mathbb{1}$, then $\mathfrak{u}_{\bar{0}} = \mathfrak{g}$, because $[\Pi(\mathbb{1}), \Pi(x)] = 2x$ for any $x \in \mathfrak{g}_{\bar{0}}$.

For example, if $A = \mathrm{Mat}(n)$, then on $(Q(A))^S$, there is an odd trace, nowadays called *queertrace*; it was first defined in [2] by the formula

$$\mathrm{qtr} : \begin{pmatrix} X & Y \\ Y & X \end{pmatrix} \mapsto \mathrm{tr}\, Y.$$

Case 2. Clearly, $\mathbb{Z}/2$-grading of A makes $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ a $\mathbb{Z}/2$-graded Lie algebra.

Actually, this is a particular case of Case 1. However, a $\mathbb{Z}/2$-grading in the Lie algebra $\mathfrak{g}$ and hence a $\mathbb{Z}/2\times\mathbb{Z}/2$-bigrading in the Lie superalgebra $\mathfrak{qg} := (Q(A))^S$ enable us to sharpen the answer.

We will denote the elements of $\mathfrak{g}_{\bar 0}$ by letters $x, y, \dots$, and the elements of $\mathfrak{g}_{\bar 1}$ by letters $a, b, c, \dots$. We have $\mathfrak{g}' = (\mathfrak{g}')_{\bar 0} \oplus (\mathfrak{g}')_{\bar 1}$, where

$$\begin{aligned}(\mathfrak{g}')_{\bar 0} &= \mathrm{Span}([x,y] := xy - yx, \;\; [a,b] := ab - ba), \;\; (\mathfrak{g}')_{\bar 1}\\ &= \mathrm{Span}([x,a] := xa - ax).\end{aligned}$$

The components of $(\mathbb{Z}/2\times\mathbb{Z}/2)$-grading of $\mathfrak{qg}$ are of the form:

$$(q\mathfrak{g})_{\bar 0,\bar 0} = \mathfrak{g}_{\bar 0},\;\; (\mathfrak{qg})_{\bar 0,\bar 1} = \mathfrak{g}_{\bar 1},\;\; (\mathfrak{qg})_{\bar 1,\bar 0} = \Pi(\mathfrak{g}_{\bar 0}),\;\; (\mathfrak{qg})_{\bar 1,\bar 1} = \Pi(\mathfrak{g}_{\bar 1}).$$

The elements that span homogeneous components of $(\mathfrak{qg})'$ are as follows:

$(\mathfrak{qg})'_{\bar 0,\bar 0}$	$(\mathfrak{qg})'_{\bar 0,\bar 1}$	$(\mathfrak{qg})'_{\bar 1,\bar 0}$	$(\mathfrak{qg})'_{\bar 1,\bar 1}$
$[x,y] = xy - yx$	$[x,a]$	$[x,\Pi(y)]$	$[x,\Pi(a)]$
$[a,b] = ab - ba$	$= xa - ax$	$= \Pi(xy - yx)$	$= \Pi(xa - ax)$
$[\Pi(a),\Pi(b)] = ab + ba$	$[\Pi(a),\Pi(x)]$	$[a,\Pi(b)]$	$[a,\Pi(x)]$
$[\Pi(x),\Pi(y)] = xy + yx$	$= ax + xa$	$= \Pi(ab - ba)$	$= \Pi(ax - xa)$

Therefore,

$$(\mathfrak{qg})'_{\bar 1,\bar 0} = \Pi(\mathfrak{g})'_{\bar 0},\quad (\mathfrak{qg})'_{\bar 1,\bar 1} = \Pi(\mathfrak{g})'_{\bar 1},\quad (\mathfrak{qg})'_{\bar 0,\bar 0} = (A_{\bar 0})^2 + (A_{\bar 1})^2,\quad (\mathfrak{qg})'_{\bar 0,\bar 1} = A_{\bar 0}A_{\bar 1},$$

where the last two equalities mean equalities as spaces.

We see that equalities (4) are satisfied. In particular, if A has unit, then

$$(\mathfrak{qg})'_{\bar 0,\bar 0} = \mathfrak{g}_{\bar 0} \text{ and } (\mathfrak{qg})'_{\bar 0,\bar 1} = \mathfrak{g}_{\bar 1}.$$

Hence, in this case, there are no supertraces on $\mathfrak{qg}$ of grades $(\bar 0,\bar 0)$ and $(\bar 0,\bar 1)$.

Case 3. Clearly, $\mathfrak{qg} = \mathfrak{g}\oplus\Pi(\mathfrak{g})$ as superspaces. It is also clear that the natural $(\mathbb{Z}/2\times\mathbb{Z}/2)$-grading on $Q(A)$ induces the same grading on $\mathfrak{qg}$, where

$$(\mathfrak{qg})_{\bar 0,\bar 0} = \mathfrak{g}_{\bar 0},\;\; (\mathfrak{qg})_{\bar 0,\bar 1} = \mathfrak{g}_{\bar 1},\;\; (\mathfrak{qg})_{\bar 1,\bar 0} = \Pi(\mathfrak{g}_{\bar 0}),\;\; (\mathfrak{qg})_{\bar 1,\bar 1} = \Pi(\mathfrak{g}_{\bar 1}).$$

Let us now compare the supercommutants of $\mathfrak{g}$ and $\mathfrak{qg}$. We will denote the elements of $\mathfrak{g}_{\bar 0}$ by letters $x, y, \dots$, and the elements of $\mathfrak{g}_{\bar 1}$ by letters $a, b, c, \dots$. We have $\mathfrak{g}' = (\mathfrak{g}')_{\bar 0}\oplus(\mathfrak{g}')_{\bar 1}$, where

$$\begin{aligned}(\mathfrak{g}')_{\bar 0} &= \mathrm{Span}([x,y] := xy - yx, \;\; [a,b] := ab + ba),\\ (\mathfrak{g}')_{\bar 1} &= \mathrm{Span}([x,a] := xa - ax).\end{aligned}$$

$(\mathfrak{qg})'_{\bar 0,\bar 0}$	$(\mathfrak{qg})'_{\bar 0,\bar 1}$	$(\mathfrak{qg})'_{\bar 1,\bar 0}$	$(\mathfrak{qg})'_{\bar 1,\bar 1}$
$[x,y]=xy-yx$ $[a,b]=ab+ba$ $[\Pi(a),\Pi(b)]=ab-ba$ $[\Pi(x),\Pi(y)]=xy+yx$	$[x,a]$ $=xa-ax$ $[\Pi(a),\Pi(x)]$ $=ax-xa$	$[x,\Pi(y)]$ $=\Pi(xy-yx)$ $[a,\Pi(b)]$ $=\Pi(ab-ba)$	$[x,\Pi(a)]$ $=\Pi(xa-ax)$ $[a,\Pi(x)]$ $=\Pi(ax+xa)$

The elements that span homogeneous components of $(\mathfrak{qg})'$ are as follows: Therefore,

$$(\mathfrak{qg})'_{\bar 0,\bar 1}=(\mathfrak{g})'_{\bar 1},\quad (\mathfrak{qg})'_{\bar 0,\bar 0}=(A_{\bar 0})^2+(A_{\bar 1})^2,\quad (\mathfrak{qg})'_{\bar 1,\bar 1}=\Pi(A_{\bar 0}A_{\bar 1}),$$

where the last two equalities mean equalities as spaces.

We see that equalities (5) are satisfied. In particular, if A has unit, then

$$(\mathfrak{qg})'_{\bar 0,\bar 0}=\mathfrak{g}_{\bar 0}\text{ and }(\mathfrak{qg})'_{\bar 1,\bar 1}=\Pi(\mathfrak{g}_{\bar 1}).$$

Hence, in this case, there are no even supertraces on $\mathfrak{qg}$, i.e., supertraces of grades $(\bar 0,\bar 0)$ and $(\bar 1,\bar 1)$. □

4 Examples of Supertraces on Queerified Algebras and Superalgebras

4.1 Clifford–Weyl Algebras and Superalgebras

Among various definitions of the Weyl and Clifford algebras, we select their description as associative (super)algebras of differential operators with polynomial coefficients on the $2n|m$-dimensional superspace with coordinates $u:=(x,\xi)$ generated by the u_i and $\frac{\partial}{\partial u_i}$ subject to the relations $[\frac{\partial}{\partial u_i},u_j]=\delta_{ij}$.

Recall that the *Clifford* algebra $\mathrm{Cliff}(2m)$ on $2m$ generators can be considered as a $\mathbb{Z}/2$-graded associative superalgebra generated by the anti-commuting elements ξ_i and $\frac{\partial}{\partial\xi_i}$, which is natural to consider as a superalgebra with the ξ_i, and hence $\frac{\partial}{\partial\xi_i}$, odd.

The *Clifford* algebra $\mathrm{Cliff}(2m-1)$ is defined as the algebra preserving an element $J\in\mathrm{Cliff}(2m)$ such that $J^2=a\,\mathrm{id}$ for any fixed $a\in\mathbb{C}^\times$. For example, one can take $J=\sqrt{-1}(\xi_1+\frac{\partial}{\partial\xi_1})$, then $J^2=-1$.

Clearly, by a linear change of indeterminates, the *Clifford* algebra $\mathrm{Cliff}(m)$ can be given for any m by relations $\theta_i^2=1$ for $i=1,\dots m$ in terms of the new indeterminates θ_i.

The Weyl algebra W_n of polynomial differential operators in n even indeterminates x_i is an associative algebra generated by n commuting indeterminates x_i and the corresponding $\partial_i:=\frac{\partial}{\partial x_i}$. More generally, define the Clifford–Weyl superalgebra $\mathrm{CW}(n|m):=W_n\otimes\mathrm{Cliff}(m)$.

4.1 Theorem *Let A be a $\mathbb{Z}/2$-graded simple associative algebra of characteristic $p\neq 2$ with supercenter Z whose elements supercommute with any $a\in A$. Let the*

Montgomery's condition

$$\text{if } u^2 \in Z, \text{ then } u \in Z \text{ for any homogeneous } u \in A_{\bar{1}} \tag{6}$$

hold. Then, there are no even supertraces on $\mathfrak{q}A$*, but there is one odd supertrace.*

Proof Observe that the superalgebra A of differential operators in any finite number of indigenously odd indeterminates (a.k.a. the *Clifford* algebra on $2n$ generators considered as a $\mathbb{Z}/2$-graded associative superalgebra) is isomorphic to the matrix superalgebra $\mathrm{Mat}(2^{n-1}|2^{n-1})$ on which there is an even supertrace, whereas on $\mathfrak{q}(2^n) := \mathfrak{q}(\mathfrak{gl}(2^{n-1}|2^{n-1})) = \mathfrak{q}((\mathrm{Mat}(2^{n-1}|2^{n-1}))^L)$ there is the (well-known today) odd *queer trace*. Therefore, by arguments in the proof of Theorem 3.1, and using Montgomery's theorem [18, Th.3.8] which states that, provided condition (6) holds, $SL(A) := (A^S)'/((A^S)' \cap Z)$ is a simple Lie superalgebra, we are done. □

Comments Vasiliev was, most probably, the first to publish that *on the Weyl algebra* W_n *of polynomial differential operators in* n *even indeterminates* x_i *considered as superalgebra with parity given by* $p(x_i) = p(\partial_i) = \bar{1}$ *for all* i*, where* $\partial_i := \frac{\partial}{\partial x_i}$*, there is an even supertrace*, see [24].

For a generalized Calogero–Moser case, see [25]; the detailed version [26] contains an elementary proof of uniqueness of the supertrace on W_n. The algebras of "matrices of complex size" first appeared as associative algebras in the book [5] and as Lie algebras in [7]. For a generalization to symplectic reflection algebras, see [11, Th.7.1.1]. Alexey Lebedev suggested a beautiful elementary proof of the existence of the supertrace on W_n, see §6.

Recall that Herstein, see [8], proved that *for any simple finite-dimensional associative algebra* A *with center* Z, *the Lie algebra* $L(A) := (A^L)'/((A^L)' \cap Z)$ *is simple, unless* $[A : Z] = 4$ *and* A *has characteristic* 2.

Obviously unaware of Vasiliev's works on supertraces, his results were rediscovered by mathematicians, see [18, Proposition 4.3] and [19]. Montgomery found out the sufficient condition (6) to the super version of Herstein's theorem (see [8]) and formulated it in the infinite-dimensional situation (in the finite-dimensional case, it is also true).

4.2 (Super)algebras of "Matrices of Complex Size"

Theorem 4.1 is applicable to the queerifications of both algebras and superalgebras A of "matrices of complex size", see [15, Subsection 2.2] with the same answer: there are no even supertraces on $\mathfrak{q}A$, but there is one odd supertrace. (Since $\mathfrak{gl}(\lambda) = \mathfrak{gl}(\lambda)' \oplus \mathbb{C}\,1$, one can define the trace on the Lie algebra $\mathfrak{gl}(\lambda)$ by any value on 1. Although we do not need it here, recall—for its beauty—that J. Bernstein defined the trace on $\mathfrak{gl}(\lambda)$ for $\lambda \in \mathbb{C}\backslash\mathbb{Z}$, see [9], so that $\mathrm{tr}(1) = \lambda$; Bernstein's trace naturally generalizes the trace on $\mathrm{Mat}(|n|)$ such that $\mathrm{tr}(1_{|n|}) = |n|$ for any $n \in \mathbb{Z}\backslash\{0\}$.)

4.3 Symplectic Reflection Algebras and Superalgebras

For the classification of traces (resp. supertraces) on these algebras and superalgebras A, see [10, Tables on pp.5,6]. Considering them as algebras (resp. superalgebras) we get the exact number of supertraces on their Lie qeerifications, according to general results established in Case 1 (resp. Case 3) of Theorem 3.1.

Open problem For a description of ideals in these algebras and superalgebras, see [11–13]. Determine when these ideals are themselves simple algebras and superalgebras, and describe (super)traces on them and on their queerifications.

4.4 (Super)Trace on the (Super)Algebra of Pseudo-Differential Operators

4.4.1 $N = 0$

Recall that the associative algebra Ψ of pseudo-differential operators of integer order is $\mathcal{F}((D^{-1}))$, where $D := \frac{d}{dx}$ and $\mathcal{F}$ is the algebra of functions in x, with multiplication given for any integer n by the Leibniz rule

$$\begin{aligned} D^n f &:= \sum_{k\geq 0} \binom{n}{k} D^k(f) D^{n-k}, \text{ where } \binom{n}{k} \\ &:= \frac{n(n-1)\dots(n-k+1)}{k!}, \text{ for any } f \in \mathcal{F}. \end{aligned}$$

Adler defined a trace on the algebra Ψ of pseudo-differential operators, see [1] and very reader-friendly reviews [16, 20], as the composition of the residue and the indefinite integral

$$\operatorname{tr}\left(\sum_{k\leq n} f_k D^k\right) = \int f_{-1} dx, \text{ where } f_k \in \mathcal{F}.$$

This trace (it vanishes on the commutators even before the integral is taken, just residue suffices, see [20]) takes values in $\mathcal{F}$. By Theorem 3.1, **there are no even supertraces on $\mathfrak{q}\Psi$, but there are ≥ 1 odd supertraces**; we conjecture there is just one odd supertrace.

4.4.2 $N = 1$

On the superalgebra $\Psi_1 := \mathcal{F}((\mathcal{D}^{-1}))$ of $N = 1$-extended pseudo-differential operators, where $\mathcal{D} := \frac{\partial}{\partial\xi} + \xi\frac{\partial}{\partial x}$ and $\mathcal{F}$ is the algebra of functions in the even x and odd ξ, Manin and Radul defined super residue, super binomial coefficients and an even supertrace, see [17]. By Theorem 3.1, **there are no even supertraces on $\mathfrak{q}\Psi_1$, but there are ≥ 1 odd supertraces**; we conjecture there is just one odd supertrace.

5 An Application of Traces: Integrals in Involution

Let A be an associative (super)algebra, $\mathfrak{g} := A^L$ or A^S, tr a (super)trace on A, and b the corresponding *invariant symmetric* bilinear form (briefly: IS form)

$$b(X, Y) := \mathrm{tr}(XY) \text{ for any } X, Y \in A.$$

Let, moreover, b be non-degenerate, briefly: NIS (for examples, see [3, 12]). Since the spaces of $\mathfrak{g}$ and A coincide, we can (and will) consider b as a form on $\mathfrak{g}$.

Let tr, and hence b, be even. Let $L := L(t) \in \mathfrak{g}$ be a point on the curve depending on parameter t interpreted as time, $P \in \mathfrak{g}$ a fixed element, called/interpreted as a Hamiltonian. Then, for the equation (P and L are in honor of Peter Lax)

$$\dot{L} = [L, P], \text{ where } L, P \in \mathfrak{g} \text{ and dot signifies the derivative with respect to time } t, \tag{7}$$

the functions $L \mapsto \mathrm{tr}(L^k)$ on $\mathfrak{g}$, identified with $\mathfrak{g}^*$ thanks to the NIS b, are *integrals in involution*, i.e., they commute with the Hamiltonian P and each other with respect to the Poisson bracket $\{-, -\}$ defined on the space $\mathcal{F}(\mathfrak{g}^*)$ of functions on $\mathfrak{g}^*$ as follows, see, e.g., [1]. We identify $\mathfrak{g}$ with the space of linear functions on $\mathfrak{g}^*$; for any functions $f, g \in \mathcal{F}(\mathfrak{g}^*)$, set

$$\{f, g\}(X) := X([\mathrm{d}f(X), \mathrm{d}g(X)]) \text{ for any } X \in \mathfrak{g}^*. \tag{8}$$

In the super setting, a more subtle version of (7) is more adequate: it involves a $1|1$-dimensional Time with even coordinate t and odd one τ, see [22]:

$$(\partial_\tau + \tau\partial_t)L = [L, H], \text{ where } L, H \in A. \tag{9}$$

Clearly, $(\partial_\tau + \tau\partial_t)^2 = \partial_t$ and setting $P = \frac{1}{2}[H, H]$ we get Eq. (7). The functions $L \mapsto \mathrm{tr}(L^k)$ are *integrals in involution* for Eq. (9) as well.

It seems, nobody considered yet the Euler equations or Lax pairs (7) related with superalgebras A on which there is an odd trace qtr, and hence an odd b. If b is odd and non-degenerate, then $\mathfrak{g} \simeq \Pi\mathfrak{g}^*$, and an antibracket, rather than a Poisson bracket, is defined on the space $\Pi\mathcal{F}(\mathfrak{g}^*)$ of functions on $\mathfrak{g}^*$. The functions $L \mapsto \mathrm{qtr}(L^k)$ are integrals in involution, i.e., they commute with the Hamiltonian and each other (themselves including) with respect to the antibracket.

On $2n|k$-dimensional superspace on which the Poisson bracket is defined (or $n|n$-dimensional superspace on which the antibracket bracket is defined), let for the dynamical system (7), there be n first integrals in involution. Then, a theorem of Shander guarantees complete integrability of the system whatever $k < \infty$ is, see [23].

The traces on (super)algebras A considered in Sect. 4.4 determine what researchers conceded to call "complete integrability" in the case of infinite-dimensional Hamiltonian system (since there are infinitely many of these traces, "this infinity is a half of the infinite dimension"). Examples: (1) the KdV equations for the case where L is

the Schrödinger operator; (2) ($N = 1$)-superextended KdV, see Sect. 4.4.2, where L is an ($N = 1$)-superextension of the Schrödinger operator.
Open problem For any simple associative algebra A considered in this note, give explicit examples of integrable systems related with $Q(A)$.

6 Supertrace on W_n

Although the statement of Lemma 6.1 is known, we find its proof due to Lebedev is very interesting and worth publishing.

6.1 Two General Facts [21]

1. If there is no (super)trace on an associative (super)algebra A, then there is no (super)trace on any product $A \otimes B$ for any associative (super)algebra B with unit: if $[a_1, a_2] = a$, then $[a_1 \otimes 1, a_2 \otimes b] = a \otimes b$ for any $a_1, a_2 \in A$ and $b \in B$, i.e., if any element of A can be represented as a linear combination of (super)commutators, then the same is true for any element of $A \otimes B$.
2. Let tr_i be a (super)trace on the associative (super)algebra A_i for $i = 1, 2$. Then,

$$\mathrm{tr}(a_1 \otimes a_2) := (\mathrm{tr}_1\, a_1)(\mathrm{tr}_2\, a_2) \text{ for any } a_i \in A_i$$

is a (super)trace on $A_1 \otimes A_2$.

6.1 Lemma *Consider W_n as a superalgebra with $p(x_i) = p(\partial_{x_i}) = \bar{1}$.*
Then, on W_n, there is an even supertrace.

Observe that if we consider W_n as an algebra, not a superalgebra, no analog of Lemma 6.1 takes place since the associative algebra W_n is simple; on the other hand, the center (constants) is given by a non-trivial cocycle on the simple Lie algebra constructed via Montgomery's theorem ([18, Th.3.8]). The proof below demonstrates existence of the supertrace; its uniqueness (up to a non-zero factor) should be proved separately. For the proof of uniqueness, see [18, 26].

Proof (A. Lebedev) Actually, $W_n = \mathbb{K}1 \oplus [W_n, W_n]$, where $[W_n, W_n]$ is the supercommutant.

Let $n = 1$ and $D := \frac{d}{dx}$. Introduce the weight function wht: let $\mathrm{wht}(x) := 1$, so $\mathrm{wht}(D) := -1$. On W_n, define the following linear function T:

$$T(P) := \begin{cases} \left(P(\frac{1}{x+1})\right)|_{x=1} & \text{if } \mathrm{wht}(P) = 0, \\ 0 & \text{if } \mathrm{wht}(P) \neq 0. \end{cases}$$

Let us prove that T is a supertrace on W_n, i.e.,

$$T(PQ) = (-1)^{p(P)p(Q)} T(QP).$$

Clearly, it suffices to prove this for the case where P and Q are monomials whose weights are opposite.

Case 1: $P = x^{n+1}D^n$ and $Q = D$. Then,

$$\begin{aligned}
T(PQ) &= (x^{n+1}(-1)^{n+1}\tfrac{(n+1)!}{(x+1)^{n+2}})|_{x=1} = -(-\tfrac{1}{2})^{n+2}(n+1)!\\
T(QP) &= (D(x^{n+1}(-1)^{n}\tfrac{n!}{(x+1)^{n+1}})|_{x=1}\\
&= ((-1)^n\tfrac{(n+1)!\, x^n}{(x+1)^{n+1}} - (-1)^n\tfrac{(n+1)!\, x^{n+1}}{(x+1)^{n+2}})|_{x=1} = (-\tfrac{1}{2})^{n+2}(n+1)!
\end{aligned}$$

This implies the answer for the case where $Q = D$ and any P of weight 1, because any such P can be represented as a linear combination of operators of the form $x^{n+1}D^n$.

Case 2: $\mathrm{wht}(P) = -1$ and $Q = x$. Then,

$$\begin{aligned}
T(QP) &= \left(xP\left(\tfrac{1}{x+1}\right)\right)|_{x=1} = \left(P\left(\tfrac{1}{x+1}\right)\right)|_{x=1},\\
T(PQ) &= \left(P\left(\tfrac{x}{x+1}\right)\right)|_{x=1} = \left(P\left(1-\tfrac{1}{x+1}\right)\right)|_{x=1} = -\left(P\left(\tfrac{1}{x+1}\right)\right)|_{x=1},
\end{aligned}$$

since $P(1) = 0$ because $\mathrm{wht}(P) < 0$.

This implies the general case where P and Q are monomials of opposite weights, because we can transplant x and D, one by one, from the end of PQ to the beginning until we get QP, and each transplantation changes the sign by the opposite; by $(-1)^{deg(Q)} = (-1)^{p(P)p(Q)}$ altogether.

Since $T(1) = \frac{1}{2}$, it follows that 1 cannot be represented as a linear combination of supercommutators.

For $n > 1$, recall that $W_n \simeq W_{n-1} \otimes W_1$ and apply general fact 2), see Sect. 6.1. □

Comment: how to guess the form of T (A. Lebedev). Let P be a differential operator of weight 0. In the basis $1, x, x^2, \ldots$, consider P as a linear operator on the space of polynomials and consider the matrix of P. Naively, ignoring possible divergence, the supertrace of this matrix is equal to

$$\sum_{n=0}^{\infty}(-1)^n(\text{the coefficient of } x^n \text{ in } Px^n).$$

Since $\mathrm{wht}(P) = 0$, then Px^n is equal to the above-mentioned coefficient of x^n. In other words, the coefficient is equal to the value of Px^n at $x = 1$. Hence, the supertrace is equal to

$$\sum_n(-1)^n Px^n|_{x=1} = P(1 - x + x^2 - x^3 + \cdots)|_{x=1} = \left(P\left(\frac{1}{1+x}\right)\right)|_{x=1}.$$

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