NORMALITY OF TWO FAMILIES OF MEROMORPHIC FUNCTIONS CONCERNING PARTIALLY SHARED VALUES

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Abstract. In this paper, normality of a family of meromorphic functions is deduced from the normality of a given family. Precisely, we have proved: Let $\mathcal{F}$ and $\mathcal{G}$ be two families of meromorphic functions on a domain D, and $a,\ b,\ c$ be three finite complex numbers such that $a \neq 0$ and $b \neq c$. Suppose that $\mathcal{G}$ is normal in D such that no sequence in $\mathcal{G}$ converges locally uniformly to infinity in D. If $n \geq 3$ and for each function $f \in \mathcal{F}$, there exists $g \in \mathcal{G}$ such that $f^{'} - af^n$ and $g^{'} - ag^n$ partially share the values b and c, then $\mathcal{F}$ is normal in D. Further, examples are given to establish the sharpness of the result.

1. Introduction and main results

Let D be a domain in the complex plane $\mathbb{C}$. A family $\mathcal{F}$ of meromorphic functions on D is said to be normal if from every sequence $\{f_n\}$ in $\mathcal{F}$ we can extract a subsequence $\{f_{n_k}\}$ which converges locally uniformly to f in D with respect to the spherical metric, where f is either a meromorphic function or identically equal to infinity in D. A family $\mathcal{F}$ is said to be normal at $z_0 \in D$ if it is normal in some neighborhood of z_0; thus $\mathcal{F}$ is normal in D if and only if it is normal at each point $z \in D$. (see [14]).

Let f and g be two meromorphic functions in D and let $a \in \mathbb{C}$. We shall denote by $E(f,a)$ the set of zeros of $f - a$ (ignoring multiplicities). We say that f and g share the value a if $E(f,a) = E(g,a)$. Further, if $E(f,a) \subset E(g,a)$, we say that f and g share the value a partially (see [18]).

According to Bloch's principle [14] any condition which reduces a meromorphic function in $\mathbb{C}$ to a constant is likely to force a family of meromorphic functions in a domain D to be normal. Although this principle as well as its converse do not hold in general (see, for example [2, 13]) still it serves as a guiding principle for obtaining normality criteria corresponding to Picard type theorems and vice-versa (see [1]).

In 1959, Hayman [5] proved that *if f is a meromorphic function in the complex plane, $a \in \mathbb{C} \setminus \{0\}$ and the differential polynomial $f^{'} - af^n,\ n \geq 5$, does not assume a finite complex value in $\mathbb{C}$, then f is constant.* This result is not true for $n = 3, 4$ as shown by Mues [10]. In view of Bloch's principle, Hayman [6] in 1967 conjectured that there exists a normality criterion corresponding to this Picard-type theorem. Over the next few decades, the following normality criterion was established thereby proving the Hayman's conjecture.

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Theorem 1.1. *Let $\mathcal{F}$ be a family of meromorphic (holomorphic) functions in a domain D, $n \in \mathbb{N}$ and a, b be two finite complex numbers such that $n \geq 3$ $(n \geq 2)$ and $a \neq 0$. If for each $f \in \mathcal{F}$, $f^{'} - af^n \neq b$, then $\mathcal{F}$ is normal in D.*

The proof of Theorem 1.1 for meromorphic functions is due to S. Li [8], X. Li [9] and Langley [7] for $n \geq 5$, Pang [11] for $n = 4$, Chen and Fang [3] and Zalcman [17] for $n = 3$ independently and the proof of Theorem 1.1 for holomorphic functions is due to Drasin [4] for $n \geq 3$ and Ye [16] for $n = 2$.

In 2008, Zhang [19] considered the idea of shared values and proved the following.

Theorem 1.2. *Let $\mathcal{F}$ be a family of meromorphic (holomorphic) functions in D, $n \in \mathbb{N}$ and a, b be two finite complex numbers such that $n \geq 4$ $(n \geq 2)$ and $a \neq 0$. If for each pair of functions f and g in $\mathcal{F}$, $f^{'} - af^n$ and $g^{'} - ag^n$ share the value b, then $\mathcal{F}$ is normal in D.*

In this paper, we consider the related problems concerning two families of meromorphic functions and prove the following theorem:

Theorem 1.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be two families of holomorphic functions on a domain D, and a, b, c be three complex numbers such that $a \neq 0$ and $b \neq c$. Suppose that $\mathcal{G}$ is normal in D such that no sequence in $\mathcal{G}$ converges locally uniformly to infinity in D. If $n \geq 2$ and for each function $f \in \mathcal{F}$, there exists $g \in \mathcal{G}$ such that $f^{'} - af^n$ and $g^{'} - ag^n$ partially share the values b and c, then $\mathcal{F}$ is normal in D.*

In the following example, we show that the condition 'partial sharing of two values b and c' in Theorem 1.3 can not be reduced to one.

Example 1.4. Consider the two families $\mathcal{F} := \{f_j(z) = e^{jz} : j \in \mathbb{N}\}$ and $\mathcal{G} := \{1\}$ of holomorphic functions on $\mathbb{D}$. Note that $g_j^{'} - g_j^2 \equiv -1$. Therefore, $f_j^{'} - f_j^2 = -1 \Rightarrow g_j^{'} - g_j^2 = -1$. But $\mathcal{F}$ fails to be normal at $z = 0$.

We demonstrate in the subsequent example that Theorem 1.3 fails to be true when $n = 1$. Therefore, the condition $n = 2$ is the best possible for Theorem 1.3.

Example 1.5. Consider the two families $\mathcal{F} := \{f_j(z) = jz : j \in \mathbb{N}\}$ and $\mathcal{G} := \{-1\}$ of holomorphic functions on $\mathbb{D}$. Then, clearly, $f_j^{'}(z) - f_j(z) = j(1 - z) \neq 0$, and for each $f_j \in \mathcal{F}$, there exists $g_j \in \mathcal{G}$ such that $f_j^{'}(z) - f_j(z) = 1 \Rightarrow g_j^{'}(z) - g_j(z) = 1$. But $\mathcal{F}$ fails to be normal at $z = 0$.

The following example illustrates that Theorem 1.3 is not valid for the family of meromorphic functions when $n = 2$.

Example 1.6. Consider the two families

$$\mathcal{F} := \left\{ f_j(z) = \frac{jz}{1 + jz^2} : j \in \mathbb{N} \right\}$$

and

$$\mathcal{G} := \{1\}$$

of meromorphic functions on $\mathbb{D}$. Take $a = -1$. Then, clearly, $f_j^{'}(z) - af_j^2(z) = \frac{j}{(1+jz^2)^2} \neq 0$ and for each $f_j \in \mathcal{F}$, there exists $g_j \in \mathcal{G}$ such that $f_j^{'}(z) - af_j^2(z) = 1 \Rightarrow g_j^{'}(z) - ag_j^2(z) = 1$. But $\mathcal{F}$ is not normal at $z = 0$ since $f_j(0) = 0$ and for $z \neq 0$, $f_j(z) \to 1/z$ as $n \to \infty$.

However, Theorem 1.3 can be extended to families of meromorphic functions provided that $n \geq 3$.

Theorem 1.7. *Let $\mathcal{F}$ and $\mathcal{G}$ be two families of meromorphic functions on a domain D, and $a,\ b,\ c$ be three finite complex numbers such that $a \neq 0$ and $b \neq c$. Suppose that $\mathcal{G}$ is normal in D such that no sequence in $\mathcal{G}$ converges locally uniformly to infinity in D. If $n \geq 3$ and for each function $f \in \mathcal{F}$, there exists $g \in \mathcal{G}$ such that $f^{'} - af^n$ and $g^{'} - ag^n$ partially share the values b and c, then $\mathcal{F}$ is normal in D.*

In the following example, we show that the condition 'partial sharing of two values b and c' in Theorem 1.7 can not be reduced to one.

Example 1.8. Consider the two families

$$\mathcal{F} := \left\{ f_j(z) = \frac{1}{jz} : j \in \mathbb{N} \right\}$$

and

$$\mathcal{G} := \left\{ \frac{1}{z + \frac{1}{j^2} - 1} : j \in \mathbb{N} \right\}$$

of meromorphic functions on $\mathbb{D}$. Then for each $f_j \in \mathcal{F}$ there exists $g_j \in \mathcal{G}$ such that $f_j^{'} - f_j^3 = 0 \Rightarrow g_j^{'} - g_j^3 = 0$. Also, $g_j(z) \to g(z) = \frac{1}{z-1} \not\equiv \infty$. But $\mathcal{F}$ fails to be normal at $z = 0$.

For $n = 2$, we have the following weak version of the Theorem 1.7.

Theorem 1.9. *Let $\mathcal{F}$ and $\mathcal{G}$ be two families of meromorphic functions on a domain D such that each $f \in \mathcal{F}$ has neither simple zeros nor simple poles. Let $a,\ b$ and c be three finite complex numbers such that $a \neq 0$ and $b \neq c$. Suppose that $\mathcal{G}$ is normal in D such that no sequence in $\mathcal{G}$ converges locally uniformly to infinity in D. If for each function $f \in \mathcal{F}$, there exists $g \in \mathcal{G}$ such that $f^{'} - af^2$ and $g^{'} - ag^2$ partially share the values b and c, then $\mathcal{F}$ is normal in D.*

Note that Example 1.6 also shows that the condition 'each $f \in \mathcal{F}$ has neither simple zeros nor simple poles' in Theorem 1.9 can not be omitted.

2. Lemmas and proof of the results

In order to prove our results, we need the following lemmas.

Lemma 2.1. [12] *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disk $\mathbb{D}$ such that all the zeros of $f \in \mathcal{F}$ are of multiplicity at least p and all the poles of $f \in \mathcal{F}$ are of multiplicity at least q. Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. Then, for every $\alpha \in (-p, q)$, there exist*

(a) *points z_n in $\mathbb{D} : z_n \to z_0$;*
(b) *functions $f_n \in \mathcal{F}$;*
(c) *positive real numbers $\rho_n : \rho_n \to 0$*

such that the re-scaled sequence $\{g_n(\zeta) = \rho_n^{\alpha} f_n(z_n + \rho_n \zeta)\}$ converges spherically locally uniformly on $\mathbb{C}$ to a non-constant meromorphic function g on $\mathbb{C}$ of finite order.

Lemma 2.2. [3] *Let f be a meromorphic function in $\mathbb{C}$, and let n be a positive integer. If $f^n f^{'}$ does not assume a non-zero finite complex number in $\mathbb{C}$, then f is constant.*

Lemma 2.3. [15] *Let f be a meromorphic function in $\mathbb{C}$ and b be a non-zero complex number. If f has neither simple zero nor simple pole and $f^{'}(z) \neq b$, then f is constant.*

Proof of the Theorem 1.3 We may consider D to be an open unit disk $\mathbb{D}$. Suppose that the family $\mathcal{F}$ is not normal at $z_0 \in \mathbb{D}$. Then by Lemma 2.1, there exist points $z_j \in \mathbb{D}$ with $z_j \to z_0$, a sequence of positive numbers $\rho_j \to 0$ and a sequence of functions $f_j \in \mathcal{F}$ such that

$$F_j(\zeta) = \rho_j^{\frac{1}{n-1}} f_j(z_j + \rho_j\zeta) \to F(\zeta) \tag{2.1}$$

locally uniformly on $\mathbb{C}$, where F is a non-constant entire function of finite order.
From (2.1), we have

$$\rho_j^{\frac{n}{n-1}}\{(f_j^{'} - af_j^n)(z_j + \rho_j\zeta) - b\} = (F_j^{'} - aF_j^n)(\zeta) - \rho_j^{\frac{n}{n-1}} b \to F^{'}(\zeta) - aF^n(\zeta) \tag{2.2}$$

and

$$\rho_j^{\frac{n}{n-1}}\{(f_j^{'} - af_j^n)(z_j + \rho_j\zeta) - c\} = (F_j^{'} - aF_j^n)(\zeta) - \rho_j^{\frac{n}{n-1}} c \to F^{'}(\zeta) - aF^n(\zeta) \tag{2.3}$$

locally uniformly on $\mathbb{C}$.

For each $f_j \in \mathcal{F}$, there exists $g_j \in \mathcal{G}$ such that $f_j^{'} - af_j^n$ and $g_j^{'} - ag_j^n$ share the values b and c partially in $\mathbb{D}$. Since $\mathcal{G}$ is normal, there exists a subsequence in $\{g_j\}$, again denoted by $\{g_j\}$, that converges uniformly to a holomorphic function $g(z) \not\equiv \infty$ in some neighborhood of z_0.

Suppose $(F^{'} - aF^n) \not\equiv 0$ otherwise $\frac{-1}{n-1}\frac{1}{F^{n-1}} \equiv a\zeta + d$, for some $d \in \mathbb{C}$, which contradicts to the fact that F is an entire function and $n \geq 2$. Further, suppose that $(F^{'} - aF^n)(\zeta) \neq 0$, $\zeta \in \mathbb{C}$. Then $\frac{F^{'}}{F^n} \neq a$. By setting $F = 1/\phi$, we have $\phi^{n-2}\phi^{'} \neq -a$. When $n \geq 3$, ϕ is constant by Lemma 2.2 and when $n = 2$, ϕ is again constant by Hayman's alternative since $\phi \neq 0$ and $\phi^{'} \neq -a$. In both cases, ϕ is constant. This implies that F is constant, a contradiction. Thus $(F^{'} - aF^n)$ has at least one zero.

Now we have two cases:

Case-I. $(g^{'} - ag^n)(z_0) \neq b$.

Suppose that $(F^{'} - aF^n)(\zeta_0) = 0$, for some $\zeta_0 \in \mathbb{C}$. From (2.2), by Hurwitz's theorem, there exists a sequence $\{\zeta_j\}$ with $\zeta_j \to \zeta_0$ such that for sufficiently large j

$$(F_j^{'} - aF_j^n)(\zeta_j) - \rho_j^{\frac{n}{n-1}} b = 0,$$

and thus

$$(f_j^{'} - af_j^n)(z_j + \rho_j\zeta_j) = b.$$

By hypothesis, we have $(g_j^{'} - ag_j^n)(z_j + \rho_j\zeta_j) = b$ and so $(g^{'} - ag^n)(z_0) = b$, a contradiction.

Case-II. $(g^{'} - ag^n)(z_0) = b$.

By using (2.3) instead of (2.2) in Case-I, we obtain $(g^{'} - ag^n)(z_0) = c$ $(\neq b)$ which is not true. This completes the proof. $\square$

Proof of the Theorem 1.7 We may consider D to be an open unit disk $\mathbb{D}$. Suppose that the family $\mathcal{F}$ is not normal at $z_0 \in \mathbb{D}$. Then there exists a sequence $\{f_n\} \subset \mathcal{F}$ which has no locally convergent subsequence at z_0. Thus, by Lemma 2.1, there exist points $z_j \in \mathbb{D}$ with $z_j \to z_0$, a sequence of positive numbers $\rho_j \to 0$, and a sequence of functions in $\{f_j\}$ again denoted by $\{f_j\}$ such that

$$F_j(\zeta) = \rho_j^{\frac{1}{n-1}} f_j(z_j + \rho_j\zeta) \to F(\zeta) \tag{2.4}$$

locally uniformly on $\mathbb{C}$ with respect to spherical metric, where F is a non-constant meromorphic function on $\mathbb{C}$ of finite order.
From (2.4), we have

$$(F_j^{'} - aF_j^n)(\zeta) - \rho_j^{\frac{n}{n-1}} b = \rho_j^{\frac{n}{n-1}}\{(f_j^{'} - af_j^n)(z_j + \rho_j\zeta) - b\} \to F^{'}(\zeta) - aF^n(\zeta) \tag{2.5}$$

and

$$(F_j^{'} - aF_j^n)(\zeta) - \rho_j^{\frac{n}{n-1}} c = \rho_j^{\frac{n}{n-1}}\{(f_j^{'} - af_j^n)(z_j + \rho_j\zeta) - c\} \to F^{'}(\zeta) - aF^n(\zeta) \tag{2.6}$$

spherically locally uniformly on $\mathbb{C}$ except possibly at the poles of F.

For each $f_j \in \mathcal{F}$, there exists $g_j \in \mathcal{G}$ such that $f_j^{'} - af_j^n$ and $g_j^{'} - ag_j^n$ partially share the values b and c in $\mathbb{D}$. Since $\mathcal{G}$ is normal, there exists a subsequence in $\{g_j\}$, again denoted by $\{g_j\}$, that converges uniformly to a meromorphic function $g(z) \not\equiv \infty$ in some neighborhood of z_0.

Claim. $(F^{'} - aF^n)(\zeta_0) = 0$, for some $\zeta_0 \in \mathbb{C}$.

Suppose that $(F^{'} - aF^n)(\zeta) \neq 0$. Then $\frac{F^{'}}{F^n} \neq a$. By setting $F = 1/\phi$, $\phi^{n-2}\phi^{'} \neq -a$. By Lemma 2.2, ϕ and so F is constant, a contradiction. This proves the claim.

Now we have three cases:

Case-I. $(g^{'} - ag^n)(z_0) \neq b, \infty$.

By Claim, $(F^{'} - aF^n)(\zeta_0) = 0$, for some $\zeta_0 \in \mathbb{C}$. Since $(F^{'} - aF^n) \not\equiv 0$, otherwise $\frac{-1}{n-1}\frac{1}{F^{n-1}} \equiv a\zeta + d$, for some $d \in \mathbb{C}$, which contradicts to the fact that F is a non-constant meromorphic function and $n \geq 3$, by (2.5), there exists a sequence $\{\zeta_j\}$ with $\zeta_j \to \zeta_0$ such that for sufficiently large j, $(f_j^{'} - af_j^n)(z_j + \rho_j\zeta_j) = b$. By assumption, we have $(g_j^{'} - ag_j^n)(z_j + \rho_j\zeta_j) = b$ and so $(g^{'} - ag^n)(z_0) = b$, a contradiction.

Case-II. $(g^{'} - ag^n)(z_0) = b$.

Using (2.6) instead of (2.5) in Case-I, we obtain $(g^{'} - ag^n)(z_0) = c$ $(\neq b)$, which is not true.

Case-III. $(g^{'} - ag^n)(z_0) = \infty$.

Then, clearly, $g(z_0) = \infty$. Suppose that z_0 is a pole of g with multiplicity $k \geq 1$. Then, for sufficiently large j, g_j has exactly $l \leq k$ distinct poles $z_j^1, \ldots, z_j^l$ in $D(z_0, r)$ with multiplicities $\alpha_1, \ldots, \alpha_l$ respectively such that $z_j^i \to z_0$ $(i = 1, \ldots, l)$ and $\sum_{i=1}^{l} \alpha_i = k$. Renumbering if possible, we may assume that the number l and multiplicities $\alpha_i, i = 1, \ldots, l$ are independent of j. Now set

$$H_j(z) := g_j(z)\prod_{i=1}^{l}(z - z_j^i)^{\alpha_i}.$$

Then the functions H_n are holomorphic in $D(z_0, r)$ and $H_n \to H$ on $D(z_0, r/2) \setminus \{z_0\}$, where $H(z) = g(z)(z - z_0)^k$ is holomorphic on $D(z_0, r)$. Note that $H(z_0) \neq 0, \infty$. Hence by maximum principle, $H_n \to H$ on $D(z_0, r/2)$.

We have

$$\begin{aligned} g_j^{'}(z) &= \left(H_j(z) \prod_{i=1}^{l} (z - z_j^i)^{-\alpha_i} \right)^{'} \\ &= H_j^{'}(z) \prod_{i=1}^{l} (z - z_j^i)^{-\alpha_i} - H_j(z) \sum_{i=1}^{l} \alpha_i (z - z_j^i)^{-\alpha_i - 1} \prod_{s \neq i} (z - z_j^s)^{-\alpha_s} \\ &= \prod_{i=1}^{l} (z - z_j^i)^{-\alpha_i - 1} \left(H_j^{'} \prod_{i=1}^{l} (z - z_j^i) - H_j(z) \sum_{i=1}^{l} \alpha_i \prod_{s \neq i} (z - z_j^s) \right). \end{aligned} \tag{2.7}$$

Then

$$g_j^{'}(z) - a g_j^n(z) - b = K_j(z) \prod_{i=1}^{l} (z - z_j^i)^{-\alpha_i - 1}, \tag{2.8}$$

where

$$\begin{aligned} K_j(z) &= H_j^{'} \prod_{i=1}^{l} (z - z_j^i) - H_j(z) \sum_{i=1}^{l} \alpha_i \prod_{s \neq i} (z - z_j^s) \\ &\quad - a H_j^n(z) \prod_{i=1}^{l} (z - z_j^i)^{-\alpha_i(n-1)+1} - b \prod_{i=1}^{l} (z - z_j^i)^{\alpha_i + 1}. \end{aligned} \tag{2.9}$$

Since $H(z_0) \neq 0, \infty$, we have

$$\begin{aligned} K_j(z) &\to H^{'}(z)(z - z_0)^l - H(z) k (z - z_0)^{l-1} - \frac{a H^n(z)}{(z - z_0)^{k(n-1)-l}} - b(z - z_0)^{k+l} \\ &= \frac{1}{(z - z_0)^{k(n-1)-l}} \left\{ H^{'}(z)(z - z_0)^{k(n-1)} - k H(z)(z - z_0)^{k(n-1)-1} - a H^n(z) - b(z - z_0)^{nk} \right\} \end{aligned} \tag{2.10}$$

and

$$\left(H^{'}(z)(z - z_0)^{k(n-1)} - k H(z)(z - z_0)^{k(n-1)-1} - a H^n(z) - b(z - z_0)^{nk} \right)_{z = z_0} = -a H^n(z_0) \neq 0. \tag{2.11}$$

Therefore, $K_j(z)$ and so $g_j^{'}(z) - a g_j^n(z) - b$ has no zeros in some neighborhood of z_0. By assumption, we find that $f_j^{'}(z) - a f_j^n(z) - b$ has no zero in some neighborhood of z_0. By Theorem 1.1, the sequence $\{f_j\}$ is normal at z_0, a contradiction. $\square$

Proof of the Theorem 1.9 Following the proof of Theorem 1.7 , we only need to prove that $F' - aF^2 \not\equiv 0$ and $F' - aF^2$ has at least one zero. Suppose that $F' - aF^2 \equiv 0$. Then $(\frac{1}{F})' \equiv a$ which implies that $\frac{1}{F} \equiv a\zeta + d$, for some $d \in \mathbb{C}$, which contradicts to the fact that F has no simple pole. Next suppose that $F' - aF^2 \neq 0$. Then $\frac{F'}{F^2} \neq a$. By setting $F = 1/\phi$, $\phi' \neq -a$. By Lemma 2.3, ϕ and so F is constant, a contradiction. □

3. Disclosure statement

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