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Classification of NMS-flows with unique twisted saddle orbit on orientable 4-manifolds

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Abstract: Topological equivalence of Morse-Smale flows without fixed points (NMS-flows) under assumptions of different generalities was studied in a number of papers. In some cases when the number of periodic orbits is small, it is possible to give exhaustive classification, namely to provide the list of all manifolds that admit flows of considered class, find complete invariant for topological equivalence and introduce each equivalence class with some representative flow. This work continues the series of such articles. We consider the class of NMS-flows with unique saddle orbit, under the assumption that it is twisted, on closed orientable 4-manifolds and prove that the only 4-manifold admitting the considered flows is the manifold $\mathbb{S}^3 \times \mathbb{S}^1$. Also, it is established that such flows are split into exactly eight equivalence classes and construction of a representative for each equivalence class is provided.

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1 Introduction and main results

In the present paper we consider *NMS-flows* f^t, namely *non-singular* (without fixed points) Morse-Smale flows which are defined on orientable-manifold M^4. Non-wandering set of such flow consist of a finite number of hyperbolic periodic orbits. Asimov proved [Asi75] that ambient manifold of such flow is a union of round handles. However, if the number of orbits is small, topology of the ambient manifold can be specified. For instance, in dimension 3 only lens spaces admit NMS-flows with two periodic orbits. Moreover, it was shown in [PS22a] that each lens space (closed orientable manifold obtained by gluing two solid tori along boundaries) admit exactly two classes of topological equivalence except for 3-sphere $\mathbb{S}^3$ and projective space $\mathbb{R}\mathrm{P}^3$ which both admit the unique equivalence class. Moreover only two 4-manifolds $\mathbb{S}^3\times\mathbb{S}^1$, $\mathbb{S}^3\tilde{\times}\mathbb{S}^1$ admit such flows and each admits exactly two topological equivalence classes.

Campos et al. [CCMAV04] argued that lens spaces are the only prime (a manifold that cannot be expressed as a non-trivial connected sum of two manifolds) 3-manifolds that are ambient for NMS-flow with unique saddle periodic orbit, but this is not so. There exists infinite series of mapping tori non-homeomorphic to lens spaces which admit such flows [Shu21]. Moreover, necessary and sufficient conditions for topological equivalence of such flows were obtained in [PS22b]. Finally, any 3-manifold admitting such flows is a lens space or connected sum[1] of two lens spaces or a small Seifert fibered[2] 3-manifold [PS22c]. The invariants constructed in these works are different from known ones, for example *scheme of the flow* constructed by Umanskii [Uma90] for Morse-Smale

[1] A *connected sum of two n-dimensional manifolds* is a manifold formed by deleting a ball inside each manifold and gluing together the resulting boundary spheres.

[2] A *Seifert fibered space* is a closed orientable 3-manifold that can be ecomposed into a disjoint union of circles (*fibers*) such that each fiber has a neigbourhood that is fiber-wise homeomorphic to standart fibered torus.

flows with finite number of singular trajectories (closed orbits, fixed points and heteroclinic orbits).

In the present paper we establish the topology of orientable 4-manifolds that are ambient for NMS-flows with exactly one saddle orbit assuming that it is *twisted* (its invariant manifolds are non-orientable). Remarkably, all such flows are suspensions over Morse-Smale diffeomorphisms on 3-manifolds, which are classified in [BGP19]. 3-diffeomorphisms are known to posess wild separatrices [Pix77], which complicates their classification. Pixton constructed an example of 3-diffeomorphism with one saddle orbit having wild unstable separatrice. Bonatti and Grines classified the class of sphere diffeomorphisms that have non-wandering set consisting of four fixed points: two sinks, a source, and a saddle [BG00]. They showed that the Pixton class contains a countable set of pairwise topologically non-conjugate diffeomorphisms.

As was shown in [PS20b] suspensions over Pixton diffeomorphisms also have wild unstable separatrices and such class contains a countable family of pairwise non-equivalent flows. However, in this case the saddle orbit of the flow is non-twisted. Surprisingly, there are no flows with unique saddle orbit that is twisted and having wild separatrix. Besides, the number of equivalence classes of such flows appear to equal 8.

Let us proceed to the formulation of the results.

Let M^4 be connected closed orientable 4-manifold. Flow $f^t : M^4 \to M^4$ is called *Morse-Smale flow* if (a) its chain-recurrent set[3] consist of finite number of periodic orbits and fixed points and (b) the unstable manifold of each chain-recurrent set component has transversal intersection with the stable manifold of any other chain-recurrent set component. Let f^t be NMS-flow (Morse-Smale flow without fixed points) and $\mathcal{O}$ be its periodic orbit. There exists tubular neighborhood $V_{\mathcal{O}}$ homeomorphic to $\mathbb{D}^3 \times \mathbb{S}^1$ such that the flow is topologically equivalent to the suspension over a linear diffeomorphism of the plane defined by the matrix which determinant is positive and eigenvalues are different

[3]Point $x \in M$ is called *chain-recurrent* for the flow f^t if for any $T,\ \varepsilon > 0$ there exist points $x_1, \dots x_n \in M$ and real numbers $t_0, \dots t_n > T$ such that $x = x_0 = x_n$ and $d(f^{t_i}(x_i), x_{i+1}) < \varepsilon$, where d is a metric on M

from ± 1 (see. Proposition 1). If absolute values of both eigenvalues are greater (less) than one, the corresponding periodic orbit is *attracting (repelling)*, otherwise it is *saddle*. The saddle orbit is called *twisted* if both eigenvalues are negative and *non-twisted* otherwise.

Consider the class $G_3^-(M^4)$ of NMS-flows $f^t: M^4 \to M^4$ with unique saddle orbit which is twisted. Since the ambient manifold M^4 is the union of stable (unstable) manifolds of its periodic orbits, the flow $f^t \in G_3^-(M^4)$ has at least one attracting and at least one repelling orbit. In Section 2 the following fact is established.

Lemma 1. *The non-wandering set of any flow $f^t \in G_3^-(M^4)$ consists of exactly three periodic orbits S, A, R, saddle, attracting and repelling, respectively.*

Unstable manifold os saddle orbit S of the flow $f^t \in G_3^-(M^4)$ can be either 3- or 2-dimensional; let $G_3^{-1}(M^4)$, $G_3^{-2}(M^4)$ denote corresponding subclasses of $G_3^-(M^4)$. Obviously, since dimension of unstable manifold is invariant under an equivalence homeomorphism no flow in $G_3^{-1}(M^4)$ is topologically equivalent to any flow in $G_3^{-2}(M^4)$. Furthermore, $G_3^{-2}(M^4) = \{f^{-t} : f^t \in G_3^{-1}(M^4)\}$ and the flows f^t, f'^t are topologically equivalent if and only if f^{-t}, f'^{-t} are topologically equivalent. This immediately implies that classification in the class $G_3^-(M^4)$ reduces to classification in the subclass $G_3^{-1}(M^4)$.

Let $f^t \in G_3^{-1}(M^4)$. Since the flow f^t in some tubular neighborhood of is topologically equivalent to the suspension over linear diffeomorphism of the plane, the topology of periodic orbits A, S, R stable and unstable manifolds is:

- $W^u_S \cong \mathbb{R}^2 \tilde{\times} \mathbb{S}^1$ (open solid Klein bottle);
- $W^s_S \cong \mathbb{R} \tilde{\times} \mathbb{S}^1$ (open Möbius band);
- $W^s_A \cong W^u_R \cong \mathbb{R}^3 \times \mathbb{S}^1$ (open solid torus);
- $W^u_A \cong W^s_R \cong \mathbb{S}^1$ (circle).

Let $\mathcal{O} \in \{S, A, R\}$. Choose the generator $\mathcal{G}_{\mathcal{O}}$ of boundary $T_{\mathcal{O}} = \partial V_{\mathcal{O}} \cong \mathbb{S}^2 \times \mathbb{S}^1$ fundamental group which is homologous to $\mathcal{O}$ in $V_{\mathcal{O}} \cong \mathbb{D}^3 \times \mathbb{S}^1$. By definition the manifold T_S is secant for all flow trajectories except the periodic ones. Since the flow in some tubular neighborhood of S is topologically equivalent to suspension, the set $K_S = W^u_S \cap T_S$ is homeomorphic to the Klein bottle. Let λ_S, μ_S be the knots (simple closed curves), which are generators of the fundamental group $\pi_1(K_S)$ with relation $[\lambda_S * \mu_S] = [\mu_S^{-1} * \lambda_S]$. We will call the curve μ_S *meridian* and the curve λ_S *longitude*. By virtue of Proposition 3 the Klein bottle longitude embedded in $\mathbb{S}^2 \times \mathbb{S}^1$ is a generator of fundamental group $\pi_1(\mathbb{S}^2 \times \mathbb{S}^1)$. Consider the longitude λ_S be oriented in such way that its homotopy type $\langle \lambda_S \rangle$ in T_S coincide with type $\langle \mathcal{G}_S \rangle$. So, the set $K_A = W^u_S \cap T_A$ is the Klein bottle with longitude λ_A which is pointwise transferred along the flow f^t orbits from λ_S.

Since the flow in some tubular neighborhood of S is topologically equivalent to suspension the set $\gamma_S = W^s_S \cap T_S$ is a knot in T_S, wrapping around $\mathcal{G}_S$ twice. We will assume that the knot γ_S is oriented in such way that its homotopy type $\langle \gamma_S \rangle$ on T_S coincides with homotopy type of $2\langle \mathcal{G}_S \rangle$. So the set $\gamma_R = W^s_S \cap T_R$ is a knot in T_R and its orientation is induced by the flow f^t from γ_S.

Lemma 2. *Let $f^t \in G_3^{-1}(M^4)$ then the following conditions hold:*

1. $\langle \lambda_A \rangle = \delta_A \langle \mathcal{G}_A \rangle$, $\delta_A \in \{-1, +1\}$ *in* T_A*;*
2. $\langle \gamma_R \rangle = \delta_R \langle \mathcal{G}_R \rangle$, $\delta_R \in \{-1, +1\}$ *in* T_R*.*

Let

$$C_{f^t} = (\delta_A, \delta_R).$$

Theorem 1. *Flows f^t, $f'^t \in G_3^{-1}(M^4)$ are topologically equivalent if and only if $C_{f^t} = C_{f'^t}$.*

Theorem 2. *For any element $C \in \mathbb{S}^0 \times \mathbb{S}^0$ there exists a flow $f^t \in G_3^{-1}(M^4)$ such that $C = C_{f^t}$.*

Theorem 3. *The only 4-manifold that is ambient for a flow of the class $G_3^-(M^4)$ is $\mathbb{S}^3 \times \mathbb{S}^1$. Moreover $G_3^-(\mathbb{S}^3 \times \mathbb{S}^1)$ consists of eight classes of topological equivalence.*

Note that weakening the saddle orbit twistedness condition fundamentally changes the picture. For example, in [PS20a] non-singular flows that are suspensions over Pixton diffeomorphisms on a three-dimensional sphere are considered. It is proved that in the class under consideration there exist flows with wildly embedded invariant manifolds of the saddle orbit. Moreover, there are an infinite number of topological equivalence classes for such flows.

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2 Flows of the class $G_3^{-1}(M^4)$

2.1 Structure of periodic orbits

This section is devoted to proof of Lemma 1: non-wandering set of any flow $f^t \in G_3^{-1}(M^4)$ consists of three periodic orbits S, A, R, saddle, attracting and repelling respectively.

Proof. The proof is based on the following representation of the ambient manifold M^4 of the NMS-flow f^t with the set of periodic orbits Per_{f^t} (see, for example, [Sma67])

$$M^4 = \bigcup_{\mathcal{O}\in Per_{f^t}} W^u_{\mathcal{O}} = \bigcup_{\mathcal{O}\in Per_{f^t}} W^s_{\mathcal{O}}, \tag{1}$$

as well as the asymptotic behavior of invariant manifolds

$$\mathrm{cl}(W^u_{\mathcal{O}}) \setminus W^u_{\mathcal{O}} = \bigcup_{\tilde{\mathcal{O}}\in Per_{f^t}\,:\, W^u_{\tilde{\mathcal{O}}}\cap W^s_{\mathcal{O}}\neq\emptyset} W^u_{\tilde{\mathcal{O}}}, \tag{2}$$

$$\mathrm{cl}(W^s_{\mathcal{O}}) \setminus W^s_{\mathcal{O}} = \bigcup_{\tilde{\mathcal{O}}\in Per_{f^t}\,:\, W^s_{\tilde{\mathcal{O}}}\cap W^u_{\mathcal{O}}\neq\emptyset} W^s_{\tilde{\mathcal{O}}}. \tag{3}$$

In particular, it follows from eq. (1) that any NMS-flow has at least one attracting orbit and at least one repulsive one. Moreover, if an NMS-flow has a saddle periodic orbit, then the basin of any attracting orbit has a non-empty intersection with an unstable manifold of at least one saddle orbit (see Proposition 2.1.3 [GMP16]) and a similar situation with the basin of a repulsive orbits.

Now let $f^t \in G_3^{-1}(M^3)$ and S be its only saddle orbit. It follows from eq. (2) that $W_S^u \setminus S$ intersects only basins of attracting orbits. Since the set $W_S^u \setminus S$ is connected and the basins of attracting orbits are open, then W_S^u intersects exactly one such basin. Denote by A the corresponding attracting orbit. Since there is only one saddle orbit, there is only one attracting orbit. Similar reasoning for W_S^s leads to the existence of a unique repulsive orbit R. □

2.2 Canonical neighborhoods of periodic orbits

Recall the definition of a suspension. Let $\varphi : M^3 \to M^3$ be a diffeomorphism of a 3-manifold. We define the diffeomorphism $g_\varphi : M^3 \times \mathbb{R}^1 \to M^3 \times \mathbb{R}^1$ by the formula

$$g_\varphi(x_1, x_2, x_3, x_4) = (\varphi(x_1, x_2, x_3), x_4 - 1).$$

Then the group $\{g_\varphi^n\} \cong \mathbb{Z}$ acts freely and discontinuously on $M^3 \times \mathbb{R}^1$, whence the orbit space $\Pi_\varphi = M^3 \times \mathbb{R}^1 g_\varphi$ is a smooth 4-manifold, and the natural projection $v_\varphi : M^3 \times \mathbb{R}^1 \to \Pi_\varphi$ is a covering. At the same time, the flow $\xi^t : M^3 \times \mathbb{R}^1 \to M^3 \times \mathbb{R}^1$, given by the formula

$$\xi^t(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4 + t),$$

induces the flow $[\varphi]^t = v_\varphi \xi^t v_\varphi^{-1} : \Pi_\varphi \to \Pi_\varphi$. The flow $[\varphi]^t$ is called the *suspension of the diffeomorphism* φ.

We define the diffeomorphisms $a_0, a_1, a_2, a_3 : \mathbb{R}^3 \to \mathbb{R}^3$ by the formulas

$$a_3(x_1, x_2, x_3) = (2x_1, 2x_2, 2x_3),\ a_0 = a_3^{-1},$$

$$a_{\pm 1}(x_1, x_2, x_3) = (\pm 2x_1, \pm 1/2x_2, 1/2x_3),\ a_{\pm 2} = a_{\pm 1}^{-1}.$$

Let

$$V_0 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 |\ 4^{x_4}x_1^2 + 4^{x_4}x_2^2 + 4^{x_4}x_3^2 \leqslant 1\},$$

$$V_{\pm 1} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 |\ 4^{-x_4}x_1^2 + 4^{x_4}x_2^2 + 4^{x_4}x_3^2 \leqslant 1\},$$

$$V_{\pm 2} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 |\ 4^{-x_4}x_1^2 + 4^{-x_4}x_2^2 + 4^{x_4}x_3^2 \leqslant 1\},$$

$$V_3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 |\ 4^{-x_4}x_1^2 + 4^{-x_4}x_2^2 + 4^{-x_4}x_3^2 \leqslant 1\}.$$

For $i \in \{0, \pm1, \pm2, 3\}$ we set $v_i = v_{a_i}$, $T_i = \partial V_i$ and $\mathbb{V}_i = v_i(V_i)$, $\mathbb{T}_i = v_i(T_i)$.

The following statement, proved by M. Irwin [Irw70], describes the behavior of flows in a neighborhood of hyperbolic periodic orbits.

Proposition 1 (M. Irwin [Irw70]). *If $\mathcal{O}$ is a hyperbolic orbit of a flow $f^t : M^4 \to M^4$ defined on an orientable 4-manifold M^4, then there exists a tubular neighborhood $V_{\mathcal{O}}$ of the orbit $\mathcal{O}$ such that the flow $f^t\big|_{V_{\mathcal{O}}}$ is topologically equivalent, by means of some homeomorphism $H_{\mathcal{O}}$, to one of the following streams:*

- $[a_0]^t|_{\mathbb{V}_0}$ *if $\mathcal{O}$ is an attracting orbit;*
- $[a_1]^t|_{\mathbb{V}_1}$ *if $\mathcal{O}$ is a non-twisted saddle orbit with a two-dimensional unstable manifold $W^u_{\mathcal{O}}$;*
- $[a_{-1}]^t|_{\mathbb{V}_{-1}}$ *if $\mathcal{O}$ is a twisted saddle orbit with a two-dimensional unstable manifold $W^u_{\mathcal{O}}$;*
- $[a_2]^t|_{\mathbb{V}_2}$ *if $\mathcal{O}$ is a non-twisted saddle orbit with an unstable 3-manifold $W^u_{\mathcal{O}}$;*
- $[a_{-2}]^t|_{\mathbb{V}_{-2}}$ *if $\mathcal{O}$ is a twisted saddle orbit with an unstable 3-manifold $W^u_{\mathcal{O}}$;*
- $[a_3]^t|_{\mathbb{V}_3}$ *if $\mathcal{O}$ is a repelling orbit.*

The neighborhood $V_{\mathcal{O}} = H_{\mathcal{O}}(\mathbb{V}_i)$, $i \in \{0, \pm1, \pm2, 3\}$ described in Proposition 1 is called *canonical neighborhood* of the periodic orbit $\mathcal{O}$.

When proving topological equivalence, we will use the following fact, which follows from the proof of Theorem 4 and Lemma 4 in [PS22a], and can also be found in [Uma90] (Theorem 1.1).

Proposition 2. *A homeomorphism $h_i : \partial\mathbb{V}_i \to \partial\mathbb{V}_i$ for $i \in \{0, 3\}$ extends to a homeomorphism $H_i : \mathbb{V}_i \to \mathbb{V}_i$ realizing the equivalence of the flows $[a_i]^t$ with itself if and only if the induced isomorphism $h_{i*} : \pi_1(\partial\mathbb{V}_i) \to \pi_1(\partial\mathbb{V}_i)$ is identical.*

2.3 Trajectory mappings

Consider a flow $f^t: M^4 \to M^4$ from the set $G_3^{-1}(M^3)$. Then $V_A = H_A(\mathbb{V}_0)$, $V_R = H_R(\mathbb{V}_3)$, $V_S = H_S(\mathbb{V}_{-2})$. Let $\Gamma = \{(x_1, x_2, x_3, x_4) \in T_{-2} | \ 4^{x_4} x_3^2 = 1/2\}$, $\Gamma^u = Ox_2x_3x_4 \cap T_{-2}$, $\Gamma^s = Ox_1x_4 \cap T_{-2}$. By construction, the set T_{-2} is homeomorphic to $\mathbb{S}^2 \times \mathbb{R}$, the set Γ consists of two surfaces, each of which is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$, dividing T_{-2} into three connected components, one of which N^u contains the cylinder $\Gamma^u \cong \mathbb{S}^1 \times \mathbb{R}$, and the union N^s the other two contains a pair of $\Gamma^s \cong \mathbb{S}^0 \times \mathbb{R}$ curves, one curve in each component. Then on T_S

- $K_S = H_S(v_{-2}(\Gamma^u)))$ is a Klein bottle;
- $\gamma_S = H_S(v_{-2}(\Gamma^s)))$ is a knot winding twice around the generator $\mathcal{G}_S$;
- $N_S^u = H_S(v_{-2}(\mathrm{cl}(N^u)))$ is a tubular neighborhood of K_S;
- $N_S^s = H_S(v_{-2}(\mathrm{cl}(N^s)))$ is a tubular neighborhood of γ_S;
- $\partial N_S^u = \partial N_S^s = H_S(v_{-2}((\Gamma)))$ is a two-dimensional torus.

Let

$$N_R^s = \left(\bigcup_{t>0,\, w \in N_S^s} f^{-t}(w) \right) \cap T_R, \ N_R^u = T_R \setminus N_R^s,$$

$$N_A^u = \left(\bigcup_{t>0,\, w \in N_S^u} f^{t}(w) \right) \cap T_A, \ N_A^s = T_A \setminus N_A^u$$

and introduce the following mappings:

- using Poincaré map between $N_R^u \subset T_R$ and N_S^s we define a continuous function $\tau_R: N_R^u \to \mathbb{R}^+$ so that $f^{\tau_R(r)}(r) \in N_S^s$ for $r \in N_R^s$. Next, we continuously extend it to $\tau_R: T_R \to \mathbb{R}^+$ and define the set $\mathcal{T} = \bigcup\limits_{r \in N_R^u} f^{\tau_R(r)}(r)$ which does not intersect the torus T_A if V_A is small enough. We set $\mathcal{T}_R = \mathcal{T} \cup N_S^s$ and define a homeomorphism $\psi_R: T_R \to \mathcal{T}_R$ by the formula $\psi_R(r) = f^{\tau_R(r)}(r)$, denote by $\mathcal{V}_R$ the closure of the connected component of the set $M^4 \setminus \mathcal{T}_R$, containing R;

- we set $\mathcal{T}_A = \mathcal{T} \cup N^u_S$. Since future orbits, that intersect $\mathcal{T}$ go towards A a continuous function $\tau_A : T_A \to \mathbb{R}^+$ such that $f^{-\tau_A(a)}(a) \in \mathcal{T}_A$ for $a \in T_A$ is uniquily defined; note that $\partial\mathcal{T} = \partial N^u_S = \partial N^s_S$. So, we define a homeomorphism $\psi_A : T_A \to \mathcal{T}_A$ by the formula $\psi_A(a) = f^{-\tau_A(a)}(a)$, denote by $\mathcal{V}_A$ the closure of the connected component of the set $M^4 \setminus \mathcal{T}_A$ containing A.

We will call the introduced homeomorphisms ψ_R, ψ_A *trajectory maps*. Note that the ambient manifold M^4 is represented as

$$M^4 = \mathcal{V}_A \cup V_S \cup \mathcal{V}_R.$$

Note that

$$\mathcal{V}_A \cap \mathcal{V}_R = \mathcal{T}, \quad \gamma_S \subset N^u_S = \mathcal{V}_A \cap V_S, \quad K_S \subset \mathcal{N}^s_S = V_R \cap V_S. \tag{4}$$

Moreover, in the manifolds $\mathcal{V}_A$ and $\mathcal{V}_R$ the flow f^t is topologically equivalent to the suspensions $[a_0]^t$ and $[a_3]^t$, respectively.

3 Homotopy types of knots λ_A, γ_R

In this section, we will prove Lemma 2. To do this, we first describe the properties of the embedding of the Klein bottle into the manifold $\mathbb{S}^2 \times \mathbb{S}^1$.

Recall that the Klein bottle $\mathbb{K}$ is the square $[0,1] \times [0,1]$ with sides glued by the relation

$$(x, 0) \sim (x, 1), \quad (0, y) \sim (0, 1 - y).$$

Let $\upsilon : [0,1] \times [0,1] \to \mathbb{K}$ be the natural projection, then the curves

$$\lambda = \upsilon([0,1] \times \{1/2\}),\ \mu = \upsilon(\{0\} \times [0,1])$$

are generators of the fundamental group $\pi_1(\mathbb{K})$ with relation

$$[\lambda * \mu] = [\mu^{-1} * \lambda],$$

where the curve λ is called *longitude* and the curve μ is called *meridian*.

It is well known that the Klein bottle does not embed into $\mathbb{R}^3$, however, it can be embeddable into $\mathbb{S}^2\times\mathbb{S}^1$, for example by defining the embedding $\tilde{e}_0 : [0,1]\times[0,1] \to \mathbb{S}^2\times\mathbb{S}^1$ by the formula

$$\tilde{e}_0(x,y) = \left(\sin\pi x\cos 2\pi y,\ \cos\pi x\cos 2\pi y,\ \sin 2\pi y, e^{2\pi i x}\right)$$

and noticing that $\tilde{e}_0(x,y) = \tilde{e}_0(x',y') \iff (x,y) \sim (x',y')$. Then (see, for example, [Kos80, Chapter 5])

$$e_0 = \tilde{e}_0 v^{-1} : \mathbb{K} \to \mathbb{S}^2\times\mathbb{S}^1$$

is the desired embedding of the Klein bottle in $\mathbb{S}^2\times\mathbb{S}^1$. Let

$$K_0 = e_0(\mathbb{K}).$$

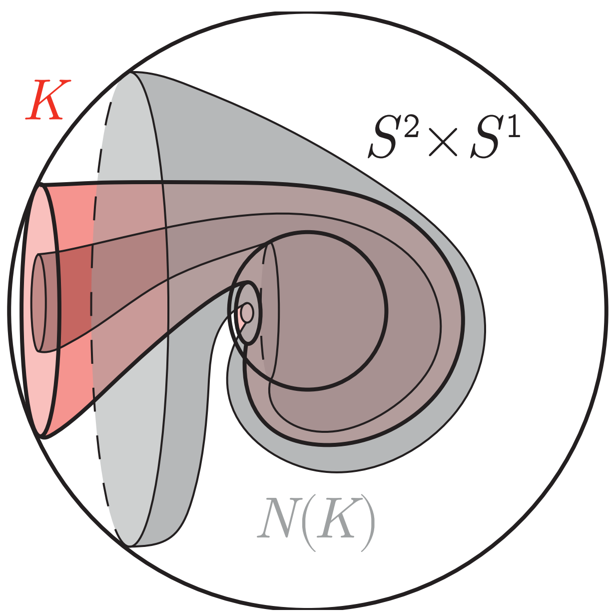

Figure 1: Bottle of Klein in $\mathbb{S}^2\times\mathbb{S}^1$

Proposition 3 (Proposition 1.4, [BGP02]). *Let $e : \mathbb{K} \to \mathbb{S}^2\times\mathbb{S}^1$ be an embedding Klein bottles $\mathbb{K}$, $K = e(\mathbb{K})$, $N(K) \subset \mathbb{S}^2\times\mathbb{S}^1$ be a tubular neighborhood of K and $V(K) = \mathbb{S}^2\times\mathbb{S}^1 \setminus \operatorname{int} N(K)$ (see Figure 1). Then:*

1) *the curve* $e(\lambda)$ *is a generator of the fundamental group* $\pi_1(\mathbb{S}^2 \times \mathbb{S}^1)$*;*

2) *the set* $V(K)$ *is a solid torus whose meridian is homotopic to the curve* $e(\mu)$*;*

3) *there exists an orientation-preserving homeomorphism* $h: \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$ *such that* $h(K) = K_0$ *and* $h_* = id: \pi_1(\mathbb{S}^2 \times \mathbb{S}^1) \to \pi_1(\mathbb{S}^2 \times \mathbb{S}^1)$.

Proposition 4 (Proposition 4.2, [GMP16])**.** *A knot* γ *in manifold* $\mathbb{S}^2 \times \mathbb{S}^1$ *is trivial if and only if there exists tubular neighbourhood* $N(\gamma)$ *in* $\mathbb{S}^2 \times \mathbb{S}^1$ *such that the manifold* $(\mathbb{S}^2 \times \mathbb{S}^1) \setminus N(\gamma)$ *is homeomorphic to solid torus.*

It remains to prove Lemma 2. To do this, recall that we have chosen $T_{\mathcal{O}} = \partial V_{\mathcal{O}} \cong \mathbb{S}^2 \times \mathbb{S}^1$, $\mathcal{O} \in \{A, S, R\}$ generator $\mathcal{G}_{\mathcal{O}}$ of the fundamental group of $T_{\mathcal{O}}$, homologous in $V_{\mathcal{O}} \cong \mathbb{D}^3 \times \mathbb{S}^1$ orbit $\mathcal{O}$. Due to the fact that canonical neighborhoods of periodic orbits can be chosen so that $M^4 = \mathcal{V}_A \cup V_S \cup \mathcal{V}_R$ (see Section 2.3), everywhere below we assume that $V_A = \mathcal{V}_A$, $V_R = \mathcal{V}_R$.

We also established (see eq. (4)) that the set $K_S = W^u_S \cap T_S$ is a Klein bottle on $T_S \cong \mathbb{S}^2 \times \mathbb{S}^1$ and oriented its parallel λ_S so that $\langle \lambda_S \rangle = \langle \mathcal{G}_S \rangle$ on T_S. Since the set $K_A = W^u_S \cap T_A$ coincides with K_S, then $\lambda_A = \lambda_S$. We also established that the set $\gamma_S = W^s_S \cap T_S$ is a knot on T_S and oriented so that $\langle \gamma_S \rangle = 2\langle \mathcal{G}_S \rangle$ to T_S. Since the set $\gamma_R = W^s_S \cap T_R$ coincides with γ_S, then $\gamma_R = \gamma_S$.

Let us show that the knots λ_A, γ_R are generators in the fundamental groups of the manifolds T_A, T_R, respectively.

Proof. Since T_A is homeomorphic to the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ and λ_A is a parallel of the Klein bottle $K_A \subset T_A$, then item 1) of Proposition 3 implies that λ_A is a generator in the fundamental group of T_A, that is, on T_A

$$\langle \lambda_A \rangle = \delta_A \langle \mathcal{G}_A \rangle$$

for $\delta_A \in \{-1, +1\}$. The set $N(K_A) = N^u_S$ is a tubular neighborhood of the Klein bottle K_A in T_A. It follows from item 2) of Proposition 3 that the set $V(K_A) = T_A \setminus \operatorname{int} N(K_A)$ is a solid torus. The set $N(\gamma_R) = N^s_S$ is a tubular neighborhood of the knot γ_R in T_R. On the other hand $T_R \setminus \operatorname{int} N(\gamma_R) = V(K_A)$. Thus, the complement to the tubular neighborhood of γ_R in T_R is a solid torus. By Proposition 4, γ_R is a generator in T_R and, therefore,

$$\langle \gamma_R \rangle = \delta_R \langle \mathcal{G}_R \rangle$$

for $\delta_R \in \{-1, +1\}$. □

4 Classification of flows of the set $G_3^{-1}(M^4)$

In this section, we will prove Theorem 1.

Proof.

Necessity. Let flows f^t and f'^t have invariants $C_{f^t} = (\delta_A, \delta_R),\ C_{f'^t} = (\delta_{A'}, \delta_{R'})$ and are topologically equivalent by the homeomorphism $H:\ M^4 \to M^4$. Let us show that $C_{f^t} = C_{f'^t}$.

Let $h_A = H\big|_{T_A}$ and $T_{A'} = h(T_A)$. Then by Proposition 2

$$\langle \mathcal{G}_{A'} \rangle = h_{A*} \langle \mathcal{G}_A \rangle.$$

It follows from item 3) of Proposition 3 that $\lambda_{A'} = h_A(\lambda_A)$ is the parallel of the Klein bottle $K_{A'}$. Since the longitude λ_A of the Klein bottle is oriented consistent with the saddle orbit S and H transforms the orbit S into the orbit S' with orientation preservation, then $\lambda_{A'}$ is oriented consistent with the saddle orbit S'. On the other side, by Lemma 2,

$$\langle \lambda_A \rangle = \delta_A \langle \mathcal{G}_A \rangle,\ \langle \lambda_{A'} \rangle = \delta_{A'} \langle \mathcal{G}_{A'} \rangle,$$

whence, by virtue of a simple chain of equalities

$$\delta_{A'} \langle \mathcal{G}_{A'} \rangle = h_{A*}(\delta_A \langle \mathcal{G}_A \rangle) = \delta_A \langle \mathcal{G}_{A'} \rangle,$$

we get that $\delta_A = \delta_{A'}$. It is proved similarly that $\delta_R = \delta_{R'}$. Thus $C_{f^t} = C_{f'^t}$.

Sufficiency. Let the flows f^t and f'^t have equal invariants $C_{f^t} = (\delta_A, \delta_R),\ C_{f'^t} = (\delta_{A'}, \delta_{R'})$. Let us show that the flows f^t and f'^t are topologically equivalent.

Proposition 1 implies that the homeomorphism

$$H\big|_{V_S} = H_{S'}H_S^{-1} : V_S \to V_{S'}$$

is topological equivalence homeomorphism of the flows $f^t\big|_{V_S}$ and $f'^t\big|_{V_{S'}}$. It remains to extend this homeomorphism to $\mathcal{V}_A$ and $\mathcal{V}_R$.

The homeomorphism H is already defined on the set $\mathcal{T}_A \cap T_S$, which is a tubular neighborhood $N(K_A)$ of the Klein bottle K_A. By item 2) of Proposition 3 the set $V(K_A) = T_A \setminus \operatorname{int} N(K_A)$ is a solid torus whose meridian is homotopic to the meridian μ_A of the Klein bottle K_A to $N(K_A)$. It follows from the properties of the homeomorphism H that $K_{A'} = H(K_A)$ and $N(K_{A'}) = H(N(K_A))$ is a tubular neighborhood of the Klein bottle $K_{A'}$. By point 2) of Proposition 3 the set $V(K_{A'}) = T_{A'} \setminus \operatorname{int} N(K_{A'})$ is a solid torus whose meridian is homotopic to the meridian $\mu_{A'}$ of the Klein bottle $K_{A'}$ in $N(K_{A'})$. Since any homeomorphism of the Klein bottle does not change the homotopy class of the meridian (see, for example, [Lic63, Lemma 5]), the homeomorphism $H : \partial V(K_A) \to \partial V(K_{A'})$ extends to the homeomorphism $H : V(K_A) \to V(K_{A'})$ (see, for example, [Rol03, Exercise 2E5]). Thus H is defined on $\mathcal{T}_A$ and $\mathcal{T}_R$.

Since the parallel λ_A ($\lambda_{A'}$) of the Klein bottle is oriented consistent with the saddle orbit S (S') and H transforms the orbit S into the orbit S' orientation-preserving, then $H_*(\langle\lambda_A\rangle) = \langle\lambda_{A'}\rangle$. Since $\delta_A = \delta_{A'}$, then $H_*(\delta_A\langle\lambda_A\rangle) = \delta_{A'}\langle\lambda_{A'}\rangle$ and hence $H_*(\langle\mathcal{G}_A\rangle) = \langle\mathcal{G}_{A'}\rangle$. By Proposition 2 the homeomorphism $H\big|_{\mathcal{T}_A}$ extends to $\mathcal{V}_A$ by a homeomorphism realizing the equivalence of flows $f^t\big|_{\mathcal{V}_A}$ and $f'^t\big|_{\mathcal{V}_{A'}}$. Similarly, H can be extended to $\mathcal{V}_R$. Thus, the homeomorphism H is defined on the whole M^4 and realizes the equivalence of the flows f^t, f'^t. □

5 Realization of flows by admissible set

In this section, we will prove Theorem 2: for any element $C \in \mathbb{S}^0 \times \mathbb{S}^0$ there is a flow $f^t \in G_3^{-1}(M^4)$ such that $C = C_{f^t}$.

Proof. Let us construct the flow $f^t:\ \mathbb{S}^2\times\mathbb{S}^1\to\mathbb{S}^2\times\mathbb{S}^1$ with the invariant $C_{f^t}=(+1,+1)$ as a suspension over the sphere diffeomorphism $\zeta:\ \mathbb{S}^2\to\mathbb{S}^2$ with three periodic orbits. To do this, we describe the construction of the diffeomorphism ζ.

Let $\chi^t:\ \mathbb{R}^3\to\mathbb{R}^3$ be the flow defined by system of equations:

$$\begin{cases}\dot{x}=-x\\ \dot{y}=-y\\ \dot{z}=-z(z-1)(z+1).\end{cases}$$

and the diffeomorphism $q:\ \mathbb{R}^3\to\mathbb{R}^3$ be defined by the formula:

$$q(x,y,z)=(x,-y,-z).$$

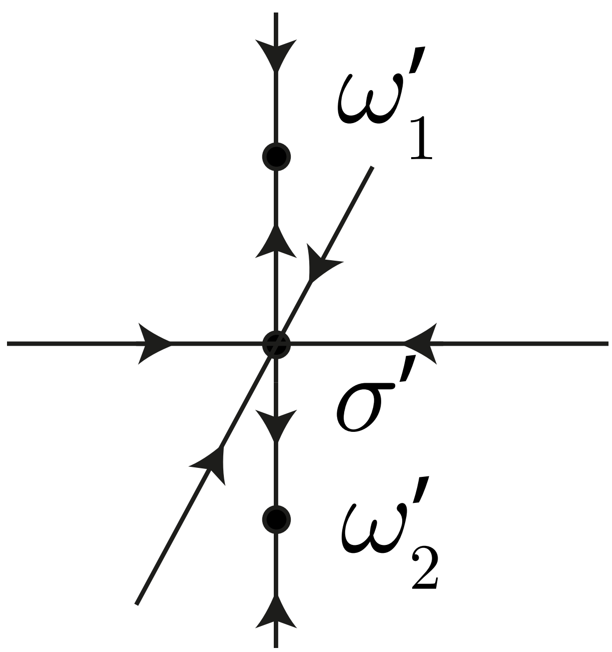

Figure 2: Flow χ^t phase portrait

Using stereographic projection (see Figure 3) $\vartheta:\ \mathbb{S}^3\setminus\{N\}\to\mathbb{R}^3$ ($N=(0,0,0,1), S=$

$(0, 0, 0, -1)$) by the given formula:

$$\vartheta(x_1, x_2, x_3, x_4) = \left(\frac{x_1}{1-x_4}, \frac{x_2}{1-x_4}, \frac{x_3}{1-x_4}\right).$$

project the diffeomorphism $q\chi^1$ onto $\mathbb{S}^3$:

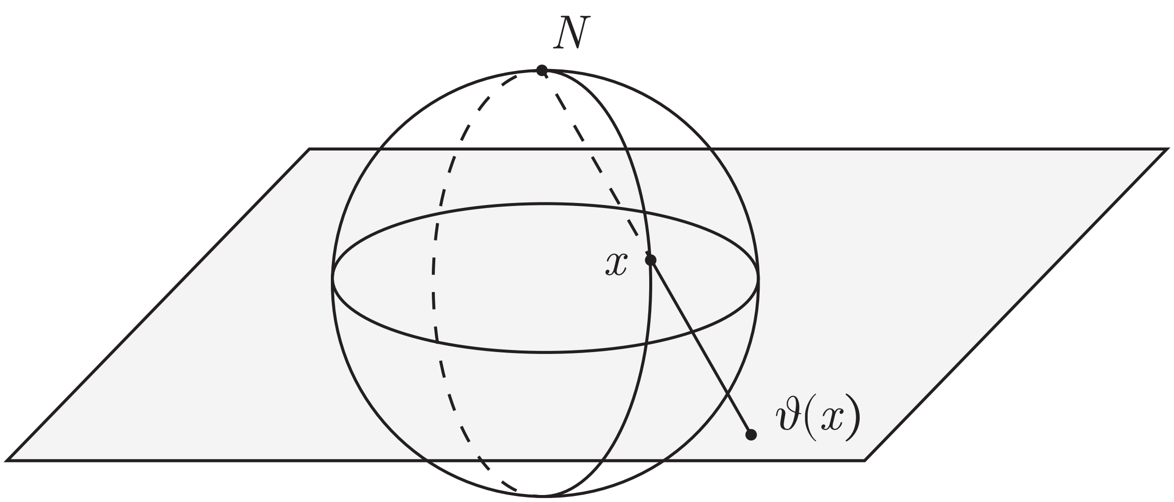

Figure 3: Stereographic projection

$$f(x) = \begin{cases} \vartheta^{-1} q\chi^1 \vartheta(x), & x \notin \{N, S\} \\ S, & x = N, \\ N, & x = S \end{cases}$$

Non-wandering set of diffeomorphism f consists of four periodic points:

- hyperbolic sink orbit ω_1, ω_2 of period 2: $\omega_1 = \vartheta^{-1}(0,\ 0,\ 1)$, $\omega_2 = \vartheta^{-1}(0,\ 0,\ -1)$;
- hyperbolic saddle $\sigma = \vartheta^{-1}(0,\ 0,\ 0)$;
- hyperbolic sourse $\alpha = N$.

Then the flow $f^t = [f]^t$ belongs to the class $G_3^{-1}(M^3)$ and $C_{f^t} = (+1, +1)$. By construction, f is an orientation-preserving diffeomorphism of the 3-sphere, and hence the ambient manifold of the suspension $[f]^t$ is homeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$.

We construct the rest of the flows of class $G_3^{-1}(M^4)$ by modifying the constructed flow f^t in neighborhoods of attracting and repelling orbits such that its their stability does not change but the orbits go in opposite direction.

Let $\mathcal{O}$ be an attractive or repelling periodic orbit of f^t and $V_{\mathcal{O}} = V_{\mathcal{O}}^0$ be its canonical neighborhood, $V_{\mathcal{O}}^t = f^t(V_{\mathcal{O}}^0)$. Without loss of generality, we assume that $V_A^{-1} \cap V_R^1 = \varnothing$. Let $\vec{v}(x)$ denote the vector field induced by the flow f^t on $\mathbb{S}^3 \times \mathbb{S}^1$.

Recall that $V_{\mathcal{O}} \cong \mathbb{R}^3 \times \mathbb{S}^1$. For points x that belong to the basin of the orbit $\mathcal{O}$. Let $\vec{n}_{\mathcal{O}}(x)$ denote the field of unit outward normals to the hypersurfaces $\partial V_{\mathcal{O}}^t \cap \{(x,y) \in V_{\mathcal{O}} \mid y = const\}$ in $\{(x,y) \in V_{\mathcal{O}} \mid y = const\}$ and let $s_{\mathcal{O}}(x) \in \mathbb{R}$ be the time such that $f^{s_{\mathcal{O}}(x)}(x) \in \partial V_{\mathcal{O}}^0$. We define the vector field $\vec{v}'(x)$ on $\mathbb{S}^3 \times \mathbb{S}^1$ by the formulas

$$\vec{v}'(x) = \begin{cases} (1 - s_A^2(x))\vec{n}(x) + s_A^2(x)\vec{v}(x), & x \in V_A^{-1} \setminus V_A^1 \\ (1 - s_R^2(x))\vec{n}(x) + s_R^2(x)\vec{v}(x), & x \in V_R^1 \setminus V_R^{-1} \\ v(x), & \text{otherwise} \end{cases}$$

and denote by f'^t the flow it induces on $\mathbb{S}^3 \times \mathbb{S}^1$.

Recall, that flow $f'^t\big|_{V_{\mathcal{A}}}$ $(f'^t\big|_{V_{\mathcal{R}}})$ is congugated to $[a_0]^t\big|_{\mathbb{V}_0}$ $([a_3]^t\big|_{\mathbb{V}_3})$ by a homeomorphism $h_A(h_R)$. For $\delta \in \{-1, +1\}$ we define the diffeomorphism $\bar{w}_\delta : \mathbb{R}^3 \to \mathbb{R}^3$ by the formula:

$$\bar{w}_\delta(x_1, x_2, x_3, x_4) = (4^{x_4}x_1, 4^{x_4}x_2, 4^{x_4}x_3, -x_4)\,.$$

Note, that $\bar{w}_\delta$ preserve V_0 and V_3. Next, define diffeomorphisms

$$w_\delta^A = h_A \bar{w}_\delta h_A^{-1}, \quad w_\delta^R = h_R \bar{w}_\delta h_R^{-1}$$

Note, that $\vec{v}'$ in invariant under w_δ^A, w_δ^R .

For $C = (\delta_A, \delta_R) \in \mathbb{S}^0 \times \mathbb{S}^0$ we induce the flow f_C^t on $\mathbb{S}^3 \times \mathbb{S}^1$ by vector field

$$\vec{v}_C(x) = \begin{cases} (dw_{\delta_A}^A)\vec{v}' w_{\delta_A}^A(x), & x \in V_A \\ (dw_{\delta_R}^R)\vec{v}' w_{\delta_R}^R(x), & x \in V_R \\ \vec{v}'(x), & \text{otherwise.} \end{cases}$$

It is easy to see that the flow's f_C^t invariant is C and its ambient manifold is $\mathbb{S}^3 \times \mathbb{S}^1$. $\square$

6 Ambient manifolds of the flow of class $G_3^-(M^4)$

In this section, we prove Theorem 3: the only 4-manifold admitting $G_3^-(M^4)$ flows is $\mathbb{S}^3 \times \mathbb{S}^1$. Moreover, the set $G_3^-(\mathbb{S}^3 \times \mathbb{S}^1)$ consists of exactly eight equivalence classes of the considered flows.

Proof. Assume that $f^t \in G_3^-(M^4)$ then by Theorem 1 f^t is topologically equivalent to either the flow f_C^t or flow f_C^{-t} for some $C \in \mathbb{S}^0 \times \mathbb{S}^0$. And since the topological equivalence of flows implies the homeomorphism of their ambient manifolds, the supporting manifold of the flow f^t is homeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$.

Since the elements of the set $C \in \mathbb{S}^0 \times \mathbb{S}^0$ correspond one-to-one to the equivalence classes of flows from $G_3^{-1}(M^4)$, the family $G_3^-(M^4) = G_3^{-2}(M^4) \sqcup G_3^{-1}(M^4)$ contains 8 topological equivalence classes. □

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