Arnold Mathematical Journal
Volume 12, Issue 1, 2026

Symmetric cubic polynomials

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Received 20 Oct 2024; Accepted 22 Jun 2025

Abstract:

We describe a model $\mathcal{M}_3^{comb}$ for the boundary of the connectedness locus $\mathcal{M}_3^{sy}$ of the parameter space of cubic symmetric polynomials $p_c(z) = z^3 - 3c^2z$. We show that there exists a monotone continuous function $\pi : \partial\mathcal{M}_3^{sy} \to \mathcal{M}_3^{comb}$ which is a homeomorphism if $\mathcal{M}_3^{sy}$ is locally connected.

AMS Classification: 37F20;37F10, 37F50

Key words and phrases: Complex dynamics; laminations; Mandelbrot set; Julia set

1 Introduction

A central problem of Complex Dynamics is to describe parameter spaces of holomorphic maps. Investigating the deceptively simple *quadratic family* $f_c(z) = z^2 + c$ led to an explosion of activity in the field. Aided by computer graphics capabilities, mathematicians

DOI: 10.56994/ARMJ.012.001.002

made many interesting discoveries concerning the connectedness locus of the quadratic family, the celebrated *Mandelbrot set* $\mathcal{M}_2$.

One of the first such discoveries, made by Douady and Hubbard [15], was that $\mathcal{M}_2$ is connected. Then the combinatorial description of the structure of the Mandelbrot set was largely carried out in the language of *laminations* introduced by Thurston [42] (see Section 3 and [42] or [6] for precise definitions and other details). Douady constructed a topological *pinched disk* model of $\mathcal{M}_2$; Thurston made this model more explicit and described it in terms of laminations. If $\mathcal{M}_2$ is locally connected (which is still an open question), then it is homeomorphic to its model. The local connectivity of the Mandelbrot set is one of the most important long standing conjectures in the field; if true, it will imply the density of hyperbolicity property of the quadratic family.

The above describes the quadratic version of what one can call the *Douady–Hubbard–Thurston program*, i.e. a two step approach to studying some complex one-dimensional parameter space of polynomials that we now fix. Similar to the quadratic family, its most interesting part is the *connectedness locus*, i.e., the locus of all polynomials in the space with connected Julia sets (in the quadratic case, this is the Mandelbrot set). One needs to prove that the connectedness locus is connected itself. Then two steps are made. On the first step one describes the laminations of the polynomials from the parameter space in question producing in the end the corresponding space of laminations, most likely described itself by a certain *parameter lamination* (like, e.g., Thurston's QML lamination [42]). On the second step one constructs a monotone map from the connectedness locus of interest to the quotient space of the closed unit disk under the parameter lamination. This quotient space can be viewed as a model for the connectedness locus.

The Douady–Hubbard–Thurston program has been implemented for quadratic polynomials, and then for *unicritical polynomials* $z^d + c$ of any degree d, cf. [1, 22, 39, 40], where c is the parameter.

We initiated this program for the space $\mathcal{SCP}$ of *symmetric cubic polynomials* $p_c(z) = z^3 - 3c^2 z$ with *marked critical point* c in papers [8, 9] that serve as a prequel to the present article. Namely, in [8] we investigated the space of *symmetric cubic laminations* and constructed the associated parameter lamination called the *cubic symmetric comajor lamination* C_sCL (see Section 3). In [9] we proved an analog of Lavaurs algorithm for this lamination. Now, let $\mathcal{M}_3^{sy}$ be the *connectedness locus* of $\mathcal{SCP}$. The aim of the present article is to complete the program for the space $\mathcal{SCP}$ and prove the following theorem.

Main Theorem. *The set $\mathcal{M}_3^{sy}$ is a full continuum. There exists a monotone continuous surjective map $\pi : \mathcal{M}_3^{sy} \to \overline{\mathbb{D}}/C_sCL$. If $\mathcal{M}_3^{sy}$ is locally connected, then π is a homeomorphism.*

We are not aware of any other articles in which the Douady–Hubbard–Thurston program is fully implemented for non-unicritical polynomials. We did recently learn of a manuscript by Xavier Buff [12] in which he studies the parameter space of symmetric polynomials of the form $P_{\lambda,d}(z) = \lambda z + z^d$ and shows that the connectedness locus M_d contains the unit disk $\mathbb{D}$ and that every component of $M_d \setminus \overline{\mathbb{D}}$ is homeomorphic to a limb of the Mandelbrot set.

Acknowledgments.

Some figures in the article have been produced with a modified version of *Mandel*, a software written by Wolf Jung. The authors are grateful to the reviewer for careful reading and useful suggestions.

2 Notation and preliminaries

We assume knowledge of basic facts and concepts of complex dynamics. We also use standard notation (such as $J(P)$ for the Julia set of a polynomial P, etc).

Consider the space $\mathcal{SCP}$ of symmetric cubic polynomials $p_c(z) = z^3 - 3c^2z$ with marked critical point c, or, more formally, the space of pairs (p_c, c), which, in turn, is uniquely parameterized by the values of c. Since $p_c = p_{-c}$, every polynomial (except for $p_0(z) = z^3$) shows in $\mathcal{SCP}$ twice. Thus, $\mathcal{SCP}$ is a (branched) two-to-one cover of the moduli space of all odd cubic polynomials, where the moduli space means the quotient space with respect to complex linear conjugacy. The critical points of p_c are c and $-c$, the corresponding critical values are $-2c^3$ and $2c^3$. The *marked cocritical point* of p_c, i.e., the other preimage of $-2c^3 = p_c(c)$, is $-2c$. Subsets A and B of the plane are said to be *mutually symmetric* if $B = -A$. If $A = -A$ we call a set A *symmetric*. Since p_c is odd, the Julia set $J(p_c)$, the filled Julia set $K(p_c)$, and their complements are symmetric. Observe that $p_c(0) = 0$ and $p_c'(z) = p_c'(-z) = 3z^2 - 3c^2$.

Let $\mathcal{M}_3^{sy}$ be the *connectedness locus* of $\mathcal{SCP}$, i.e., the set of all c for which the Julia set of p_c is connected. It is known that the Julia set of p_c is connected if and only if all forward orbits of critical points of p_c are bounded. Since p_c has mutually symmetric critical orbits, we conclude that $c \in \mathcal{M}_3^{sy}$ if and only if the orbit of c or, equivalently of $-2c$ or $2c$, is bounded.

For any $r > 0$ set $\mathbb{D}_r = \{z \in \mathbb{C} \mid |z| < r\}$, and write $\mathbb{D}$ for $\mathbb{D}_1$. Let $\mathbb{S}^1$ be the unit circle. For a set $A \subset \mathbb{C}$, let $\overline{A}$ be its closure and ∂A be its boundary. We use the terms *periodic orbit* and *cycle* interchangeably. External rays to the Julia set of a polynomial P are denoted $R_P(\theta)$ where θ is the argument of the ray (if there is no ambiguity we may omit the polynomial from our notation; also, we write $R_c(\theta)$ instead of $R_{p_c}(\theta)$).

Let X, Y be topological spaces and $f : X \to Y$ be continuous. Then f is said to be *monotone* if $f^{-1}(y)$ is connected for each $y \in Y$. It is known that if f is monotone and X is a continuum then $f^{-1}(Z)$ is connected for every connected $Z \subset f(X)$.

3 Symmetric cubic laminations

Invariant laminations were introduced in [42]; they play a major role in polynomial dynamics. The preceding papers [8, 9] of this series contain an overview, based on [42] and [6]. Here we follow [8] and [9] (see Section 2 of [8] for a detailed discussion).

If a monic polynomial P has a locally connected Julia set $J(P)$, then $P|_{J(P)}$ is topologically conjugate to a suitable quotient of the d-tupling map $\sigma_d : \mathbb{S}^1 \to \mathbb{S}^1$ (where $\sigma_d(z) = z^d$ if

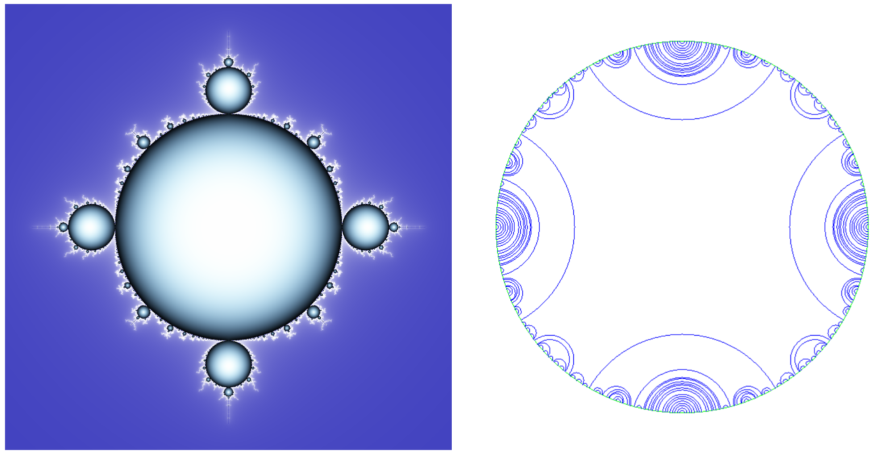


Figure 1: The parameter space of symmetric cubic polynomials $\mathcal{SCP}$ on the left and the Symmetric Cubic Comajor Lamination C_sCL on the right.

$\mathbb{S}^1$ is viewed as the unit circle in $\mathbb{C}$ and as $\sigma_d(\theta) = d\theta$ if $\mathbb{S}^1$ is identified with $\mathbb{R}/\mathbb{Z}$). The quotient is with respect to an equivalence relation $\sim_P$; the *leaves* of the corresponding lamination $\mathcal{L}_P$ are by definition the edges of the convex hulls of all $\sim_P$-classes.

A chord of $\overline{\mathbb{D}}$ with endpoints a, b is denoted by $\overline{ab}$; it is *critical* if $\sigma_d(a) = \sigma_d(b)$ while $a \neq b$. A lamination is normally denoted by $\mathcal{L}$ while the union of all its leaves and $\mathbb{S}^1$ by $\mathcal{L}^*$. For $G \subset \overline{\mathbb{D}}$, denote $G \cap \mathbb{S}^1$ by $\mathcal{V}(G)$ and call the elements of $\mathcal{V}(G)$ *vertices* of G. Call G a *gap* of a lamination $\mathcal{L}$ if it is the closure of a component of $\mathbb{D} \setminus \mathcal{L}^*$. A gap G is said to be *critical* if its image is not a gap or the degree of $\sigma_d|_{\partial G}$ is greater than 1. A *critical set* is a critical leaf or a critical gap. If G is a leaf or a gap of $\mathcal{L}$, then G coincides with the convex hull of $\mathcal{V}(G)$. A gap G is called *infinite (finite)* if and only if $\mathcal{V}(G)$ is infinite (finite).

Let $\mathcal{L}$ be a lamination. The equivalence relation $\sim_\mathcal{L}$ induced by $\mathcal{L}$ is defined by declaring that $x \sim_\mathcal{L} y$ if and only if there exists a finite concatenation of leaves of $\mathcal{L}$ joining x to y. A lamination $\mathcal{L}$ is called a *q-lamination* if the convex hulls of $\sim_\mathcal{L}$-classes are precisely finite gaps or leaves of $\mathcal{L}$. Two distinct chords are called (σ_d-)*siblings* if they have the same σ_d-image.

3.1 Symmetric laminations

Let τ be the rotation of $\overline{\mathbb{D}}$ (or of $\mathbb{S}^1$) by 180° around its center $\mathcal{O}$. Also, given a map $f : X \to X$ we call $x \in X$ *preperiodic of preperiod* $k > 0$ or k-*preperiodic* if $f^k(x)$ is f-periodic while $f^{k-1}(x)$ is not periodic.

Definition 3.1 (Symmetric laminations)**.** A σ_3-invariant lamination $\mathcal{L}$ is called a *symmetric (cubic) lamination* if $\ell \in \mathcal{L}$ implies $\tau(\ell) \in \mathcal{L}$.

Definition 3.2 (length and majors)**.** Given a non-diameter chord ℓ in $\overline{\mathbb{D}}$, define the arc $h(\ell)$ as the shortest arc of $\mathbb{S}^1$ joining the endpoints of ℓ. If ℓ' and ℓ'' are chords such that $h(\ell') \subset h(\ell'')$, then we say that ℓ' is *under* ℓ''. Define the *length* $|\ell|$ of ℓ as the length of $h(\ell)$ divided by 2π in the case when ℓ is not a diameter; if ℓ is a diameter, set $|\ell| = \frac{1}{2}$. Given a symmetric lamination $\mathcal{L}$, call a leaf M a *major of* $\mathcal{L}$ if there are no leaves of $\mathcal{L}$ closer in length to $\frac{1}{3}$ than M.

Let $\Gamma : [0, \frac{1}{2}] \to [0, \frac{1}{2}]$ be a piecewise linear function with slope ± 3 defined as $\Gamma(x) = 3x$ if $0 \le x \le \frac{1}{6}$ and as $\Gamma(x) = |3x - 1|$ if $\frac{1}{6} \le x \le \frac{1}{2}$. Then $|\sigma_3(\ell)| = \Gamma(|\ell|)$. Simple analysis of the dynamics of Γ shows that for any leaf ℓ an eventual image of ℓ has length 0, or length $\frac{1}{2}$, or length which is between $\frac{1}{4}$ and $\frac{5}{12}$.

Suppose that $\frac{1}{4} \le |\ell| \le \frac{5}{12}$. Then there exists a sibling chord ℓ' of ℓ such that the strip S of $\overline{\mathbb{D}}$ between ℓ and ℓ' has two circle arcs on its boundary, each at most $\frac{1}{6}$ long. We also consider chords $\tau(\ell)$ and $\tau(\ell')$ as well as the $\overline{\mathbb{D}}$-strip $\tau(S)$ between them. The union $S \cup \tau(S)$, denoted $\mathrm{SH}(\ell)$, is called the *short strips set* of ℓ.

If a major M of a symmetric lamination $\mathcal{L}$ is critical, there is a unique point $x \in \mathbb{S}^1$ that is not an endpoint of M with the same σ_3-image as M. This point x is called a *comajor (of* $\mathcal{L}$*)*. If M is not critical, then a leaf M' (similar to ℓ' from the above) and leaves $\tau(M)$ and $\tau(M')$ are also majors of $\mathcal{L}$. We set $\mathrm{SH}(\mathcal{L}) = \mathrm{SH}(M)$ in this case and call this set the *short strips set* of $\mathcal{L}$. Let us stress again that $\mathrm{SH}(\mathcal{L})$ is formed by *two* strips and that $\mathrm{SH}(M) = \mathrm{SH}(\tau(M))$. The third sibling $\overline{c}$ of M that is disjoint from $M \cup M'$, is of length at most $\frac{1}{6}$. It is called a *comajor (of* $\mathcal{L}$*)*. Similarly we define a *cocritical set* $\overline{\mathrm{co}}(U)$ of a critical set U as the gap, or the leaf, or the point disjoint from U but with the same image as U.

Because of the symmetry, comajors, majors, etc., come in pairs. A pair of comajors $\overline{c}, \tau(\overline{c})$ of a symmetric lamination $\mathcal{L}$ is called a symmetric *comajor pair*. It is *degenerate* if its elements are points and *non-degenerate* otherwise. For a symmetric lamination $\mathcal{L}$ we often assume that one of its majors is marked; we denote this major by $M_{\mathcal{L}}$ and the corresponding comajor by $\overline{\mathrm{co}}_{\mathcal{L}}$. Observe that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are symmetric laminations such that $\mathrm{SH}(\mathcal{L}_1) \subset \mathrm{SH}(\mathcal{L}_2)$ (e.g., if $\mathcal{L}_2 \subset \mathcal{L}_1$) then the comajors of $\mathcal{L}_1$ are located under the comajors of $\mathcal{L}_2$.

Lemma 3.3 ([8]). *Let ℓ be a leaf of a symmetric lamination $\mathcal{L}$ with $|\ell| \geq \frac{1}{4}$. If $n > 0$ is the least such that $\sigma_3^n(\ell) \subset \mathrm{SH}(\ell)$, then the leaf $\sigma_3^n(\ell)$ non-strictly separates (in $\overline{\mathbb{D}}$) either ℓ from ℓ', or $\tau(\ell)$ from $\tau(\ell')$. Thus, either $\sigma_3^n(\ell)$ equals one of the leaves $\ell, \ell', \tau(\ell), \tau(\ell')$, or it is closer to $\frac{1}{3}$ in length than ℓ. In particular, forward images of majors/comajors of $\mathcal{L}$ never enter the open circle arcs on the boundary of the set $\mathrm{SH}(\mathcal{L})$.*

Lemma 3.3 motivates the next definition.

Definition 3.4 (Legal pairs). If a symmetric pair $\{\overline{c}, \tau(\overline{c})\}$ is either degenerate or satisfies the following conditions:

(a) no two iterated forward images of $\overline{c}, \tau(\overline{c})$ cross, and

(b) no forward image of $\overline{c}$ crosses the interior of $\mathrm{SH}(M_{\overline{c}})$,

then $\{\overline{c}, \tau(\overline{c})\}$ is said to be a *legal pair*.

Lemma 3.5 ([8]). *A legal pair $\{c, \tau(c)\}$ is the comajor pair of the symmetric lamination $\mathcal{L}(c)$. A symmetric pair $\{c, \tau(c)\}$ is a comajor pair if and only if it is legal.*

A symmetric lamination with an infinite gap such that the map σ_3 on it is of degree greater than 1 is called a *Fatou lamination*.

Lemma 3.6 ([8]). *A symmetric lamination is Fatou if and only if it has a preperiodic comajor of preperiod 1.*

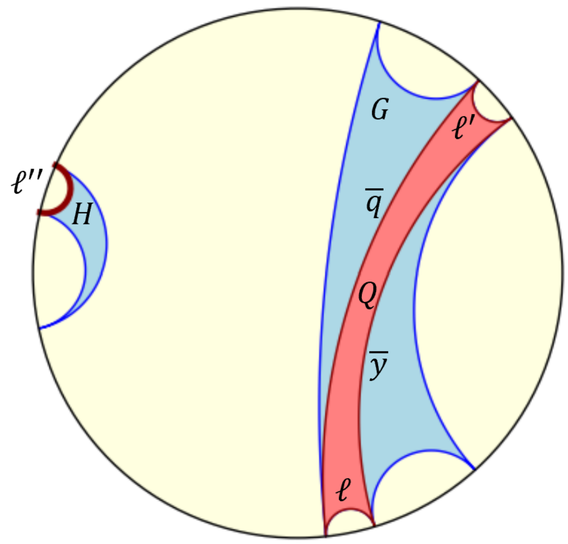


Figure 2: Chords $\overline{q}$ and $\overline{y}$ inserted into a gap G are majors of $\mathcal{L}'$ corresponding to the comajor ℓ''. Note: this figure is intentionally unrealistic; it is not a part of any symmetric cubic lamination. More realistic figures will have Q too narrow.

Suppose that $\mathcal{L}$ is a symmetric q-lamination with two finite critical gaps each of which is preperiodic of preperiod at least two. Then $\mathcal{L}$ is called a *symmetric Misiurewicz lamination*. A symmetric Misiurewicz lamination has a well defined pair of comajors. Suppose that the critical $\sim_{\mathcal{L}}$-classes are gaps G and $\tau(G)$ with at least 6 vertices each. Then there are two cocritical gaps $H \neq G$ and $\tau(H) \neq \tau(G)$ of $\mathcal{L}$ such that $\sigma_3(H) = \sigma_3(G)$ and $\sigma_3(\tau(H)) = \sigma_3(\tau(G))$. One edge of H and one edge of $\tau(H)$ are the comajors of $\mathcal{L}$. Two majors of $\mathcal{L}$ are edges of G that are siblings of the comajor edge of H; two other majors of $\mathcal{L}$ are edges of $\tau(G)$ that are siblings of the comajor edge of $\tau(H)$. While other edges of G and $\tau(G)$ are not siblings of the comajors, they can generate majors of other laminations that are finite tunings of $\mathcal{L}$.

Indeed, suppose that ℓ and ℓ' are two sibling edges of G that are not majors. The convex hull of $\ell \cup \ell'$ is a 4-gon Q with two extra edges $\overline{y}$ and $\overline{q}$ not equal to ℓ or ℓ', see Fig. 2. Construct a new lamination $\mathcal{L}'$ (not a q-lamination) by inserting $\overline{y}$ and $\overline{q}$ in G, pulling them back along the backward orbit of G and then doing the same with $\tau(G)$ and its backward orbit. The majors of $\mathcal{L}'$ are $\overline{y}$ and $\overline{q}$ and their τ-images. If ℓ'' is a leaf of $\mathcal{L}$ which is not an edge of G and is such that $\sigma_3(\ell'') = \sigma_3(\ell)$ then ℓ'' and $\tau(\ell'')$ are the two comajors of $\mathcal{L}'$. Repeating this construction for all pairs of sibling edges of G but the majors, we see that every edge of the cocritical gap H or $\tau(H)$ is a comajor of a certain symmetric lamination which is a tuning of the original symmetric Misiurewicz lamination $\mathcal{L}$. Call cocritical sets of Misiurewicz laminations *Misiurewicz* cocritical sets. By [8, Theorem 3.9], symmetric laminations have no wandering gaps. Therefore, the above is a full description of finite gaps formed by comajors. The cocritical gaps H and $\tau(H)$ described above will be called *Misiurewicz cocritical gaps*; similarly, if a symmetric Misiurewicz q-lamination has critical 4-gons (not 6-gons or higher as was assumed above) we call its comajors *Misiurewicz cocritical leaves*.

Theorem 3.7 ([8, 9]). *The set of non-degenerate comajors of symmetric laminations is a q-lamination $\mathcal{L}$ invariant under τ that induces an equivalence relation $\sim_{\mathcal{L}}$ on $\mathbb{S}^1$. For any non-degenerate comajor $\overline{c}$ (i.e., a leaf of $\mathcal{L}$) one of the following holds.*

1. *It is a two-sided limit leaf in $\mathcal{L}$ which is not eventually periodic.*
2. *It is a preperiodic leaf of $\mathcal{L}$ with preperiod at least 2 which is either a two-sided limit leaf of $\mathcal{L}$ (in which case $\overline{c}$ is a Misiurewicz cocritical leaf), or an edge of a finite gap H of $\mathcal{L}$ whose edges are limits of leaves in $\mathcal{L}$ disjoint from H (in which case H is a Misiurewicz cocritical gap).*
3. *It is a 1-preperiodic comajor of a Fatou lamination and is disjoint from all other leaves of $\mathcal{L}$; all such comajors $\overline{c}$ are dense in $\mathcal{L}$ and all 1-preperiodic angles are endpoint of such comajors.*

Since comajors are leaves of q-laminations, their endpoints are either both not preperiodic, or both preperiodic with the same preperiod and the same period, or both periodic with the same period. All classes of $\sim_{\mathcal{L}}$ from Theorem 3.7 are finite. By Theorem 3.7, periodic points of $\mathbb{S}^1$ are degenerate comajors.

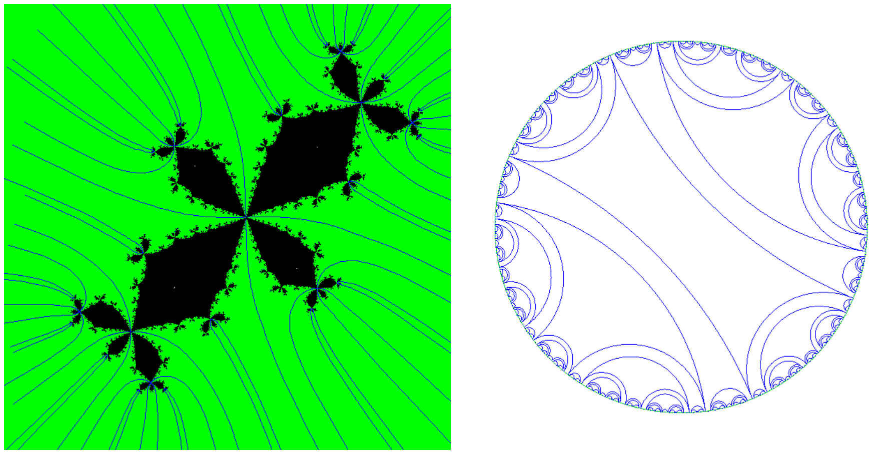


Figure 3: The symmetric cubic lamination with type B comajor $(\frac{5}{48}, \frac{7}{48})$ on the right and the Julia set of a corresponding polynomial with external rays on the left.

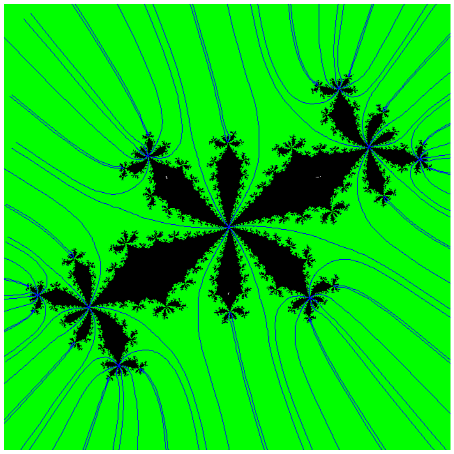
Definition 3.8. The q-lamination from Theorem 3.7 is called the *Cubic Symmetric Comajor Lamination* and is denoted by C_sCL. It induces an equivalence relation denoted $\sim_{sy}$. The $\sim_{sy}$-classes corresponding to symmetric Misiurewicz laminations are called *Misiurewicz* $\sim_{sy}$*-classes*. Denote by $\mathcal{M}^{sy}_{3,comb}$ the quotient space $\overline{\mathbb{D}}/\sim_{sy}$. Let $\mathfrak{pr}_{comb} : \overline{\mathbb{D}} \to \mathcal{M}^{sy}_{3,comb}$ be the corresponding quotient projection.

Theorem 3.7 verifies the *density of hyperbolicity* conjecture for C_sCL.

Lemma 3.9. *Let $\mathcal{L}$ be a symmetric non-empty Fatou lamination. Then one of the following holds.*

(B) There is only one cycle $\mathcal{A}$ of Fatou gaps of $\mathcal{L}$. It has even period $2m$, and is τ-symmetric. The periodic majors M and $\sigma_3^m(M) = \tau(M)$ of $\mathcal{L}$ are edges of critical gaps $U \in \mathcal{A}$ and $V = \sigma_3^m(U) = \tau(U)$. Non-periodic majors of $\mathcal{L}$ are siblings of M and $\tau(M)$ and edges of U and V, respectively. The remaining (i.e., not belonging to a major) $2m$-periodic vertices x, y of U are such that $\sigma_3^m(x) = \tau(y)$ while $\sigma_3^m(y) = \tau(x)$.

(D) There are exactly two cycles of Fatou gaps of the same period, interchanged by τ. Critical gaps U, $V = \tau(U)$ belong to different cycles. The periodic majors M and $\tau(M)$



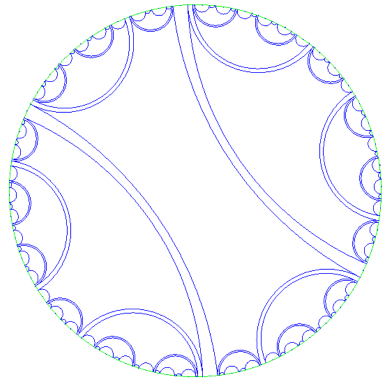


Figure 4: The symmetric cubic lamination with type D comajor $(\frac{7}{78}, \frac{4}{39})$ on the right and the Julia set of a corresponding polynomial with external rays on the left.

of $\mathcal{L}$ are edges of U and V, respectively. Non-periodic majors are siblings of M and $\tau(M)$ and edges of U and V, respectively.

In either case all edges of infinite gaps are eventually mapped to periodic majors. The only periodic orbit of edges of a Fatou gap of $\mathcal{L}$ is the orbit of a major of $\mathcal{L}$ and it has the same period as the Fatou gap.

Lemma 3.9 summarizes the results of [8] (compare [8], Lemma 3.8) dealing with symmetric non-empty (i.e., having some non-degenerate leaves) Fatou laminations. Figures 3 and 4 show examples of polynomials of type B and D and the corresponding laminations.

Proof. All claims of the lemma except for the last one are immediate; observe that the claims concerning the period of the majors follow from Lemma 3.3. To prove the last claim consider an edge ℓ of a critical gap U from a cycle $\mathcal{T}$ of Fatou gaps. It is well-known that any edge of U eventually maps to a periodic or a critical edge. Since, evidently, U has no critical edges, it suffices to prove that the only periodic edge of U is M. Indeed, let $N \neq M$ be a periodic edge of U. Then no image of N can be a point (since N is periodic) or a diameter of $\overline{\mathbb{D}}$ (since otherwise N itself is a diameter invariant under σ_3, hence $N = M$).

Take the closest approach N' in length to $\frac{1}{3}$ among the images of N. By Lemma 3.3 an eventual image of N' that is an edge of U must coincide with M, a contradiction. □

The terminology below is adopted from [31, 34], see also [7].

Definition 3.10. Symmetric Fatou laminations with properties from Lemma 3.9(B) (respectively, Lemma 3.9(D)) are said to be *of type B* (respectively, *of type D*).

We will also need an immediate corollary of Lemma 6.1 of [8].

Corollary 3.11 ([8], Lemma 6.1)**.** *Distinct symmetric Fatou laminations have disjoint comajors.*

Next we consider infinite gaps of C_sCL. One of them plays a special role. Recall that $\mathcal{O}$ is the center of $\mathbb{D}$. Each comajor is of length at most $\frac{1}{6}$. Hence $\mathcal{O}$ does not belong to any comajor; it must then lie inside a gap. The *main gap* G_{main} is by definition the gap of C_sCL that contains $\mathcal{O}$ in its interior.

Theorem 3.12. *The gap G_{main} is infinite, and $\tau(G_{main}) = G_{main}$. Each edge ℓ of G_{main} is a comajor with the same image as the longest edge of a σ_3-invariant symmetric finite rotational gap H and is associated with the symmetric Fatou lamination $\mathcal{L}_{\mathcal{H}}$ formed by H, Fatou gaps of degree greater than 1 attached to H and "rotating" around H, and their pullbacks.*

Proof. If G_{main} is finite, then, by Theorem 3.7, it is a Misiurewicz cocritical gap of preperiod at least 2 of a symmetric lamination $\mathcal{L}$. This is a contradiction, since then the other cocritical set of $\mathcal{L}$ contains $\mathcal{O}$ and intersects the interior of G_{main}. Thus, G_{main} is infinite.

Let ℓ be an edge of G_{main} and the marked comajor of a symmetric lamination $\mathcal{L}$. Since $\mathcal{O} \in G_{main}$, then ℓ cannot be located under another comajor. By Theorem 3.7, the leaf ℓ can only be a 1-preperiodic comajor of a Fatou lamination $\mathcal{L}$. Let U be the marked critical Fatou gap of $\mathcal{L}$ with periodic major M. If M is the limit of leaves of $\mathcal{L}$ (necessarily from outside of U), then ℓ is the limit of leaves ℓ_i so that ℓ is located under ℓ_i for any i. By Lemma 6.6 of [8], this implies that ℓ is the limit of the comajors under which ℓ

is located, a contradiction. Hence M is an edge of a periodic gap and is isolated in $\mathcal{L}$. Clearly, so is $\tau(M)$. Let us remove the grand orbits of M and $\tau(M)$ from $\mathcal{L}$ and consider the resulting family of leaves $\mathcal{L}'$. It easily follows (essentially, by definition) that $\mathcal{L}'$ is again a symmetric lamination. If $\mathcal{L}' \subsetneq \mathcal{L}$ is non-empty, then, evidently, it has a comajor ℓ' such that ℓ is under ℓ', a contradiction. Thus, $\mathcal{L}'$ is the empty lamination and, so, the grand orbits of M and $\tau(M)$ form the entire $\mathcal{L}$.

Since $\mathcal{L}$ must have a finite invariant gap H, it follows that $\mathcal{L}$ consists of H, Fatou gaps attached to H and "rotating" around H, and their iterated pullbacks. Observe that H itself must be symmetric under τ. By Lemma 3.9, the lamination $\mathcal{L}$ can be of type B or D. By definition, the two periodic majors of $\mathcal{L}$ are the closest to criticality edges of H (in this case it is equivalent to being the longest). There are two comajors of H; just like in the case of symmetric polynomials, either of them can be marked, and so in C_sCL the lamination $\mathcal{L}$ is reflected twice. □

The following notion will allow us to deal with type B and D laminations in a unified fashion.

Definition 3.13 (First (half-)return)**.** Let $\mathcal{L}$ be a symmetric Fatou lamination and U be a critical gap of $\mathcal{L}$. If $\mathcal{L}$ is of type D and the critical gap U is of period n then set $\eta = \sigma_3^n|_U$. If $\mathcal{L}$ is of type B and the critical gap U is of period $2m$ set $\eta = \tau \circ \sigma_3^m|_U$. Thus, η is a self-map of $U \cap \mathbb{S}^1$; it can also be extended linearly over the edges of U and, using a barycentric construction, inside U. The map η is called the *first (half-)return map* of U.

Strictly speaking, η depends on the choice of a symmetric lamination and its gap, however, we will not reflect it in writing to lighten the notation. Lemma 3.14 is left to the reader.

Lemma 3.14. *Let U be a critical gap of a symmetric Fatou lamination $\mathcal{L}$. Then η maps ∂U onto ∂U in a 2-to-1 fashion and is semiconjugate to σ_2 by a monotone map ϕ collapsing edges of U to points. The fixed point set of $\eta|_{\partial U}$ is the periodic major of $\mathcal{L}$. If $\ell \subset \overline{\mathbb{D}}$ is a chord* *whose endpoints are never mapped to the σ_2-fixed point, then the ϕ-preimage of $\ell \cap \mathbb{S}^1$ spans a chord in U that has a unique sibling $\ell' \subset \overline{\mathrm{co}}(U)$.*

The leaf/point ℓ' from the last claim of Lemma 3.14 is said to be *induced* by ℓ.

A parabolic quadratic polynomial from the Main Cardioid has a lamination $\mathcal{L}_2$ called *central*; the major of $\mathcal{L}_2 \neq \emptyset$ is an edge shared by a finite invariant gap and a critical periodic Fatou gap.

Theorem 3.15. *Let G be an infinite gap of C_sCL not containing $\mathcal{O}$. Then, for some Fatou lamination $\mathcal{L}$, a cocritical Fatou gap V of $\mathcal{L}$ contains G, and ∂G consists of single points and chords in V corresponding to majors of σ_2-invariant central laminations. In particular, edges of G are 1-preperiodic while other vertices of G have infinite orbits.*

Proof. Let us consider the *ceiling* of the gap G, that is the unique edge ℓ that separates G from $\mathcal{O}$. In other words, the gap G is located under ℓ. By Theorem 3.7, the edge ℓ is a comajor of a Fatou lamination $\mathcal{L}$. Denote by U the critical Fatou gap of $\mathcal{L}$ with periodic major M such that $\sigma_3(M) = \sigma_3(\ell)$. Then all edges of G are associated with symmetric laminations $\mathcal{L}'$ that tune $\mathcal{L}$. Evidently, $\mathcal{L}'|_U$ is invariant under η, the first (half-)return map introduced in Definition 3.13. Recall that η is of degree 2 and is modeled by σ_2; the map $\phi : \partial U \to \mathbb{S}^1$ collapses the edges of U and semiconjugates η with σ_2, as explained above.

Set $V = \overline{\mathrm{co}}(U)$. Periodic majors M' of σ_3 in U correspond under ϕ to periodic majors M'' of σ_2 in $\mathbb{D}$. It follows that the minor $\overline{c}'' = \sigma_2(M'')$ of σ_2 defines a chord $\overline{c}'$ in V that is mapped under σ_3 to the σ_3-image of M'. As is immediate from the definitions, $\overline{c}'$ is a comajor corresponding to M'. Thus, there is a natural correspondence between the quadratic minors and the cubic comajors in V. Under this correspondence, G pairs with the central gap of the quadratic minor lamination. Hence, the edges of G are associated with central quadratic laminations. □

4 Connectedness of $\mathcal{M}_3^{sy}$

Recall that h^k denotes the k-th iteration of a map h.

Lemma 4.1. *The set $\mathcal{M}_3^{sy}$ is invariant under the multiplication by i.*

Proof. We claim that if $f(z) = z^3 - az$ and $g(z) = z^3 + az$ then f^2 and g^2 are conjugate. Indeed, f and $-g$ are conjugate by the map $I : z \longmapsto iz$; hence f^2 and $(-g)^2$ are conjugate by I. Since g is an odd function, we have $(-g)^2 = g^2$. Thus, I conjugates f^2 and g^2. Since $p_{ic} = z^3 + 3c^2z$ while $p_c(z) = z^3 - 3c^2z$, then p_c^2 and p_{ic}^2 are conjugate. □

We now need a construction similar to that for quadratic polynomials; in our description below we follow the exposition from [29]. Take a topological disk Δ_c around infinity in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ that contains no critical points of p_c, does not contain 0, and is such that $p_c(\Delta_c) \subset \Delta_c$ (in particular, $0 \notin p_c^n(\Delta_c)$ for $n \geq 0$). Define Böttcher function $B_c(z) = \lim_{n\to\infty}(p_c^n(z))^{1/3^n}$ on Δ_c, where the root is taken so that the corresponding functions are tangent to the identity at infinity. The existence of a single valued branch follows from the fact that Δ_c is simply connected, and that 0 does not belong to $p_c^n(\Delta_c)$. Recall that the Green function $g_c : \mathbb{C} \to \mathbb{R}$ is defined as $g_c(z) = \lim_{n\to\infty} 3^{-n} \log_+ |p_c^n(z)|$, where $\log_+(t)$ is the maximum of 0 and $\log t$. Then $g_c(z) = \log |B_c(z)|$ for all $z \in \Delta_c$. The *equipotential* $E_c(t) \subset \mathbb{C}$ is defined as the level set $\{g_c = t\}$ of the Green function; this is a real analytic curve for $t > 0$, possibly singular. Note that B_c conjugates p_c and z^3 near infinity.

For $c \in \mathbb{C} \setminus \mathcal{M}_3^{sy}$, set Δ_c to be the exterior of the equipotential of p_c passing through $\pm c$ (the equipotential $E_c(g_c(c))$ has singularities but the exterior of it is a topological disk). Then the required properties of Δ_c are fulfilled. It is easy to see (from the continuous dependence of the Böttcher coordinate on parameters) that the union $\mathcal{U}$ of $\{c\} \times \Delta_c$, where c runs through the complement of $\mathcal{M}_3^{sy}$, is open. Standard arguments show that $B_c(z)$ is analytic in both c and z on $\mathcal{U}$.

If $c \in \mathbb{C} \setminus \mathcal{M}_3^{sy}$, then $(c, -2c)$ is always on the boundary of $\mathcal{U}$, since the value of the Green function of p_c at the cocritical point $-2c$ coincides with those at $\pm c$. However, the map $(c, z) \longmapsto B_c(z)$ extends analytically to a neighborhood of $-2c$. Moreover, $-2c$ is a regular point of this analytic extension in the sense that $z \longmapsto B_c(z)$ is a conformal injection in a neighborhood of this point. From now on we will assume that $B_c(z)$ is defined in this neighborhood of $-2c$.

For a fixed $c \notin \mathcal{M}_3^{sy}$ the map B_c is a conformal isomorphism between Δ_c and the set $\{z \in \mathbb{C} : |z| > g_c(c)\} = \mathbb{C} \setminus \overline{\mathbb{D}}_{g_c(c)}$. This defines *initial segments of (dynamical) external rays* $R_c(\theta)$ of p_c, i.e. B_c-preimages of the radial rays of argument θ in $\mathbb{C} \setminus \overline{\mathbb{D}}_{g_c(c)}$. Evidently, these initial segments of external rays of p_c are orthogonal to all equipotentials $E_c(t), t > g_c(c)$. Moreover, as we mentioned above equipotentials can be defined for any $t > 0$. This allows one to give the following definition: a *smooth external ray* R of p_c is a smooth unbounded curve that crosses every equipotential orthogonally and terminates in the Julia set of p_c. All but countably many initial segments of external rays defined above extend as smooth external rays. However countably many initial segments will hit critical points or their eventual preimages (in what follows such points are called *(pre)critical* or eventually critical) and, therefore, will not extend as smooth external rays.

Let $\Psi(c) = B_c(-2c)$ be the Böttcher coordinate of the marked cocritical point in the sense of the analytic continuation mentioned above. Then Ψ is well-defined and holomorphic on $\mathbb{C} \setminus \mathcal{M}_3^{sy}$. Theorem 4.2 is analogous to the corresponding statement for the Mandelbrot set; it proves the first claim of the Main Theorem stated in the Introduction. In what follows we denote the Riemann sphere by $\overline{\mathbb{C}}$.

Theorem 4.2. *The symmetric connectedness locus $\mathcal{M}_3^{sy}$ is a full continuum.*

Proof. Let us show that Ψ maps $\mathbb{C} \setminus \mathcal{M}_3^{sy}$ onto $\mathbb{C} \setminus \overline{\mathbb{D}}$. First we prove that $\Psi(c) \sim \sqrt[3]{2}c$ as $c \to \infty$. Set $z_n = p_c^n(-2c)$ and let

$$r_n = z_{n+1}/z_n^3 = 1 - 3c^2/z_n^2.$$

If $|c| \geq 2$ and $|z| \geq 2|c|$, then $|p_c(z)/z| \geq 4|c|^2 - 3|c|^2 = |c|^2 \geq 4$ and hence, $|p_c(z)| \geq 4|z|$. Thus, $|z_n| \geq 2 \cdot 4^{n-1}|c|^3 \geq 2|c|^3$ for $n \geq 1$ (which implies that $\mathcal{M}_3^{sy} \subset \mathbb{D}_2$) and for any $c \in \mathbb{D}_2$ we have $J(p_c) \subset \mathbb{D}_4$.

We conclude that $|c| \geq 2$ yields

$$1 - \frac{3}{4|c|^4} \leq |1 - 3c^2/z_n^2| = |r_n| \leq 1 + \frac{3}{4|c|^4}.$$

On the other hand,

$$z_n = r_{n-1} z_{n-1}^3 = r_{n-1} r_{n-2}^3 z_{n-2}^9 = \cdots = r_{n-1} r_{n-2}^3 \cdots r_2^{3^{n-3}} r_1^{3^{n-2}} (2c^3)^{3^{n-1}}$$

which yields that

$$\sqrt[3^n]{z_n} = (\sqrt[3^n]{r_{n-1}} \sqrt[3^{n-1}]{r_{n-2}} \ldots \sqrt[27]{r_2} \sqrt[9]{r_1})(\sqrt[3]{2})c.$$

Using the above bounds on $|r_n|$, the formula for the sum of the geometric series, and the fact that $\Psi(c) = \lim_{n\to\infty} \sqrt[3^n]{z_n}$ we see that

$$\left(\sqrt[6]{1 - \frac{3}{4|c|^4}}\right)\sqrt[3]{2}|c| \leq \sqrt[3^n]{|z_n|} \leq \left(\sqrt[6]{1 + \frac{3}{4|c|^4}}\right)\sqrt[3]{2}|c|,$$

that yields the following: as $c \to \infty$, $\Psi(c) = \sqrt[3]{2}c(1 + O(1/c^{2/3}))$. Thus, the map Ψ can be continuously extended to ∞ with $\Psi(\infty) = \infty$ implying that it can be done holomorphically and that the local degree of Ψ at ∞ is 1. In particular, ∞ is in the interior of the range of Ψ.

By the above, $\mathcal{M}_3^{sy} \subset \mathbb{D}_2$, in particular, $\mathcal{M}_3^{sy}$ is compact. Let M be the maximum of the continuous function $(z, c) \longmapsto g_c(z)$ on the compact set $\overline{\mathbb{D}}_2 \times \overline{\mathbb{D}}_4$. It follows that $|B_c(z)| \leq e^M$ for all $c \in \overline{\mathbb{D}}_2$ and $z \in \overline{\mathbb{D}}_4$ such that $(c, z) \in \overline{\mathcal{U}}$.

We claim that if $c_n \to \mathcal{M}_3^{sy}$ then $|\Psi(c_n)| \to 1$. Indeed, otherwise there exists a sequence $c_n \to c_0 \in \mathcal{M}_3^{sy}$ with $|\Psi(c_n)| > e^m > 1$. Take k such that $3^k m > M$. Since $|B_c(p_c^k(-2c))| = |(B_c(p_c(-2c))^{3^k}| = |\Psi(c)|^{3^k}$, then $|B_{c_n}(p_{c_n}^k(-2c_n))| > e^M$ which, by the choice of M, implies $|p_{c_n}^k(-2c_n)| > 4$ for all sufficiently large n (indeed, $|c_n| \leq 2$ for large n). By continuity, $p_{c_0}^k(-2c_0) \geq 4$; this shows that the cocritical point of p_{c_0} escapes to infinity contradicting the choice of c_0.

Since ∞ is not on the boundary of the range of Ψ it follows from the above that Ψ is a proper holomorphic map from $\overline{\mathbb{C}} \setminus \mathcal{M}_3^{sy}$ onto $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Hence it is a branched covering with a well-defined degree. However, the point ∞ has exactly one preimage of degree 1; hence Ψ has degree 1 and is actually a conformal isomorphism. □

The function Ψ gives us an analogue of Böttcher coordinates for the complement of $\mathcal{M}_3^{sy}$. In particular, we can define *external parameter rays* (or simply *parameter rays*) as preimages of radial straight lines under Ψ, namely, $\mathcal{R}_\theta(r) = \Psi^{-1}(re^{2\pi i\theta})$ with $r \in (1, \infty)$. The parameter ray $\mathcal{R}_\theta$ *lands* at a parameter w if $\lim_{r\to 1} \mathcal{R}_\theta(r) = w$. Note that, by definition, the parameter ray $\mathcal{R}_\theta$ consists of the parameters c such that the marked cocritical point $-2c$ of p_c belongs to the dynamical external ray $R_c(\theta)$ of p_c. Also note, that the map Ψ is a conformal isomorphism between $\mathbb{C} \setminus \mathcal{M}_3^{sy}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$ tangential to the map $z \longmapsto \sqrt[3]{2}z$ at infinity.

5 Hyperbolic components of $\mathcal{M}_3^{sy}$ and their roots

Hyperbolic components of polynomial parameter spaces play an important role in complex dynamics. Here we study them for the parameter space of symmetric cubic polynomials.

5.1 Preliminaries

We start by recalling basic definitions. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function. The *multiplier* $\rho(z)$ of a periodic point z of minimal period n under f is defined as the derivative of the first return map to z, that is, $\rho(z) = (f^n)'(z)$. A periodic point z is said to be *super-attracting* if $\rho(z) = 0$ (which implies that the orbit of z contains a critical point), *attracting* if $|\rho(z)| < 1$, and *parabolic* if $\rho(z)$ is a root of unity (which implies that $(f^{kn})'(z) = 1$ for some k).

A polynomial $p_c \in \mathcal{M}_3^{sy}$ (and the parameter c) is *hyperbolic*/*parabolic* if both of its *finite* critical points are attracted to *finite* attracting/parabolic cycles. To characterize such polynomials we need a lemma.

Lemma 5.1. *If U is an open topological disk and $-U = U$, then $0 \in U$. If A is a full continuum and $-A = A$, then $0 \in A$.*

Proof. Set $s(z) = z^2$; then $U = s^{-1}(s(U))$, and $s : U \to s(U)$ is a branched covering. The claim now follows from the Riemann–Hurwitz formula applied to this covering. Take a tight symmetric Jordan neighborhood V of A and set $U = s^{-1}(V)$; then $0 \in U$ by the above. Since A is the intersection of all such U, it follows that $0 \in A$. □

We can now describe hyperbolic polynomials p_c more explicitly.

Lemma 5.2. *A polynomial p_c is hyperbolic if and only if it possesses one of the following:*

(a) an invariant symmetric attracting Fatou domain on which p_c is 3-to-1; this happens if and only if $|c| < \sqrt{1/3}$, or

(b) a unique symmetric cycle of attracting Fatou domains of period $2n \geq 2$; there are exactly two mutually symmetric domains in the cycle containing critical points $\pm c$, or

(d) two mutually symmetric attracting cycles of Fatou domains.

Thus, if p_c has an attracting cycle, then p_c is hyperbolic. Also, case (a) is the only case when a hyperbolic polynomial p_c has a unique bounded periodic Fatou domain.

Proof. If there exists a Fatou domain U on which the map is 3-to-1, then U must be symmetric (otherwise, $-U$ is another Fatou domain on which the map is 3-to-1, which is impossible). By Lemma 5.1, this implies that $0 \in U$ and U is invariant. Since there must exist a unique fixed point in U and this point must be attracting, then 0, being a fixed point, must be attracting. Since $p_c'(0) = -3c^2$, the corresponding hyperbolic component of $\mathcal{M}_3^{sy}$ is the round disk of radius $\sqrt{1/3}$ centered at the origin. This corresponds to case (a) and covers $c = 0$ so from now on we assume that $c \neq 0$ and hence p_c has distinct critical points c and $-c$ with mutually symmetric orbits.

If c (resp., $-c$) is attracted to an attracting cycle, then so is $-c$ (resp., c) which implies that if p_c has an attracting cycle, then p_c is hyperbolic. We can also assume that there are no 3-to-1 Fatou domains. Now, if p_c has a cycle A of attracting Fatou domains then by symmetry $-A$ is also a cycle of attracting Fatou domains. Suppose that $A = -A$. By the assumption critical domains in A contain exactly one critical point; since p_c is symmetric, the critical domains in A are mutually symmetric. Moreover, the fact that p_c is symmetric implies that the first iterate p_c^n that maps either critical domain from A to the other one is the same for both critical domains which implies that the period of A is $2n$. This corresponds to case (b). Otherwise A and $-A$ are distinct cycles of Fatou domains which corresponds to case (d). □

Lemma 5.3 is similar to Lemma 5.2 and its proof is left to the reader.

Lemma 5.3. *Suppose that a polynomial p_c has a parabolic cycle. Then one of the following holds:*

(b) a unique symmetric cycle of parabolic Fatou domains of p_c of period $2n \geq 2$ has exactly two mutually symmetric critical domains;

(d) two parabolic cycles of Fatou domains of p_c are mutually symmetric.

For brevity, a Cremer/Siegel point (cycle) of a polynomial will be referred to as a *CS-point (cycle)*. Now we show that for CS-cycles the situation with symmetric polynomials is similar to that in Lemma 5.2. First we state a part of Theorem 4.3 from [4] combined with results from [18] and [23]. Define a *rational cut* as the union of two external rays with rational arguments that land on the same point called the *vertex* of the cut. If the vertex is a repelling (parabolic) periodic point, then we call the cut *repelling (parabolic)*.

Theorem 5.4 ([4, 18, 23]). *Let P be a polynomial and T be CS-cycle. There exists a recurrent critical point c of P and a point $q \in T$ that are not separated by any rational cut of P. Two different objects, each of which is a CS-point, a parabolic domain, or an attracting domain, are always separated by a rational cut.*

Theorem 5.4 is used in the proof of the following lemma.

Lemma 5.5. *Suppose that a polynomial p_c has a CS-cycle T. Then one of the following holds:*

(a) the only non-repelling cycle of p_c is $T = \{0\}$, and neither of the critical points $\pm c$ is separated from T by a rational cut;

(b) the only non-repelling cycle of p_c is T, it is a symmetric cycle of period $2n$;

(d) there are exactly 2 non-repelling cycles, namely, T and $-T$.

Proof. Assume that 0 is a CS-point. Then, by Theorem 5.4, there is a recurrent critical point c not separated from 0 by any rational cut, and the same holds for $-c$. This corresponds to case (a) of the lemma.

If 0 is parabolic or attracting then it attracts at least one critical point of p_c and hence, by symmetry, both of them. In this case p_c has no other non-repelling cycles. So, from now on we assume that 0 is repelling.

If p_c has a symmetric CS-cycle T of period $2n$ then, by Theorem 5.4, it has a point $w \in T$ not separated from a recurrent critical point, say, c, of p_c by a rational cut; hence, $-w \in T$ is not separated from a recurrent critical point $-c$ of p_c by a rational cut. This, again by Theorem 5.4, implies that there are no other non-repelling cycles of p_c. This corresponds to case (b) of the lemma.

Finally, let T and $-T$ be distinct mutually symmetric CS-cycles of p_c. By Theorem 5.4, we may assume that T has a point w not separated from a recurrent critical point, say, c of p_c by a rational cut; hence, $-w \in -T$ is not separated from a recurrent critical point $-c$ of p_c by a rational cut. This implies that there are no other non-repelling cycles of p_c. This corresponds to case (d) of the lemma. □

5.2 Hyperbolic components and multipliers

Let p_{c_0} have a periodic point w of period n such that $(p_{c_0}^n)'(w) \neq 1$. By the implicit function theorem applied to the equation $p_c^n(z) = z$, there is a holomorphic function $\alpha(c)$ defined on an open Jordan disk around c_0 such that $\alpha(c_0) = w$, and $\alpha(c)$ is a periodic point of p_c of period n. Also, the multiplier $(p_c^n)'(\alpha(c))$ is a holomorphic function of c. Hence the set of hyperbolic parameters is an open subset of $\mathcal{M}_3^{sy}$; a connected component $\mathcal{H}$ of this set is called a *hyperbolic component* of $\mathcal{M}_3^{sy}$. For any $c \in \mathcal{H}$ the ω-limit set of the marked critical point is the unique *marked* attracting cycle Q_c; the *period* of $\mathcal{H}$ is the period of Q_c.

Conjecturally, every connected component of the interior of $\mathcal{M}_3^{sy}$ is hyperbolic.

Theorem 5.6 ([41]). *Every component of the Fatou set of a rational function is eventually periodic. In particular, any bounded Fatou domain of a hyperbolic symmetric cubic polynomial eventually maps into a cycle of Fatou domains that contains an attracting cycle.*

Recall that, by Lemma 5.2, the set $\mathbb{D}_{\sqrt{1/3}}$ is a hyperbolic component of $\mathcal{M}_3^{sy}$.

Definition 5.7. The set $\mathbb{D}_{\sqrt{1/3}}$ is called the *main hyperbolic component* of $\mathcal{M}_3^{sy}$ and is denoted $\mathcal{H}_{main}$.

Corollary 5.8 follows from Lemmas 5.2, 5.3 and 5.5.

Corollary 5.8. *A polynomial p_c has one symmetric non-repelling cycle, or two mutually symmetric non-repelling cycles with equal multipliers, or no non-repelling cycles at all.*

Let $\mathcal{H} \neq \mathcal{H}_{main}$ be a hyperbolic component and $c \in \mathcal{H}$. Denote by $F_c^{\pm}$ the critical Fatou domains of p_c that correspond to the critical Fatou gaps $U_{\mathcal{H}}^{\pm}$, respectively, of $\mathcal{L}_{\mathcal{H}}$ (we always assume that the marked critical point c belongs to F_c^+). Let $M_{\mathcal{H}}$ and $M'_{\mathcal{H}}$ be the majors of $\mathcal{L}_{\mathcal{H}}$ that are edges of $U_{\mathcal{H}}^+$; then $\sigma_3(M_{\mathcal{H}}) = \sigma_3(M'_{\mathcal{H}})$ by Lemma 3.9, and we always assume that $M_{\mathcal{H}}$ is periodic. Let $\overline{\mathrm{co}}_{\mathcal{H}}^+$ be the marked comajor (i.e., $\sigma_3(\overline{\mathrm{co}}_{\mathcal{H}}^+) = \sigma_3(M_{\mathcal{H}})$) and let $\overline{\mathrm{co}}_{\mathcal{H}}^-$ be the other comajor of $\mathcal{L}_{\mathcal{H}}$. Similar objects can be defined for any hyperbolic or parabolic parameter c yielding such notation as $U_c^{\pm}, M_c, M'_c$.

Let a hyperbolic component $\mathcal{H}$ be given. All $c \in \mathcal{H}$ satisfy the same option (a), (b) or (d) from Lemma 5.2. According to these three cases, $\mathcal{H}$ is said to be of *type* A, B or D, respectively. Also, given a polynomial p_c with parabolic (or attracting) periodic point z, let $m_r(c)$ be the minimal number such that $p_c^{m_r(c)}$ fixes dynamical external rays landing on z (or the point z itself if it is attracting); note that it does not depend on the choice of a particular ray and is called the *ray period* of p_c. The number $(p_c^{m_r(z)})'(z)$ is called the *ray multiplier* (of p_c) and is denoted $r\!p(c)$. Notice that the period of a parabolic point z may be strictly smaller than $m_r(c)$ so that $m_r(c)$ is a multiple of the period.

A similar concept can be defined for a hyperbolic component $\mathcal{H}$. Namely, let $r\!p_{\mathcal{H}} : \mathcal{H} \to \mathbb{D}$ be defined as $r\!p_{\mathcal{H}}(c) = r\!p(c)$. As we show in Theorems 5.9 and 5.10, the function

$rp_{\mathcal{H}}$ can be extended over $\overline{\mathcal{H}}$. For a parabolic parameter $c \in \partial\mathcal{H}$, this extended function $rp_{\mathcal{H}}$ may not be equal to $rp(c)$. To emphasize that, we use the subscript $_{\mathcal{H}}$ in the notation, and call $rp_{\mathcal{H}}$ the *ray multiplier based on* $\mathcal{H}$. This difference is not present for parameters $c \in \mathcal{H}$, but does show for parameters $c \in \partial\mathcal{H}$.

Theorem 5.9. *For a type D hyperbolic component $\mathcal{H}$ of $\mathcal{M}_3^{sy}$ the map $rp_{\mathcal{H}}$ can be extended onto $\overline{\mathcal{H}}$ so that $rp_{\mathcal{H}} : \overline{\mathcal{H}} \to \overline{\mathbb{D}}$ is a homeomorphism conformal on $\mathcal{H}$.*

Proof. The result essentially follows from Theorem C of [21], however, we need to explain how the terminology of Inou–Kiwi relates to ours. Let $\mathcal{L}$ be a cubic invariant q-lamination with at least one cycle of Fatou gaps. With $\mathcal{L}$, one associates a *reduced mapping schema* $T(\mathcal{L})$. Instead of giving a general definition of mapping schemata, we give an explicit description of $T(\mathcal{L})$ in the case when $\mathcal{L}$ is symmetric of type D, that is, when $\mathcal{L}$ has two distinct cycles of Fatou gaps. In this case, $T(\mathcal{L})$ can be represented as the graph with two vertices and two (directed) edges that are loops based at both vertices. Every edge of $T(\mathcal{L})$ is in general equipped with a positive integer called the *degree*; in our specific case, the degrees of both loops are equal to 2. Intuitively, the arrows of $T(\mathcal{L})$ represent the first return maps to the critical Fatou gaps of $\mathcal{L}$. The space $\mathcal{C}(T(\mathcal{L}))$ in our specific case consists of all pairs (q_0, q_1) of monic centered quadratic polynomials q_0, q_1 with connected Julia sets — informally, the two loops of $T(\mathcal{L})$ are replaced with q_0 and q_1.

By definition, the space $\mathfrak{R}(\mathcal{L})$ consists of all monic cubic polynomials f such that

- the filled Julia set $K(f)$ is connected;
- for any (pre)periodic leaf of $\mathcal{L}$ with endpoints α, β, the corresponding external rays $R_f(\alpha)$ and $R_f(\beta)$ land on the same (pre)periodic point of $K(f)$;
- let U_0, U_1 be the critical Fatou gaps of f; the corresponding subcontinua K_0 and K_1 of $K(f)$ are polynomial-like filled Julia sets of certain polynomial-like restrictions of f^n, where n is the period of U_0 and U_1.

Here, one needs to explain in which sense K_0 corresponds to U_0, and similarly with K_1 and U_1. For each $\ell \in \mathcal{L}$, let α and β be the endpoints of ℓ, and write Γ_ℓ for the cut formed by the external rays $R_f(\alpha)$, $R_f(\beta)$, and their common landing point. Then K_0 corresponding to U_0 means that K_0 lies on the same side of Γ_ℓ as U_0 relative to ℓ, for every $\ell \in \mathcal{L}$. By the Douady–Hubbard straightening theorem, f^n is hybrid equivalent to a unique monic quadratic polynomial q_i near K_i, where $i = 0, 1$. The *Inou–Kiwi straightening map* (abbreviated as *IK-straightening map*) $\chi_{\mathcal{L}} : \mathfrak{R}(\mathcal{L}) \to \mathcal{C}(T(\mathcal{L}))$ takes f to (q_0, q_1). (The fact that q_i are quadratic yields some simplification: in higher degree cases one needs additional normalization called *internal angles assignment* in order to make q_i unique).

Theorem C of [21] can now be formulated as follows. *Denote by* $\mathrm{Hyp}(\mathcal{C}(T(\mathcal{L})))$ *the set of hyperbolic maps contained in* $\mathcal{C}(T(\mathcal{L}))$ *(in our specific case,* $(q_0, q_1) \in \mathcal{C}(T(\mathcal{L}))$ *being hyperbolic means both* q_0 *and* q_1 *are hyperbolic). Then* $\chi_{\mathcal{L}}(\mathfrak{R}(\mathcal{L})) \supset \mathrm{Hyp}(\mathcal{C}(T(\mathcal{L})))$*, the inverse image of* $\mathrm{Hyp}(\mathcal{C}(T(\mathcal{L})))$ *under* $\chi_{\mathcal{L}}$ *is an open set, and the restriction of* $\chi_{\mathcal{L}}$ *onto this open set is biholomorphic.* Now let $\mathcal{H}$ be a given type D hyperbolic component of $\mathcal{M}_3^{sy}$; it lies in some hyperbolic component $\hat{\mathcal{H}}$ of the connectedness locus of all monic centered cubic polynomials. All polynomials from $\hat{\mathcal{H}}$ have the same lamination, say, $\mathcal{L}$. By Theorem C of [21], the restriction of $\chi_{\mathcal{L}}$ to $\hat{\mathcal{H}}$ is a biholomorphic isomorphism between $\hat{\mathcal{H}}$ and the product of the interior Ca of the main cardioid with itself. The image of $\mathcal{H}$ under $\chi_{\mathcal{L}}$ is then the diagonal in $\mathrm{Ca} \times \mathrm{Ca}$, and the restriction of $\chi_{\mathcal{L}}$ is a biholomorphic map between $\mathcal{H}$ and this diagonal (the latter is isomorphic to $\mathbb{D}$ under the multiplier map). It follows that $r\rho_{\mathcal{H}} : \mathcal{H} \to \mathbb{D}$ is a conformal isomorphism. Since the boundary of any hyperbolic component is contained in a real algebraic curve and, as such, is locally connected, then, clearly, it extends to a homeomorphism $r\rho_{\mathcal{H}} : \overline{\mathcal{H}} \to \overline{\mathbb{D}}$. Observe that the situation here is similar to the quadratic case [15]. □

Let a polynomial p_c have a parabolic or attracting cycle X_c of type B. Then the period of X_c is an even number $2n$, and $p_c^n(x) = -x$ for every $x \in X_c$. The number $-(p_c^n)'(x)$ does not depend on the point $x \in X_c$, is denoted $r\tilde{\rho}(c)$, and is called the *ray half-multiplier* (of p_c). Note that $r\tilde{\rho}(c)$ can be interpreted as the multiplier of the fixed point x of the map $-p_c^n$. As before, the ray half-multiplier depends only on the parameter and is defined as long as p_c is of type B. If $c \in \mathcal{H}$, where $\mathcal{H}$ is of type B, then we write $r\tilde{\rho}_{\mathcal{H}}$ for the restriction of the function $\tilde{r\rho}_{\mathcal{H}}(c)$ to $\mathcal{H}$. The *ray (half-)multiplier* of c is defined as either $r\rho(c)$ or $\tilde{r\rho}(c)$ depending on whether c is type D or type B.

Theorem 5.10. *Let $\mathcal{H}$ be a hyperbolic component of $\mathcal{M}_3^{sy}$ of type B. The map $\tilde{r\rho}_{\mathcal{H}}$ can be extended over $\overline{\mathcal{H}}$ so that $\tilde{r\rho}_{\mathcal{H}} : \overline{\mathcal{H}} \to \overline{\mathbb{D}}$ is a homeomorphism which is conformal on $\mathcal{H}$ while the ray multiplier $r\rho_{\mathcal{H}} = \tilde{r\rho}^2_{\mathcal{H}} : \overline{\mathcal{H}} \to \overline{\mathbb{D}}$ is a double-covering.*

Proof. Similarly to Theorem 5.9, Theorem 5.10 also follows from Theorem C of [21]. Let $\hat{\mathcal{H}}$ be the hyperbolic component in the full space of monic centered cubic polynomials containing $\mathcal{H}$. All polynomials from $\hat{\mathcal{H}}$ have the same lamination $\mathcal{L}$ so that the corresponding reduced mapping schema $T(\mathcal{L})$ has two vertices and two directed edges connecting the two vertices in opposite directions; each edge has degree 2. The space $\mathcal{C}(T(\mathcal{L}))$ consists of pairs (q_0, q_1) of monic centered quadratic polynomials such that the Julia set of $q_1 \circ q_0$ is connected. The straightening map of Inou–Kiwi provides a biholomorphic isomorphism between $\hat{\mathcal{H}}$ and the *principal hyperbolic component* in $\mathcal{C}(T(\mathcal{L}))$ consisting of all (q_0, q_1) such that $q_1 \circ q_0$ has an attracting fixed point, and $J(q_1 \circ q_0)$ is a Jordan curve. Clearly, symmetric cubic polynomials are mapped to pairs (q_0, q_1), for which $q_0 = q_1$. The corresponding slice of the principal hyperbolic component is biholomorphic to $\mathbb{D}$, and the corresponding conformal isomorphism is given by the multiplier of the attracting fixed point of $q_0 = q_1$. □

Let us now define the center and the root of a hyperbolic component.

Definition 5.11 (Center and root)**.** Let $\mathcal{H}$ be a hyperbolic component of type B or D. The *center* of $\mathcal{H}$ is the point $c \in \mathcal{H}$ such that p_c has superattracting cycle(s). The *root* of $\mathcal{H}$ is the point $r_{\mathcal{H}} \in \partial\mathcal{H}$ such that $r\rho_{\mathcal{H}}(r_{\mathcal{H}}) = 1$ (if $\mathcal{H}$ is of type D) or $\tilde{r\rho}_{\mathcal{H}}(r_{\mathcal{H}}) = 1$ (if $\mathcal{H}$ is of type B).

Lemma 5.12 justifies the above definition of the roots.

Lemma 5.12. *Suppose that z is a parabolic point of p_c of ray period n. Then there is a major $\overline{\theta\theta'}$ of $\mathcal{L}_c$ such that both $R_c(\theta)$ and $R_c(\theta')$ land on $p_c^k(z)$, for some k with $0 \le k < n$. Moreover, the ray (half-)multiplier of c equals 1.*

Proof. Let U be a period n parabolic domain attached to z, and F be the Fatou gap of $\mathcal{L}_c$ corresponding to U. Replacing F with a suitable Fatou gap from the same cycle, we may assume that F has a unique periodic major $M = \overline{\theta\theta'}$. If U is of type D then the only p_c^n-fixed point in ∂U is z and the only σ_3^n-fixed points in ∂F form the major M. Hence $R_c(\theta)$ and $R_c(\theta')$ land on z. If U is of type B then $n = 2m$ and there are three p_c^n-fixed points in ∂U associated with M and two vertices x, y of F. Let $x', y' \in \partial U$ be points associated with vertices x, y of F. We claim that x', y' are not parabolic. Suppose that x' is parabolic. Then $p_c^m(y') = -x'$ by part (B) of Lemma 3.9, which implies that y' is also parabolic, a contradiction. Hence only M can be associated with z as desired. Let us now prove that the ray (half-)multiplier of c equals 1.

(B) Let p_c be of type B, $z \in \partial F_c^+$ be parabolic, and F_c^+ be of period $2m$; then $-p_c^m(z) = z$, and, by Lemma 3.14, the map $-p_c^m$ fixes the two external rays of p_c landing on z. Hence $(-p_c^m)'(z) = 1$, which implies that $r\tilde{\rho}(c) = 1$.

(D) Similar to (B) (details are left to the reader). □

The main component $\mathcal{H}_{main}$ of $\mathcal{M}_3^{sy}$ (for which 0 is attracting) is a round disk of radius $\sqrt{3}/3$. For $|c| < \sqrt{3}/3$ the Julia set $J(p_c)$ is a Jordan curve and $p_c|_{J_c}$ is conjugate to σ_3. In other words, the map is neither of type B nor of type D, and neither Theorem 5.9 nor Theorem 5.10 applies. Each polynomial p_c with $|c| = \sqrt{3}/3$ has a neutral fixed point at 0 of multiplier $-3c^2$. If $c_{1,2} = \pm\frac{i}{\sqrt{3}}$ then the corresponding polynomials $p_{c_{1,2}} = z^3 + z$ have multiplier 1 at fixed point 0. It is easy to see that the lamination associated with $z^3 + z$ has the leaf $\overline{0\,\frac{1}{2}}$, which is the horizontal diameter of the unit circle, and two invariant Fatou gaps located, respectively, above and below $\overline{0\,\frac{1}{2}}$; the presence of these invariant gaps completely determines the lamination. In particular, the laminations at these two parameters are different from the empty lamination which corresponds to any polynomial in $\mathcal{H}_{main}$. We will not consider these points (or any other points) as roots of $\mathcal{H}_{main}$ so that $\mathcal{H}_{main}$ has the center (at the origin) but does not have a root.

6 Parabolic polynomials

The following stability property will be used repeatedly. We state it only for symmetric cubic polynomials.

Theorem 6.1 (Lemma B.1 of [18]). *Let w be a repelling periodic point of p_{c_0} such that an external dynamical ray $R_{c_0}(\theta_0)$ with rational argument $\theta_0 \in \mathbb{R}/\mathbb{Z}$ lands on w. Let θ be an angle such that an eventual σ_3-image of θ belongs to the σ_3-orbit of θ_0 and the rays from the (finite) p_{c_0}-orbit of $R_{c_0}(\theta)$ are all smooth and do not land on critical points. Then, for c sufficiently close to c_0, the dynamical external rays from the p_c-orbit of $R_c(\theta)$ are smooth, move continuously with c, and land on preperiodic or repelling periodic points of p_c close to the landing points of the dynamical external rays of p_{c_0} with the same argument.*

Recall that the lamination $\mathcal{L}_c$ is defined for all c such that $K(p_c)$ is locally connected. In particular, if $\mathcal{H}$ is a hyperbolic component, then for any $c \in \mathcal{H}$ the lamination $\mathcal{L}_c$ exists and is independent of the choice of c. Denote it by $\mathcal{L}_{\mathcal{H}}$; clearly, $\mathcal{L}_{\mathcal{H}}$ is a symmetric Fatou lamination.

Definition 6.2 (Geometrically finite and sub-hyperbolic [20]). A polynomial f is *geometrically finite* if all its critical points are preperiodic points or attracted by parabolic or attracting cycles. If, moreover, f has no parabolic points, then it is said to be *sub-hyperbolic*.

Theorem 6.3 based on [20] and additional arguments.

Theorem 6.3 ([20]). *The following holds.*

1. *A parabolic symmetric polynomial p_c is accessible from a hyperbolic component $\mathcal{H}$ of $\mathcal{M}_3^{sy}$ such that $\mathcal{L}_{\mathcal{H}} = \mathcal{L}_c$.*

2. *If $\mathcal{H}'$ is a hyperbolic component and $c' \in \partial\mathcal{H}'$ is such that $\mathcal{L}_{c'} = \mathcal{L}_{\mathcal{H}'}$, then $c' = r_{\mathcal{H}'}$ and $\overline{\mathrm{co}}^+_{c'} = \overline{\mathrm{co}}^+_{\mathcal{H}'}$.*

Proof. (1) Let p_c be the given parabolic polynomial. It is of type B or D. By the main theorem of [20], there exists a path f_t of monic centered cubic polynomials such that $f_0 = p_c$, f_t is sub-hyperbolic for $t > 0$, and $f_t|_{J(f_t)}$ is topologically conjugate to p_c on its Julia set. Since p_c is of type B or D, the polynomial f_t is hyperbolic for any $t > 0$ and, hence, there exists a hyperbolic component $\widehat{\mathcal{H}}$ in the space of all monic centered cubic polynomials that contains the path $\{f_t\}_{t>0}$.

We now use the description of $\widehat{\mathcal{H}}$ given in Theorems 5.9 and 5.10. In both cases (type D or type B), the straightening map (ρ or $\tilde{\rho}$, resp.) yields a biholomorphic parametrization of $\widehat{\mathcal{H}}$ by pairs (q_0, q_1) of monic centered quadratic polynomials. A path in $\widehat{\mathcal{H}}$ converging to p_c can be represented as $(q_0(t), q_1(t))$, where $t \in (0, 1]$, and the latter path converges as $t \to 0$ to some (q, q) since both multipliers or half-multipliers have the same limit. It follows that the path in $\widehat{\mathcal{H}}$ represented by $(q_0(t), q_0(t))$ also converges to p_c. On the other hand, this new path consists of symmetric cubic polynomials. Thus p_c is accessible from $\mathcal{H} = \widehat{\mathcal{H}} \cap \mathcal{SCP}$, as desired.

(2) The claim follows from the assumption that $\mathcal{L}_{c'} = \mathcal{L}_{\mathcal{H}'}$ and Lemma 5.12. □

The following is a combinatorial description of hyperbolic components of $\mathcal{M}_3^{sy}$.

Theorem 6.4. *The map taking a hyperbolic component $\mathcal{H}$ (other than $\mathcal{H}_{main}$) of $\mathcal{M}_3^{sy}$ to the corresponding marked comajor $\overline{\mathrm{co}}_{\mathcal{H}}^{+}$ is a bijection between the hyperbolic components of $\mathcal{M}_3^{sy}$ (with the exception of $\mathcal{H}_{main}$) and 1-preperiodic comajors of C_sCL. Two distinct marked parabolic polynomials must have distinct marked comajors.*

Proof. Bijection follows from results of A. Poirier [35], which in turn extend earlier work of Bielefeld–Fisher–Hubbard [2]. Namely, injectivity is Theorem 1.1 and surjectivity is Theorem 1.3. More precisely, the map defined in [35] sends $\mathcal{H}$ to the corresponding *critical portrait*. On the other hand, there is a bijection between the marked comajors of C_sCL and symmetric cubic Fatou critical portraits of period > 1.

It remains to prove the last claim. Suppose that two parabolic marked polynomials p_c and $p_{c'}$ are associated with the same marked comajor ℓ. By Theorem, 6.3, c is the root of a unique hyperbolic domain $\mathcal{H}$ such that $\overline{\mathrm{co}}^+_{\mathcal{H}} = \ell$, and, similarly, $p_{c'}$ is the root of a unique hyperbolic domain $\mathcal{H}'$ such that $\overline{\mathrm{co}}^+_{\mathcal{H}'} = \ell$. By the above, $\mathcal{H} = \mathcal{H}'$. Hence $c = c'$ is the root of $\mathcal{H}$, and $p_c = p_{c'}$ is the same marked polynomial. □

We will now study the place of a parabolic parameter c in the parameter space. More precisely, given a hyperbolic component $\mathcal{H}$ we show that $\mathcal{L}_{\mathcal{H}} = \mathcal{L}_{r_{\mathcal{H}}}$. We also consider parameter rays $\mathcal{R}_\theta$ with 1-preperiodic argument θ, show that they land on a parabolic parameter c, and relate θ and the marked comajor associated with p_c.

We will interchangeably use notation 3θ and $\sigma_3(\theta)$ for any $\theta \in \mathbb{S}^1$. Recall also that τ denotes the rotation of $\overline{\mathbb{D}}$ or $\mathbb{S}^1$ by 180°.

Lemma 6.5. *If $\mathcal{L}' \subset \mathcal{L}$ are two symmetric laminations such that $\mathcal{L}' \neq \emptyset$ is Fatou then $\mathcal{L}$ cannot contain a finite periodic gap G located inside a Fatou gap U' of $\mathcal{L}'$ with an edge which is a major of $\mathcal{L}'$.*

Proof. Suppose that the gap G as described in the lemma exists and $M = \overline{xy}$ is a major of $\mathcal{L}'$ and an edge of G. Consider the first (half-)return map η (see Definition 3.13) of U' with respect to $\mathcal{L}'$. Then $\eta|_{\partial U'}$ is two-to-one. If G exists, it is η-invariant, which is impossible. Indeed, $\eta|_{\partial U'}$ is two-to-one and semiconjugate to σ_2 by the map collapsing the edges of U' to points; by Lemma 3.9 all edges of U' are preimages of its major. Hence the existence of G implies the existence of an invariant leaf or gap of σ_2 with a σ_2-fixed endpoint, which is absurd. □

Lemma 6.6. *Every parameter ray $\mathcal{R}_\theta$ at a 1-preperiodic angle θ lands on a parabolic parameter c_0 on the boundary of $\mathcal{M}_3^{sy}$. Moreover, one of the rays $R_{c_0}(\theta \pm \frac{1}{3})$ lands on a parabolic periodic point of p_{c_0}.*

Proof. We claim that every parameter ray $\mathcal{R}_\theta$ at a 1-preperiodic angle θ lands on a parabolic parameter $c_0 \in \partial\mathcal{M}_3^{sy}$, and the dynamical ray $R_{c_0}(3\theta)$ lands on a p_{c_0}-parabolic point. Indeed, let $c_0 \in \partial\mathcal{M}_3^{sy}$ be in the accumulation set of $\mathcal{R}_\theta$. Recall that for $c \in \mathcal{R}_\theta$, the dynamical external ray $R_c(\theta)$ passes through the cocritical point $-2c$ escaping to infinity under the action of p_c. Thus, the dynamical external ray $R_c(3\theta)$ has a periodic argument 3θ and, therefore, is not smooth because an eventual σ_3-image of 3θ is the argument of an external ray that terminates at the critical point c. In particular, $R_c(3\theta)$ is not smooth.

On the other hand, the dynamical external ray $R_{c_0}(3\theta)$ of p_{c_0} is periodic and lands on a repelling or parabolic periodic point z_0 of p_{c_0}. Hence $R_{c_0}(\theta)$ lands on a non-periodic point z_1 such that $p_{c_0}(z_1) = z_0$. Since $c_0 \in \mathcal{M}_3^{sy}$, all dynamical external rays of p_{c_0} are smooth. If z_0 is a repelling point of p_{c_0}, then, by Theorem 6.1, the dynamical external ray $R_c(3\theta)$ is smooth and lands on a repelling periodic point for all c close to c_0. However, by the previous paragraph, if $c \in \mathcal{R}_\theta$ then $R_c(3\theta)$ is not smooth. This shows that z_0 is a parabolic point of p_{c_0}.

Since the dynamical ray $R_{c_0}(3\theta)$ lands on z_0, the period of z_0 divides the σ_3-period N of 3θ. Since the multiplier of z_0 with respect to $p_{c_0}^N$ is 1, there are finitely many candidates for an accumulation parameter c_0. Indeed, parabolic parameters c_0, for which there exists an N-periodic point of multiplier 1, form an algebraic subset of $\mathcal{SCP}$; this algebraic set is either finite or the entire plane, the latter option being clearly nonsensical. As the accumulation set of a ray is connected, it consists of exactly one such parabolic parameter. It follows also that one of the dynamical external rays $R_{c_0}(\theta \pm \frac{1}{3})$ lands on a parabolic point of p_{c_0}. □

Recall that if $J(f)$ is locally connected, then there is a well-defined σ_d-invariant lamination $\mathcal{L}_f$.

Definition 6.7 (Repelling leaves)**.** Let f be a degree $d > 1$ monic polynomial. For such f, define *repelling* (or *f-repelling*) leaves of $\mathcal{L}_f$ as leaves corresponding to pairs of rays landing on the same repelling periodic point of f or an iterated preimage thereof. Similarly, we can talk of *parabolic* (or *f-parabolic*) leaves of $\mathcal{L}_f$.

Repelling leaves and related laminations were used in [10] in the proof of continuity of the constructed there monotone (except for one point) model of the entire cubic connectedness locus.

Lemma 6.8. *Suppose that a sequence of monic degree $d > 1$ polynomials f_n converges to a polynomial f (necessarily monic of degree d) as $n \to \infty$. Then the following holds.*

1. *Let the dynamical external rays of f_n with periodic arguments $\theta_1, \dots, \theta_m$ land on the same point (neither m nor the angles θ_i depend on n). If the landing points of $R_f(\theta_1)$, $\dots$, $R_f(\theta_m)$ are repelling, then they coincide.*
2. *If $J(f)$ and all $J(f_n)$ are locally connected, $\mathcal{L}_{f_n} = \mathcal{L}$ for some lamination $\mathcal{L}$, and no critical point of f is mapped to a repelling periodic point, then all repelling leaves of $\mathcal{L}_f$ belong to $\mathcal{L}$.*

Proof. The lemma follows from Theorem 6.1. □

We are going to apply Lemma 6.8 to the situation where f is on the boundary of a hyperbolic domain $\mathcal{H}$ in some parameter space of polynomials and $f_n \in \mathcal{H}$ for each n.

Lemma 6.9. *Suppose that a parameter $c \in \mathcal{SCP} \setminus \mathcal{M}_3^{sy}$ is such that the rays $R_c(\theta \pm \frac{1}{3})$ hit a critical point. Let T be a σ_3-periodic polygon whose iterated forward σ_3-images are disjoint from the critical leaves $\overline{\theta + \frac{1}{3}\theta - \frac{1}{3}}$ and $\overline{\theta - \frac{1}{6}\theta + \frac{1}{6}}$. Then all dynamical external rays of arguments that are vertices of T land on the same point.*

This statement is not new: see [17]; it can also be deduced, e.g., from a more general Theorem 7.1 of [14], which gives a model for the landing pattern of all external rays of p_c (cf. also [13, Theorem 5.4] for a restatement of this result in the language of invariant laminations). For completeness, we give a sketch in our specific situation.

Sketch of a proof. Consider all external rays in the dynamical plane of p_c whose arguments correspond to vertices of T. All these rays land, perhaps at different points. Let $\hat{T}$ be the union of these rays with some continuum so that $\hat{T}$ is connected and disjoint from the forward orbits of the critical points of p_c. Denote by q the period of T. For every $n = 0, 1, \dots$, let $\hat{T}_n$ be the p_c^{qn}-pullback of $\hat{T}$ that contains the same collection of external rays as $\hat{T}$ and the σ_3^{qn}-pullback of the connecting continuum along the orbit of T.

Since p_c is hyperbolic, the sequence $\hat{T}_n$ converges to a connected set, comprising the original periodic rays and a continuum containing their landing points. However, all points of the limiting continuum must be inside of the Julia set, so this continuum has to be a singleton since the Julia set is totally disconnected. □

We can now prove a key theorem describing the limit transition of laminations in our situation.

Theorem 6.10. *If c is parabolic then the following holds.*

1. *If $c = r_{\mathcal{H}}$ for a hyperbolic component $\mathcal{H} \neq \mathcal{H}_{main}$ then $\mathcal{L}_c = \mathcal{L}_{\mathcal{H}}$ and $\overline{\text{co}}^+_{r_{\mathcal{H}}} = \overline{\text{co}}^+_{\mathcal{H}}$.*
2. *If c is the landing point of a parameter ray $\mathcal{R}_\theta$ and θ is 1-preperiodic, then θ is an endpoint of the marked comajor $\overline{\alpha\beta}$ of $\mathcal{L}_c$.*

Proof. (1) By the main result of [19], there exists a parabolic parameter $c_\circ$ with $\mathcal{L}_{c_\circ} = \mathcal{L}_{\mathcal{H}}$ (the surgery of [19] is local, hence it can be performed in a symmetric fashion to yield a symmetric polynomial $p_\circ$). This parabolic parameter $c_\circ$ is the root point of a certain hyperbolic component $\mathcal{H}_\circ$ such that $\mathcal{L}_{c_\circ} = \mathcal{L}_{\mathcal{H}_\circ}$, by Theorem 6.3. Moreover, $c_\circ$ and $\mathcal{H}_\circ$ have the same marked (co)majors. It remains only to establish that $\mathcal{H} = \mathcal{H}_\circ$, and the latter follows from Theorem 6.4.

Alternatively, remove all parabolic leaves from $\mathcal{L}_c$ to get a new symmetric hyperbolic lamination $\mathcal{L}' \subset \mathcal{L}_c$. By Lemma 6.8, $\mathcal{L}' \subset \mathcal{L}_{\mathcal{H}}$. Hence $\mathcal{L}_c$ is a tuning of $\mathcal{L}'$ done in two steps: (I) consistently add to $\mathcal{L}'$ a cycle (in the B case) or two cycles (in the D case) of finite gaps (of the same period as the corresponding cycles of hyperbolic gaps of $\mathcal{L}'$) with attached hyperbolic gaps; (II) pull this finite collection of gaps back. Moreover, $\mathcal{L}_{\mathcal{H}}$ is obtained similarly. Since by Pommerenke-Levin-Yoccoz inequality the combinatorial rotation number of the inserted cycle(s) of finite gaps in both $\mathcal{L}_c$ and $\mathcal{L}_{\mathcal{H}}$ cases is the same, $\mathcal{L}_c = \mathcal{L}_{\mathcal{H}}$.

(2) Let V^+_c be the closure of the maximal open subset of $\mathbb{D}$ containing U^+_c and disjoint from all repelling leaves of $\mathcal{L}_c$. Since there are no fixed return triangles of σ_3 by Lemma

4.4 of [8], either $V^+ = \mathbb{D}$, or the boundary of V_c^+ consists of a Cantor subset of $\mathbb{S}^1$ and countably many pairwise disjoint leaves of $\mathcal{L}_c$. Clearly, $U_c^+ \subset V_c^+$. We claim that if $V_c^+ \neq U_c^+$ then there exists a gap $G \subset V_c^+$ such that all images of U_c^+ inside V_c^+ share an edge with G. Indeed, suppose otherwise. Then it follows from [23] that there are repelling cutpoints of $J(p_c)$ such that the convex hulls of the arguments of dynamical external rays of p_c landing on them separate V_c^+, a contradiction with the definition of V_c^+. This proves the existence of G with the listed properties. Moreover, it follows that the edges of G are the periodic leaves of $\mathcal{L}_c$ inside V_c^+.

By Lemma 6.8, the angles $\theta \pm \frac{1}{3}$ are vertices of V_c^+. By Lemma 6.6, one of the rays $R_c(\theta \pm \frac{1}{3})$ (say, $R_c(\theta + \frac{1}{3})$) lands on a parabolic periodic point z_c of p_c. By way of contradiction, suppose that $\theta \notin \{\alpha, \beta\}$. Set $\overline{AB}$ to be the marked major of $\mathcal{L}_c$; by assumption $\theta + \frac{1}{3} \notin \{A, B\}$. Then $U_c^+ \neq V_c^+$ and we can consider the gap G defined in the previous paragraph. Since at least two rays land on each parabolic point of p_c, the dynamical external rays whose arguments are vertices of G land on z_c. Thus, the three angles A, B and $\theta + \frac{1}{3}$ are vertices of G and the dynamical external rays of arguments A, B and $\theta + \frac{1}{3}$ land on z_c. Note that the major edge of G coincides with the marked major $\overline{AB}$ of $\mathcal{L}_c$.

Similar to the construction of the degree two first (half-)return map η (see Definition 3.13), define a degree two map $\eta' : V_c^+ \to V_c^+$ semiconjugate to σ_2 by collapsing edges of V_c^+ to points. Under this semiconjugacy G maps to a finite σ_2-invariant gap (or leaf) G', and the critical leaf $\overline{\theta - \frac{1}{3}, \theta + \frac{1}{3}}$ projects to a critical leaf ℓ in $\overline{\mathbb{D}}$ with a σ_2-periodic endpoint which is a vertex of G' but not an endpoint of the Thurston major of G' (see [42]). Therefore there exists a finite σ_2-invariant gap T' disjoint from ℓ. Then the lifting of T' gives a finite η-invariant gap $T \subset V_c^+$. Note that G and T have disjoint sets of vertices.

Now consider a point c_t on the parameter ray $\mathcal{R}_\theta$, where the parameter t corresponds to the value of the Green function for $\mathcal{M}_3^{sy}$ at c_t so that c_t converges to c as $t \to 0$. Then the rays $R_{c_t}(\theta \pm \frac{1}{3})$ both hit the marked critical point c_t. By Lemma 6.9, the dynamical external rays of p_{c_t} whose arguments correspond to the vertices of T land on the same repelling periodic point w_t. The point w_t has a well-defined limit w_0 as $t \to 0$. On the other hand, for every angle γ corresponding to a vertex of T, the ray $R_c(\gamma)$ lands on a repelling periodic point of p_c. By Theorem 6.1 and by Lemma 6.8 this point is close to w_t for small t, hence it must coincide with w_0. Thus, w_0 is repelling for p_c, and the gap or leaf T corresponding to w_0 must belong to $\mathcal{L}_c$, a contradiction. □

Observe that, by Lemma 6.6, there is a dense set of 1-preperiodic angles such that the corresponding parameter rays land on parabolic parameters in $\mathcal{M}_3^{sy}$.

Theorem 6.11. *Let $\overline{\alpha\beta}$ be the marked comajor of a symmetric Fatou lamination $\mathcal{L}$. Then there exists a unique hyperbolic component $\mathcal{H} \neq \mathcal{H}_{main}$ such that the parameter rays $\mathcal{R}_\alpha$, $\mathcal{R}_\beta$ land on the root point $r_\mathcal{H}$ and $\mathcal{L}_{r_\mathcal{H}} = \mathcal{L}_\mathcal{H} = \mathcal{L}$. Moreover, let $c \in \partial\mathcal{H}, c \neq r_\mathcal{H}$ be a parabolic parameter. Then $\mathcal{L}_c \supsetneqq \mathcal{L}_\mathcal{H}$, and the marked comajor of $\mathcal{L}_c$ is located under the marked comajor of $\mathcal{L}_\mathcal{H}$.*

Proof. Consider the marked comajor $\overline{\alpha\beta}$ of a symmetric Fatou lamination $\mathcal{L}$. Then α and β are 1-preperiodic. By Lemma 6.6, the parameter ray $\mathcal{R}_\alpha$ lands on a parabolic parameter c. The angle α is an endpoint of the marked comajor $\overline{\mathrm{co}}_c^+$ of the lamination $\mathcal{L}_c$, by Theorem 6.10. Since distinct 1-preperiodic comajors are disjoint, $\overline{\mathrm{co}}_c^+ = \overline{\alpha\beta}$. By Theorem 6.3, there exists a hyperbolic component $\mathcal{H}$ such that $c = r_\mathcal{H}$ is the root of $\mathcal{H}$. By Theorem 6.10, the lamination $\mathcal{L}_\mathcal{H}$ coincides with $\mathcal{L}_c$. By Theorem 6.4, the set $\mathcal{H}$ is a unique hyperbolic component such that $\overline{\mathrm{co}}_\mathcal{H}^+ = \overline{\alpha\beta}$. It follows that $\mathcal{R}_\beta$ lands on c, too. The fact that $\mathcal{L}_{r_\mathcal{H}} = \mathcal{L}_\mathcal{H} = \mathcal{L}$ follows from Theorem 6.10.

To prove the last claim of the theorem, let $c \in \partial\mathcal{H}, c \neq r_\mathcal{H}$, be a parabolic parameter. Since $c \neq r_\mathcal{H}$, the repelling periodic points of polynomials $p_{c^*} \in \mathcal{H}$ associated with the marked major $M_\mathcal{H}$ of $\mathcal{L}_\mathcal{H}$ converge to a repelling periodic point of p_c of the same period as $c^* \to c$. Hence all the edges of the gaps $U_\mathcal{H}^\pm$ of $\mathcal{L}_\mathcal{H} = \mathcal{L}_{c^*}$ remain edges of $\mathcal{L}_c$. Thus, the critical gaps of $\mathcal{L}_c$ are contained in $U_\mathcal{H}^\pm$, and, hence, $\mathcal{L}_c \supset \mathcal{L}_\mathcal{H}$, and the comajors of $\mathcal{L}_c$ are located under those of $\mathcal{L}_\mathcal{H}$ (this yields the claim of the theorem about the marked comajors). The hyperbolic component with the same marked comajor as $\mathcal{L}_c$ cannot coincide with $\mathcal{H}$, since c is not a root point of $\mathcal{H}$. Hence, $\mathcal{L}_c \neq \mathcal{L}_\mathcal{H}$. □

Call the parameter rays from Theorem 6.11 *characteristic rays of a hyperbolic component* $\mathcal{H} \neq \mathcal{H}_{main}$. Recall that by an arc $(a, b) \subset \mathbb{S}^1$ we always mean the *positively oriented circle arc* with endpoints $a, b \in \mathbb{S}^1$.

Lemma 6.12. *The characteristic rays are the only two strictly preperiodic rays that accumulate on a parabolic parameter c.*

Proof. By Theorems 6.3 and 6.11, c is the landing point of the parameter rays $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ where $\overline{\alpha\beta}$ is a 1-preperiodic comajor. Thus, all 1-preperiodic comajors give rise to cuts in the parameter plane. Hence, if a comajor separates $\overline{\alpha\beta}$ from an angle γ in $\overline{\mathbb{D}}$, then $\mathcal{R}_\gamma$ cannot accumulate on c. Since, by Theorem 3.7, the 1-preperiodic comajors are dense in C_sCL and disjoint from all other comajors, it follows that the only way a parameter ray $\mathcal{R}_\gamma$ can accumulate on c is when γ is a vertex of an infinite gap G with an edge $\overline{\alpha\beta}$. However, by Theorem 3.15, the fact that γ is preperiodic implies that γ is actually 1-preperiodic. Again by Theorem 6.3, this implies that $\mathcal{R}_\gamma$ cannot land on c, as desired. □

Theorem 6.13. *Each parabolic parameter $c \in \partial\mathcal{H}_{main}$ is associated to a comajor $\overline{\alpha\beta}$ which is an edge of G_{main} and, accordingly, to a point of $\overline{\mathbb{D}}/C_sCL$. The parameter rays $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ land on c. For every hyperbolic domain $\mathcal{H} \neq \mathcal{H}_{main}$ the corresponding marked comajor $\overline{\mathrm{co}}^+_{\mathcal{H}} = \overline{\theta_1\theta_2}$ is associated to the parameter rays $\mathcal{R}_{\theta_1}$ and $\mathcal{R}_{\theta_2}$ that land on the root $r_{\mathcal{H}}$ of $\mathcal{H}$ and separate $\mathcal{H}$ from $\mathcal{H}_{main}$.*

Proof. Let $c \in \partial\mathcal{H}_{main}$ be a parabolic parameter. Then 0 is a parabolic point associated to a finite invariant gap T whose vertices are the arguments of dynamical external rays of p_c landing on 0. It follows that there is a unique symmetric lamination $\mathcal{L}$ associated with T which has the gap T, Fatou gaps of degree greater than 1 attached to T and "rotating" around T, and pullbacks of all these gaps (this fully describes $\mathcal{L}$). Let $\overline{\alpha\beta}$ be the marked comajor of $\mathcal{L}$ associated with the marked cocritical point $-2c$ of p_c; by Theorem 6.11, the parameter rays $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ land on c. By Theorem 3.12, all parabolic parameters $c \in \partial\mathcal{H}_{main}$ correspond to edges of the main gap G_{main} of C_sCL and in the end map to the corresponding points of $\overline{\mathbb{D}}/C_sCL$ that belong to the main domain D_{main} of $\overline{\mathbb{D}}/C_sCL$. The rest of the theorem easily follows. □

7 Misiurewicz parameters

A number $c \in \mathbb{C}$ is a *Misiurewicz parameter* if the p_c-orbits of critical values are strictly preperiodic. If c is a Misiurewicz parameter, then K_c is connected ($c \in \mathcal{M}_3^{sy}$), all p_c-periodic points are repelling (see Theorem 5.4), and $J(p_c)$ is a dendrite (recall that a dendrite is a locally connected continuum that contains no Jordan curves). Recall that a cubic symmetric lamination $\mathcal{L}$ is called a *Misiurewicz* lamination if its critical sets are strictly preperiodic. Such laminations and their comajors are discussed in detail right after Lemma 3.6. By Theorem 3.7(2), Misiurewicz cocritical leaves (gaps) are leaves (gaps) of C_sCL approached from all sides by 1-preperiodic comajors. Since for each polynomial from $\mathcal{SCP}$ a critical point is marked, then each Misiurewicz lamination is considered twice, with either cocritical set marked. Finally, recall that the lamination C_sCL defines a laminational equivalence relation $\sim_{sy}$. For brevity by "$\sim_{sy}$-class" we will mean an equivalence class of $\sim_{sy}$.

Lemma 7.1. *Let p_c be a cubic symmetric polynomial with a dendritic Julia set, and T be the marked critical set of $\mathcal{L}_c$. If $c_n \in \mathcal{SCP} \setminus \mathcal{M}_3^{sy}$ converge to c as $n \to \infty$, then the arguments of the external rays of p_{c_n} hitting the marked critical point c_n converge to vertices of T.*

Proof. Every leaf of $\mathcal{L}_c$ that is not an edge of a gap is approximated by (pre)periodic leaves from both sides, and every edge of a gap G in $\mathcal{L}_c$ is approximated by preperiodic leaves from outside of G. Choose a neighborhood W of T in $\mathbb{D}$ whose boundary is formed by (pre)periodic leaves close to the edges of T, and appropriate circle arcs. By Theorem 6.1, there is a neighborhood $\mathcal{W}$ of c in $\mathcal{SCP}$ such that for $c^* \in \mathcal{W}$, the leaves forming the boundary of W in $\mathbb{D}$ are associated with cuts formed by the dynamical external rays of p_{c^*} (it does not matter whether $J(p_{c^*})$ is connected or not). The part of the dynamical plane of p_{c^*} bounded by all these cuts contains the point c^*. This implies the desired. □

The following theorem is a special case of [26, Theorem 1] but we give a proof for completeness, and also since our special case is much simpler than the general one.

Theorem 7.2. *Let $\mathcal{L}$ be a Misiurewicz lamination. Parameter rays whose arguments are vertices of the marked cocritical set of $\mathcal{L}$ land on a Misiurewicz parameter $\hat{c}$ such that $\mathcal{L}_{\hat{c}} = \mathcal{L}$.*

Proof. Let θ be a k-preperiodic angle with $k > 1$. Choose $c_0 \in \partial\mathcal{M}_3^{sy}$ in the accumulation set of $\mathcal{R}_\theta$. Then the periodic dynamical external ray $R_{c_0}(3^k\theta)$ lands on a periodic point z_0. By Lemma 6.12, the point z_0 is repelling. For a parameter c, consider the union R_c of the rays from the forward orbit of the closure of $R_c(\theta)$; it consists of finitely many rays and their landing points. By Lemma 7.1, there are no critical points among their landing points provided that c is close to c_0.

By Theorem 6.1, for some neighborhood $\mathcal{W}$ of c_0 in $\mathcal{SCP}$, the set R_c depends continuously on $c \in \mathcal{W}$, consists only of smooth rays and their landing points, and contains no critical points of p_c. In particular, $R_{c_0}(\theta)$ lands on the cocritical point $-2c_0$. Thus, p_{c_0} is a symmetric Misiurewicz polynomial, and $\mathcal{L}_{c_0}$ is a symmetric Misiurewicz lamination. Since there are countably many symmetric Misiurewicz polynomials, and the accumulation set of $\mathcal{R}_\theta$ is either a non-degenerate continuum (hence uncountable) or a point, it follows that $\mathcal{R}_\theta$ lands on c_0 and that θ is a vertex of a cocritical set of $\mathcal{L}_{c_0}$.

By Theorem 3.7, all preperiodic angles of preperiod > 1 are partitioned into vertex sets of various Misiurewicz cocritical sets. Thus, for a preperiod > 1 angle θ there exists a unique symmetric lamination $\mathcal{L}$ such that θ is a vertex of a cocritical set of $\mathcal{L}$. Taking into account the fact that each polynomial is counted in $\mathcal{SCP}$ twice (depending on the choice of the marked critical point), and choosing marked cocritical sets accordingly, we see that if $\mathcal{L}$ is a Misiurewicz lamination whose marked cocritical set has vertices $\theta_1, \dots, \theta_m$, then the parameter rays $\mathcal{R}_{\theta_1}, \dots, \mathcal{R}_{\theta_m}$ land on a Misiurewicz parameter $\hat{c}$ such that $\mathcal{L}_{\hat{c}} = \mathcal{L}$. □

8 The Structure of $\mathcal{M}_3^{sy}$

Recall that $\mathcal{M}_{3,comb}^{sy}$ is the quotient space of $\overline{\mathbb{D}}$ under $\sim_{sy}$ and $\mathfrak{pr}_{comb} : \overline{\mathbb{D}} \to \mathcal{M}_{3,comb}^{sy}$ is the corresponding quotient projection (see Definition 3.8). Given a continuum $K \subset \mathbb{C}$, define the *topological hull of* K as the complement to the unbounded complementary component of K. Recall also that a *monotone* map is a map whose point-preimages (fibers) are connected. Theorem 8.4 completes the proof of the Main Theorem.

However first we introduce tools from continuum theory developed in [3] (for basic continuum theory facts see, e.g., [36]) and applied in [3] to the problem of modeling connected polynomial Julia sets. The tools apply to all planar continua.

Let A be a continuum. Then an onto map $\varphi : A \to Y_{\varphi,A}$ is said to be a *finest (monotone) map (onto a locally connected continuum)* if for any other monotone map $\psi : A \to L$ onto a locally connected continuum L there exists a monotone map $h : Y_{\varphi,A} \to L$ such that $\psi = h \circ \varphi$. Observe, that in this situation the map h is automatically monotone because for $x \in L$ we have $h^{-1}(x) = \varphi(\psi^{-1}(x))$. It is easy to see that all sets $Y_{\varphi,A}$ are homeomorphic and all finest maps φ are the same up to a homeomorphism. Thus from now on we may talk of ***the** finest model* $Y_A = Y$ *of* A and ***the** finest map* $\varphi_A = \varphi$ *of* A *onto* Y.

A planar continuum $Q \subset \mathbb{C}$ is said to be *unshielded* if it coincides with the boundary of its topological hull. Thus, unshielded continua are the boundaries of *full* planar continua.

Theorem 8.1 (Theorem 1 of [3]). *Let* Q *be an unshielded continuum. Then there exist the finest map* φ *and the finest model* Y *of* Q*. Moreover,* φ *can be extended to a map* $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *which maps* ∞ *to* ∞*, in* $\overline{\mathbb{C}} \setminus Q$ *collapses only those complementary domains to* Q *whose boundaries are collapsed by* φ*, and is a homeomorphism elsewhere in* $\overline{\mathbb{C}} \setminus Q$*.*

It may happen that the finest model is a point (e.g., this is so if the continuum is *indecomposable*, i.e. cannot be represented as the union of two non-trivial subcontinua). In [3] we establish useful sufficient conditions for this not to be the case, then apply Theorem 8.1 to a polynomial P with connected Julia set, and explicitly construct the finest models (see Theorem 8.3 below). Moreover, in [3, Theorem 2] we show that for a connected polynomial Julia set $J(P)$ the model is *dynamical*, i.e. admitting a self-mapping to which $P|_{J(P)}$ is semiconjugate. The self-mapping in question is actually a topological polynomial. This gives an alternative proof of results of [25] and extends them onto all connected polynomial Julia sets. Denote by $I_\theta(Q)$ the *impression* of the Riemann ray $R_Q(\theta)$ to an unshielded continuum Q which by definition is the same as the impression of the corresponding prime end (abusing the terminology we will call them *impressions of angles* θ).

Definition 8.2. Let Q be an unshielded planar continuum. Declare two points $x, y \in Q$ equivalent if they belong to a finite connected union of impressions of angles, and denote this equivalence relation on Q by $\asymp_Q$. Then consider the intersection $\approx_Q$ of all closed equivalence relations on Q that contain $\asymp_Q$. Declare two angles $\alpha, \beta \in \mathbb{S}^1$ equivalent if their impressions are contained in one $\approx_Q$-class, and denote this equivalence relation on $\mathbb{S}^1$ by $\sim_Q$.

Definition 8.2 offers a constructive version of Theorem 8.1. Moreover, it also suggests a laminational interpretation of the finest model of Q (the second part of Theorem 8.3 is not explicitly proven in [3] but immediately follows and can be established repeating verbatim a part of the proof of [3, Theorem 2]).

Theorem 8.3 (Theorem 18 of [3])**.** *The quotient map $Q \to Q/\approx_Q$ is the finest map of the continuum Q. The equivalence relation $\sim_Q$ is laminational, and the finest model $Q/\approx_Q$ of Q is homeomorphic to $\mathbb{S}^1/\sim_Q$.*

We can now prove Theorem 8.4.

Theorem 8.4. *There is a monotone continuous surjective map $\pi : \mathcal{M}_3^{sy} \to \mathcal{M}_{3,comb}^{sy}$; if $\mathcal{M}_3^{sy}$ is locally connected, π is a homeomorphism.*

Proof. Set $\approx_{\mathcal{M}_3^{sy}}=\approx$ and $\sim_{\mathcal{M}_3^{sy}}=\sim$. By Theorem 8.3, the spaces $\mathcal{M}_3^{sy}/\approx$ and $\mathbb{S}^1/\sim$ are homeomorphic. Thus, it suffices to show that equivalence relations $\sim$ and $\sim_{sy}$ on $\mathbb{S}^1$ coincide. By Theorem 3.7 and the results in Sections 6 and 7, if $\mathbf{g}$ is a $\sim_{sy}$-class whose convex hull does not intersect an infinite gap of C_sCL, then **g** is in fact a $\sim$-class. It remains to consider $\sim_{sy}$-classes **h** whose convex hulls intersect infinite gaps of C_sCL. In what follows we use notation and terminology from right before Theorem 3.15. In particular, recall that $\mathcal{O}$ is the center of $\overline{\mathbb{D}}$.

Let **h** be a $\sim_{sy}$-class such that CH(**h**) intersects ∂G, where G is an infinite gap of C_sCL. If $\mathcal{O} \notin G$, then, by Theorems 3.7 and 3.15, there is an associated with G Fatou lamination $\mathcal{L}$ with critical Fatou gap U and the (half-)return map $\eta : \partial U \to \partial U$ semiconjugate to $\sigma_2 : \mathbb{S}^1 \to \mathbb{S}^1$ by a map ϕ collapsing all edges of U. We have $G \subset U$, and the ϕ-images of the edges of G are the majors of laminations from the Main Cardioid of σ_2.

Let ℓ be the edge of G separating the rest of G from $\mathcal{O}$. By the properties of the Main Cardioid, an infinite gap of C_sCL is attached to G at any edge $\ell' \neq \ell$ of G while the points of ∂G that are not endpoints of edges of G are C_sCL-classes. By Theorem 3.7, ℓ may be an edge of another infinite gap of C_sCL or it may be non-isolated in C_sCL.

By Theorem 6.4, there is a unique hyperbolic component $\mathcal{H}$ with $\overline{\mathrm{co}}^+_{\mathcal{H}} = \ell$. By Theorem 6.11, a dense subset of $\partial\mathcal{H}$ consists of parabolic parameters c with two parameter rays corresponding to the endpoints of an edge of G landing on $c \in \partial\mathcal{H}$. Properties of impressions and the fact that $\partial\mathcal{H}$ is a Jordan curve imply that for all edges of G other than ℓ the corresponding $\sim$-class and $\sim_{sy}$-class coincide.

The situation with ℓ is different and has three cases. Firstly, if ℓ is an edge of another infinite gap of C_sCL not containing $\mathcal{O}$, then the above arguments show that the $\sim$-class associated with ℓ consists only of the endpoints of ℓ. Secondly, ℓ may be the limit of other edges of C_sCL converging to ℓ from outside of G. Let Γ_ℓ be the parameter cut corresponding to ℓ. By Theorem 3.7, Γ_ℓ is approximated by 1-preperiodic parabolic cuts separating Γ_ℓ from impressions of other parameter rays in the complementary component of Γ_ℓ not containing $\mathcal{H}$; thus, in this case, too, the $\sim$-class and the corresponding $\sim_{sy}$-class coincide.

The remaining case is when ℓ is the shared edge of G and the gap G_{main} of C_sCL corresponding to $\mathcal{H}_{main}$. Recall that $\partial\mathcal{H}_{main}$ is the circle of radius $\frac{\sqrt{3}}{3}$ centered at the origin.

For $c \in \partial\mathcal{H}_{main}$, the multiplier at the neutral point 0 is $-3c^2$. Hence there is a dense in $\partial\mathcal{H}_{main}$ set of parabolic parameters c associated to the comajors like ℓ above. The above arguments show that for all edges of G_{main} and for all points of ∂G_{main} that are not endpoints of an edge of G_{main} the same conclusion holds: the $\sim$-classes and the $\sim_{sy}$-classes are the same. □

From now on, we will always denote the modeling projection from Theorem 8.4 by π.

Definition 8.5. Let $\mathcal{H}$ be a hyperbolic component associated with a comajor $\overline{\alpha\beta}$. Then the parameter rays $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ land on the root $r_\mathcal{H}$ of $\mathcal{H}$. Denote by $\mathcal{W}(\overline{\alpha\beta})$ the component of $\mathbb{C} \setminus [\mathcal{R}_\alpha \cup \mathcal{R}_\beta]$ that does not contain $\mathcal{H}_{main}$ and will call this set the *wake generated by* $\overline{\alpha\beta}$.

The next lemma describes wakes in a more dynamical fashion.

Lemma 8.6. *Let $\overline{\alpha\beta}$ be a comajor corresponding to a major $\overline{\alpha'\beta'}$.*

1. *If $c \in \mathcal{W}(\overline{\alpha\beta})$, then the external rays $R_c(\alpha')$ and $R_c(\beta')$ are smooth and land at the same repelling periodic point.*

2. *If $R_c(\alpha')$ and $R_c(\beta')$ are smooth and land at the same repelling periodic point, then $c \in \mathcal{W}(\overline{\alpha\beta}) \cup \mathcal{W}(\tau(\overline{\alpha\beta}))$.*

Proof. (1) The argument given below follows [31]. Set $\mathcal{W} := \mathcal{W}(\overline{\alpha\beta})$, and let q be the period of α and β. The claim holds for the interior of the corresponding hyperbolic domain $\mathcal{H}$. By the properties of C_sCL, parabolic parameters on the boundary of $\mathcal{H}$, and hence the entire $\mathcal{H}$, are separated from $\mathcal{H}_{main}$ by the parameter rays $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ that land on the root $r_\mathcal{H}$ of $\mathcal{H}$. Thus, $\mathcal{H} \subset \mathcal{W}$, and, by Theorem 6.11, for all parameters $c \in \mathcal{H}$ the dynamical external rays $R_c(\alpha')$ and $R_c(\beta')$ land on the same repelling periodic point.

We claim that the rays $R_c(\alpha')$ and $R_c(\beta')$ land for all $c \in \mathcal{W}$. Indeed, by the properties of comajors the circle arc $(\sigma_3^2(\alpha'), \sigma_3^2(\beta'))$ contains no images of $\sigma_3(\alpha')$ or $\sigma_3(\beta')$. Hence points from the orbits of $\sigma_3(\alpha')$ and $\sigma_3(\beta')$ cannot be endpoints of critical chords with the other endpoint in (α', β'). However, these are precisely the critical chords defining the critical cuts for polynomials $p_c, c \in \mathcal{W}$. It follows that for all $c \in \mathcal{W}$ the rays $R_c(\alpha')$ and $R_c(\beta')$ remain smooth (never pass through a cut) and land as claimed.

The landing point of $R_c(\alpha')$ is repelling except for finitely many values of $c \in \mathcal{W}$, for which it may become parabolic; the multiplier of this parabolic point must be a q-th root of unity. By the maximum modulus principle, such values of c are impossible. It follows that $R_c(\alpha')$ always lands on a repelling point, for $c \in \mathcal{W}$, and the same holds for $R_c(\beta')$. Since the two landing points coincide in $\mathcal{H}$, they also coincide everywhere in $\mathcal{W}$, by Theorem 6.1.

(2) Suppose that $R_c(\alpha')$ and $R_c(\beta')$ are smooth and land at the same repelling periodic point. By Theorem 6.1, this property is stable under small perturbations of c. In particular, if the Julia set of p_c is connected, then we may replace c with a postcritically finite value c' with the same property. Also, if $J(p_c)$ is disconnected, then, upon a slight perturbation of c, one may assume that c lies on a rational ray. Let c' be the landing point of this ray, and replace c' with a nearby postcritically finite parameter c''. Thus, in any case, it is safe to assume that c is postcritically finite, in particular, $J(p_c)$ is locally connected, and the corresponding lamination $\mathcal{L}_c$ determines the topological dynamics of p_c on $J(p_c)$. All periodic leaves of $\mathcal{L}_c$ are repelling.

By the assumption, $\overline{\alpha'\beta'} \in \mathcal{L}_c$. Let S' be the strip formed by $\overline{\alpha'\beta'}$ and its sibling chord so that $S' \cup \tau(S')$ is a short strips set. Some major M_c of $\mathcal{L}_c$ must be contained in S'; it follows that a comajor of $\mathcal{L}_c$ is under $\overline{\alpha\beta}$. Either this is the marked comajor of $\mathcal{L}_c$, in which case $c \in \mathcal{W}(\overline{\alpha\beta})$, or the symmetric one, in which case $c \in \mathcal{W}(\tau(\overline{\alpha\beta}))$. □

Recall that, by Theorem 3.7(3) (i.e., by Theorem 4.15 of [9]), the 1-preperiodic comajors are dense in C_sCL. Using this, we will now relate the dynamics of certain symmetric cubic polynomials with their location in the parameter space. To do this, we will need the original results of Kiwi [25] that we have already mentioned in the beginning of this section. As far as we know, these were the first results where connected but not necessarily locally connected Julia sets of certain polynomials were modeled. The polynomials in question are polynomials with connected Julia set and all cycles repelling. However, when stating Kiwi's theorem, we use the approach of [3] described in the beginning of this section.

Theorem 8.7 (Theorem 5.12 of [25])**.** *Let P be a complex polynomial with connected Julia set $J(P)$ and all cycles repelling. Then $\sim_{J_P}$ is a σ_d-invariant laminational equivalence relation defining a q-lamination $\mathcal{L}_{\sim_P}$. Moreover, $P : J(P) \to J(P)$ is monotonically semiconjugate to a topological polynomial $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$, the topological Julia set $J_{\sim_P}$ is a dendrite, and the semiconjugacy p_P between $P|_{J(P)}$ and $f_{\sim_P}|_{J_{\sim_P}}$ is one-to-one on all (pre)periodic points of P. If $J(P)$ is locally connected, p_P is a homeomorphism.*

Let P be a symmetric polynomial with connected Julia set and all cycles repelling. Theorem 8.7 allows one to define, for such P, its q-lamination $\mathcal{L}_{\sim_P}$ and, therefore, the associated marked cocritical set C_P. Let us show that then C_P determines the fiber of the modeling projection π from Theorem 8.4 that contains P. To this end, we need a well-known fact concerning the dynamics of dendritic topological Julia sets $J_\sim$ defined by a laminational equivalence relation $\sim$. Still, for the sake of completeness we sketch its proof.

Lemma 8.8. *Let $\sim$ be a σ_d-invariant laminational equivalence relation such that $J_\sim$ is a dendrite. Suppose that $\overline{ab}$ is a periodic chord whose forward σ_d-orbit consists of pairwise unlinked chords that do not cross edges of critical sets of $\mathcal{L}_\sim$. Then $a \sim b$.*

Sketch of the proof. Consider preimages of critical gaps or leaves of $\mathcal{L}_\sim$; choose only those preimages that are themselves gaps of leaves of $\mathcal{L}_\sim$. Take the closure of their union. By the assumptions made, there is a unique component U of the complement to this union that contains $\overline{ab}$ in its closure. The set of all classes of points from $\overline{U} \cap \mathbb{S}^1$ corresponds to a non-degenerate continuum $T \subset J_\sim$ whose orbit, by the assumption, has no iterated images that contain critical points of $f_\sim$ as their cutpoints. On the other hand, by Theorem C of [5], the continuum T is non-wandering and has two distinct iterated images that intersect. The union of the appropriate iterated images of T will then yield connected n-periodic subset of $J_\sim$ whose closure A contains no critical cutpoints. This implies that there are periodic attracting points of $f^n_\sim$ in A, which contradicts the expanding properties of σ_d. □

We can now describe some *fibers*, i.e., point preimages, of π.

Theorem 8.9. *If all cycles of $P \in \mathcal{M}_3^{sy}$ are repelling, then the π-fiber of C_P contains P. Moreover, for any P' in the same fiber, $C_{P'} = C_P$ and therefore $\mathcal{L}_{\sim_{P'}} = \mathcal{L}_{\sim_P}$. If, in addition, P is finitely renormalizable, then the fiber is $\{P\}$.*

Proof. By Theorem 6.13, we may assume that C_P is not a 1-preperiodic comajor. Suppose now that C_P is a finite gap. Then, by Theorem 3.7(3), all edges of C_P are approximated from outside of C_P arbitrarily well by 1-preperiodic comajors that give rise to the associated wakes in the parameter space. Let us consider the edge ℓ of C_P that separates the interior of C_P from the center of the circle. Choose a 1-preperiodic comajor $\overline{y}$ close to ℓ that also separates C_P from the center of the circle; it corresponds to a periodic major ℓ_y, and, by the properties of majors, the iterated σ_3-images of ℓ_y never enter a critical strip defined by ℓ_y or its symmetric counterpart. Thus, ℓ_y satisfies the assumptions of Lemma 8.8. By Lemma 8.8, this implies that the endpoints of ℓ_y are $\sim_P$-equivalent and, hence, are associated with a repelling cut Y of $J(P)$. Therefore, by Lemma 8.6, the parameter of P is located in the wake $\mathcal{W}_{\overline{y}}$ for any $\overline{y}$ with the listed properties.

On the other hand, let $\ell' \neq \ell$ be another edge of C_P. We can approximate it (again by Theorem 3.7(3)) by 1-preperiodic comajors, necessarily from the outside of C_P. These define wakes as before, yet this time the corresponding periodic leaves will not coexist with the critical sets of $\mathcal{L}_P$. Hence, P cannot belong to those wakes, which implies that it belongs to their complements in the plane.

By definition, the intersection of the wakes (described in the first paragraph of the proof), the complements of wakes (described in the second paragraph of the proof), and $\mathcal{M}_3^{sy}$ is the fiber of the modeling map π from Theorem 8.4; this completes the proof of the first claim in the case when C_P is a gap. If C_P is a leaf, a similar (almost verbatim) argument implies the same conclusion. Observe that in this case Theorem 3.7 implies that C_P is approximated by 1-preperiodic comajors from both sides.

The case when $C_P = \{x\}$ is a singleton is a bit different. Recall, that $J_{\sim_P}$ is a dendrite. Denote by $\zeta(X)$ the point in the topological Julia set $J_{\sim_P}$ that corresponds to the convex hull X of a $\sim_P$-class. By [8, Lemma 3.3], there is a well defined invariant central (i.e. containing the center of the circle) gap (or leaf) $\mathrm{CG}_{\mathcal{L}_{\sim_P}}$ of $\mathcal{L}_\sim$. Let $a = \zeta(\mathrm{CG}_{\mathcal{L}_{\sim_P}})$. Connect a and $\zeta(\{x\})$ with an arc I and consider dynamics of points of I that are located close to $\zeta(\{x\})$. Let $y \in I, y = \zeta(Y)$ be a point close to $\zeta(\{x\})$. Denote by Z_y the component of $J_{\sim_P} \setminus \{y\}$ that contains $\zeta(\{x\})$. Then by Lemma 3.8 (the so-called Short Strips Lemma) of [8] translated into the language of dynamics on $J_{\sim_P}$ we see that the forward orbit of y either never enters Z_y, or, if it does, the first time it enters Z_y is in I again. In the former case it follows from the definitions that the edge ℓ'' of Y that separates the rest of Y from x (i.e., the edge of Y that “faces” x) is a comajor. By Theorem 3.7(3) this implies that there are 1-preperiodic comajors very close to ℓ'' and the arguments from the beginning of the proof apply in this case, too.

Hence we may assume that orbits of all points $y \in I$ close to $\zeta(\{x\})$ always eventually enter their sets Z_y. However there are (pre)periodic points in I arbitrarily close to $\zeta(\{x\})$, and these cannot keep getting mapped to points of I closer and closer to $\zeta(\{x\})$. Thus, there are comajors associated to points of I arbitrarily close to $\zeta(\{x\})$ which implies the desired.

By the above, for every polynomial P' in the fiber of C_P, the associated lamination $\mathcal{L}_{\sim P'}$ has C_P as a marked comajor. It follows that $\mathcal{L}_{\sim P} = \mathcal{L}_{\sim P'}$ as critically marked laminations, cf. Corollary 3.11. To prove the last claim of the lemma, observe that in the case of finitely renormalizable symmetric polynomials the claim follows from [27, Theorem 1.2] stating, in a special case, that, if $\mathcal{L}_{\sim_P} = \mathcal{L}_{\sim_{P'}}$ is at most finitely many times renormalizable, and both P and P' have all cycles repelling, then $P = P'$, up to affine conjugacy. □

To conclude we would like to say that we expect the following claims to hold for the fibers of π:

1. On the union of the boundaries of all hyperbolic components, π is injective.

2. Every nontrivial fiber contains a polynomial whose Julia set has positive area and supports an invariant measurable line field; there is a J-stable component inside

the fiber consisting of such polynomials.

However, we do not prove these claims here.

Acknoledgements

The first named author was partially supported by NSF grant DMS-2349942. The second named author was partially supported by NSF grant DMS–1807558. The work of the fourth named author was supported by the HSE University Basic Research Program.

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