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Monodromy Groups and the Insolvability of Transcendental Equations in Quadratures

Allan Allemand   Alexey Kanel-Belov   Rodion Zaytsev

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Abstract:

This paper presents the authors' results in applying Arnold's method to compute the monodromy groups of certain trigonometric complex equations, and also provides a survey of other results in this area.

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1 Introduction

Our primary goal in this paper is to compute the monodromy groups for several equations of interest. The transcendental equations investigated here are not arbitrary; they are

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motivated by their appearance in mathematical physics and spectral theory. Equations of the type $\tan(z) - z = 0$ are canonical in this regard: they arose in Fourier's 19th-century analysis of the heat equation, where their roots correspond to the eigenvalues of a Sturm-Liouville operator, as noted by F. Klein [5]. A similar result was obtained by Fourier, but here we present a new perspective and proof using modern methods, which interestingly leads to the same conclusion. A. Elishev demonstrated this fact in modern terms [2], and we include his argument.

Modern analogues, such as the equation studied by Heifetz for acoustic wave propagation, further highlight the physical importance of both the real roots (propagating modes) and the complex roots (resonances) [3]. We include a full summary of his results, as the original was published in a difficult-to-access Soviet journal. This motivates a deeper study of the complete root structure of such equations under parameter variation, which is what their monodromy groups describe.

Our primary tool is the topological Galois theory developed by V. I. Arnold and later by A. G. Khovanskii [4]. This theory connects the solvability of an equation to the algebraic properties of its monodromy group. The notion of solvability can be made precise through a hierarchy of function classes. Starting with elementary functions (built from rational functions, $\exp(z)$, $\log(z)$, and algebraic operations), one can define functions solvable in quadratures (allowing integration), and recursively, in k**-quadratures** (meaning functions obtained by applying up to k nested integrations). The union of all such classes forms the class of functions solvable in generalized quadratures. The main theorem of topological Galois theory, in its strong form due to Khovanskii, addresses this entire hierarchy:

Theorem. *If the solution $z(a)$ of an equation $f(z) = a$, considered as a function of the parameter a, is representable in generalized quadratures, then the monodromy group of the equation is solvable.*

In this paper, we use the contrapositive of this theorem. By showing these monodromy groups to be unsolvable (e.g., containing A_5 or A_∞), we prove that these equations are not solvable even in the broad class of generalized quadratures.

Our approach follows the constructive spirit of Arnold's method. The core idea is to analyze the behavior of the roots z_i of the equation $f(z) = a$ as the parameter a traverses a closed loop in the complex plane. The roots themselves trace continuous paths, which may not be closed, resulting in a permutation of their initial positions. This permutation defines an element of the monodromy group. Non-trivial permutations are generated when these loops encircle the critical values of $f(z)$ — the images of points where the derivative $f'(z)$ vanishes. Examples of application of this method with illustrations can be found in our previous paper [1].

Therefore, our work in the subsequent sections will be to identify these critical values and deliberately construct paths around them to generate a set of permutations sufficient to prove the unsolvability of the corresponding monodromy group.

2 The Case of a Self-Adjoint Function

Let the function $f(z)$ have the following properties:

1. The function $f(z)$ is self-adjoint, i.e., $f(\overline{z}) = \overline{f(z)}$.
2. The critical points of $f(z)$ (where $f'(z) = 0$) have multiplicity 3.
3. The equation $f(x) = a$ for a real a has a unique real root.

From complex analysis, we know that traversing a critical point must permute three roots in a cycle. Since the function is self-adjoint, its set of roots consists of one real root and pairs of complex conjugate roots. Consequently, when traversing a critical point, the unique real root and a pair of complex conjugate roots must be permuted.

By traversing one critical point, we obtain the permutation of roots $(1\,2\,3)$, where 1 denotes the unique real root. By traversing another critical point, we get the permutation $(1\,4\,5)$. Together, these two permutations generate the alternating group A_5. Thus, the monodromy group of the equation is unsolvable, which implies that the equation is not solvable in elementary functions.

It remains for us to verify that the following equations satisfy these initial conditions, from which their insolvability will immediately follow. Note that although we do not provide the proof here, the same method is applicable to simpler equations like $\sin(z) - z = a$ and $\cos(z) - z = a$.

2.1 Insolvability of $\sin(\sin(z)) - z = a$

Let us prove the insolvability of the equation

$$\sin(\sin(\dots \sin(z) \dots)) - z = a$$

where the function $f(z)$ is the difference between a composition of sines and the identity function. For simplicity, we will show that conditions 1–3 are met for the simple composition

$$\sin(\sin(z)) - z = a$$

as the proof for more complex compositions is completely analogous. First, let us find the critical points of the function:

$$f(z) = \sin(\sin(z)) - z$$
$$f'(z) = \cos(\sin(z))\cos(z) - 1 = 0 \Rightarrow \cos(\sin(z))\cos(z) = 1$$

Clearly, both $\cos(\sin(z))$ and $\cos(z)$ must be simultaneously equal to either 1 or -1. The latter is impossible, because at the points $z = \pi + 2\pi k$, the sine is zero, and the cosine of zero is one. Consequently, the critical points of the function will be

$$z_k = 2\pi k, k \in \mathbb{Z}$$

At these points,

$$f''(z_k) = -\sin(\sin(z_k))\cos(z_k) - \cos(\sin(z_k))\sin(z_k) = -0 \cdot 1 - 1 \cdot 0 = 0$$

$$f'''(z_k) = -\cos(\sin(z_k))\cos^2(z_k) - \sin(\sin(z_k))\sin(z_k) +$$

$$+\sin(\sin(z_k))\sin(z_k)\cos(z_k) - \cos(\sin(z_k))\cos(z_k) = -1 - 0 + 0 - 1 = -2 \neq 0$$

We find that our points have multiplicity 3. Furthermore, a real root exists for any real a and is unique due to the strict monotonicity of the function, since

$$f'(x) = \cos(\sin(x))\cos(x) - 1 < 0$$

at all points except $2\pi k$. The other roots will come in complex conjugate pairs because the composition of two self-adjoint functions is self-adjoint.

2.2 Insolvability of $\sin(\cos(z)) - z = a$

Now we will show that conditions 1–3 are satisfied for

$$\sin(\cos(z)) - z = a$$

Let us find the critical points and check their multiplicity:

$$f'(z) = -\cos(\cos(z))\sin(z) - 1 = 0 \Rightarrow \cos(\cos(z))\sin(z) = -1 \Rightarrow$$

$$\Rightarrow z_k = -\pi/2 + 2\pi k, k \in \mathbb{Z}$$

Furthermore,

$$f''(z_k) = \sin(\cos(z_k))\sin^2(z_k) - \cos(\cos(z_k))\cos(z_k) = 0 - 0 = 0$$

$$f'''(z_k) = -\cos(\cos(z_k))\sin^3(z_k) + 2\sin(\cos(z_k))\sin(z_k)\cos(z_k) -$$

$$-\sin(\cos(z_k))\sin(z_k)\cos(z_k) + \cos(\cos(z_k))\sin(z_k) = -1 - 0 - 0 - 1 = -2 \neq 0$$

In this case, there is a unique real root because

$$-\cos(\cos(x))\sin(x) - 1 < 0$$

at all points except $-\pi/2 + 2\pi k$. As before, the other roots will come in complex conjugate pairs because the composition of two self-adjoint functions is self-adjoint.

3 Insolvability of $\tan(z) - z = a$

Let us prove the insolvability of the equation

$$\tan(z) - z = a$$

First, we find the critical points:

$$f'(z) = \frac{1}{\cos^2(z)} - 1 = 0 \Rightarrow \cos^2(z) = 1 \Rightarrow z_k = \pi k, k \in \mathbb{Z}$$

Let us check the multiplicity of these points:

$$f''(z_k) = \frac{2\sin(z_k)}{\cos^3(z_k)} = 0$$
$$f'''(z_k) = \frac{2}{\cos^2(z_k)} + \frac{6\sin^2(z_k)}{\cos^4(z_k)} = 2 \neq 0$$

A consequence of Rouché's theorem is that the equation $\tan(z) - z = 0$ has only real roots (a detailed proof is given in [6]). Consequently, if we let the parameter a in the equation $\tan(z) - z = a$ approach zero, three roots will merge into one point.

The function $f(z) = \tan(z) - z$ is self-adjoint, which implies that its roots must come in complex conjugate pairs. In this case, the function $f(x) = \tan(x) - x$ is strictly monotonic on each interval $(-\pi/2 + \pi k, \pi/2 + \pi k)$, which means that the equation has a unique root on each such interval. The situation is similar to the previous cases, except that the real root is not unique. Nevertheless, due to the multiplicity of 3, in the vicinity of a critical point, one real root and a pair of complex conjugate roots will also be permuted. We will construct a path for the parameter in such a way that a real root and a complex conjugate root from the upper half-plane are exchanged.

We modify the method used in the previous section. Let us fix the value $a = -1$ and denote two complex roots as c_1 and $c_2 = \overline{c_1}$, where c_1 is in the upper half-plane. Let us denote the real roots as $z_0, z_1, z_2, \ldots$, where $z_0 \to 0$ as $a \to 0$.

We start with the configuration of real roots:

$$(z_1, z_2, z_3, \ldots).$$

Consider a path in the parameter plane of a that starts at $a = -1$ and moves left along the real axis, bypassing each critical value along a small semicircle (we choose the one that causes an exchange between a real root and c_1).

When a reaches the n-th critical point, it loops around it in a small circle and returns along the same path. Let us examine what happens to the roots.

First, c_1 swaps places with z_1, then z_1 with z_2, and so on.

By the time we approach the n-th critical point, the arrangement of roots becomes:

$$(c_1, z_1, z_2, \dots, z_{n-2}, z_n, \dots).$$

The root z_{n-1} moves into the upper half-plane. Traversing the critical point causes a cyclic permutation (z_n, z_{n-1}, c_2). Repeating this traversal twice, we get:

$$(c_1, z_1, \dots, z_{n-3}, z_{n-2}, c_2, \dots).$$

Now z_n is in the upper half-plane, and z_{n-1} is in the lower. As we move back, all previous permutations are reversed: z_n and z_{n-2} swap places, then z_{n-2} and z_{n-3}, and so on.

As a result, we get the configuration:

$$(z_1, z_2, \dots, z_{n-2}, z_n, c_2, \dots),$$

where c_1 returns to its original place, while z_{n-1} remains below.

If we now repeat the same operation for the $(n+1)$-th critical point, we get the permutation:

$$(z_1, z_2, \dots, z_{n-2}, z_n, z_{n+1}, z_{n-1}, \dots),$$

and both complex roots will return to their places. This corresponds to the cycle $(n-1\ n+1\ n)$.

Such 3-cycles for all $n \geq 5$ generate the **infinite alternating group**, denoted A_∞. By analogy with S_∞, this group is a group of finitely supported permutations. More precisely, A_∞ is the subgroup of S_∞ consisting of all *even* permutations on the countable set of roots.

Since any 3-cycle is an even permutation, and any finite alternating group A_k is generated by 3-cycles, our method allows us to construct any permutation from A_k for any k. As the group A_k is unsolvable for $k \geq 5$, the group A_∞ which contains all of them is also unsolvable.

4 Insolvability of an Equation related to $z^z = a$

We now turn to another classical transcendental equation, $z^z = a$. While a full analysis of its monodromy group is possible, for the purpose of demonstrating the method with simpler derivatives, we will analyze the related, but distinct, equation $e^z + z = a$. It is important to note that these two equations are not equivalent. The analysis below pertains strictly to $e^z + z = a$.

Let us find the critical points of $f(z) = e^z + z$:

$$f'(z) = e^z + 1 = 0 \Rightarrow z_k = (2k+1)\pi i, k \in \mathbb{Z}$$

In this case,

$$f''(z_k) = e^{z_k} = -1 \neq 0$$

Thus, the critical points have multiplicity 2. The function $f(x)$ of a real argument is strictly monotonic because

$$f'(x) = e^x + 1 > 0$$

which means the equation has a unique real root.

First, for simplicity, let us consider the starting point $a = 0$.

Let us consider a path $z(t)$ that leads from this real root to the n-th critical point, then makes a semi-loop around it, swapping the roots, after which the new root returns along the same path.

Now let us derive the trajectory described by the parameter a while the real root moves along the specified path. On the first segment of the path:

$$z(t) = x + it,$$

where x is the real root (i.e., $e^x = -x$), so

$$a(t) = x + it + e^{x+it} = x(1 - \cos t) + i(t - x \sin t).$$

It is important to note that the loops bypass the "dangerous" points a_k, since at $t = (2k+1)\pi$:

$$a = 2x + i(2k+1)\pi.$$

Since $x < -\frac{1}{2}$ (which is easy to prove), and the real part of a_k is -1, the point a is to the left of a_k. This also shows that at the end of the first path segment, a is located to the left of a_n.

Let us denote $y_n = (2n+1)\pi$. Then the second segment of the path is given by the equation:

$$z(s) = s + iy_n + e^{s+iy_n} \Rightarrow a(s) = s - e^s + iy_n.$$

Thus, $a(s)$ also moves along a straight line parallel to the real axis.

On the third segment of the path, after the permutation of the roots, the parameter a describes a loop around a_n, which follows from the general theory. This trajectory is homotopically equivalent to a simple loop around the corresponding point.

Since a plane with a discrete set of points removed is homotopically equivalent to a wedge of circles, any loop around these points can be decomposed into a product of simple loops like the one shown above.

This means that any permutation can be realized by successive transpositions of the real root with the others. On the other hand, the permutation group is transitive, which is true in the general case.

Let us consider the decomposition of a permutation σ that maps r_n to the real root (let us number the roots with natural numbers so that $r_1 = x$ is the real root). As shown above, it will have the form:

$$\sigma = (1\,n_1)(1\,n_2)\dots(1\,n_N).$$

Obviously, the transposition $(1\,n)$ must be part of this product, otherwise the root r_n would remain in its place. Consequently, the group contains all transpositions $(1\,n)$, which generate the group S_∞ of all permutations of the countable set of roots that affect only a finite number of elements, known as the group of finitely supported permutations. S_∞ is countable and contains all finite symmetric groups S_n. The unsolvability of S_n for $n \geq 5$ implies the unsolvability of S_∞.

A Related Work in Functional Analysis

The equations we have studied are not merely mathematical curiosities; they arise in various physical and analytical contexts. In this appendix, we review two such connections.

A.1 Asymptotic Solutions

For the reader's convenience and due to the difficulty of accessing the original Soviet-era source [3], this section provides a detailed summary of Heifetz's work on the roots of a dispersion equation. The formulas obtained are used for numerical calculations and asymptotic investigation of the roots.

Heifetz considered a point acoustic source located in a homogeneous liquid layer on a homogeneous liquid half-space. This model, despite its simplicity, proves to be very useful in some cases (e.g., shallow sea) and has been the subject of many studies. The calculation of the eigenvalues of the corresponding boundary value problem reduces to solving the transcendental equation

$$mz \cot z = \sqrt{a^2 - z^2},$$

where $m = \rho_1/\rho$, ρ and ρ_1 are the densities of the liquid layer and the lower half-space, respectively, $a = kh\nu$, h is the layer thickness, $k = \omega/c$ is the wave number of the layer, ω is the angular frequency of the source, c is the speed of sound in the layer, c_1 is the speed of sound in the bottom, $\nu^2 = 1 - c^2/c_1^2$. Throughout, it is assumed that $\rho < \rho_1$, $c < c_1$, and the wave number for the bottom $k_1 = \omega/c_1$ is real. Similar equations arise in other problems as well.

Depending on the method of calculating the integral representing the sound pressure field, it is necessary to consider either only the real roots lying on the so-called physical sheet of the Riemann surface of the radical (they generate normal, or propagating modes, and it is these roots that correspond to the eigenvalues of the problem), or also the complex roots lying on the non-physical sheet of the Riemann surface (virtual modes or resonances). The roots of the transcendental equation were previously investigated by Heifetz using asymptotic and numerical methods, and its parametric solution is known. However, the development of algorithms for the numerical solution of this equation, in particular for calculating complex roots, continued to attract the attention of researchers. This work is devoted to refining the location of the roots of the equation, deriving explicit formulas for all roots, and their investigation. Heifetz considered the case $a > 0$ and $m > 1$. The case $0 < m < 1$ can be studied similarly.

Let us describe the location of the roots of the equation. It is obvious that there are no roots on the rays $x = \operatorname{Re} z > a$ and $x < -a$, so to isolate the single-valued branches of the radical, it is convenient to make cuts along these rays. The complex plane thus cut is denoted by D_0. We denote by $(\sqrt{a^2 - z^2})_\pm$ the single-valued and continuous branches of the radical in D_0, defined by the conditions $(\sqrt{a^2 - z^2})|_{z=0} = \pm a$, and consider in D_0 two holomorphic functions $f_\pm(z) = (\sqrt{a^2 - z^2})_\pm \sin z - mz \cos z$. The roots of $f_-(z)$ lie on the physical sheet, and the roots of $f_+(z)$ on the non-physical sheets of the Riemann surface.

To solve the two equations $f_\pm(z) = 0$ in D_0, we transform the original transcendental equation to the form

$$z = \frac{1}{2i} \ln \frac{\sqrt{a^2 - z^2} + imz}{\sqrt{a^2 - z^2} - imz} + \pi j, \quad j = 0, \pm 1, \pm 2, \dots .$$

Thus, we replace our original equation with a countable set of equations

$$l_j^\pm(z) = 0, \quad j = 0, \pm 1, \dots ,$$

where

$$l_j^\pm(z) = z + \frac{i}{2} \ln \frac{(\sqrt{a^2 - z^2})_\pm + imz}{(\sqrt{a^2 - z^2})_\pm - imz} - \pi j.$$

It is easy to see that on the rays $z = iy$, $|y| > \frac{a}{\sqrt{m^2-1}}$, the original equation also has no roots. We denote the region D_0, cut along these rays, by D, and select the branch of the logarithm in D by the condition $\ln 1 = 0$ (the choice of any other branch of the logarithm only leads to a renumbering of the roots). In the simply connected region D, $l_j^{\pm}(z)$ are single-valued analytic functions, the number of whose roots can be found by the argument principle. We find that on the physical sheet, the functions $l_j^-(z)$ for $0 < |j| < \frac{a}{\pi} + \frac{1}{2}$ have one real root x_j^{v-}, with $x_0^{v-} = 0$ and $x_{-j}^{v-} = -x_j^{v-}$, $j > 0$, while for $|j| \geq \frac{a}{\pi} + \frac{1}{2}$, $l_j^-(z)$ have no roots. In other words, the original physical problem has $[a/\pi + 1/2]$ propagating modes. (Here $[x]$ denotes the integer part of the number x.)

On the non-physical sheet, the functions $l_j^+(z)$ for $1 \leq |j| < \frac{a}{\pi} + \frac{1}{2}$ have one real root x_j^+, with $x_{-j}^+ = -x_j^+$, $j > 0$, and for $|j| > \frac{a}{\pi} + \frac{3}{2}$, $l_j^+(z)$ have two complex-conjugate roots $z_j^{\pm}$, $\bar{z}_j^{\pm}$, with $\operatorname{Im} z_j^+ > 0$, $z_{-j}^{\pm} = -\bar{z}_j^{\pm}$, $j > 0$. The function $l_0^+(z)$ has three roots for $a \geq \pi/2$: $z_0 = 0$ and symmetric purely imaginary roots $\pm iy_0$, $y_0 > 0$, and for $0 < a < \pi/2$ – five roots: $z_0 = 0$ and $\pm z'$, $\pm \bar{z}'$. Finally, the function $l_{j_a}^+(z)$, where $j_a = [a/\pi + 1/2]$ (the case of $l_{j_a}^-(z)$ is analogous), also has two roots, which for $a > a_{j_a} = \pi(j_a - 1/2)$ but close to a_{j_a}, are complex-conjugate, approach the real axis as a increases, and at some value of the parameter $a = a_{kr}$ merge to form a double "non-physical" real root x_{kr}, $x_{kr} > a$. With a further increase in a, it splits into two simple real roots, the larger of which moves towards the edge of the cut – the point $x = a$, reaches the edge at $a = \pi(j_a + 1/2)$ and then passes to the physical sheet – thus a new propagating mode is born. For the values a_{kr} and x_{kr}, Heifetz indicated that one can write down transcendental equations, which were not detailed there.

To obtain explicit formulas for the roots, another method is used: we consider only the roots $z_j^{\pm}$ of the functions $l_j^+(z)$, $|j| \geq a/\pi + 1/2$; formulas for all other roots are derived analogously. Regardless of whether $z_j^{\pm}$ is complex or real (for $j = [a/\pi + 1/2]$ and $a \geq a_{kr}$ – see above), it is a root of a quadratic trinomial $z^2 + p_j z + q_j$ with real coefficients. Consider the integrals

$$I_k = \oint_\gamma \frac{z^2 + p_j z + q_j}{z^k l_j^{+2}(z)} dz, \quad k = 3, 4,$$

where γ is a closed contour in D containing the point $z = 0$ inside. By calculating these integrals, first, by the residue theorem and, second, by deforming the contour γ so that it coincides with the boundary of the region D, we obtain a system of two linear equations with respect to p_j and q_j, from which $z_j = 1/2(-p_j + \sqrt{p_j^2 - 4q_j})$, where

$$p_j = \frac{\delta_1\delta_2 - \delta_0\delta_3}{\delta_1\delta_3 - \delta_2^2}, \quad q_j = \frac{\delta_0\delta_2 - \delta_1^2}{\delta_1\delta_3 - \delta_2^2},$$
$$\delta_0 = d_0 + \pi j B_j - \pi j A_j, \quad \delta_1 = d_1 - \frac{1}{\pi j}C_j + \frac{1}{\pi j}D_j,$$
$$\delta_2 = d_2 - jE_j - \pi j F_j, \quad \delta_3 = d_3 - \frac{1}{\pi j}G_j - \frac{1}{\pi j}H_j.$$

The coefficients $d_k, A_j, \dots, H_j$ are defined through integrals:

$$d_0 = -\frac{1}{\pi j}, \quad d_1 = \frac{m-a}{a\pi^2 j^2}, \quad d_2 = -\frac{(a-m)^2}{a\pi^3 j^3},$$
$$d_3 = \frac{m(3-2m^2)}{6a\pi^2 j^2} - \frac{(a-m)^3}{a^3\pi^4 j^4}.$$

$$A_j = \int_a^\infty \frac{(2x-\pi)f(x)dx}{xg(x)}, \quad C_j = \int_a^\infty \frac{f(x)h(x)dx}{x^2 g(x)},$$
$$E_j = \int_a^\infty \frac{(2x-\pi)f(x)dx}{x^3 g(x)}, \quad G_j = \int_a^\infty \frac{f(x)h(x)dx}{x^4 g(x)},$$

where

$$f(x) = \ln\frac{mx + \sqrt{x^2-a^2}}{mx - \sqrt{x^2-a^2}}, \quad h(x) = \left(x - \frac{\pi}{2}\right)^2 + \pi^2 j^2 + \frac{1}{4}f^2(x),$$
$$g(x) = h^2(x) - 4\pi^2 j^2 (x - \pi/2)^2.$$

$$B_j = 2\int_{a/\sqrt{m^2-1}}^\infty \frac{b(y)dy}{yd(y)}, \quad D_j = \int_{a/\sqrt{m^2-1}}^\infty \frac{c(y)dy}{yd(y)},$$
$$F_j = 2\int_{a/\sqrt{m^2-1}}^\infty \frac{b(y)dy}{y^3 d(y)}, \quad H_j = \int_{a/\sqrt{m^2-1}}^\infty \frac{c(y)dy}{y^4 d(y)},$$

where

$$b(y) = y - \frac{1}{2}\ln\frac{my + \sqrt{a^2+y^2}}{my - \sqrt{a^2+y^2}},$$
$$c(y) = b^2(y) - \pi^2(j^2 - 1/4), \quad d(y) = c^2(y) + 4\pi^2 j^2 b^2(y).$$

These integrals converge slowly, but changes of variables $t = 1/\sqrt{2}f(x)$ or $t = \frac{1}{2}\ln\frac{my+\sqrt{a^2+y^2}}{my-\sqrt{a^2+y^2}}$ transform them into integrals with an exponentially decaying integrand, the calculation of which on a computer poses no difficulties. This method does not require the choice of an initial approximation and provides the same accuracy for roots with any numbers.

Table 1: Values of resonances and absorption coefficients

j	ξ_j	$\bar{\xi}_j$	$\bar{B}_j$	B_j	B_{jT}
6	$0.365801 + 0.730159 \cdot 10^{-3}i$	$0.3659 + 0.68 \cdot 10^{-3}i$	6.342	5.90	6.40
7	$0.346401 + 1.911244 \cdot 10^{-3}i$	$0.3464 + 1.92 \cdot 10^{-3}i$	16.6009	16.7	16.8
8	$0.322795 + 2.903837 \cdot 10^{-3}i$	$0.3228 + 2.90 \cdot 10^{-3}i$	25.2224	25.2	25.7

The table presents the values of the first three resonances $\xi_j = \frac{1}{h\sqrt{(kh)^2-z_j^2}}$ of the considered model, calculated using the formulas above and rounded to 10^{-8}, for the parameter values $h = 100$ m, $m = 2$, $c = 1500$ m/s, $c_1 = 1700$ m/s, $\omega = 2\pi f$, $f = 100$ Hz. They correspond to the complex roots z_j of equation (1) with numbers $j = 6, 7, 8$. We denote by $B_j = (20\lg e)\operatorname{Im}\xi_j$ the absorption coefficient of the j-th mode. For comparison, the table also shows $\bar{\xi}_j$ and $\bar{B}_j$ – the corresponding values of the same quantities, calculated by another method in other works. In addition to numerical calculations, the formulas obtained allow for the study of the asymptotic behavior of the roots. Along with the known asymptotics of large-modulus complex roots $z_j \approx \pi j + \pi/2 + (i/2)\ln\frac{m+1}{m-1} + O(1)$, $j \to \pm\infty$, we obtain, for example, for the roots corresponding to propagating modes, the relation $x_j \approx \pi j + (m/a)\pi j + O(a^{-2})$ as $a \to +\infty$ and for a fixed j, $1 \le j < a/\pi + 1/2$, as well as an asymptotic for complex roots useful in the case of a shallow sea, when there are no normal modes, as $a \to 0$ and for a fixed j: $z_j \approx \pi j + \pi/2 + (i/2)\ln\frac{m+1}{m-1} + O(a)$, with the estimate $O(a)$ being uniform in $j = 1, 2, \ldots$.

A.2 Sturm-Liouville Operators

In a related work, A. Elishev [2] described another way to prove the reality of the roots of the equation $\tan(z) - z = a$. We summarize his argument here, with the author's permission.

First, it is necessary to prove that all zeros of $\phi(z) = \tan(z) - z$ are in fact given by the square roots of the eigenvalues of some regular Sturm-Liouville operator. Indeed, let us define the space $L^2([0,1], dx)$ as the space of complex-valued square-integrable functions defined on the unit interval $[0,1]$, with the inner product given by

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dx.$$

Suppose that H_0 is the subspace of $L^2([0,1], dx)$ consisting of twice-differentiable functions f for which f' and f'' are square-integrable, and such that

$$f(0) = 0$$

and

$$f(1) - f'(1) = 0$$

Then the operator $H = -d^2/dx^2$, defined on H_0 is a positive semi-definite operator corresponding to a regular Sturm-Liouville problem with separated boundary conditions. Consequently, by a fundamental result of Sturm-Liouville theory, together with positive semi-definiteness, the eigenvalues of H form an increasing sequence.

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots.$$

Note that each eigenvalue has multiplicity one, and also that H is self-adjoint — this can be checked directly by integration by parts. We establish its positive semi-definiteness.

The explicit form of the eigenfunctions (unique up to a scalar factor for each eigenvalue) is a trivial exercise. First, we obtain for $\lambda_0 = 0$

$$\psi_0(x) = x.$$

Then, for $\lambda \neq 0$, $\lambda = k^2$, the boundary value problem is

$$\begin{aligned} \psi'' + k^2\psi = 0, &\quad 0 < x < 1, \\ \psi(0) = 0, & \\ \psi(1) - \psi'(1) = 0. & \end{aligned}$$

The general solution is given by $\psi(x) = A\cos(kx) + B\sin(kx)$. The first boundary condition implies $A = 0$, while the second yields the equation

$$\sin(k) - k\cos(k) = 0$$

or (if $\cos(k) \neq 0$)

$$\tan(k) - k = 0$$

Consequently, the spectrum of H is given by the set

$$\{0\} \cup \{\lambda \in \mathbb{C} : \tan(\sqrt{\lambda}) - \sqrt{\lambda} = 0\}.$$

Now, since H is self-adjoint, its spectrum is real, which means that any solution to

$$\tan(z) - z = 0$$

is either real or purely imaginary, $z = iy$. The purely imaginary case leads to the equation for $y \in \mathbb{R}$:

$$\tanh(y) - y = 0$$

which has no (real) solutions other than $y = 0$. Thus, the proof of the lemma is complete.

Elishev notes that each positive eigenvalue λ corresponds to two solutions of the equation, given by $+\sqrt{\lambda}$ and $-\sqrt{\lambda}$. This is expected since $\phi(z)$ is an odd function.

The situation described in this comment is typical for the realization of the zeros of meromorphic functions as spectra of self-adjoint operators. Often they correspond to the Hamiltonians of quantum-mechanical systems; several other transcendental equations can be handled in this way.

The example provided here by Elishev (along with other easily obtainable examples of meromorphic functions whose zeros are spectra of self-adjoint operators) is quite straightforward; at the same time, on the opposite end of the complexity scale lies, as is well known, the conjectured spectral realization of the non-trivial zeros of the Riemann zeta function. In fact, as Elishev notes, there is a significant amount of evidence in favor of a spectral approach to the Riemann hypothesis, developed, in particular, in the deep and far-reaching work of Alain Connes and his school. Simple examples, such as the one considered in this comment, can, in Elishev's opinion, serve as illustrations of the powerful methods of operator theory and its potential applications to long-standing open problems.

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AUTHORS

Allan Allemand
Lomonosov Moscow State University,
Higher School of Economics
Suleyman Rustam, 33A, 64
1022 Baku, Azerbaijan
email: allansuleykin@gmail.com

Alexey Kanel-Belov
Bar-Ilan University
Ramat Gan, 5290002
52900 Ramat Gan, Israel
email: kanelster@gmail.com

Rodion Zaytsev
California Institute of Technology
1200 E California Blvd, Pasadena, CA
91125, United States
email: r.zayt@mail.ru