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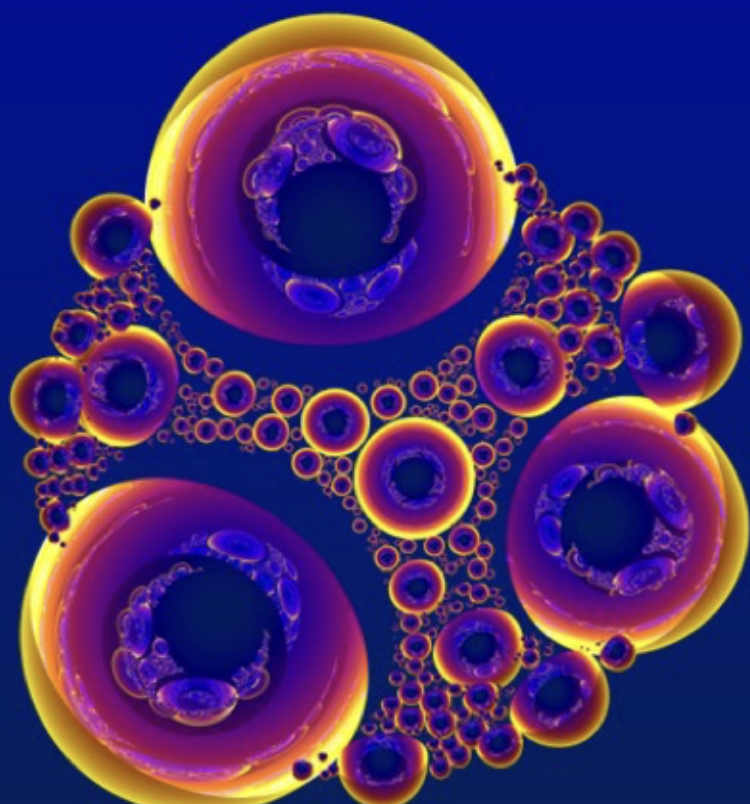

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A word from the Editors.

Dear Readers, welcome to issue 1 of volume 11 of the Arnold Mathematical Journal.

Our journal has changed the publisher: starting with this issue, it is published by the Association for Mathematical Research (AMR); it is still owned by the Institute for Mathematical Sciences (IMS) at the Stony Brook University. The journal is published electronically, following the model known as Diamond Open Access, that is, it is openly accessible to readers, authors, and libraries without charging fees to either.

We have also switched to EditFlow, the software service optimized to streamline managing the submission and peer review workflow.

We intend to continue publishing at the same rate as before: four issues a year, about 150 pages an issue; but if need be, we shall increase the rate to avoid backlogs. The papers accepted for publication will appear online right away. The full archive of the journal is available at the journal's web site at IMS.

We have expanded the Editorial Board, and we also have new Advisors: Artur Avila, Etienne Ghys, and Dennis Sullivan.

The philosophy and the scope of the journal remain the same: we publish interesting results in all areas of mathematics in three categories: research contributions, research expositions, and problem contributions. For more detail, please visit the journal's web site at https://armj.math.stonybrook.edu

Happy reading!
Sergei Tabachnikov, Editor-in-Chief Pennsylvania State University
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A Simple Construction of the Field of Witt Vectors

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Abstract: We present a short, hopefully pedagogical construction of the field and ring of Witt vectors. It uses a natural binary operation on polynomials of one variable, which we call *convolution*.

AMS Classification: 13F35

Key words and phrases: Arithmetic rings, Witt vectors

Introduction

Witt vectors form a field of characteristic 0 constructed out of a field of finite characteristic p. This construction suggested by E. Witt [Wit37] in 1936 generalizes the field $\mathbb{Q}_p$ of p-adic rationals. His construction has a reputation to be complicated and counter-intuitive. We suggest a very concise version of construction of Witt vectors. It is inspired by a paper by D.Kaledin [Kal12] who observed a relation between Witt vectors and the tame symbol in disguise of the so-called Japanese cocycle.

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In the wikipedia article on Witt vectors (july 2024) it is indicated that "they have a highly non-intuitive structure". The aim of this note is to refute this claim.

A very similar construction is essentially contained in the notes by Michiel Hazewinkel [Haz09], mainly in the section 9 and the section 14. This review contains a lot of information for further reading on the subject.

Convolution

We are going to define a binary operation on polynomials of one variable modifying the definition of a resultant.

Let $f(t) = 1 + a_1 t + \cdots$ and $g(t) = 1 + b_1 t + \cdots$ belong to the multiplicative semi-group $1 + t\mathbb{F}[t]$ of polynomials with coefficients in a field $\mathbb{F}$ and the constant term equal to 1. Define a *convolution* $f \star g$ as a polynomial with the constant term 1 and having as roots the products of one root of f and one of g. In other words, suppose that $f(t) = \prod_i (1 - \lambda_i t)$ and $g(t) = \prod_j (1 - \mu_j t)$ with λ_i, μ_j belonging to the algebraic closure of the field $\mathbb{F}$. Then

$$f \star g\,(t) = \prod_{ij} (1 - t\lambda_i \mu_j) = \prod_i g(\lambda_i t) = \prod_j f(\mu_j t).$$

The convolution can also be expressed in term of the resultant, namely

$$f \star g\,(t) = \mathtt{res}_z(f(z), z^{\deg g} g(t/z)).$$

To give an equivalent definition, consider the ring $\mathbb{F}[x,y]/(f(x)) + (g(y))$ and denote by $\hat{x}^{-1}$ and $\hat{y}^{-1}$ the multiplication in this ring by x^{-1} and y^{-1}, respectively. In the standard basis they are given by matrices with entries in $\mathbb{F}$. Then

$$f \star g\,(t) = \det(1 - t\hat{x}^{-1}\hat{y}^{-1}).$$

In this definition it is explicit that the coefficients of $f \star g$ are polynomial functions of those of f and g.

The fourth definition works for $\mathbb{F} = \mathbb{C}$ and shows the relation to the tame symbol. Consider a curve γ around zero on the complex plane sufficiently small in order not to surround any root of $f(z)$. The convolution can be defined by the formula (see P.Deligne [Del91], formula 2.7.2)

$$f \star g\,(t) = \{f(z), g(t/z)\}_\gamma = \exp\left(\frac{1}{2\pi i}\int_\gamma \ln f(z) d \ln g(t/z)\right)$$

valid for t so small that all roots of $g(t/z)$ are inside the curve γ.

The convolution enjoys the following properties obvious from the definitions:

1. $\deg(f \star g) = \deg f \deg g$,
2. $f \star 1 = 1$,
3. $f \star (1-t) = f$,
4. $(1-at) \star (1-bt) = (1-abt)$,
5. $f \star g = g \star f$,
6. $f \star (g_1 g_2) = (f \star g_1)(f \star g_2)$.

These properties imply that the semi-group $1 + t\mathbb{F}[t]$ is a commutative semi-ring with respect to the multiplication as a semi-ring addition and convolution as a semi-ring multiplication. The multiplicativity property 6 is just the expression of the distributive law of the semi-ring.

The following property is also an easy consequence of the definition:

- The set $1 + t^n\mathbb{F}[t]$ is an ideal.

This property implies that the convolution can be extended to the group of formal power series $1 + t\mathbb{F}[[t]]$ providing it with a ring structure. This ring is called the ring of the *universal* or *big* Witt vectors and is denoted by $W(\mathbb{F})$, see [Haz09], section 9.

Witt vectors

The aim of this paragraph is to give a concise definition of the Witt ring.

For that we need just another property of the universal Witt ring obviously following from the definition of the convolution:

- The set $1 + t^n\mathbb{F}[[t^n]]$ is also an ideal.

Let $\mathbb{F}$ be a field of characteristic p and let $W(\mathbb{F})$ be the corresponding universal Witt ring.

Define the *Witt ring* $W_{\mathbb{F}}$ as a quotient

$$W_{\mathbb{F}} = W(\mathbb{F}) / \prod_{n>1|n\neq p^k} (1 + t^n\mathbb{F}[[t^n]]).$$

Here we used the property and denoted the sum of ideals multiplicatively since it corresponds to the product of the series.

Observe that any element of the group $1 + t\mathbb{F}[[t]]$ can be presented either as a sum $1 + \alpha_1 t + \alpha_2 t^2 + \cdots$ or as a product $(1 - a_1 t)(1 - a_2 t^2)(1 - a_3 t^3)\cdots$.

Using the latter presentation the elements of the ring $W_{\mathbb{F}}$ can be uniquely represented as formal products

$$f(t) = \prod_{i=0}^{\infty} (1 - a_i t^{p^i}).$$

In this presentation certain properties of the Witt vectors become obvious. In particular, it follows from the property 4 that the correspondence $a \longmapsto (1 - at)$ gives an embedding of multiplicative groups $\mathbb{F}^\times \to W_{\mathbb{F}}^\times$. The images of the elements of $\mathbb{F}^\times$ are called their *Teichmüller representatives.* It is also obvious that the ring multiplication by p in the ring $W_{\mathbb{F}}$ amounts to the shift of the coefficients a_i composed with the Frobenius automorphism:

$$\prod_{i=0}^{\infty} (1 - a_i t^{p^i}) \longmapsto \prod_{i=1}^{\infty} (1 - a_{i-1}^p t^{p^i}).$$

This property allows to identify the field of fractions of the ring $W_{\mathbb{F}}$ with the expressions of the form

$$\prod_{i=N}^{\infty} (1 - a_i t^{p^i})$$

with possibly negative N.

Recall that for a field $\mathbb{F}_p$ of p elements the ring $W_{\mathbb{F}}$ coincides with the ring $\mathbb{Z}_p$ of p-adic integers.

Relation to the standard definition of the Witt vectors

Consider the ring of formal series $\mathbb{C}[[t]]^{+\circ}$ with respect to addition and coefficentwise (Hadamard) multiplication denoted by $\circ$ defined as

$$(\sum_{k=0}^{\infty} a_k t^k)\circ(\sum_{k=0}^{\infty} b_k t^k) = \sum_{k=0}^{\infty} a_k b_k t^k$$

Clearly this ring is just a direct sum of infinitely many copies of the ring $\mathbb{C}$.

The map $f \longmapsto -f'/f$ gives an isomorphism between the rings $(1+\mathbb{C}[[t]])^{\cdot\star}$ and $\mathbb{C}[[t]]^{+\circ}$. Indeed

$$(-f'/f)\circ(-g'/g) = \left(\sum_{k=1}^{\infty}(\sum_i \lambda_i^k)t^{k-1}\right)\circ\left(\sum_{k=0}^{\infty}(\sum_j \mu_j^k)t^{k-1}\right) =$$

$$= \left(\sum_k(\sum_{ij}\lambda_i^k\mu_j^k)t^{k-1}\right) = -(f\star g)'/(f\star g),$$

and obviously

$$-f'/f - g'/g = -(fg)'/(fg).$$

In the explicit coordinates we have

$$\prod(1-a_i t^i) \longmapsto \sum_i\sum_k a_i^k t^{ik-1} = \sum_l\sum_{i|l} i a_i^{l/i} t^{l-1}.$$

The expressions

$$S_l(a_1, a_2, \ldots) = \sum_{i|l} i a_i^{i/l}$$

are called the *universal Witt polynomials* (P. Cartier [Car67] and [Haz09] section 9). We see that each of the Witt polynomials gives a homomorphism from the universal Witt ring to the ring of complex numbers and a collection of all such polynomials gives an isomorphism of the universal Witt ring to the infinite sum of complex numbers. The standard construction uses this isomorphism to define the ring structure in terms of the coefficients a_i. Then one proves that the product and the sum are in fact given by algebraic expressions with integer coefficients and thus are defined over any field.

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On Intersection of Lemniscates of Rational Functions

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Abstract: For a non-constant complex rational function P, the *lemniscate* of P is defined as the set of points $z \in \mathbb{C}$ such that $|P(z)| = 1$. The lemniscate of P coincides with the set of real points of the algebraic curve given by the equation $L_P(x, y) = 0$, where $L_P(x, y)$ is the numerator of the rational function $P(x+iy)\overline{P}(x-iy)-1$. In this paper, we study the following two questions: under what conditions two lemniscates have a common component, and under what conditions the algebraic curve $L_P(x, y) = 0$ is irreducible. In particular, we provide a sharp bound for the number of complex solutions of the system $|P_1(z)| = |P_2(z)| = 1$, where P_1 and P_2 are rational functions.

AMS Classification: 30C10, 14P25, 11D61

Key words and phrases: Lemniscate, Unimodular Points, Bezout Theorem, Blaschke Products, Separated Variables Curves

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1 Introduction

Let P be a non-constant complex rational function on the Riemann sphere $\widehat{\mathbb{C}}$. The *lemniscate* of P is defined as

$$\mathcal{L}_P = \{z \in \mathbb{C} : |P(z)| = 1\}. \tag{1}$$

The paper is devoted to two more or less independent problems concerning the lemniscates. The first problem is to determine the maximal number of intersection points of $\mathcal{L}_{P_1}$ and $\mathcal{L}_{P_2}$, i.e., the number of solutions of the system of equations

$$|P_1(z)| = |P_2(z)| = 1, \tag{2}$$

where P_1 and P_2 are rational functions of given degrees n_1 and n_2. The second problem is to find out when a lemniscate (considered as a real algebraic curve in $\mathbb{R}^2$) is irreducible.

Our result about the intersection of lemnisicates is the following statement.

Theorem 1.1. *Let P_1 and P_2 be non-constant complex rational functions of degrees n_1 and n_2. The following three conditions are equivalent:*

(i) *$\mathcal{L}_{P_1}$ and $\mathcal{L}_{P_2}$ have more than $2n_1n_2$ common points.*

(ii) *$\mathcal{L}_{P_1} \cap \mathcal{L}_{P_2}$ is infinite.*

(iii) *$P_1 = B_1 \circ W$ and $P_2 = B_2 \circ W$ for some rational functions W, B_1, B_2 such that each of B_1, B_2 maps the unit circle $\mathbb{T}$ to itself.*

Furthermore, for any natural n_1 and n_2 there exist rational functions of degrees n_1 and n_2 such that $\mathcal{L}_{P_1}$ and $\mathcal{L}_{P_2}$ have exactly $2n_1n_2$ intersection points.

We remark that for a rational function B, the condition $B(\mathbb{T}) \subset \mathbb{T}$ can be written in the form

$$B(z)\overline{B}(1/z) = 1 \qquad \text{for all } z \in \widehat{\mathbb{C}}. \tag{3}$$

Such functions are known in the complex analysis under the name of *quotients of finite Blaschke products.*

In a recent paper [13], the equivalence of conditions (i), (ii), (iii) was proven with a weaker bound $(n_1 + n_2)^2$ instead of the bound $2n_1n_2$. Notice that the result of [13] implies the main result of the paper [1] by Ailon and Rudnick, which is equivalent to the following statement: if P_1 and P_2 are complex polynomials, then

$$\#\bigcup_{k=1}^{\infty}\{z \in \mathbb{C} \ : \ P_1(z)^k = P_2(z)^k = 1\} \leq C(P_1, P_2), \tag{4}$$

for some constant $C(P_1, P_2)$ that depends only on P_1 and P_2, unless for some non-zero integers m_1 and m_2 the equality

$$P_1^{m_1}(z)P_2^{m_2}(z) = 1 \tag{5}$$

holds (see Section 2.4 below). In addition, the result of [13] answers the question of Corvaja, Masser, and Zannier ([5]) about the intersection of an irreducible curve $\mathcal{C}$ in $\mathbb{C}^* \times \mathbb{C}^*$ with $\mathbb{T} \times \mathbb{T}$, in case $\mathcal{C}$ has genus zero and is parametrized by rational functions P_1, P_2 (see [13] for more details). These applications of [13] stem from the fact that the numbers $C(P_1, P_2)$ and $\#(\mathcal{C} \cap \mathbb{T} \times \mathbb{T})$ obviously are bounded from above by the number of solutions of (2). Thus, a sharp bound for the last number is of great interest, and our Theorem 1.1 provides it.

In brief, our proof of Theorem 1.1 given in Section 2 goes as follows. If P is a complex rational function of degree n, then under the standard identification of $\mathbb{C}$ with $\mathbb{R}^2$ the lemniscate $\mathcal{L}_P$ coincides with the set of real points of the affine algebraic curve of degree $2n$ given by the equation $L_P(x, y) = 0$, where $L_P(x, y)$ is the numerator of the rational function

$$P(x + iy)\overline{P}(x - iy) - 1. \tag{6}$$

After the linear change of variables $z = x + iy$, $w = x - iy$ (in $\mathbb{C}^2$) the Newton polygon of L_P becomes the square $n \times n$. Thus, the Bézout Theorem for bihomogeneous polynomials ([14, §4.2.1]), which is also a simplest case of the Bernstein-Kushnirenko Theorem [2], implies that if (i) holds, then L_{P_1} and L_{P_2} have a common factor, i.e., the system $L_{P_1} = L_{P_2} = 0$ has infinitely many *complex* solutions. This is not yet (ii), but we prove a kind of "real" version of the Bézout theorem (Proposition 1), which implies in particular that if (i) holds, then the system $L_{P_1} = L_{P_2} = 0$ has infinitely many *real* solutions (Corollary 2(c)). This gives us the implication (i) ⇒ (ii). In turn, the implication (ii) ⇒ (iii) is deduced from a "rational" version of the Cartwrite Theorem [3] (Corollary 1). Finally, since (iii) implies that $\mathcal{L}_W \subset \mathcal{L}_{P_1} \cap \mathcal{L}_{P_2}$, the implication (iii) ⇒ (i) is obvious.

Notice that the equivalence of conditions (i), (ii), (iii) can be proven also by modifying the approach of the paper [13] (see Section 2.3).

Another problem considered in this paper (in Section 3) is the following. Given a rational function P, under what conditions the curve $L_P(x, y) = 0$ is irreducible over $\mathbb{C}$? It is not hard to see that the following "composition condition" is sufficient for the reducibility: there exists a quotient of finite Blaschke products B of degree at least 2 and a rational function W such that $P = B \circ W$. Notice that if P is a *polynomial*, the composition condition reduces to the condition that $P = W^k$ for some polynomial W and $k \geq 2$. Its necessity for the reducibility of $L_P(x, y)$ in the polynomial case was established by the first author in [12] (notice that this result has found applications to complex dynamics, see [11]). In Section 3.2, we show however that for *rational* P the reducibility of $L_P(x, y) = 0$ does not imply in general the composition condition.

Our approach to the problem of irreducibility of algebraic curves $L_P(x, y) = 0$ is based on the observation that a change of variables allows us to consider this problem in the context of the more general problem of irreducibility of "separated variables" curves $P(x) - Q(y) = 0$, where $P(z)$ and $Q(z)$ are rational functions. In particular, we deduce our examples of reducible $L_P(x, y)$ for P not satisfying the composition condition from examples of reducible separated variables curves found in [4]. On the other hand, modifying the arguments from [12], we provide a handy *sufficient* condition for the irreducibility of separated variables curves in case *one* of rational functions $P(z)$ and $Q(z)$ is a polynomial (Theorem 3.4).

2 Intersection of lemniscates

2.1 The Cartwright theorem

For a simple closed curve Γ in $\mathbb{C}$, we denote by $\mathcal{M}_\Gamma$ the set of all non-constant functions meromorphic on $\mathbb{C}$ having modulus one on Γ. The following result was proved by Cartwright in [3].

Theorem 2.1. *Let Γ be a simple closed curve in $\mathbb{C}$ such that $\mathcal{M}_\Gamma$ is nonempty. Then there exists a function $\varphi \in \mathcal{M}_\Gamma$ such that each function f in $\mathcal{M}_\Gamma$ may be written in the form $f = B\circ\varphi$, where B is a quotient of finite Blaschke products.*

A detailed discussion and generalizations of the Cartwright theorem can be found in the papers [15], [16]. Below we need the following specialization of the Cartwright theorem, where the notation $\mathcal{R}_\Gamma$ stands for the set of all non-constant complex rational functions on $\mathbb{C}$ having modulus one on Γ.

Corollary 1. *Let Γ be a simple closed curve in $\mathbb{C}$ such that $\mathcal{R}_\Gamma$ is nonempty. Then there exists a rational function $W \in \mathcal{R}_\Gamma$ such that each function P in $\mathcal{R}_\Gamma$ may be written in the form $P = B\circ W$, where B is a quotient of finite Blaschke products.*

Proof. Applying Theorem 2.1, we conclude that there exists a meromorphic function $W \in \mathcal{M}_\Gamma$ such that if $P \in \mathcal{R}_\Gamma$, then $P = B\circ W$ for some quotient of finite Blaschke products B. However, since the great Picard theorem implies that any non-rational function W meromorphic on $\mathbb{C}$ takes all but at most two values in $\mathbb{CP}^1$ infinitely often, this equality implies that W is rational. □

2.2 A real version of the Bézout Theorem.

Let us recall that the classical Bézout theorem about intersections of curves in $\mathbb{CP}^2$ implies that the number of intersection points of two affine algebraic curves $F(x,y) = 0$ and $G(x,y) = 0$ of degrees m and n in $\mathbb{C}\times\mathbb{C}$ does not exceed mn, unless the polynomials $F(x,y)$

and $G(x, y)$ have a non-constant common factor in $\mathbb{C}[x, y]$. In case the bidegrees (m_1, m_2) and (n_1, n_2) of $F(x, y)$ and $G(x, y)$ are relatively small with respect to m and n, a better bound can be obtained from the bihomogeneous Bézout theorem, which implies that the number of intersection points of $F(x, y) = 0$ and $G(x, y) = 0$ does not exceed $n_1m_2 + n_2m_1$, unless $F(x, y)$ and $G(x, y)$ have a non-constant common factor (see [14, §4.2.1]). In the proof of Theorem 1.1, we will use a real version of this last bound provided by Corollary 2(c) below.

Let X be a compact non-singular complex algebraic (or analytic) variety. A *real structure* on X is an anti-holomorphic involution $\sigma : X \to X$. We denote the variety X endowed with the real structure σ by X^σ. A point p of X^σ is called *real* if $\sigma(p) = p$. The set of real points of X^σ (the *real locus* of X^σ) is denoted by $\mathbb{R}X^\sigma$. A basic example is a projective variety in $\mathbb{CP}^n$ defined by polynomial equations with real coefficients endowed with the involution of complex conjugation. In this case $\mathbb{R}X^\sigma$ is the subset of $\mathbb{RP}^n$ defined by the same equations.

If $X = \mathbb{CP}^1 \times \mathbb{CP}^1$, then there are two non-isomorphic real structures on X:

$$\sigma_h : (x, y) \longmapsto (\bar{x}, \bar{y}), \qquad \sigma_e : (x, y) \longmapsto (\bar{y}, \bar{x}).$$

The real loci $\mathbb{R}X^{\sigma_h}$ and $\mathbb{R}X^{\sigma_e}$ are, respectively, $\mathbb{RP}^1 \times \mathbb{RP}^1$ (a torus) and the image of $\mathbb{CP}^1$ in X under the embedding $z \longmapsto (z, \bar{z})$ (a sphere). Note that X^{σ_h} and X^{σ_e} are isomorphic to the complexifications of, respectively, hyperboloid and ellipsoid in $\mathbb{CP}^3$ endowed with the usual complex conjugation:

$$X^{\sigma_h} \cong \{z \mid z_0^2 + z_1^2 = z_2^2 + z_3^2\}, \qquad X^{\sigma_e} \cong \{z \mid z_0^2 = z_1^2 + z_2^2 + z_3^2\};$$

here z stands for $(z_0 : z_1 : z_2 : z_3)$. Indeed, it is straightforward to check that the following mapping $X \to \mathbb{CP}^3$ defines the required isomorphism:

$$(x, y) \longmapsto (x_0y_0 + x_1y_1 : \alpha(x_0y_1 - x_1y_0) : x_0y_0 - x_1y_1 : x_0y_1 + x_1y_0),$$

where $\alpha = 1$ for σ_h and $\alpha = i$ for σ_e; here x and y stand for $(x_0 : x_1)$ and $(y_0 : y_1)$.

Let us recall that if C is an irreducible curve on a smooth compact algebraic surface X, then by the genus formula (see e.g. [14, §4.4.1]) the number of singular points of C is bounded from above by its *arithmetic genus* $p_a(C)$, which in this case can be computed by the formula

$$2p_a(C) = 2 + C \cdot (C + K_X),$$

where K_X is the canonical class of X.

Proposition 1. *Let X be a smooth compact complex algebraic surface endowed with a real structure σ. Let A and B be σ-invariant algebraic curves on X. Suppose that $\mathbb{R}A \cap \mathbb{R}B$ is finite and $C^2 \geq p_a(C)$ for each irreducible component C of $A \cup B$. Then*

$$\#(\mathbb{R}A \cap \mathbb{R}B) \leq A \cdot B. \tag{7}$$

Proof. We say that a σ-invariant curve C on X is *σ-irreducible* if $C = D \cup \sigma(D)$ where D is irreducible.

Let us first consider the case when A and B are σ-irreducible. If $A \neq B$, then A and B have no common components, and hence

$$\#(\mathbb{R}A \cap \mathbb{R}B) \leq \#(A \cap B) \leq A \cdot B.$$

Suppose now that $A = B$. Let us show that $\mathbb{R}A \subset \mathrm{Sing}(A)$, where $\mathrm{Sing}(A)$ is the set of singular points of A. Indeed, for any $p \in \mathbb{R}A$ one can choose local coordinates (z, w) in a neighborhood of p such that $\mathbb{R}X = \{\operatorname{Im} z = \operatorname{Im} w = 0\}$ and $A = \{f(z, w) = 0\}$, where f is a polynomial with real coefficients. Thus, if A were non-singular at p, then $\mathbb{R}A$ would be infinite by the Implicit Function Theorem, which contradicts our hypothesis that $\mathbb{R}A \cap \mathbb{R}B$ is finite. Thus, $\mathbb{R}A \subset \mathrm{Sing}(A)$.

It follows that if $A = B$ is irreducible (in the usual sense), we have

$$\#(\mathbb{R}A \cap \mathbb{R}B) = \#\mathbb{R}A \leq \#\,\mathrm{Sing}(A) \leq p_a(A) \leq A^2 = A \cdot B.$$

On the other hand, if $A = B$ is reducible, then $A = C \cup \sigma(C)$ where C is irreducible and $\sigma(C) \neq C$. Therefore, we have $\mathbb{R}A \subset C \cap \sigma(C)$, and hence

$$\#(\mathbb{R}A \cap \mathbb{R}B) = \#\mathbb{R}A \leq C \cdot \sigma(C) \leq C^2 + \sigma(C)^2 + 2C \cdot \sigma(C) = A^2 = A \cdot B.$$

This completes the proof in the case where A and B are σ-irreducible.

Now we proceed to the general case. Let $A = A_1 \cup \cdots \cup A_k$ and $B = B_1 \cup \cdots \cup B_l$ where each A_i and each B_j is σ-invariant and σ-irreducible. Then

$$\#(\mathbb{R}A \cap \mathbb{R}B) \leq \sum_{i,j} \#(\mathbb{R}A_i \cap \mathbb{R}B_j) \leq \sum_{i,j} A_i \cdot B_j = A \cdot B. \qquad \square$$

Corollary 2. (a). *Let $F(x_1, x_2)$ and $G(x_1, x_2)$ be polynomials with real coefficients of degrees m and n. Then the number of real solutions of the system $F = G = 0$ is either infinite or bounded above by mn.*

(b). *Let $F(x_1, x_2)$ and $G(x_1, x_2)$ be polynomials with real coefficients such that $\deg_{x_k} F = m_k$ and $\deg_{x_k} G = n_k$, $k = 1, 2$. Then the number of real solutions of the system $F = G = 0$ is either infinite or bounded above by $m_1 n_2 + m_2 n_1$.*

(c). *Let $F(z, w)$ and $G(z, w)$ be polynomials with complex coefficients such that $F(z, w) = \overline{F}(w, z)$ and $G(z, w) = \overline{G}(w, z)$. Let $\deg_z F = m$ and $\deg_z G = n$. Then the number of solutions of the system $F = G = 0$ belonging to the set $\{(z, w) \mid w = \bar{z}\}$ is either infinite or bounded above by $2mn$.*

Proof. (a). We apply Proposition 1 to the curves in $\mathbb{CP}^2$ (with the standard real structure) defined by the homogeneous equations

$$x_0^m F(x_1/x_0, x_2/x_0) = 0, \qquad x_0^n G(x_1/x_0, x_2/x_0) = 0,$$

and observe that, for a curve C of degree d in $\mathbb{CP}^2$, we have (see [14, §4.2.3])

$$p_a(C) = (d-1)(d-2)/2 \leq d^2 = C^2.$$

(b). We apply Proposition 1 to $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ endowed with the real structure σ_h (see above) and to the curves defined by the bihomogeneous equations

$$u_1^{m_1} u_2^{m_2} F(x_1/u_1, x_2/u_2) = 0, \qquad u_1^{n_1} u_2^{n_2} G(x_1/u_1, x_2/u_2) = 0.$$

In this case, for a curve C of bidegree (d_1, d_2) in X, we have (see [14, §4.2.3])

$$p_a(C) = (d_1 - 1)(d_2 - 1) \leq 2d_1 d_2 = C^2.$$

(c). The same proof as in (b), but with σ_e instead of σ_h. (Note that our hypothesis about F and G implies that their bidegrees are (m,m) and (n,n) respectively.) □

Notice that higher-dimensional analogs of Corollary 2 do not hold. Indeed, let $F_1 = P(x)+P(y)$, $F_2 = F_3 = z$, where $P(x) = \prod_{k=1}^{n}(x-k)^2$, $n \geq 3$ (see [8, Example 13.6]). Then the number of real solutions of the system of equations $F_1 = F_2 = F_3 = 0$ is finite but greater than $\prod \deg F_i$.

2.3 Proof of Theorem 1.1

The implication (i) ⇒ (ii) immediately follows from Corollary 2(c), because the embedding $\mathbb{C} \to \mathbb{C}^2$, $z \longmapsto (z, \bar{z})$ identifies $\mathcal{L}_{P_k}$ with

$$\{(z,w) \ : \ P_k(z)\overline{P}_k(w) = 1\} \cap \{(z,w) \ : \ w = \bar{z}\}, \quad k = 1, 2.$$

The implication (ii) ⇒ (iii) follows from Corollary 1. Indeed, suppose that the intersection $\mathcal{L}_{P_1} \cap \mathcal{L}_{P_2}$ is infinite. Then the complexifications of $\mathcal{L}_{P_1}$ and $\mathcal{L}_{P_2}$ have a common real component A whose real locus $\mathbb{R}A$ is also infinite. By composing P_1 and P_2 with a Möbius transformation, we may assume that $|P_j(\infty)| \neq 1$, $j = 1, 2$, so that $\mathbb{R}A$ is compact. Recall that the real locus of a real algebraic curve in neighborhood of every its non-isolated point is homeomorphic to a "star" with an even number of branches (see e.g. [9], p. 104). Thus, the set of non-isolated points of $\mathbb{R}A$ is homeomorphic to a graph in $\mathbb{R}^2$, all of whose vertices have even valency. Such a graph necessarily have a cycle D, and, by construction, the rational functions P_1 and P_2 have modulus one on D. Applying now Corollary 1 to D, we obtain (iii).

To prove (iii) ⇒ (i), it is enough to observe that if B is a quotient of finite Blaschke products, then $B^{-1}(\mathbb{T})$ contains $\mathbb{T}$. Thus, (iii) implies that both $\mathcal{L}_{P_1}$ and $\mathcal{L}_{P_2}$ contain the infinite set $\mathcal{L}_W$ as a subset.

To prove the last part of the theorem, let us fix a Möbius transformation ν that maps the real line to the unit circle, and observe that the lemniscate of $P_1 = \nu \circ z^{n_1}$ is a union of n_1 lines on $\mathbb{C}$ passing through the origin. Under the identification of $\widehat{\mathbb{C}}$ with S^2 via the stereographic projection, $\mathcal{L}_{P_1}$ becomes a union of n_1 big circles passing through two antipodal points a_1, b_1. Let δ be a Möbius transformation of $\widehat{\mathbb{C}}$ corresponding to an isometry of S^2, and $P_2 = \nu\circ z^{n_2}\circ\delta$. Then $\mathcal{L}_{P_2}$ is a union of n_2 big circles passing through two antipodal points a_2, b_2. Any two distinct big circles intersect at two points. Hence, if δ is chosen generically (so that $\{a_1, b_1\}\cap\mathcal{L}_{P_2} = \{a_2, b_2\}\cap\mathcal{L}_{P_1} = \varnothing$), then $\#(\mathcal{L}_{P_1}\cap\mathcal{L}_{P_2}) = 2n_1n_2$. $\square$

Note that Theorem 1.1 can be proved somewhat shorter using the approach of the paper [13], where instead of the intersection $\mathcal{L}_{P_1}\cap\mathcal{L}_{P_2}$ an algebraic curve $\mathcal{C}$ parametrized by P_1 and P_2 is considered. Under this approach, the proof of Theorem 1.1 reduces to proving that if an algebraic curve $\mathcal{C}$ parametrized by rational functions P_1 and P_2 of degrees n_1 and n_2 has more than $2n_1n_2$ points whose coordinates have modulus one, then it can be parametrized by some quotients of finite Blaschke products. On the other hand, considering instead of the functions P_1, P_2 the functions

$$\widehat{P}_1 = T\circ P_1\circ T^{-1}, \quad \widehat{P}_2 = T\circ P_2\circ T^{-1},$$

where

$$T(z) = i\frac{1+z}{1-z},$$

one can easily see that the last statement is equivalent to the statement that if an algebraic curve $\mathcal{C}$ parametrized by rational functions P_1 and P_2 of degrees n_1 and n_2 has more than $2n_1n_2$ real points, then it can be parametrized by some rational functions with real coefficients. Finally, the last statement follows from the bihomogeneous Bézout Theorem since if $\mathcal{C}$ is not defined over $\mathbb{R}$, then the set

$$\mathcal{C}\cap\mathbb{R}^2 = \mathcal{C}\cap\overline{\mathcal{C}}$$

is finite and contains at most $\mathcal{C}\cdot\overline{\mathcal{C}} = 2n_1n_2$ points (in [13] the usual version of the Bézout Theorem was used).

Our proof of Theorem 1.1, while longer, considers the lemniscates directly and relates the initial problem with a real version of the Bézout Theorem, which might have an independent interest.

2.4 Intersection of polynomial lemniscates

The polynomial version of Theorem 1.1 is the following statement.

Theorem 2.2. *Let P_1 and P_2 be non-constant complex polynomials of degrees n_1 and n_2. The following three conditions are equivalent:*

(i) *$\mathcal{L}_{P_1}$ and $\mathcal{L}_{P_2}$ have more than $2n_1n_2$ common points.*

(ii) *$\mathcal{L}_{P_1} \cap \mathcal{L}_{P_2}$ is infinite.*

(iii)* *$P_1 = P^{n_1}$, $P_2 = cP^{n_2}$ for some polynomial P, natural n_1, n_2, and $c \in \mathbb{C}$ with $|c| = 1$.*

Proof. It is clear that Condition (iii)* is a particular case of Condition (iii) for $B_1 = z^{n_1}$, $B_2 = cz^{n_2}$, and $W = P$. Thus, in view of Theorem 1.1, it is enough to show that if P_1 and P_2 are polynomials, then (iii) reduces to (iii)*.

To prove the last statement, let us observe that if P_1 and P_2 are polynomials, then each of the functions B_1 and B_2 has a unique pole, and this pole is common for B_1 and B_2. By condition (3), each pole of B_k, $k = 1, 2$, is symmetric with respect to $\mathbb{T}$ to a zero of the same multiplicity and vice versa. Therefore, there exists $a \in \widehat{\mathbb{C}} \setminus \mathbb{T}$ such that

$$B_1 = c_1\Big(\frac{z-a}{1-\bar{a}z}\Big)^{n_1}, \qquad B_2 = c_2\Big(\frac{z-a}{1-\bar{a}z}\Big)^{n_2}$$

for some $n_1, n_2 \geq 1$ and $c_1, c_2 \in \mathbb{C}$ with $|c_1| = |c_2| = 1$, implying that

$$P_1 = c_1\widetilde{P}^{n_1}, \qquad P_2 = c_2\widetilde{P}^{n_2}, \tag{8}$$

where

$$\widetilde{P} = \frac{z-a}{1-\bar{a}z} \circ W.$$

Finally, it is easy to see that $\widetilde{P}$ is a polynomial and (iii)* holds for $P = c_1^{\frac{1}{n_1}}\widetilde{P}$ and $c = c_2 c_1^{-n_2/n_1}$. □

Notice that in the paper [13] it is erroneously claimed that in the polynomial case Condition (iii) of Theorem 1.1 simply reduces to the condition that $P_1 = P^{m_1}$, $P_2 = P^{m_2}$ for some polynomial P. This inaccuracy however does not affect the main application of results of [13] in the polynomial case: the result of Ailon and Rudnick mentioned in the introduction. Indeed, if c in (iii)* is not a root of unity, then the system

$$P_1(z)^k = 1, \quad P_2(z)^k = 1 \tag{9}$$

has no solutions for any $k \geq 1$, since (iii)* and (9) imply that

$$P^{n_1 n_2 k}(z) = c^{n_1 k} P^{n_1 n_2 k}(z) = 1.$$

On the other hand, if $c^l = 1$, then (5) holds for $m_1 = n_2 l$ and $m_2 = -n_1 l$.

2.5 On sharpness of the bound $2n_1n_2$ in the polynomial case

The last statement of Theorem 1.1 states that the bound $2n_1n_2$ is sharp if we speak of all rational functions. However, it does not seem so when we restrict ourselves to polynomials only. The maximal number of intersection points of two polynomial lemniscates that we succeeded to realize, is given in the following statement.

Proposition 2. *Let* $1 \leq n_1 \leq n_2$ *and* $d = \gcd(n_1, n_2)$*. Then there exist polynomials* P_1 *and* P_2 *such that* $\deg P_k = n_k$, $k = 1, 2$*, and*

$$\#(\mathcal{L}_{P_1} \cap \mathcal{L}_{P_2}) = M(n_1, n_2) := n_1 n_2 + n_2 + de, \qquad e = \begin{cases} 0, & \text{if } n_1/d \text{ is odd,} \\ 1, & \text{if } n_1/d \text{ is even.} \end{cases}$$

Proof. Let $P_k(z) = (z/r_k)^{n_k} - 1$, $k = 1, 2$, where $r_k > 0$. The lemniscate $\mathcal{L}_{P_k}$ is "flower-shaped", i.e. it is a union of n_k loops ("petals") outcoming from 0. It is clear that $\mathcal{L}_{P_k}$ tends to a union of n_k lines (we denote it by L_k) when $r_k \to \infty$. Let us fix r_1. It is not difficult to show that for a suitably chosen rotation R we have

$$\#\big(L_2 \cap R(\mathcal{L}_{P_1}) \setminus \{0\}\big) = n_2 + de.$$

A further small shift of $R(\mathcal{L}_{P_1})$ produces $n_1 n_2$ additional crossings near 0. Finally, we approximate L_2 by $\mathcal{L}_{P_2}$ by choosing $r_2 \gg r_1$. □

Note that the number $M(n_1, n_2)$ in Proposition 2 coincides with the upper bound $2n_1n_2$ given by Theorem 1.1 if and only if $n_1 = 1$.

3 Irreducibility of lemniscates

3.1 Irreducibility of separated variables curves and lemniscates

Let us recall that a *separated variable curve* is an algebraic curve in $\mathbb{C}^2$ given by the equation $E_{P,Q}(x, y) = 0$, where $E_{P,Q}(x, y)$ is the numerator of $P(x) - Q(y)$ for some non-constant complex rational functions $P(z)$ and $Q(z)$. The irreducibility problem for separated variable curves is quite old and still widely open (see [7] for an introduction to the subject). One of the most complete results in this direction is a full classification of reducible curves $E_{P,Q}(x, y) = 0$ in the case when P and Q are indecomposable polynomials. All possible ramifications of such P and Q were described by Fried in [6], and polynomials themselves were found by Cassou-Noguès and Couveignes in [4]. For further progress, we refer the reader to the recent paper [10] and the bibliography therein. Here and below, by the irreducibility we always mean the irreducibility over $\mathbb{C}$.

Let us recall that $L_P(x, y)$ is the numerator of the rational function (6), and set

$$\widehat{L}_P(x, y) = E_{P,1/\overline{P}}(x, y).$$

The irreducibility problem for curves $L_P(x, y) = 0$ defining lemniscates is a particular case of the irreducibility problem for separated variables curves due to the following statement, which is immediate from the identity $L_P(x, y) = \widehat{L}_P(x + iy, x - iy)$.

Lemma 3.1. *Let P be a non-constant complex rational function. Then the curve $L_P(x, y) = 0$ is irreducible if and only if the curve $\widehat{L}_P(x, y) = 0$ is irreducible.* □

The following theorem was proved in the paper [12].

Theorem 3.2. *Let P and Q be non-constant complex polynomials. Then the curve $E_{P,1/Q}(z, w) = 0$ is reducible if and only if*

$$P(z) = P_1(z)^d \qquad \textit{and} \qquad Q(w) = Q_1(w)^d$$

for some $d > 1$ and polynomials $P_1(z)$ and $Q_1(w)$.

It is easy to see that combined with Lemma 3.1, Theorem 3.2 implies the following result also proved in [12].

Theorem 3.3. *Let P be a non-constant complex polynomial. Then the polynomial $L_P(x,y)$ is reducible if and only if*

$$P(z) = P_1(z)^d$$

for some $d > 1$ and polynomial $P_1(z)$.

Notice that since $|P_1(z)^d| = 1 \Leftrightarrow |P_1(z)| = 1$, this theorem implies that any polynomial lemniscate is the zero set of an irreducible polynomial in x and y.

Definition 1. A complex rational function $P(z)$ satisfies the *Composition Condition* if there exist a quotient of finite Blaschke products B of degree at least two and a non-constant rational function W such that $P = B \circ W$.

Theorem 3.3 shows that the Composition Condition is necessary and sufficient for the reducibility of $L_P(x,y)$ in the polynomial case. Moreover, it is easy to see that the Composition Condition is *sufficient* for the reducibility of $L_P(x,y)$ for any rational function P. Indeed, it follows from (3) that for any quotient of finite Blaschke products B and a rational function W the equality $W(x) = 1/\overline{W}(y)$ for some $x, y \in \mathbb{C}$ implies the equality

$$P(x) = B(W(x)) = \frac{1}{\overline{B}(1/W(x))} = \frac{1}{\overline{B}(\overline{W}(y))} = \frac{1}{\overline{P}(y)}.$$

Therefore, the curve $\widehat{L}_W(x,y) = 0$ is a component of the curve $\widehat{L}_P(x,y) = 0$, implying that the latter curve is reducible since the considered curves have different degrees (in view of the assumption $\deg B > 1$). Thus, the curve $L_P(x,y) = 0$ is reducible by Lemma 3.1.

In the next section, we show that Composition Condition is not necessary for the reducibility of L_P, while in this section, modifying the idea of [12], we establish a sufficient condition for the irreducibility of an algebraic curve $E_{P,Q}(x,y) = 0$ in the case when one of the rational functions P, Q is a polynomial.

Namely, we prove the following result.

Theorem 3.4. *Let P be a polynomial of degree $n \geq 1$, and Q a rational function. Assume that multiplicities $q_1, q_2, \ldots, q_l$ of poles of Q satisfy the condition $\gcd(q_1, q_2, \ldots, q_l, n) = 1$. Then the curve $E_{P,Q}(x, y) = 0$ is irreducible.*

Notice that like Theorem 3.2, Theorem 3.4 easily implies Theorem 3.3.

First proof of Theorem 3.4 (cf. the proof of [12, Theorem 1]). Let C be the closure of the curve $E_{P,Q} = 0$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$, that is, the curve defined by the bihomogeneous equation $u^n v^m E_{P,Q}(x/u, y/v) = 0$, $m = \deg Q$. We identify $\mathbb{CP}^1$ with $\widehat{\mathbb{C}}$ denoting $(x : 1)$ by x and $(1 : 0)$ by ∞.

Suppose there exists a proper factor $E'(x, y)$ of $E_{P,Q}$. Let C' be the corresponding subset of C. Let $y_1, \ldots, y_l \in \mathbb{CP}^1$ be the poles of Q of multiplicities $q_1, \ldots, q_l$ respectively. The germ of C at (∞, y_i) has the form $U^n = Y^{q_i}$ in some local analytic coordinates (U, Y). Indeed, the equation of C in the affine coordinates $\widehat{U} = u/x$, $\widehat{Y} = y/v - y_i$ ($\widehat{Y} = v/y$ if $y_i = \infty$) has the form

$$\widehat{U}^n f_i(\widehat{U}) = \widehat{Y}^{q_i} g_i(\widehat{Y}), \qquad f_i(0)g_i(0) \neq 0,$$

thus it has the desired form in the local coordinates $U = \widehat{U} f_i^{1/n}$, $Y = \widehat{Y} g_i^{1/q_i}$ for any choice of single-valued branches of the roots of f_i and g_i near 0.

The binomial $U^n - Y^{q_i}$ factorizes as $\prod_j^{k_i}(U^{b_i} - \omega^j Y^{a_i})$, where $k_i = \gcd(q_i, n)$, $a_i = q_i/k_i$, $b_i = n/k_i$, and ω is a primitive k_i-th root of unity. Thus the germ of C at (∞, y_i) has k_i local analytic branches, which we denote by γ_{ij}, $j = 1, \ldots, k_i$. Let k_i' be the number of those that belong to C'.

Let $L_i = \mathbb{CP}^1 \times \{y_i\}$. For local intersections, we have $(\gamma_{ij} \cdot L_i)_{(\infty, y_i)} = b_i$ for each i, j. Hence

$$k_i' b_i = (C'.L_i)_{(\infty, y_i)} = n' := \deg_x E', \qquad i = 1, \ldots, l,$$

and we obtain

$$\frac{k_i' a_i}{q_i} = \frac{k_i'}{k_i} = \frac{k_i' b_i}{n} = \frac{n'}{n}, \qquad 1 \leq i \leq l. \tag{10}$$

Let d'/d be the reduced form of this fraction, i.e., $d'/d = n'/n$ and $\gcd(d', d) = 1$. Then $d > 1$ (because $n' < n$) and d divides all the denominators in (10), which implies that $\gcd(q_1, \ldots, q_l, n) > 1$. □

Now we expose more or less the same proof using the language of field extensions. The notations in both proofs are consistent. Of course, the proof of [12, Theorem 1] can be reinterpreted similarly.

Second proof of Theorem 3.4. For a compact Riemann surface C, we denote the field of meromorphic functions on C by $\mathcal{M}(C)$. Given a meromorphic function $\theta : C \to \mathbb{CP}^1$, we denote the local multiplicity of θ at the point z by $e_\theta(z)$.

Assume that the curve $E_{P,Q}(x,y) = 0$ is reducible, and let C be a desingularization of some of its components. Then there exist meromorphic functions $\varphi : C \to \mathbb{CP}^1$ and $\psi : C \to \mathbb{CP}^1$ of degrees $m' < m = \deg Q$ and $n' < n$ such that

$$P\circ\varphi = Q\circ\psi \tag{11}$$

and the compositum of the fields $\varphi^*\mathcal{M}(\mathbb{CP}^1) \subseteq \mathcal{M}(C)$ and $\psi^*\mathcal{M}(\mathbb{CP}^1) \subseteq \mathcal{M}(C)$ is the whole field $\mathcal{M}(C)$. Furthermore, if $\theta : C \to \mathbb{CP}^1$ is a meromorphic function defined by any of the sides of equality (11), then by the Abhyankar Lemma (see e. g. [17], Theorem 3.9.1) for every point t_0 of C the following equality holds:

$$e_\theta(t_0) = \operatorname{lcm}\big(e_P(\varphi(t_0)), e_Q(\psi(t_0))\big) \tag{12}$$

Let $Q^{-1}\{\infty\} = \{y_1, y_2, \dots, y_l\}$, where $e_Q(y_i) = q_i$, $1 \le i \le l$, and

$$\psi^{-1}\{y_i\} = \{z_{i1}, z_{12}, \dots, z_{ik'_i}\}, \quad 1 \le i \le l.$$

Let us set

$$k_i = \gcd(q_i, n), \quad a_i = q_i/k_i, \quad b_i = n/k_i.$$

Since $P^{-1}\{\infty\} = \infty$, by (12) we have

$$e_\theta(z_{ij}) = \operatorname{lcm}(q_i, n), \quad 1 \le j \le k'_i, \quad 1 \le i \le l.$$

Therefore,

$$e_\psi(z_{ij}) = \frac{\operatorname{lcm}(q_i, n)}{q_i} = b_i, \qquad 1 \le j \le k'_i, \quad 1 \le i \le l,$$

implying that, for each $i = 1, \dots, l$, we have $k'_i b_i = \deg\psi = n'$, whence we obtain the equation (10) and conclude in the same way as in the first proof. □

3.2 A counterexample

In this subsection we show that the Composition Condition (see Definition 1) is not necessary for reducibility of the curve $L_P(x,y)=0$. Let us set

$$T(z)=\frac{z-i}{z+i}. \tag{13}$$

We recall that this Möbius transformation (called the Cayley transform) maps $\widehat{\mathbb{R}}:=\mathbb{R}\cup\{\infty\}$ to the unit circle $\mathbb{T}$.

Lemma 3.5. *Let S be a non-constant complex rational function. Then the curve $\widehat{L}_{T\circ S}(x,y)=0$ is irreducible if and only if the curve $E_{S,\overline{S}}(x,y)=0$ is irreducible.*

Proof. If $S=S_1/S_2$, where S_1 and S_2 have no common roots, then

$$(T\circ S)(z)=\frac{S_1(z)-iS_2(z)}{S_1(z)+iS_2(z)},$$

where $S_1(z)-iS_2(z)$ and $S_1(z)+iS_2(z)$ also have no common roots. Thus,

$$\begin{aligned}\widehat{L}_{T\circ S}(x,y)&=(S_1(x)-iS_2(x))(\overline{S}_1(y)+i\overline{S}_2(y))-(S_1(x)+iS_2(x))(\overline{S}_1(y)-i\overline{S}_2(y))\\&=2i(S_1(x)\overline{S}_2(y)-S_2(x)\overline{S}_1(y))=2iE_{S,\overline{S}}(x,y).\end{aligned}$$ □

To show that the Composition Condition is not necessary for the reducibility of $L_P(x,y)$, one can use examples of reducible curves of the form $E_{S,\overline{S}}(x,y)=0$ found in [4]. For instance, we can take

$$\begin{aligned}S(z)=\frac{1}{11}z^{11}-(a+1)z^9+2z^8+(3a-9)z^7-16(a+1)z^6+(21a+36)z^5\\+(30a-90)z^4-63az^3+(100a+120)z^2+(24a-117)z-18(a+1),\end{aligned} \tag{14}$$

where a satisfies $a^2+a+3=0$. It is shown in [4] that for $S(z)$ the following conditions hold. First, the curve $S(x)-\overline{S}(y)=0$ is reducible. Second,

$$\overline{S}(z)\neq S(cz+b), \tag{15}$$

for any $c\in\mathbb{C}^*$, $b\in\mathbb{C}$ (the last condition makes this example non-trivial since every curve of the form $S(x)-S(cy+b)=0$ obviously has a factor $x-cy-b=0$).

Let us consider a rational function $P = T \circ S$, where T and S are defined by (13) and (14). By Lemma 3.5, the curve $\widehat{L}_P(x, y) = 0$ is reducible, implying by Lemma 3.1 that the curve $L_P(x, y) = 0$ is also reducible. Since the degree of P is a prime number, if the Composition Condition were satisfied by P, we would have

$$P = T \circ S = B \circ \mu \tag{16}$$

for some quotient of finite Blaschke products B and Möbius transformation μ. Let now ν be a Möbius transformation that maps $\widehat{\mathbb{R}}$ to $\mu^{-1}(\mathbb{T})$. Then $(\mu \circ \nu)(\widehat{\mathbb{R}}) = \mathbb{T}$ and it follows from (16) combined with the equalities $B(\mathbb{T}) = \mathbb{T}$ and $T(\widehat{\mathbb{R}}) = \mathbb{T}$ that

$$(S \circ \nu)(\widehat{\mathbb{R}}) = (S \circ \mu^{-1})(\mathbb{T}) = (T^{-1} \circ B)(\mathbb{T}) = T^{-1}(\mathbb{T}) = \widehat{\mathbb{R}}. \tag{17}$$

Therefore the rational function $\overline{(S \circ \nu)} - (S \circ \nu)$ identically vanishes on $\mathbb{R}$, and hence on $\mathbb{C}$. Thus,

$$S \circ \nu = \overline{S \circ \nu} = \overline{S} \circ \overline{\nu},$$

and denoting the Möbius transformation $\nu \circ \overline{\nu}^{-1}$ by δ, we rewrite this equality as

$$\overline{S} = S \circ \delta. \tag{18}$$

Since S is a polynomial, (18) implies that δ also is a polynomial. Moreover, $\deg \delta = 1$ (because δ is a Möbius transformation), thus (18) contradicts (15). The obtained contradiction shows that P does not satisfy the Composition Condition.

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Hodge Theory on Tropical Curves

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Abstract: We construct an analog of the Hodge theory on complex manifolds in the case of tropical curves. We use the analytical approach to the problem, it is based on language of tropical differential forms and methods of L^2−cohomologies. In particular, the cohomology groups of a tropical curve can be defined via the de Rham complex of tropical differential forms. We translate standard notions of the complex Hodge theory: the Kähler form, the Hodge star operator, the Laplace-Beltrami operator to the tropical case. The main result of the article is that the tropical Laplace-Beltrami operator is a self-adjoint unbounded operator and the cohomology groups of a tropical curve are isomorphic to the spaces of harmonic forms on this curve.

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1 Introduction

In this paper we construct a tropical analog of the classical Hodge theory for Kähler manifolds. We study only the case of one-dimensional tropical varieties, it is a tropical analog of the Hodge theory on smooth complex curves. The main purpose of this paper is to show that the classical analytical construction of the Hodge theory via harmonic forms, Laplace-Beltrami operator and related objects and methods can be translated to the tropical case practically word by word. So it is mostly an illustration of application the method, though some of the results, like a computation of cohomology group of a tropical curve, can be obtained using much simpler technique.

Let us very briefly recall the classical Hodge theory for complex curves. For general references on the Hodge theory see [4], [5]. Given a smooth compact complex curve C of genus n. Let $\mathcal{E}^{p,q}(C)$ be the space of smooth (p,q)–differential forms on C. The Dolbeault cohomology $H^{p,q}_{\overline{\partial}}(C)$ is the cohomology group of the complex $(\mathcal{E}^{p,*}(C), \overline{\partial})$, where $\overline{\partial}$ is a differential

$$\overline{\partial} : \mathcal{E}^{p,q}(C) \to \mathcal{E}^{p,q+1}(C).$$

Suppose g is a hermitian metric on C and ω is the corresponding Kähler form. The metric g induces a scalar product on $\mathcal{E}^{p,q}(C)$ and defines the Hodge star operator

$$*\colon \mathcal{E}^{p,q}(C) \to \mathcal{E}^{1-p,1-q}(C).$$

Let $\overline{\partial}^*$ be a metric adjoint to $\overline{\partial}$. The Laplace-Beltrami operator is defined as follows:

$$\Delta = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^* : \mathcal{E}^{p,q}(C) \to \mathcal{E}^{p,q}(C).$$

The space of harmonic forms $\mathcal{H}^{p,q}(C)$ is by definition the kernel of $\Delta : \mathcal{E}^{p,q}(C) \to \mathcal{E}^{p,q}(C)$.

Our main goal is to prove a tropical analog of the following statement.

Theorem 1.1. *Every harmonic form $\varphi \in \mathcal{H}^{p,q}(C)$ is $\overline{\partial}$–closed and, consequently, defines a cohomology class $[\varphi] \in H^{p,q}_{\overline{\partial}}(C)$. The map $\varphi \to [\varphi]$ is an isomorphism between $\mathcal{H}^{p,q}(C)$ and $H^{p,q}_{\overline{\partial}}(C)$. The Hodge star operator is an isomorphism between the spaces of harmonics* *forms* $*\colon \mathcal{H}^{p,q}(C) \simeq \mathcal{H}^{1-p,1-q}(C)$. *The dimensions of cohomology groups are equal to*

$$\dim_{\mathbb{C}} H^{0,0}_{\bar\partial}(C) = \dim_{\mathbb{C}} H^{1,1}_{\bar\partial}(C) = 1,$$

$$\dim_{\mathbb{C}} H^{1,0}_{\bar\partial}(C) = \dim_{\mathbb{C}} H^{0,1}_{\bar\partial}(C) = n,$$

where n *is the genus of* C.

A one-dimensional tropical variety is essentially a metric graph Γ with some additional features. An analog of smooth (p, q)–differential forms is a special class of tensor fields on edges of Γ satisfying some boundary conditions at vertices. We denote this class of tensor by $\mathcal{E}^{p,q}(\Gamma)$ and call it the *space of regular tropical super* of degree (p, q). There is a differential

$$d'' : \mathcal{E}^{p,q}(\Gamma) \to \mathcal{E}^{p,q+1}(\Gamma)$$

and the cohomology group $H^{p,q}_{d''}(\Gamma)$ of the complex $(\mathcal{E}^{p,*}, d'')$.

For references on and tropical cohomologies see [7], [8], [9]: notion of tropical cohomologies was introduced in [7] using methods of algebraic topology, notion of tropical superforms, which play in the tropical case the role of smooth differential forms, was introduced in [9], and in [8] differential topological approach to tropical cohomologies was developed, i.e., an analog of the de Rham or Dolbeault cohomology theory. In [6] the de Rham cohomology of metric graphs is considered in a fashion quite similar to our paper. In [10] metric graphs was studied as a tropical limits of a degeneration of a family of complex holomorphic curves and tropical holomorphic forms were obtained as tropical limits of usual holomorphic forms.

The space $\mathcal{E}^{p,q}(\Gamma)$, the operator d'', and the cohomology group $H^{p,q}_{d''}(\Gamma)$ play in the tropical case the same role as $\mathcal{E}^{p,q}(C), \bar\partial$, and $H^{p,q}_{\bar\partial}(C)$ in the complex case. Actually, the space of tropical superforms on the interval $[a, b] \subset \mathbb{R}$ can be identified with the space of $U(1)$–invariant differential forms on the complex annulus $\mathrm{Log}^{-1}([a, b]) \subset \mathbb{C} \setminus \{0\}$, see Section 2.6. Various operations on this tropical space: integration, taking the differential, etc., can be interpreted in term of usual complex operations on this space of $U(1)$–invariant differential forms.

There are tropical analogs of a Kähler form and a hermitian metric on Γ. This tropical Kähler form induces a scalar product on $\mathcal{E}^{p,q}(\Gamma)$. Let d''^* be the metric adjoint operator to d'' with respect to this scalar product. Then we can define the Laplace-Beltrami operator

$$\Delta = d''d''^* + d''^*d''.$$

The space of *harmonic forms* $\mathcal{H}^{p,q}(\Gamma)$ is by definition the kernel of Δ.

The genus of tropical curve Γ is, by definition, the rank of $H^1(\Gamma)$, where $H^1(\Gamma)$ is the usual topological cohomology group of the graph Γ. The main result of this paper is the following

Theorem 1.2. *Let Γ be a tropical curve of genus n. Every harmonic superform $\varphi \in \mathcal{H}^{p,q}(\Gamma)$ is d''−closed and, consequently, defines the cohomology class $[\varphi] \in H^{p,q}_{d''}(\Gamma)$. The map $\varphi \to [\varphi]$ is an isomorphism between $\mathcal{H}^{p,q}(\Gamma)$ and $H^{p,q}_{d''}(\Gamma)$. The Hodge star operator maps harmonic superform to harmonic superform and the map $*\colon \mathcal{H}^{p,q}(\Gamma) \to \mathcal{H}^{1-p,1-q}(\Gamma)$ is an isomorphism. There are isomorphisms*

$$H^{1,1}(\Gamma) \simeq H^{0,0}(\Gamma) \simeq H^0(\Gamma, \mathbb{R}) \cong \mathbb{R}$$

and

$$H^{1,0}(\Gamma) \simeq H^{0,1}(\Gamma) \simeq H^1(\Gamma, \mathbb{R}) \cong \mathbb{R}^n.$$

We consider our results in the first place as a toy model and a proof of a concept for the tropical Hodge theory, only then we consider it as a results about topology of tropical curves. Indeed, one can compute cohomology of a tropical curve using much simpler methods without any functional analysis or differential topology. In particular, since, by definition, $H^{0,q}(\Gamma)$ coincide with usual topological cohomologies of the graph Γ and can be easily computed, one can apply the tropical Poincaré duality duality [8, Theorem 4.33] and get $H^{0,q}(\Gamma) \simeq H^{1,1-q}(\Gamma)$. Also the cohomologies $H^{p,q}(\Gamma)$ were computed in [6, Proposition 2.4.2.] in terms of tropical differential forms using quite simple methods, actually in that paper a bit more general case was consider, that case is a tropical analog of curves with punctures.

The actual problem is to construct the tropical Hodge theory in higher dimensions. To do so one can follow the way of classical complex Hodge theory and methods of this paper, but it seems that there are many technical obstacles in this way. The main source of these obstacles are: nonsmoothness of tropical varieties, which are locally behave like a polyhedral complexes, and complicated behavior of various analytical objects at infinity, which is typical in L^2-cohomology theories. Both problems are illustrated in this paper: the proofs of Lemma 3.5, Theorem 3.10, Proposition 3.13 require us to do a tedious analysis of the infinite length edges case, and in Theorem 3.10 we deal with the combinatorial aspects of metric graphs. The article [11] on a PL-Hodge theory was a great source of inspiration for our work.

The paper is organized as follows. In the second section we introduce the main objects and work with differential-topological part of the problem. In the third section section we develop methods related to the functional analysis and L^2-cohomology theory. For general references on L^2-methods in the complex Hodge theory see, for example, [4, Chapter VIII].

It is interesting that the topic of this paper is closely related to quantum graphs. The main idea of quantum graphs is to study the Schrödinger equation and the Laplace equation over a metric graph [2]. In this case various boundary conditions at vertices of the graph arise. We do not know any source in the literature where methods of quantum graphs were applied to the tropical geometry.

2 Tropical curves and tropical superforms

In this section we introduce main objects of our paper: Tropical curves, Tropical superforms, Tropical cohomologies, tropical Kähler form and operations on them. We study their properties, and prove some results of differential-topological nature. Also, we show the relation of this objects to the complex geometry.

2.1 Tropical curves.

Definition 2.1. A *compact connected tropical curve* Γ is a connected metric graph with the set of vertexes V and the set of edges E satisfying the following condition:

1. The sets E and G are finite and non-empty.
2. The length $l(e)$ of an edge $e \in E$ is a positive real number or $+\infty$.
3. The length $l(e)$ is equal to $+\infty$ if and only if e is incident on a degree one vertex.
4. A finite length edge e is isometric to the closed interval $[-l(e), 0]$ with the standard Euclidean metric.
5. If an infinite length edge e is incident to a degree one vertex and to a vertex of a higher degree, then the edge is isometric to the closed interval $[-\infty, 0]$ with the standard Euclidean metric, where $-\infty$ is the image of the degree one vertex.
6. If an infinite length edge e is incident to two degree one vertices, then this edge is isometric to $[-\infty, +\infty]$. Since Γ is connected it have to be a graph with one edge and two vertices and the whole graph is isometric to $[-\infty, +\infty]$.

In this paper we will address a compact connected tropical curve as just a curve. The *genus* of a curve Γ is defined as the rank of cohomology group $H^1(\Gamma)$.

Example 2.2. Let us consider several examples of curves.

- Any metric graph with finite-length edges such that all vertices has degree ≥ 2 can be considered as a tropical curve.
- The closed interval $[-\infty, +\infty]$ is an example of a genus 0 curve. We can consider this curve as the tropical projective line $\mathbb{TP}^1 := [-\infty, +\infty]$.
- A finite number of disjoint copies of $[-\infty, 0]$ glued together along 0

$$\Gamma = [-\infty, 0] \sqcup \cdots \sqcup [-\infty, 0]/\sim$$

is another example of a genus 0 curve.

Remark 2.3. Usually in tropical geometry a tropical variety is defined in terms of polyhedral complexes in $\mathbb{TR}^n = [-\infty, +\infty)^n$. One can consider these polyhedral complexes as varieties embedded to some ambient space. In our definition we do not use any ambient space or embeddings. For any compact connected tropical curve Γ in the sense of our definition one can construct an isomorphic tropical curve in terms of polyhedral complexes in $\mathbb{TR}^n$. The similar approach to tropical curves via metric graphs were used in [10], [6].

Remark 2.4. We can consider $S^1 = \mathbb{R}/a\mathbb{Z}$, where $a \in (0, +\infty)$, as an example of a tropical curve of genus 1. Informally it is a metric graph with one edge and no vertexes and it is not consistent with our definition of a curve. To resolve this problem one can put a vertex to this curve and consider this as a metric graph with one vertex and one loop. Another approach to deal with this problem is to extend the definition of a tropical curve using the way similar to the definition a topological manifold in terms of charts and transition maps. According to this approach a tropical curve is a space locally isomorphic to a metric graph and transition maps are given by affine functions. We are not going to use this approach in the paper.

2.2 Tropical superforms over $\mathbb{R}$.

Definition 2.5. A tropical superforms of degree (p, q), $p, q = 0, 1$, on $\mathbb{R}$ is a smooth section of the line bundle

$$\Lambda^{p,q}T^*\mathbb{R} := \bigwedge^p T^*\mathbb{R} \otimes \bigwedge^q T^*\mathbb{R}.$$

We denote the linear space of $(p, q)-$tropical superforms by $\mathcal{E}^{p,q}(\mathbb{R})$.

The notions of a tropical superform on $\mathbb{R}^n$ and related objects consider in this subsection were initially introduced in [9].

Here $\bigwedge^0 T^*\mathbb{R}$ is a trivial line bundle. In particular, $(0, 0)$-tropical superforms are smooth functions on $\mathbb{R}$, the spaces of $(1, 0)$ and $(0, 1)$ tropical superforms can be identified with differential $1-$forms, i.e., with tensor fields of valency $(0, 1)$, and $(1, 1)$-tropical superforms can be identified with with tensor fields of valency $(0, 2)$.

Let x be a cartesian coordinate on $\mathbb{R}$. We denote by $d'x$ and by $d''x$ the differential dx which we consider, correspondingly, as a $(1,0)-$tropical form or $(0,1)-$tropical form, and we denote by $d'x \wedge d''x$ the tensor field $dx \otimes dx$ which we consider as a $(1,1)-$form. Then any $(0,0),(1,0),(0,1),(1,1)-$tropical superform can be written, correspondingly, as $\varphi(x), \varphi(x)d'x, \varphi(x)d''x, \varphi(x)d'x \wedge d''x$ for some smooth function $\varphi(x)$.

There is a natural wedge product

$$\wedge : \mathcal{E}^{p,q}(\mathbb{R}) \otimes \mathcal{E}^{p',q'}(\mathbb{R}) \to \mathcal{E}^{p+p',q+q'}(\mathbb{R}).$$

The wedge product of the $(1,0)-$tropical form $d'x$ and the $(0,1)-$form $d''x$ is defined to be equal to the $(1,1)-$form $d'x \wedge d''x$. The wedge product satisfies the alternation condition

$$d'x \wedge d''x = -d''x \wedge d'x,$$

$$d'x \wedge d'x = d''x \wedge d''x = 0.$$

There is the differential $d'' : \mathcal{E}^{p,q}(\mathbb{R}) \to \mathcal{E}^{p,q+1}(\mathbb{R})$. It is defined on $(0,0)-$forms, i.e., functions, as

$$d''(\varphi(x)) = \frac{\partial \varphi(x)}{\partial x} d''x$$

and on $(1,0)-$forms as

$$d''(\varphi(x)d'x) = d''(\varphi(x)) \wedge d'x = -\frac{\partial \varphi(x)}{\partial x} d'x \wedge d''x,$$

In all other cases d'' is equal to zero for dimensional reasons.

In the same way we define the differential operator

$$d' : \mathcal{E}^{p,q}(\mathbb{R}) \to \mathcal{E}^{p+1,q}(\mathbb{R}) :$$

$d'(\varphi(x)) = \frac{\partial}{\partial x}\varphi(x)d'x$ and $d'(\varphi(x)d''x) = d'(\varphi(x)) \wedge d''x = \frac{\partial}{\partial x}\varphi(x)d'x \wedge d''x$, in all other cases d' is equal to zero.

The *tropical integral* over $\mathbb{R}$ of a $(1,1)$-forms $\varphi(x)d'x \wedge d''x$ is defined as

$$\int_{\mathbb{R}} \varphi(x)d'x \wedge d''x = \int_{\mathbb{R}} \varphi(x)dx,$$

where the right hand side is the usual integral. The integral over an interval I of $\mathbb{R}$ is defined in the same way $\int_I \varphi(x)d'x \wedge d''x = \int_I \varphi(x)dx$.

Remark 2.6. From an abstract point of view to define the tropical integral of $(1,1)$-tropical superform ω over a 1-dimensional $\mathbb{R}$−linear space L we need to choose a volume form μ with a constant coefficient. We can identify this form with a non-zero element of $T^*_0 L$, since a constant section of T^*L is defined by its value at 0.

This tropical integral depends on the choice of μ, we denote it by $\int_{(L,\mu)} \omega$. The form $\mu \otimes \mu$ defines a trivialization of $T^*L \otimes T^*L$. Hence any $(1,1)$-tropical superforms ω can be written as $\omega = f(x)\mu \otimes \mu$ for some function $f(x)$ on L.

The integral is defined as

$$\int_{(L,\mu)} \omega = \int_L f(x)\mu,$$

where the right-hand side is the usual integral of a differential form of the top degree over an orientated linear space, the orientation of L is induced by the form μ.

Notice that for the form $\mu' = -\mu$ we obtain the same integral as for the form μ. Indeed, we have $\omega = f(x)\mu \otimes \mu = f(x)(-\mu) \otimes (-\mu) = f(x)\mu' \otimes \mu'$ and the integrals

$$\int_{(L,\mu)} \omega = \int_L f(x)\mu = -\int_L f(x)(-\mu) = -\int_L f(x)\mu' = \int_{\overline{L}} f(x)\mu' = \int_{(L,\mu')} \omega$$

are the same, where $\overline{L}$ is the space L with an opposite orientation, i.e., the orientation induced by μ'.

On the other hand, scaling of the volume form changes the value of the integral. Indeed, if $\mu' = c\mu, c > 0$, then $f(x)\mu \otimes \mu = \frac{1}{c^2} f(x)\mu' \otimes \mu'$. Thus we get

$$\int_{(L,\mu)} \omega = \int_L f(x)\mu,$$

$$\int_{(L,\mu')} \omega = \int_L \frac{1}{c^2} f(x)\mu' = \int_L \frac{1}{c} f(x)\mu = \frac{1}{c} \int_{(L,\mu)} \omega.$$

Let Λ be a lattice in L, then it determines the form μ uniquely up to the sign by the condition $\mu(e) = 1$, where e is a generator of Λ. Here $\mu(e)$ is a contraction of $\mu \in T^*_0 L$ and $e \in T_0 L \simeq L$. Therefore the lattice Λ defines the tropical integral uniquely.

It the case $L = \mathbb{R}$ we choose Λ to be equal to $\mathbb{Z}$ and μ to be equal to the differential of the cartesian coordinate dx, this give us the initial definition of the tropical integral.

2.3 Tropical superforms over tropical curve

Let e be an edge of Γ then the space $\mathcal{E}^{p,q}(e)$ of (p,q)–tropical superforms over e is defined as the restriction of $\mathcal{E}^{p,q}(\mathbb{R})$ to either $[-l(e),0]$ or $(-\infty,0]$ or $(-\infty,+\infty)$ if e is, consequently, isometric to $[-l(e),0]$ or $[-\infty,0]$ or $[-\infty,+\infty]$. An integral $\int_e \omega$ over the edge e of a form $\omega \in \mathcal{E}^{1,1}(e)$ is defined as an tropical integral over the corresponding interval of $\mathbb{R}$.

Definition 2.7. The linear space $\widetilde{\mathcal{E}}^{p,q}(\Gamma)$ *of tropical superforms of degree* (p,q), $p,q=0,1$ *on a curve* Γ is defined as follows

$$\widetilde{\mathcal{E}}^{p,q}(\Gamma) = \bigoplus_{e\in E} \mathcal{E}^{p,q}(e).$$

We denote by ω_e the component over the edge e of the form $\omega \in \widetilde{\mathcal{E}}^{p,q}(\Gamma)$.

The integral of a form $\omega \in \widetilde{\mathcal{E}}^{1,1}(\Gamma)$ over Γ is defined as the sum of the tropical integrals over all edges:

$$\int_\Gamma \omega = \sum_{e\in E} \int_e \omega_e.$$

Notice that for a form from $\widetilde{\mathcal{E}}^{p,q}(\Gamma)$ there are no conditions over values of the form at the ends of different edges which represent the same vertex of Γ. Actually, this space is not an adequate analog of the smooth differential form on a Riemann surface and will play supplementary role in the paper. The tropical analog of smooth forms is *regular tropical superforms* which is defined below.

Definition 2.8. The *space of regular tropical superforms* $\mathcal{E}^{p,q}(\Gamma)$ is a subspace of $\widetilde{\mathcal{E}}^{p,q}(\Gamma)$. Elements of $\mathcal{E}^{p,q}(\Gamma)$ should satisfy the following conditions:

1. **Continuity.** A $(0,0)$-form $\varphi \in \mathcal{E}^{0,0}(\Gamma)$ is continuous if for any two edges $e,e' \in E$ incident to the same vertex v, the values of φ_e and $\varphi_{e'}$ at the points corresponding to v coincide. In the other words, φ have to be a continuous function on the metric graph Γ.

2. **Kirchhoff's law.** Given a vertex $v \in V$. Let $E_v \subset E$ be a set of edges incident to this vertex. Suppose that an edge $e \in E_v$ is identified with the interval $[-l(e),0]$ and

the point $0 \in [-l(e), 0]$ corresponds to the vertex v. Let $\varphi \in \widetilde{\mathcal{E}}^{1,0}(\Gamma)$ be a $(1,0)$-form, $\varphi_e = \varphi_e(x)d'x$. We say that the form φ satisfies Kirchhoff's law at the vertex v if

$$\sum_{e \in E_v} \varphi_e(0) = 0.$$

The form φ satisfies Kirchhoff's law on the curve Γ if it satisfies Kirchhoff's law at every vertex of Γ of degree ≥ 2.

3. **Regularity at infinity.** We say that a superform ω is regular at infinity if for any degree one vertex v of Γ there is a neighborhood U of v such that the restriction of ω to U is a constant function if ω has degree $(0,0)$ and identically equal to zero otherwise, i.e., if ω has degrees $(1,0), (0,1), (1,1)$.

Thus a form is regular if is regular at infinity and, in addition to this, is continuous in case of $(0,0)$–forms, or satisfies the Kirchhoff's law in the case of $(1,0)$–form.

Remark 2.9. Very similar but slightly different definition of smooth tropical forms were introduced in [6]. It is more common to define tropical superforms on tropical varieties as a restriction of tropical superforms form $\mathbb{TR}^n$ to charts in tropical varieties, the idea is the same as in definition of smooth forms on a smooth manifold X as pullback of smooth forms on $\mathbb{R}^n$ via a embedding of X to $\mathbb{R}^n$, this approach was used in [8]. In [6, Proposition 4.3.2] it was shown that both approaches are equivalent.

Proposition 2.10. *The space of regular tropical superforms $\mathcal{E}^{*,*}(\Gamma)$ is closed under the wedge product and the d''-differential.*

The proof is straightforward.

Theorem 2.11 (Stokes' theorem)**.** *If $\omega \in \mathcal{E}^{1,0}(\Gamma)$, then $\int_\Gamma d''\omega = 0$. Consequently, if $\varphi \in \mathcal{E}^{p,0}(\Gamma)$ and $\psi \in \mathcal{E}^{1-p,0}(\Gamma)$, then*

$$\int_\Gamma d''\varphi \wedge \psi = (-1)^{p+1} \int_\Gamma \varphi \wedge d''\psi.$$

Here [8, Theorem 4.9], [6, Theorem 2.3.5] you can find practically the same theorem.

Proof. Let ω be an element of $\mathcal{E}^{1,0}(\Gamma)$. Let e be an edge of Γ and $\omega_e(x)d'x$ be a restriction of ω to e. Using the Newton-Leibniz formula we get $\int_e d''(\omega_e(x)d'x) = -\omega_e(0) + \omega_e(l(e))$. Since the integral over Γ is a sum of integrals over edges, combining the Newton-Leibniz formula, the Kirchhoff's law, and Regularity at infinity we obtain the first statement.

The second statement follows from the first statement and the Leibniz's rule. □

2.4 Tropical cohomologies

Remark 2.12. The notion of tropical homology was introduced in [7]. The de Rham (or Dolbeault) approach to tropical cohomology was developed in [8], in this subsection we repeat constructions from this article. This approach is a tropical rewriting of the standard constructions of differential topology.

Let U be an open set in Γ. We define $\mathcal{E}_\Gamma^{p,q}(U)$ as a linear space of smooth $(p,q)-$forms on U regular in the sense of Definition 2.8. The correspondence $U \to \mathcal{E}_\Gamma^{p,q}(U)$ defines the sheaf $\mathcal{E}_\Gamma^{p,q}$ of smooth tropical regular superforms over Γ, the space $\mathcal{E}_\Gamma^{p,q}(U)$ is the space of sections of $\mathcal{E}_\Gamma^{p,q}$ over U.

Let Ω_Γ^1 be the subsheaf of $d''-$closed forms the sheaf $\mathcal{E}_\Gamma^{1,0}$. Let us notice that $(1,0)-$form φ is $d''-$closed if its restriction to an edge e

$$\varphi_e(x)d'x$$

has a locally constant coefficient $\varphi_e(x)$.

Let us describe the sheaf Ω_Γ^1 more explicitly. Given a vertex v of Γ of degree $d \geq 2$. Consider a small $\varepsilon-$neighborhood U_ε of v. It is isometric to

$$U_\varepsilon = \bigsqcup_{d\text{-times}} (-\varepsilon, 0]/\sim, \tag{1}$$

where points 0 of different intervals are identified by the equivalence relation. The equivalence class of 0 is identified with the vertex v. A section of Ω_Γ^1 over U_ε is a collection of $(1,0)-$forms with constant coefficient

$$\varphi_j d'x,\ \varphi_j \in \mathbb{R},\ j = 1, \ldots, d,$$

where the form $\varphi_j d'x$ is defined on the $j-$th interval $(-\varepsilon, 0] \subset U_\varepsilon$, and the coefficients satisfy the Kirchhoff's law:

$$\sum_{j=1}^{d} \varphi_j = 0.$$

Suppose v is a degree 1 vertex its $\varepsilon-$neighborhood U_ε is isometric to $U_\varepsilon = [-\infty, -\varepsilon)$, since section of Ω^1_Γ are regular at infinity, this section are identically equal to zero on U_ε.

Let $\mathbb{R}_\Gamma$ be a subsheaf of locally constant functions on Γ of the sheaf $\mathcal{E}^{0,0}_\Gamma$. Sections of $\mathbb{R}_\Gamma$ are locally constant functions over edges satisfying the continuity property at vertices. Obviously, the subsheaf $\mathbb{R}_\Gamma$ coincides with the subsheaf of $d''-$closed functions the sheaf $\mathcal{E}^{0,0}_\Gamma$.

Remark 2.13. The sheaves $\mathbb{R}_\Gamma, \Omega^1_\Gamma, \mathcal{E}^{p,q}_\Gamma$ play in the tropical theory the same role as, correspondingly, the sheaves of holomorphic functions $\mathcal{O}_C$, holomorphic $1-$forms Ω^1_C, and smooth $(p, q)-$differential forms $\mathcal{E}^{p,q}_C$ on a smooth curve C in the complex case. The differential d'' play the same role as the $\overline{\partial}$ operator.

The space Ω^1_Γ was introduced in [10, Definition 2.25, Tropical 1-form]. In that paper it is related to degeneration of complex curves to tropical, and related degeneration of holomorphic forms on curves.

Proposition 2.14. *There are exact sequences of sheaves*

$$0 \to \Omega^1_\Gamma \xrightarrow{i} \mathcal{E}^{1,0}_\Gamma \xrightarrow{d''} \mathcal{E}^{1,1}_\Gamma \to 0,$$

$$0 \to \mathbb{R}_\Gamma \xrightarrow{i} \mathcal{E}^{0,0}_\Gamma \xrightarrow{d''} \mathcal{E}^{0,1}_\Gamma \to 0,$$

where i is the natural inclusion of subsheaves.

Proof. The map i is injective by definition. The kernel of d'' consists of forms with coefficients constant on edges. These are exactly forms either from Ω^1_Γ or from $\mathbb{R}_\Gamma$. Therefore $\operatorname{im} i = \ker d''$.

The surjectivity of d'' follows from the Newton-Leibniz formula. Let U_ε be an $\varepsilon-$neighborhood of a vertex v as defined in (1). Given a $(1, 1)-$form ω. Let

$$\omega_j(x) d'x \wedge d''x$$

be a component of ω over the j−th edge of U_ε. Let $\varphi_j(x) = \int_x^0 \omega_j(t)dt$ and $\varphi_j(x)d'x$ be a component of the $(1,0)$−form φ over j−th edge. The form φ is regular. Indeed, $\varphi_j(0) = 0$, therefore it satisfies the Kirchhoff's law at v. We have $d''\varphi = \omega$.

Let $U_\varepsilon = [-\infty, -\varepsilon)$ be an ε−neighborhood of a degree 1 vertex and $\omega(x)d'x \wedge d''x$ be a regular $(1,1)$−form on U_ε. Suppose

$$\varphi(x) = -\int_{-\infty}^{x} \omega(t)dt$$

then $\varphi(x)d'x$ is a regular form on U_ε. Indeed, since ω is regular at infinity it is zero at some neighborhood of $-\infty$. Hence the integral is convergent and $\varphi(x)$ is equal to zero at the same neighborhood of $-\infty$.

For an interior point of an edge there is a neighborhood isometric to a bounded interval $U_\varepsilon = (-\varepsilon, \varepsilon)$. The exactness of sequences over this neighborhood follows from the Newton-Leibniz formula. Thus we checked all possible cases and proved that in a neighborhood of any point $x \in \Gamma$ the operator d'' is surjective.

The proof of the surjectivity of d'' in the second sequences repeats the above arguments.

□

Let us define the bigraded cohomology group $H^{p,q}(\Gamma)$ of Γ as

$$H^{1,q}(\Gamma) = H^q(\Gamma, \Omega^1_\Gamma),$$

$$H^{0,q}(\Gamma) = H^q(\Gamma, \mathbb{R}_\Gamma).$$

Since $\mathbb{R}_\Gamma$ is the sheaf of locally constant functions, the group $H^{0,q}(\Gamma)$ is isomorphic to the usual topological cohomology group $H^q(\Gamma, \mathbb{R})$ of the graph Γ.

Proposition 2.15. *The sheaves $\mathcal{E}^{p,q}_\Gamma$ are fine and acyclic. There is an isomorphism $H^{p,q}(\Gamma) \cong H^q(\mathcal{E}^{p,*}(\Gamma), d'')$, where $H^q(\mathcal{E}^{p,*}(\Gamma), d'')$ is the cohomology group of the complex*

$$0 \to \mathcal{E}^{p,0}(\Gamma) \xrightarrow{d''} \mathcal{E}^{p,1}(\Gamma) \to 0.$$

Proof. The proof of this statement repeats the proof of acyclicity of the sheaf of smooth forms on a smooth manifold and the Čech to de Rham isomorphism on a smooth manifold.

For any open cover $\mathfrak{U}$ of Γ there is a smooth partition of unity for the sheaf of regular tropical $(0,0)$-superforms $\mathcal{E}_{\Gamma}^{0,0}$. Since the sheaf $\mathcal{E}_{\Gamma}^{p,q}$ is an $\mathcal{E}_{\Gamma}^{0,0}$–module, there is a partition of unity on it and $\mathcal{E}_{\Gamma}^{p,q}$ is a fine sheaf and, consequently, is acyclic.

Let $\mathfrak{U} = \{U_i\}$ be a finite acyclic open cover of Γ, i.e., $\mathfrak{U}$ such a cover that for any intersection U of elements of $\mathfrak{U}$ the sequences of section corresponding to the sequences (2.14) of sheaves are exact. Using the standard construction of the Čech to de Rham isomorphism we prove the proposition. □

2.5 Kähler form, inner product and Hodge star operator.

Let $g = g(x)d'x \wedge d''x \in \mathcal{E}^{1,1}(\mathbb{R})$ be a positive tropical $(1,1)$–superform over $\mathbb{R}$. We say that form is positive if $g(x) > 0$ for every $x \in \mathbb{R}$. Since $g(x)d'x \wedge d''x$ stands for $g(x)dx \otimes dx$, we can consider g as a Riemannian metric on $\mathbb{R}$. The Riemannian metric g defines the pointwise scalar product $(\varphi, \psi)_g(x)$ between elements of $\varphi, \psi \in \mathcal{E}^{p,q}(\mathbb{R})$. Indeed, we can consider elements of $\mathcal{E}^{p,q}(\mathbb{R})$ as tensor fields, a Riemannian metric defines the pointwise scalar product on tensor fields. Let us define the scalar product $(\varphi, \psi)_g$ between two forms $\varphi, \psi \in \mathcal{E}^{p,q}(\mathbb{R})$ as

$$(\varphi, \psi)_g = \int_{\mathbb{R}} (\varphi, \psi)_g(x) g,$$

where g is consider as a tropical $(1,1)$–form and the right-hand side is a tropical integral. At this moment we are not concerned with convergence of this integral. Usually we will omit subscript in $(\cdot,\cdot)_g$ and write $(\cdot,\cdot)$ instead.

Let us describe the scalar product in coordinate terms for the various p, q :

$$(f(x), h(x)) = \int_{\mathbb{R}} g(x) f(x) h(x) dx$$

$$(f(x)d'x, h(x)d'x) = (f(x)d''x, h(x)d''x) = \int_{\mathbb{R}} f(x)h(x)dx$$

$$(f(x)d'x \wedge d''x, h(x)d'x \wedge d''x) = \int_{\mathbb{R}} \frac{1}{g(x)} f(x)h(x)dx$$

The Hodge star operator

$$*_g \colon \mathcal{E}^{p,q}(\mathbb{R}) \to \mathcal{E}^{1-p,1-q}(\mathbb{R})$$

is defined by the relation

$$\int_{\mathbb{R}} \varphi \wedge *_g \psi = (\varphi, \psi)_g$$

for every $\varphi, \psi \in \mathcal{E}^{p,q}(\mathbb{R})$. Usually we will omit the subscript in $*_g$ and write $*$ instead.

The Hodge star is an isometry, i.e., for any $\varphi, \psi \in \mathcal{E}^{p,q}(\mathbb{R})$ holds

$$(\varphi, \psi) = (*\,\varphi, *\,\psi).$$

Also, for any $\psi \in \mathcal{E}^{p,q}(\mathbb{R})$ holds

$$** \,\psi = (-1)^{p+q}\psi.$$

In terms of coordinate the Hodge star looks as follows:

$$\begin{aligned} *\, f(x) &= f(x)g(x)d'x \wedge d''x, \\ *\, f(x)d'x \wedge d''x &= \frac{1}{g(x)} f(x), \\ *\, f(x)d'x &= f(x)d''x, \\ *\, f(x)d''x &= -f(x)d'x. \end{aligned} \tag{2}$$

Remark 2.16. In the differential geometry a Riemannian metric defines the standard scalar product on the space of sections of tensor fields. Since the form g is a symmetric tensor field of valence $(0,2)$ we can consider it as a Riemannian metric on $\mathbb{R}$. Also, we can consider the space $\mathcal{E}^{p,q}(\mathbb{R})$ as a space of tensor fields. Therefore, this Riemannian metric induces the standard scalar product on the space $\mathcal{E}^{p,q}(\mathbb{R})$, but this scalar product does not coincide with the tropical scalar product defined above.

Indeed, let $f(x), h(x)$ be functions on $\mathbb{R}$ then the standard scalar product on the space of functions equals

$$(f(x), h(x)) = \int_{\mathbb{R}} f(x)h(x)\sqrt{g(x)}dx.$$

On the space of 1−forms which can be identified with $\mathcal{E}^{1,0}(\mathbb{R})$ or $\mathcal{E}^{0,1}(\mathbb{R})$ the standard scalar is equal to

$$(f(x)dx, h(x)dx) = \int_{\mathbb{R}} f(x)h(x)\frac{1}{\sqrt{g(x)}}dx.$$

The reason for this is that the tropical superforms over $\mathbb{R}$ correspond to the usual differential form over $\mathbb{C} \setminus \{0\}$, not on $\mathbb{R}$, and the scalar product on the space of tropical superforms is consistent with the standard scalar product on $\mathbb{C} \setminus \{0\}$. This correspondence is described in the next subsection.

If we identify an edge e of Γ with an interval of $\mathbb{R}$, then a Kähler form on this interval defines the scalar product $(,)$ and the Hodge star operator on this edge of Γ.

Definition 2.17. A *Kähler form* g on the curve Γ is a $(1,1)-$form $g \in \widetilde{\mathcal{E}}^{p,q}(\Gamma)$ such that

1. g is positive, i.e., in local coordinates it is given by $g = g(x)d'x \wedge d''x$ with positive $g(x)$;
2. $\int_\Gamma g < +\infty$;
3. on any infinite length edge e the integral $\int_e x^2 g(x) d'x \wedge d''x$ converges.

We define the scalar product for $\varphi, \psi \in \widetilde{\mathcal{E}}^{p,q}(\Gamma)$ as follows:

$$(\varphi, \psi)_g = \int_\Gamma \varphi \wedge *_g \psi.$$

Remark 2.18. The last condition in the definition, convergence of $\int_e x^2 g(x) d'x \wedge d''x$, play its role in the study of L^2-theory in the next section. It allow us to get some estimates on convergence of various integrals. It is not clear for us what is necessary and sufficient condition here or how this condition can be weakened.

As Example 2.22 shows tropical Kähler forms that arises from the complex geometry are, actually, have rapidly decreasing at infinity coefficients, which is, actually, much higher rate of convergence that we required in the definition.

Remark 2.19. In the tropical setting a Kähler form g plays the role of a Kähler form and a hermitian metric in the classical complex geometry. Also the Kähler form g defines a Riemannian metric on each edge of Γ. Notice that, this Riemannian metric is unrelated to the metric structure on Γ, i.e., to the length of the edges. In general, the Hodge star operator does not preserve the regularity conditions, i.e., there is a regular form $\varphi \in \mathcal{E}^{p,q}(\Gamma)$ such that $*\varphi$ is not regular. This is important and unfortunate difference between the tropical and the classical settings. Indeed, the Hodge star of a smooth from on a manifold is again a smooth form.

We can summarize the results of this section as follows

Theorem 2.20. *Let* g *be a* Kähler form *on the curve* Γ. *Then* $\mathcal{E}^{p,q}(\Gamma)$ *is a differential bigraded algebra with the nondegenerate pairing*

$$\langle \cdot, \cdot \rangle : \mathcal{E}^{p,q}(\Gamma) \otimes \mathcal{E}^{1-p,1-q}(\Gamma) \to \mathbb{R},$$

$$\langle \varphi, \psi \rangle = \int_\Gamma \varphi \wedge \psi,$$

and the scalar product

$$(\varphi, \psi) = \int_\Gamma \varphi \wedge * \psi.$$

In this theorem the scalar product and and pairing are well-defined for all elements, i.e., all integrals are convergent. The convergence follows from the regularity at infinity condition for regular forms and the convergence of the tropical integral $\int_\Gamma g$ which is required by the definition of the Kähler form g.

2.6 Tropical superforms and complex geometry

At the first glance the tropical superforms and related objects may seem to be a bit artificial constructions. In this section we show that these objects can be naturally interpreted in terms of the classical complex geometry.

The real line $\mathbb{R}$ can be considered as a tropical analog of the complex torus $\mathbb{C}^*$. There is the map

$$\log|z| : \mathbb{C}^* \to \mathbb{R}$$

between them.

Let $\mathcal{E}^{p,q}(\mathbb{C}^*)$ be a space of smooth differential $\mathbb{C}$-valued forms of bidegree (p, q) over $\mathbb{C}^*$. Let us define the bigraded algebra homomorphism $\Theta : \mathcal{E}^{p,q}(\mathbb{R}) \to \mathcal{E}^{p,q}(\mathbb{C}^*)$. On the generators it is defined as follows

$$\Theta(\varphi(x)) = \varphi(\log|z|),\ \varphi(x) \in \mathcal{E}^{0,0}(\mathbb{R}),$$

$$\Theta(d'x) = \frac{1}{2\sqrt{\pi}}\frac{dz}{z},\ \Theta(d''x) = \frac{i}{2\sqrt{\pi}}\frac{d\overline{z}}{\overline{z}}.$$

Since $\mathcal{E}^{p,q}(\mathbb{R})$ is an $\mathbb{R}-$algebra, we consider Θ as an $\mathbb{R}-$algebra homomorphism.

Consider the unitary group $U(1) = \{t \in \mathbb{C} : |t| = 1\}$ and the standard action $U(1) \times \mathbb{C}^* \to \mathbb{C}^*$, i.e., $(t, z) \to t \cdot z$. This action induces action of $U(1)$ on $\mathcal{E}^{p,q}(\mathbb{C}^*)$. The image of $\mathcal{E}^{p,q}(\mathbb{R})$ under Θ is an $\mathbb{R}$-linear subspace in the space $\mathcal{E}^{p,q}_{U(1)}(\mathbb{C}^*)$ of $U(1)-$invariant forms. Its complexification $\Theta(\mathcal{E}^{p,q}(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}$ coincides with $\mathcal{E}^{p,q}_{U(1)}(\mathbb{C}^*)$.

Let $g(x)d'x \wedge d''x$ be a tropical Kähler form. One can check that its image

$$\Theta(g(x)d'x \wedge d''x) = \frac{i}{4\pi}g(\log|z|)\frac{dz \wedge d\overline{z}}{|z|^2}$$

is a Kähler form on $\mathbb{C}^*$. A Kähler form $\omega = \frac{i}{2}h(z)dz \wedge d\overline{z}$ determines the hermitian metric $h = h(z)dz \otimes d\overline{z}$ on $\mathbb{C}^*$. Thus for the Kähler form $\omega = \Theta(g(x)d'x \wedge d''x)$ the corresponding hermitian metric is

$$h = \frac{1}{2\pi}\frac{g(\log|z|)}{|z|^2}dz \otimes d\overline{z}.$$

Let $*_h$ be the Hodge star operator and $(,)_h$ be the scalar product on $\mathcal{E}^{p,q}(\mathbb{C}^*)$ corresponding to the metric h.

Proposition 2.21. *Suppose $\varphi, \psi \in \mathcal{E}^{p,q}(\mathbb{R})$, then the following relations hold:*

$$*_h\, \Theta = \Theta\, *_g,$$

$$(\Theta\varphi, \Theta\psi)_h = (\varphi, \psi)_g,$$

$$\Theta(d''\varphi) = \frac{i}{\sqrt{\pi}}\overline{\partial}\Theta(\varphi),$$

$$\Theta(d'\varphi) = \frac{1}{\sqrt{\pi}}\partial\Theta(\varphi).$$

Let $\omega \in \mathcal{E}^{1,1}(\mathbb{R})$, then

$$\int_{\mathbb{R}} \omega = \int_{\mathbb{C}^*} \Theta(\omega).$$

The tropical integral $\int_I \omega$ over an interval $I = (a, b)$ is equal to the integral of $\int_U \Theta(\omega)$ over the annulus $U = \{z \in \mathbb{C}^* : e^a < |z| < e^b\}$.

The proof is a straightforward computation.

Thus tropical superform can be reinterpreted as an $\mathbb{R}-$subalgebra of $U(1)-$invariant forms of the algebra $\mathcal{E}^{p,q}(\mathbb{C}^*)$.

Example 2.22. Let us consider the Fubini-Study metric and its Kähler form

$$\omega = \frac{i}{2\pi}\partial\overline{\partial}\log(1+|z|^2) = \frac{i}{2\pi}\frac{dz \wedge d\overline{z}}{(1+|z|^2)^2}$$

on $\mathbb{C}^* \subset \mathbb{CP}^1$.

There is the tropical form

$$\omega' = 2\frac{e^{2x}}{(1+e^{2x})^2}d'x \wedge d''x$$

such that $\Theta(\omega') = \omega$. Moreover, $\omega' = \frac{1}{2}d'd''\log(1+e^{2x})$ thus

$$\omega = \Theta(\frac{1}{2}d'd''\log(1+e^{2x})) = \frac{1}{2}\frac{1}{\sqrt{\pi}}\partial\frac{i}{\sqrt{\pi}}\overline{\partial}\log(1+|z|^2).$$

Since ω' satisfies all condition of Definition 2.17, we can consider ω' as a Kähler from on the tropical projective space $\mathbb{TP}^1 = [-\infty, +\infty]$.

Remark 2.23. The tropical form ω' from the example above is not a regular tropical form according to our definition of regularity since the regularity at infinity condition does not hold. On the other hand, the form $\Theta\omega'$ can be extended to a smooth form on $\mathbb{CP}^1$. Let us also notice that the coefficient $\frac{e^{2x}}{(1+e^{2x})^2}$ of the form ω' is a rapidly decreasing function on $\mathbb{R}$ in the sense of Schwartz space.

Moreover, since the coefficients of any Kähler form g a curve Γ are everywhere positive, a Kähler form g fails to be regular at infinity if there are infinite length edges on Γ. On the other hand, its coefficients should decrease fast enough near infinity since we require the convergence of the integral $\int_\Gamma g$.

Also, notice that the map

$$\log|z| : \mathbb{C}^* \to \mathbb{R}$$

can be extend to the map

$$\log|z| : \mathbb{C} \to \mathbb{R} \cup \{-\infty\}$$

These observations lead us to the idea to extend the notion of regularity at infinity as follows. We may call a tropical form $\varphi \in \mathcal{E}^{p,q}(\mathbb{R})$ regular at infinity if $\Theta\varphi \in \mathcal{E}^{p,q}(\mathbb{C}^*)$ can be extended to a smooth form over whole $\mathbb{C}$. This extension seems to be consistent but we do not develop this idea further in this paper.

3 L^2-theory, Laplace-Beltrami operator and harmonic form

In this section we prove the main statements of the paper. We introduce the notions of tropical superforms with L^2-coefficients, weak d''−differential, the Laplace-Beltrami operator, and harmonic tropical superforms. Main methods of this parts are in style of L^2−cohomology theory: functional analysis, unbounded differential operators, distributions, Sobolev spaces, various analytical estimations. For reference, the way how the Hodge on PL-manifolds was treated via L^2-cohomology theory in [11] is ideologically very close to our paper, [4, Chapter VIII] can be used as a general reference in L^2-methods in complex geometry.

Let us note that the main source of complications in our work is a treatment of infinite length edges which require to use some tedious analysis.

3.1 Tropical superforms with L^2-coefficients and weak d''−differential.

Let Γ be a tropical curve with a Kähler from g. Let us denote by $\mathcal{L}^{p,q}(\Gamma)$ the Hilbert space of (p,q)−form on Γ with L^2-coefficients with the scalar product $(\cdot,\cdot)_g$ defined in the subsection 2.5. This space is the metric completion of $\widetilde{\mathcal{E}}^{p,q}(\Gamma)$. Obviously, the space of regular form $\mathcal{E}^{p,q}(\Gamma)$ is a subspace of $\mathcal{L}^{p,q}(\Gamma)$, and $\mathcal{L}^{p,q}(\Gamma)$ is also the metric completion of $\mathcal{E}^{p,q}(\Gamma)$.

There is a continuous linear extension of the Hodge star operator $*$ from $\widetilde{\mathcal{E}}^{p,q}(\Gamma)$ to $\mathcal{L}^{p,q}(\Gamma)$ which we also denote by $*$. The Hodge star operator is an isometry between $\mathcal{L}^{p,q}(\Gamma)$ and $\mathcal{L}^{1-p,1-q}(\Gamma)$.

Definition 3.1. A form $\omega \in \mathcal{L}^{p,1}(\Gamma)$ is called the *weak d''–differential* of a form $\psi \in \mathcal{L}^{p,0}(\Gamma)$ if for any regular form $\varphi \in \mathcal{E}^{1-p,0}(\Gamma)$ holds

$$\int_\Gamma \omega \wedge \varphi = (-1)^{p+1} \int_\Gamma \psi \wedge d''\varphi. \tag{3}$$

We denote the weak d''–differential of a form ψ by $d''\psi$.

Obviously, the d''–differential of a regular form is also the weak d''–differential.

The weak d''–differential of a from ψ is unique if it exists. Indeed, suppose there is two such differentials ω_1 and ω_2. Then using (3) we obtain

$$\int_\Gamma (\omega_1 - \omega_2) \wedge \varphi = (-1)^{p+1} \int_\Gamma (\psi - \psi) \wedge d''\varphi = 0$$

.

$$\int_\Gamma (\omega_1 - \omega_2) \wedge \varphi = \pm \int_\Gamma ** (\omega_1 - \omega_2) \wedge \varphi = \pm(* (\omega_1 - \omega_2), \varphi) = 0$$

Since it holds for any $\varphi \in \mathcal{E}^{1-p,0}(\Gamma)$ and $\mathcal{E}^{1-p,0}(\Gamma)$ is dense in $\mathcal{L}^{1-p,0}(\Gamma)$, we get the equality $\omega_1 = \omega_2$.

Thus, there is the densely defined unbounded operator

$$d'' : \mathcal{L}^{p,0}(\Gamma) \to \mathcal{L}^{p,1}(\Gamma).$$

We denote its domain by $\mathcal{D}(d'')$ or by $\mathcal{D}^{p,0}(\Gamma)$.

In the sequel we denote the domain of an unbounded operator A by $\mathcal{D}(A)$.

3.2 Presheaves $\mathcal{L}^{p,q}$ and $\mathcal{D}^{p,0}$.

Definition 3.2. Restrictions of $\mathcal{L}^{p,q}(\Gamma)$ to an open subsets of Γ defines the presheaf $\mathcal{L}^{p,q}$ of (p, q)–superform with L^2-coefficients on Γ. Let us define the subpresheaf $\mathcal{D}^{p,0}$ of the presheaf $\mathcal{L}^{p,0}$. For an open subset $U \subset \Gamma$ the $(p, 0)$–form $\psi \in \mathcal{L}^{p,0}(U)$ belongs to $\mathcal{D}^{p,0}(U)$ if there is $\omega \in \mathcal{L}^{p,1}(U)$ such that for any $\varphi \in \mathcal{E}^{1-p,0}(\Gamma)$ with a compact support on U holds the equation (3).

Example 3.3. In the definition above the from φ has a compact support in the open set U. Let us clarify the structure of topology on the infinite edges and what is considered to be a compact support in that case. Consider the tropical projective space $\mathbb{TP}^1 = [-\infty, +\infty]$. Then, for example, $U = [-\infty, \infty)$ is an open subset of $\mathbb{TP}^1$, the set $[-\infty, a], a \in \mathbb{R}$ is a compact subset of U and the set $[a, +\infty), a \in \mathbb{R}$ is not compact.

Remark 3.4. We should warn that the sheafification of $\mathcal{L}^{p,q}$ is the sheaf $\mathcal{L}^{p,q}_{loc}$ of (p,q)–superform with locally L^2-coefficients on Γ, i.e., sections of $\mathcal{L}^{p,q}_{loc}(U)$ over an open set U are (p,q)–superform such that their coefficients are L^2-integrable functions over every compact set of U. Since Γ is compact, we have $\mathcal{L}^{p,q}_{loc}(\Gamma) = \mathcal{L}^{p,q}(\Gamma)$. To avoid complications related to the locally L^2-coefficients we will work with the presheaf $\mathcal{L}^{p,q}$.

In the other hand, $\mathcal{L}^{p,q}$ and $\mathcal{D}^{p,0}$ are almost sheaves, to be sheaves they have to satisfy Locality and Gluing axioms. Let us recall these axioms for a sheaf $\mathcal{F}$.

(Locality) Suppose U is an open set, $\{U_i\}_{i\in I}$ is an open cover of U, and $s, t \in \mathcal{F}(U)$ are sections. If $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.

(Gluing) Suppose U is an open set, $\{U_i\}_{i\in I}$ is an open cover of U, and $\{s_i \in \mathcal{F}(U_i)\}_{i\in I}$ is a family of sections. If all pairs of sections agree on the overlap of their domains, that is, if $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$, then there exists a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

Presheaves $\mathcal{L}^{p,q}, \mathcal{D}^{p,0}$ satisfy Locality axiom for any open cover and Gluing axiom only for finite covers. Indeed, if $\{U_i\}_{i\in I}$ is an infinite cover of U, then it may happen that the norms of restriction to each U_i are finite but the norm of the element on U is infinite, therefore it does not belong to $\mathcal{L}^{p,q}(U)$.

It is possible to glue sections of $\mathcal{D}^{p,0}$, because there is partition of unity in the space of regular tropical $(0,0)$–forms. Using that partition of unity one can check that the equation (3) holds for the glued section.

Indeed, suppose U is an open set, $\{U_i\}_{i\in I}$ is an open finite cover of U, and $\{\psi_i \in \mathcal{D}^{p,0}(U_i)\}_{i\in I}$ is a family of sections such that $\psi_i|_{U_i\cap U_j} = \psi_j|_{U_i\cap U_j}$ for all $i, j \in I$. Then there is a section $\psi \in \mathcal{L}^{p,0}(U)$ such that $\psi|_{U_j} = \psi_j$. Since forms $\omega_i = d''\psi_j \in \mathcal{L}^{p,1}(U_i)$ agree on the overlaps of their domains, we can glue them to the global form $\omega \in \mathcal{L}^{p,1}(U)$. Let us show that ψ is an element of $\mathcal{D}^{p,0}(U)$, that is, for any $\varphi \in \mathcal{E}^{1-p,0}(\Gamma)$ with a compact support on U holds:

$$\int_U \omega \wedge \varphi = (-1)^{p+1} \int_U \psi \wedge d''\varphi.$$

Let $\rho_i \in \mathcal{E}^{0,0}(\Gamma), i \in I$ be a partition of unity subordinate to the open cover $\{U_i\}_{i\in I}$. Then

$$\int_U \omega \wedge \varphi = \sum_{i\in I} \int_{U_i} \omega \wedge \rho_i\varphi =$$

because $\rho_i\varphi$ has a compact support on U_i, we get

$$= (-1)^{p+1} \sum_{i\in I} \int_{U_i} \psi \wedge d''(\rho_i\varphi) = (-1)^{p+1} \int_U \psi \wedge d''\varphi.$$

3.3 The main technical lemma.

Lemma 3.5. *Let $U \cong [-\infty, a)$ be an open neighborhood of a degree 1 vertex of Γ, the vertex is identified with the point $-\infty$. Given a form $\omega \in \mathcal{L}^{p,1}(U)$.*

1. *If $p = 0$ and $\omega = \omega(x)d''x$, let us define a $(0,0)$–form $\psi = \psi(x)$, i.e., a function as follows:*

$$\psi(x) = -\int_x^a \omega(t)dt.$$

The function ψ is well-defined and belongs to $\mathcal{L}^{0,0}(U)$, and the following estimates holds:

$$|\psi(x)| \leq \sqrt{a-x}||\omega(x)d''x||.$$

2. *If $p = 1$ and $\omega = \omega(x)d'x \wedge d''x$, let us define a $(1,0)$–form $\psi = \psi(x)d'x$ as follows:*

$$\psi(x) = -\int_{-\infty}^x \omega(t)dt.$$

The form ψ is well-defined and belongs to $\mathcal{L}^{0,1}(U)$, and the following estimates holds:

$$|\psi(x)| \leq \sqrt{\int_{-\infty}^x g(t)dt}||\omega(x)d'x \wedge d''x||.$$

3. *The map $\omega \to \psi$ is a bounded linear operator from $\mathcal{L}^{p,1}(U)$ to $\mathcal{D}^{p,0}(U)$ and $d''\psi = \omega$. Let us denote this operator T_U.*

4. *Suppose there is a form $\widetilde{\psi} \in \mathcal{D}^{p,1}(U)$ such that $d''\widetilde{\psi} = \omega$. Then, if $p = 1$, $\psi = \widetilde{\psi}$, and, if $p = 0$, $\psi = C + \widetilde{\psi}, C \in \mathbb{R}$.*

Remark 3.6. In other words, this lemma says that starting from a form $\omega \in \mathcal{L}^{p,1}(U)$ we can find a form $\psi \in \mathcal{L}^{p,1}(U)$ such that $d''\psi = \omega$. The value and the norm of ψ can be estimated using the norm of ω. Also, the form ψ is a unique form such that $d''\psi = \omega$ if $p = 1$, or unique up to addition of a constant if $p = 0$.

Proof. Let us consider the case of $(1,1)$–form. Let $I_{\leq x}$ be the indicator function of the set $[-\infty, x]$. Given a form $\omega = \omega(x)d'x \wedge d''x \in \mathcal{L}^{1,1}(U)$. Then the $(0,1)$–form $\psi = \psi(x)d'x$ is defined as follows:

$$\psi(x) = -\int_U I_{\leq x}(t)\omega(t)dt.$$

This integral is well-defined, indeed, it can be written using the scalar product on the space $\mathcal{L}^{1,1}(U)$:

$$\int_U I_{\leq x}(t)\omega(t)dt = -(\omega, I_{\leq x}g),$$

and the form $I_{\leq x}g$ is an element of $\mathcal{L}^{1,1}(U)$.

We have to show that $\psi \in \mathcal{L}^{1,0}(U)$. In particular, that

$$||\psi||^2 = \int_U \psi^2(x)dx < +\infty.$$

Recall that

$$||\omega||^2 = \int_U \frac{1}{g(x)}\omega^2(x)dx$$

Using the Cauchy-Schwarz inequality we get

$$|\psi(x)| = \left|\int_U I_{\leq x}(t)\omega(t)dt\right| = \left|\int_U \sqrt{g(t)}I_{\leq x}(t)\frac{1}{\sqrt{g(t)}}\omega(t)\psi\right| \leq ||\omega||\sqrt{\int_U g(t)I_{\leq x}(t)dt}.$$

So we obtained the estimation:

$$|\psi(x)| \leq ||\omega||\sqrt{\int_{-\infty}^{x} g(t)dt}.$$

Then

$$||\psi||^2 = \int_{-\infty}^{a} \psi^2(x)dx \le ||\omega||^2 \int_{-\infty}^{a} dx(\int_{-\infty}^{x} g(t)dt) =$$

changing the order of integration we get

$$= ||\omega||^2 \int_{-\infty}^{a} g(t)dt(\int_{t}^{a} dx) = ||\omega||^2 \int_{-\infty}^{a} (a-t)g(t)dt$$

By the definition of the Kähler metric (Definition 2.17) the integral $\int_{-\infty}^{a}(a-t)g(t)dt$ converges. Thus

$$||\psi|| \le C||\omega||$$

and the constant C does not depend on the choice of ω. So there is a bounded linear operator

$$T_U : \mathcal{L}^{1,1}(U) \to \mathcal{D}^{1,0}(U)$$

such that $T_U\omega = \psi$.

Let us check that $d''\psi$ is equal to ω. It means that for any for regular $(0,0)-$form φ with the compact support in U holds:

$$\int_U \omega \wedge \varphi = \int_U \psi \wedge d''\varphi.$$

Consider the right hand side of the equality

$$\int_U \psi \wedge d''\varphi = \int_U \psi(x)\varphi'(x)dx = \int_{-\infty}^{a} (-\int_{-\infty}^{x} \omega(t)dt)\varphi'(x)dx =$$

changing the order of integration we get

$$= \int_{-\infty}^{a} (-\int_{t}^{a} \varphi'(x)dx)\omega(t)dt =$$

by the Newton-Leibniz formula, since $\varphi(x)$ is equal to zero in a neighborhood of a, we obtain

$$= \int_{-\infty}^{a} \varphi(t)\omega(t)dt = \int_U \omega \wedge \varphi.$$

Thus, ψ belongs to $\mathcal{D}^{1,0}(U)$ and $d''\psi = \omega$.

Let us consider the case of $(0,1)$–forms. This case is quite similar to the previous one, but there are some minor differences. Given a form

$$\omega = \omega(x) d''x \in \mathcal{L}^{0,1}(U).$$

Let $I_{x\leq}$ be the indicator function of the set $[x,a)$. Consider the function

$$\psi(x) = -\int_U I_{x\leq}(t)\omega(t)dt.$$

This integral is well-define since it is equal to the scalar product of two elements in $\mathcal{L}^{0,1}(U)$:

$$\psi(x) = -(\omega, I_{x\leq}(t)d''x).$$

We have to show that $\psi \in \mathcal{L}^{0,0}(U)$, thus we need to check that

$$||\psi||^2 = \int_U g(x)\psi^2(x)dx < +\infty.$$

Since

$$||\omega||^2 = \int_U \omega^2(x)dx$$

using the Cauchy-Schwarz inequality we get

$$|\psi(x)| = \left|\int_U I_{x\leq}(t)\omega(t)dt\right| \leq ||\omega||\sqrt{\int_U I_{x\leq}(t)dt} = ||\omega||\sqrt{a-x}.$$

Then

$$||\psi||^2 = \int_U g(x)\psi^2(x)dx \leq ||\omega|| \int_{-\infty}^{a} (a-x)g(x)dx$$

By the definition of Kähler metric (Definition 2.17) the integral $\int_{-\infty}^{a}(a-x)g(x)dx$ converges. Thus

$$||\psi|| \leq C||\omega||$$

and the constant does not depends on the choice of ω. So there is a bounded linear operator

$$T_U : \mathcal{L}^{0,1}(U) \to \mathcal{D}^{0,0}(U)$$

such that $T_U\omega = \psi$.

Let us check that $d''\psi$ is equal to ω. It means that for any regular $(1,0)-$form $\varphi = \varphi(x)d'x$ with the compact support in U holds:

$$\int_U \omega \wedge \varphi = -\int_U \psi \wedge d''\varphi.$$

Consider the right hand side of the equality:

$$-\int_U \psi \wedge d''\varphi = -\int_{-\infty}^{a} (-\int_{x}^{a} \omega(t)dt)(-\varphi'(x))dx =$$

changing the order of integration we obtain

$$= -\int_{-\infty}^{a} (\int_{-\infty}^{t} \varphi'(x)dx)\omega(t)dt =$$

Since φ is regular it equal to zero in a neighborhood of $-\infty$. Thus, applying the Newton-Leibniz formula we obtain

$$= -\int_{-\infty}^{a} \varphi(t)\omega(t)dt = \int_U \omega \wedge \varphi.$$

Thus, ψ belongs to $\mathcal{D}^{0,0}(U)$.

Suppose there is a form $\widetilde{\psi} \in \mathcal{D}^{p,0}(U)$ such that $d''\widetilde{\psi} = \omega$. Then $d''(\widetilde{\psi} - \psi) = 0$, hence the coefficient of $\widetilde{\psi} - \psi$ should be constant, i.e., $\widetilde{\psi} - \psi$ is equal, if $p = 1$, to $cd''x$, or, if $p = 0$, to c where $c \in \mathbb{R}$. In the case $p = 1$,

$$||cd''x||^2 = \int_{-\infty}^{a} c^2 dx = +\infty,$$

hence $c = 0$ and $\widetilde{\psi} = \psi$. In the case $p = 0$, the integral

$$||c||^2 = \int_{-\infty}^{a} c^2 g(x)dx$$

converges by the definition of the Kähler form (Definition 2.17) and c can be any real number.

□

3.4 Relation to the Sobolev space.

Lemma 3.7. *Suppose U is an open subsets of Γ and it is isometric to an open interval of the finite length $U \cong (a,b)$. Given a form $\psi \in \mathcal{D}^{p,0}(U)$, in terms of coordinates it is equal either to $\psi = \psi(x)d'x$ or to $\psi = \psi(x)$. Then the coefficient $\psi(x)$ belongs to the Sobolev space* $H^1(U) = H^1(a,b)$.

Proof. Suppose $\psi \in \mathcal{D}^{0,0}(U)$. Let $\omega = \omega(x)d''x = d''\psi$. Then the norms of these elements are equal:

$$||\psi||^2 = \int_a^b \psi^2(x)g(x)dx,$$

$$||\omega||^2 = \int_a^b \omega^2(x)dx.$$

Since $g(x)$ is a nonnegative continuous function on the closure of U, there are constants $0 < c, C$ such that $c < g(x) < C$ for any $x \in U$. Therefore, the norm on $\mathcal{L}^{0,0}(U)$ is equivalent to the standard norm on $L^2(a,b)$:

$$c\int_a^b \psi^2(x)dx < ||\psi||^2 < C\int_a^b \psi^2(x)dx.$$

The norm of the $(1,1)$–form $\omega \in \mathcal{L}^{0,1}(U)$ is equal to the standard norm on $L^2(a,b)$ of its coefficient the function $\omega(x)$.

Consider equation (3):

$$\int_U \omega \wedge \varphi = -\int_U \psi \wedge d''\varphi.$$

Since $\psi \in \mathcal{D}^{0,0}(U)$, it holds for any regular from $\varphi = \varphi(x)d'x$ with a compact support on U. Therefore, the coefficient $\varphi(x) \in C_0^\infty(U)$ is a smooth function with a compact support on the interval $U = (a,b)$. In the terms of coefficient the equation looks like:

$$\int_a^b \omega(x)\varphi(x) = -\int_a^b \psi(x)\varphi'(x),$$

where $\psi(x), \omega(x) \in L^2(a,b)$, and $\varphi(x) \in C_0^\infty(a,b)$. It is exactly the definition of the Sobolev space, hence $\omega(x)$ is the weak derivative of $\psi(x)$, and the function $\psi(x)$ belongs to the Sobolev space $H^1(U)$. □

Corollary 3.8. *Given a form $\psi \in \mathcal{D}^{p,0}(\Gamma)$ and a vertex v of Γ of degree ≥ 2. There are well-defined boundary values of the coefficients of ψ at v along the edges incident to v.*

Proof. Let e be an edge of Γ incident to v. Let U be a finite open interval such that $U \subset e$ and the closure of U contains v. Then by Lemma 3.7 the coefficient of the restriction $\psi|_U$ is a function from the Sobolev space $H^1(U)$, and, therefore, has a well-defined trace at the boundary of U, in particular, at v. □

3.5 d'' is a closed.

Proposition 3.9. *The operator d'' is a closed operator.*

Proof. An operator is closed if its graph is closed. Suppose $\omega_n \in \mathcal{D}^{p,0}(d'')$, $\omega_n \to \omega$ in $\mathcal{L}^{p,0}(\Gamma)$ and $d''\omega_n \to \psi$ in $\mathcal{L}^{p,1}(\Gamma)$. Then the relation

$$\int_\Gamma d''\omega_n \wedge \varphi = (-1)^{p+1} \int_\Gamma \omega_n \wedge d''\varphi$$

holds for any $\varphi \in \mathcal{E}^{1-p,0}(\Gamma)$.

Since $**= (-1)^{p+q}\mathrm{Id}$ on the space of (p,q)–forms and $(\alpha,\beta) = \int_\Gamma \alpha \wedge * \beta$, we get

$$\int_\Gamma d''\omega_n \wedge \varphi = (-1)^{p-1} \int_\Gamma d''\omega_n \wedge ** \varphi = (-1)^{p-1}(d''\omega_n, * \varphi),$$

and

$$(-1)^{p+1} \int_\Gamma \omega_n \wedge d''\varphi = - \int_\Gamma \omega_n \wedge ** d''\varphi = -(\omega_n, * d''\varphi).$$

Thus

$$(-1)^p(d''\omega_n, * \varphi) = (\omega_n, * d''\varphi).$$

Taking limit as $n \to \infty$ we get

$$(-1)^p(\psi, * \varphi) = (\omega, * d''\varphi).$$

This equation is equivalent to

$$\int_\Gamma \psi \wedge \varphi = (-1)^{p+1} \int_\Gamma \omega \wedge d''\varphi.$$

Hence, ψ is the weak d''–differential of ω. □

3.6 L^2-cohomology and the Čech to de Rham isomorphism.

Let $H^q(\mathcal{L}^{p,*}(\Gamma), d'')$ be the cohomology group of the complex

$$0 \to \mathcal{D}^{p,0}(\Gamma) \to \mathcal{L}^{p,1}(\Gamma) \to 0,$$

where $\mathcal{D}^{p,0}(\Gamma) \subset \mathcal{L}^{p,0}(\Gamma)$ is the domain of the operator d''.

Theorem 3.10. *For a sufficiently small neighborhood U of a point $x \in \Gamma$ there are exact sequences:*

$$\begin{aligned} 0 \to \mathbb{R}_\Gamma(U) \xrightarrow{i} \mathcal{D}^{0,0}(U) \xrightarrow{d''} \mathcal{L}^{0,1}(U) \to 0, \\ 0 \to \Lambda^1_\Gamma(U) \xrightarrow{i} \mathcal{D}^{1,0}(U) \xrightarrow{d''} \mathcal{L}^{1,1}(U) \to 0, \end{aligned} \tag{4}$$

where i is a natural inclusion of subpresheaves.

Proof. Firstly, we will prove the following statement.

Lemma 3.11. *Let U be a sufficiently small neighborhood of a point in Γ then there is a bounded operator*

$$T_U : \mathcal{L}^{p,1}_\Gamma(U) \to \mathcal{D}^{p,0}_\Gamma(U)$$

such that $d''T_U = \mathrm{Id}$.

Proof. To prove the statement we will consider several distinct cases: the neighborhood U can be a neighborhood of a degree 1 vertex, or of a degree $n \geq 2$ vertex, or of an internal point of an edge; p can be equal to 0 or 1.

Let $U \simeq [-\infty, a)$ be a neighborhood of a degree 1 vertex. We identify $-\infty$ with this vertex. In this case the required operator $T_U : \mathcal{L}^{p,1}_\Gamma(U) \to \mathcal{D}^{p,0}_\Gamma(U)$ was constructed in Lemma 3.5.

Given a vertex v of Γ of degree $d \geq 2$. Consider a neighborhood U of v :

$$U = \bigsqcup_{d\text{-times}} (-a, 0]/\sim,$$

where points 0 of the different intervals are all identified by the equivalence relation $\sim$. The class of 0 is identified with the vertex v.

Given a form $\omega \in \mathcal{L}^{1,1}(U)$. Suppose $e_i \simeq (-a, 0] \subset U, i = 1, \dots, d$ is the i–th edge of U and $\omega_i = \omega_i(x) d'x \wedge d''x$ is the restriction of ω to e_i. Let us define $\psi = T_U \omega$, where $\psi_i = \psi_i(x) d'x$ is the restriction of ψ to e_i, as follows:

$$\psi_i(x) = \int_x^0 \omega_i(t) dt.$$

These functions $\psi_i(x)$ are well-defined since they can be expressed as a scalar product

$$\psi_i(x) = (\omega, I_{i,x} g),$$

where $\omega, I_{i,x} g \in \mathcal{L}^{1,1}(U)$, and $I_{i,x}$ is the indicator function of the set $[x, 0] \subset e_i$.

Since $\psi_i(0) = 0, i = 1, \dots, d$, the Kirchhoff's law holds for ψ. Using the same arguments as in Lemma 3.5 we can check that $\psi \in \mathcal{D}^{1,0}(U)$, $d''\psi = \omega$, and T_U is bounded.

Finally, if $\omega \in \mathcal{L}^{0,1}(U)$, and $\omega_i = \omega_i(x) d''x$ is the restriction of ω to e_i. Let us define $\psi = T_U \omega$, where $\psi_i = \psi_i(x)$ is the restriction of ψ to e_i, as follows:

$$\psi_i(x) = -\int_x^0 \omega_i(t) dt.$$

Since $\psi_i(0) = 0$, the Continuity property hold at the vertex v. Again, using the same arguments as above we can check that $\psi \in \mathcal{D}^{0,0}(U)$, $d''\psi = \omega$, and T_U is bounded.

The case of a neighborhood of an internal point of an edge is equivalence to the case of a neighborhood of degree 2 vertex. □

Let U be a sufficiently small neighborhood of a point in Γ. The kernel of $d'' : \mathcal{D}_\Gamma^{p,0}(U) \to \mathcal{L}_\Gamma^{p,1}(U)$ coincides with $\mathbb{R}_\Gamma(U)$ or $\Lambda_\Gamma^1(U)$. By Lemma 3.11 the map $d'' : \mathcal{D}_\Gamma^{0,0}(U) \to \mathcal{L}_\Gamma^{0,1}(U)$ is surjective. Therefore, the sequences (4) are exact. □

Proposition 3.12. *There is an isomorphism*

$$H^{p,q}(\Gamma) \cong H^q(\mathcal{L}^{p,*}(\Gamma), d'').$$

Proof. Using the exact sequences (4) we can repeat the proof of the Proposition 2.15. □

3.7 Integration by parts for weakly d''−differentiable forms.

Proposition 3.13. *If $\psi \in \mathcal{D}^{0,0}(\Gamma)$ and $\varphi \in \mathcal{D}^{1,0}(\Gamma)$ are two weakly d''−differentiable froms, then the equation of integration by parts holds:*

$$\int_\Gamma d''\psi \wedge \varphi = -\int_\Gamma \psi \wedge d''\varphi. \tag{5}$$

Proof. Firstly, let us notice that both integrals in (5) are well-defined, i.e., convergent. Indeed, consider the integral $\int_\Gamma d''\psi \wedge \varphi$, using the property of the Hodge star $**= \pm \mathrm{Id}$ we can rewrite it as $-\int_\Gamma d''\psi \wedge ** \varphi$. Thus, it is equal to $-(d''\psi, * \varphi)$. Since $d''\psi \in \mathcal{L}^{0,1}(\Gamma)$ and the Hodge star is, in this case, is an isomorphisms between $\mathcal{L}^{1,0}(\Gamma)$ and $\mathcal{L}^{0,1}(\Gamma)$, this scalar product is well-defined and, consequently, the integral is well-defined. We can apply the same argument for the second integral.

Let us choose a function $\rho_0 \in \mathcal{E}^{0,0}(\Gamma)$ such that

1. its values between 0 and 1;
2. it is equal to 1 on each finite-length edge and in a neighborhood of any vertex of degree ≥ 2;
3. it is equal to 0 on a neighborhood of any vertex of degree 1, i.e., in neighborhoods of infinite tails of the tropical curve.

Let us denote $\rho_1 = 1 - \rho_0$. These two function, ρ_0, ρ_1 give us a partition of unity such that one of them is nonzero on a finite part of the curve another on the infinite tails.

Consider the integral

$$\int_\Gamma \psi \wedge d''\varphi = \int_\Gamma \rho_0 \psi \wedge d''\varphi + \int_\Gamma \rho_1 \psi \wedge d''\varphi.$$

The support of ρ_0 is a union of finite-length edges and compact parts of infinite-length edges. Using Lemma 3.7 we obtain that ψ and φ has H^1−coefficients in a neighborhood of $\operatorname{supp} \rho_0$. For the H^1−function we can apply integration by parts and the boundary terms at vertices vanish by the same reasons as in Stokes' Theorem (Theorem 2.11).

The second integral $\int_\Gamma \rho_1\psi\wedge d''\varphi$ is a sum of integrals over infinite-length edges. Suppose an infinite-length edge e is isomorphic to $e \cong [-\infty, 0]$, the function ρ_1 is equal to 0 at a neighborhood of the point 0 and equal to 1 at a neighborhood of $-\infty$. From this moment let write $\overline{\psi}$ instead of $\rho_1\psi$. We are going to prove that

$$\int_{[-\infty,0]} d''\overline{\psi} \wedge \varphi = -\int_{[-\infty,0]} \overline{\psi} \wedge d''\varphi.$$

Then we can take a sum over all infinite-length edges this will prove the statement of the proposition.

The tropical integrals by definition equals:

$$\int_e \overline{\psi} \wedge d''\varphi = -\int_{-\infty}^{0} \overline{\psi}(t)\varphi'(t)dt$$

and

$$\int_e d''\overline{\psi} \wedge \varphi = -\int_{-\infty}^{0} \overline{\psi}'(t)\varphi(t)dt.$$

Therefore, we have to show that

$$\int_{-\infty}^{0} \overline{\psi}(t)\varphi'(t)dt = -\int_{-\infty}^{0} \overline{\psi}'(t)\varphi(t)dt. \tag{6}$$

Let us split both parts of the equality to sums of integrals:

$$\int_x^0 \overline{\psi}(t)\varphi'(t)dt + \int_{-\infty}^{x} \overline{\psi}(t)\varphi'(t)dt = -\int_x^0 \overline{\psi}'(t)\varphi(t)dt - \int_{-\infty}^{x} \overline{\psi}'(t)\varphi(t)dt,$$

where $x \in (-\infty, 0]$. Since our initial integrals are convergent, we have

$$\lim_{x\to-\infty} \int_{-\infty}^{x} \overline{\psi}(t)\varphi'(t)dt = 0$$

and

$$\lim_{x\to-\infty} \int_{-\infty}^{x} \overline{\psi}'(t)\varphi(t)dt = 0.$$

By Lemma 3.7 the restrictions of $\overline{\psi}(x), \varphi(x)$ to any interval $(x, 0) \subset [-\infty, 0] \cong e, x \in \mathbb{R}$ are functions form the Sobolev space $H^1(x, 0)$. Thus, we can apply integration by parts on $(x, 0)$:

$$\int_x^0 \overline{\psi}(t)\varphi'(t)dt = \overline{\psi}(t)\varphi(t)|_x^0 - \int_x^0 \overline{\psi}'(t)\varphi(t)dt.$$

Consider the term $\overline{\psi}(t)\varphi(t)|_x^0 = \overline{\psi}(0)\varphi(0) - \overline{\psi}(x)\varphi(x)$. The first summand $\overline{\psi}(0) = \rho_1(0)\psi(0) = 0$ is equal to zero. We are going to show that

$$\lim_{x\to-\infty} \overline{\psi}(x)\varphi(x) = 0.$$

Consider the forms $\omega = d''\varphi$ and $\tau = d''\overline{\psi}$. In the local coordinates we have:

$$\overline{\psi} = \overline{\psi}(x),\ \varphi = \varphi(x)d'x,\ \omega = \omega(x)d'x \wedge d''x,\ \tau = \tau(x)d''x.$$

Suppose $U = [-\infty, 0) \subset e$. The operator

$$T_U : \mathcal{L}^{p,1}_\Gamma(U) \to \mathcal{D}^{p,0}_\Gamma(U)$$

was defined in Lemma 3.11. It has the following properties $d''T_U = \mathrm{Id}$, $T_U d''\varphi = T_U\omega = \varphi$, and $T_U d''\overline{\psi} = T_U\tau = \overline{\psi} + c$ where c is some constant. By the definition of T_U we get:

$$\overline{\psi}(x) + c = T_U\tau = -\int_x^0 \tau(t)dt,$$

and

$$\varphi(x)d'x = T_U\omega = -(\int_{-\infty}^x \omega(t)dt)d'x.$$

Since $\overline{\psi}(0) = \rho_1(0)\psi(0) = 0$ and $\psi(0) + c = \int_0^0 \tau(t)dt$, the constant c is equal to 0, and $\overline{\psi} = T_U\tau$.

The the following estimates was proven for T_U :

$$|\varphi(x)| \le ||\omega||\sqrt{\int_{-\infty}^x g(t)dt},$$

$$|\overline{\psi}(x)| \le ||\tau||\sqrt{|x|}$$

By definition of Kähler metric (Definition 2.17) the integral $\int_{-\infty}^0 t^2 g(t)dt$ converges which is equivalent to

$$\lim_{x\to-\infty} \int_{-\infty}^x t^2 g(t)dt = 0.$$

For $t \le x < 0$ we have $x^2 g(t) \le t^2 g(t)$ and

$$x^2 \int_{-\infty}^x g(t)dt \le \int_{-\infty}^x t^2 g(t)dt$$

which is equivalent to

$$\int_{-\infty}^{x} g(t)dt \le \frac{1}{x^2}\int_{-\infty}^{x} t^2 g(t)dt.$$

Finally, combining all our estimates we get

$$|\varphi(x)\overline{\psi}(x)| \le ||\omega|| \cdot ||\tau||\sqrt{|x|}\sqrt{\int_{-\infty}^{x} g(t)dt} \le ||\omega|| \cdot ||\tau||\frac{1}{\sqrt{|x|}}\int_{-\infty}^{x} t^2 g(t)dt.$$

The right hand side tends to 0 as x tends to $-\infty$. □

3.8 The adjoint of d''.

Let us consider the adjoint operator

$$d''^* : \mathcal{L}^{p,1}(\Gamma) \to \mathcal{L}^{p,0}(\Gamma).$$

By definition, $d''^*\omega = \psi$ if for any $\varphi \in \mathcal{D}^{p,0}(\Gamma)$ holds

$$(d''\varphi, \omega) = (\varphi, \psi).$$

Proposition 3.14. *The adjoint operator*

$$d''^* : \mathcal{L}^{p,1}(\Gamma) \to \mathcal{L}^{p,0}(\Gamma)$$

is densely defined and closed. It is equal to

$$d''^* = - * d * .$$

In particular, $\psi \in \mathcal{D}(d''^*)$ *if and only if*

$$* \psi \in \mathcal{D}^{1-p,0}(\Gamma).$$

Its adjoint d''^{**} *equals* d''.

Proof. From the general properties of unbounded operators follows that the adjoint A^* of a closed densely defined operator A is a closed densely defined operator and its adjoint A^{**} equals A, [1, Theorem 3.1 and Theorem 3.3]. Thus we have to prove that $d''^* = - * d *$.

If $A : H_1 \to H_2$ is an unbounded operator with a domain $\mathcal{D}(A) \subset H_1$ then its adjoint A^* is an unbounded operator $A^* : H_1 \to H_2$ such that for any $x \in \mathcal{D}(A) \subset H_1$ holds

$$(x, A^*y)_{H_1} = (Ax, y)_{H_2},$$

the domain of A^* is a maximal subspace of elements in H_2 satisfying that relation, i.e.,

$$\mathcal{D}(A^*) = \{x \in H_2 : \exists z \in H_1, \forall y \in \mathcal{D}(A) : (y, z)_{H_1} = (Ay, x)_{H_2}\}.$$

Suppose a form $\psi \in \mathcal{L}^{p,1}(\Gamma)$ is in the domain of d''^*, $\psi \in \mathcal{D}(d''^*)$ and $\omega = d''^*\psi$. Then by the definition of the adjoint operator for any form $\varphi \in \mathcal{D}^{p,0}(\Gamma)$ the following equality holds:

$$(\varphi, \omega) = (d''\varphi, \psi).$$

We can rewrite it as follows:

$$\int_\Gamma \varphi \wedge * \omega = \int_\Gamma d''\varphi \wedge * \psi.$$

Since $\mathcal{E}^{p,0}(\Gamma)$ is a subspace of $\mathcal{D}^{p,0}(\Gamma)$ this equality hold for any regular form $\varphi \in \mathcal{E}^{p,0}(\Gamma)$. Hence, by the definition, $* \psi$ is $d''-$weakly differentiable form and its differential is equal to

$$d'' * \psi = (-1)^{p+1} * \omega.$$

Applying the Hodge star operator to both parts of the previous equality we get

$$* d'' * \psi = (-1)^{p+1} ** \omega = -\omega.$$

Hence

$$d''^*\psi = \omega = - * d'' * \psi.$$

At this moment we proved that if $\psi \in \mathcal{D}(d''^*)$ then $* \psi$ is $d''-$weakly differentiable. Let us prove the converse: if $* \psi$ is $d''-$weakly differentiable then $\psi \in \mathcal{D}(d''^*)$. By Proposition 3.13, we can integrate by parts a product of two $d''-$weakly differentiable forms, i.e., if form $* \psi$ is $d''-$weakly differentiable, then for any form $\varphi \in \mathcal{D}^{p,0}(\Gamma)$ holds:

$$\int_\Gamma d''\varphi \wedge * \psi = (-1)^{p+1} \int_\Gamma \varphi \wedge d'' * \psi = \int_\Gamma \varphi \wedge * (- * d'' * \psi).$$

That equality can be written as

$$(d''\varphi,\psi) = (\varphi, - * d'' * \psi).$$

Thus we proved that the domain of $\mathcal{D}(d''^*)$ coincides with the space of forms such that theirs Hodge stars are $d''-$weakly differentiable. □

3.9 The Laplace-Beltrami operator and harmonic tropical superforms.

Let us define the *Laplace-Beltrami operator* as follows

$$\Delta = d''d''^* + d''^*d'' : \mathcal{L}^{p,q}(\Gamma) \to \mathcal{L}^{p,q}(\Gamma).$$

It's domain equals

$$\mathcal{D}(\Delta) = \{\omega \in \mathcal{L}^{p,q}(\Gamma) : \omega \in \mathcal{D}(d''^*), \omega \in \mathcal{D}(d''), (d''\omega) \in \mathcal{D}(d''^*), (d''^*\omega) \in \mathcal{D}(d'')\}.$$

By the dimensional reasons one of the summands in Δ is identically equal to zero, so Δ is either equals $\Delta = d''d''^*$ or $\Delta = d''^*d''$.

Proposition 3.15. *Let ω be an element of $\mathcal{L}^{p,q}(\Gamma)$, then $\Delta\omega = 0$ if and only if $d''\omega = 0$ and $d''^*\omega = 0$.*

Proof. Suppose $\Delta\omega = 0$. If $\omega \in \mathcal{D}(\Delta)$, then $\omega \in \mathcal{D}(d''^*) \cap \mathcal{D}(d'')$. By Proposition 3.14 we have $d''^{**} = d''$. From the definition of an adjoint operator we get

$$0 = (\Delta\omega, \omega) = (d''d''^* + d''^*d''\omega, \omega) = (d''^*\omega, d''^*\omega) + (d''\omega, d''^{**}\omega) = (d''^*\omega, d''^*\omega) + (d''\omega, d''\omega).$$

Hence $||d''^*\omega|| = 0, ||d''\omega|| = 0$, and $d''^*\omega = d''\omega = 0$. The converse follows directly from the definition of the Laplace-Beltrami operator. □

Definition 3.16. Let us denote the kernel of the Laplace-Beltrami operator $\Delta : \mathcal{L}^{p,q}(\Gamma) \to \mathcal{L}^{p,q}(\Gamma)$ by $\mathcal{H}^{p,q}(\Gamma)$. We call this space $\mathcal{H}^{p,q}(\Gamma)$ the space of *harmonic tropical superform* of degree (p,q) on Γ.

By Proposition 3.15 any harmonic superform is closed, hence there is the map $i : \mathcal{H}^{p,q}(\Gamma) \to H^{p,q}(\Gamma)$, that maps any harmonic form to its class in the cohomology group.

Proposition 3.17. *The map* $i : \mathcal{H}^{p,q}(\Gamma) \to H^{p,q}(\Gamma)$ *is an isomorphism.*

Proof. Let ω be an element of $\mathcal{D}^{p,0}(\Gamma)$. By Proposition 3.15, $d''\omega = 0$ if and only if $\Delta\omega = 0$. Thus, $H^{p,0}(\Gamma) = \ker d'' = \mathcal{H}^{p,0}(\Gamma)$.

Lemma 3.18. *The range* $\operatorname{im} d''$ *of* d'' *is closed.*

Proof. Let $\mathfrak{U} = \{U_i\}_i$ be a cover of Γ. Let us denote $U_{ij} = U_i \cap U_j$.

Since Γ is compact, we may choose $\mathfrak{U}$ in such a way that:

1. $\mathfrak{U}$ is finite cover;
2. the sequences of sections (4) are exact over any U_i and $U_{ij} = U_i \cap U_j$;
3. there is bounded operator T_U as in Lemma 3.11 for any U_i and $U_{ij} = U_i \cap U_j$.

Let $C_i(\mathcal{S})$ be the Čech complex of a presheaf $\mathcal{S}$ and the cover $\mathfrak{U}$ with the differential δ. In particular,

$$C_0(\mathcal{S}) = \bigoplus_i \mathcal{S}(U_i),\ C_1 = \bigoplus_{i<j} \mathcal{S}(U_{ij}),$$

and $\delta : C_0(\mathcal{S}) \to C_1(\mathcal{S})$.

Since $C_i(\mathcal{L}^{p,q})$ is a direct sum of $\mathcal{L}^{p,q}(U)$ it has a structure of a Hilbert space induced from the summands. Then δ is a continuous linear operator. The kernel $\ker\delta : C_0(\mathcal{L}^{p,q}) \to C_1(\mathcal{L}^{p,q})$ coincides with $\mathcal{L}^{p,q}(\Gamma)$. Actually, the norm on this kernel does not coincide with the norm on $\mathcal{L}^{p,q}(\Gamma)$, but these two norms are equivalent. Since $\mathcal{L}^{p,q}(\Gamma)$ is the kernel of a bounded operator, it is a closed subspace of $C_0(\mathcal{L}^{p,q})$. We will consider $\mathcal{L}^{p,q}(\Gamma)$ as a subspace of $C_0(\mathcal{L}^{p,q})$

The bounded operator $T_U : \mathcal{L}^{p,1}(U) \to \mathcal{L}^{p,0}(U)$ was defined in Lemma 3.11. Let $T :$ $C_0(\mathcal{L}^{p,1}) \to C_0(\mathcal{L}^{p,0})$ be a direct sum of T_{U_i}. The composition of operators $\delta T : \mathcal{L}^{p,1}(\Gamma) \to$

$C_1(\mathcal{L}^{p,0})$ is a continuous linear operator. Combining the facts that $d''T\varphi = \varphi$, the operator δ commutes with d'', and $\ker\delta = \mathcal{L}^{p,q}(\Gamma)$, for any $\omega \in \mathcal{L}^{p,1}(\Gamma)$ we get

$$d''\delta T\omega = \delta d''T\omega = \delta\omega = 0.$$

Therefore δT is a bounded operator from $\mathcal{L}^{p,1}(\Gamma)$ to the kernel of

$$d'' : C_1(\mathcal{L}^{p,0}) \to C_1(\mathcal{L}^{p,1}),$$

which is equal to either $C_1(\mathbb{R}_\Gamma)$ or $C_1(\Lambda^1_\Gamma)$. Both spaces $C_1(\mathbb{R}_\Gamma)$ and $C_1(\Lambda^1_\Gamma)$ are finite dimensional. There are the quotient maps

$$\varepsilon : C_1(\mathbb{R}_\Gamma) \to H^1(\Gamma, \mathbb{R}_\Gamma) = C_1(\mathbb{R}_\Gamma)/\delta C_0(\mathbb{R}_\Gamma)$$

and

$$\varepsilon : C_1(\Lambda^1_\Gamma) \to H^1(\Gamma, \Lambda^1_\Gamma) = C_1(\Lambda^1_\Gamma)/\delta C_0(\Lambda^1_\Gamma).$$

These maps are continuous because these are linear maps between finite-dimensional vector spaces.

The kernel of $\varepsilon\delta T$ coincides with $\operatorname{im}\ d''$ in $\mathcal{L}^{p,1}(\Gamma)$. Indeed, assume $\omega \in \mathcal{L}^{0,1}(\Gamma)$ and $\varepsilon\delta T\omega = 0$, then there is a cochain $\psi \in C_0(\mathbb{R}_\Gamma)$ such that $\delta T\omega = \delta\psi$. Since $\delta(T\omega - \psi) = 0$, we get $T\omega - \psi \in \mathcal{L}^{0,0}(\Gamma)$ and $d''(T\omega - \psi) = \omega$. The same arguments works for the case $\omega \in \mathcal{L}^{1,1}(\Gamma)$. Since $\varepsilon\delta T$ is continuous, the kernel is closed, and, consequently, $\operatorname{im} d''$ is closed. □

Since $\operatorname{im}\ d''$ is closed, there is the decomposition

$$\mathcal{L}^{p,1}(\Gamma) = \operatorname{im}\ d'' \oplus (\operatorname{im}\ d'')^\perp.$$

Hence

$$H^1(\mathcal{L}^{p,*}(\Gamma), d'') = \mathcal{L}^{p,1}(\Gamma)/\operatorname{im}\ d'' \cong (\operatorname{im}\ d'')^\perp$$

The kernel of a closed densely defined operator coincides with the orthogonal complement of the range of the adjoint. Thus by Proposition 3.14 we get $(\operatorname{im}\ d'')^\perp = \ker d''^*$. By

Proposition 3.15 an element $\omega \in \mathcal{L}^{p,1}(\Gamma)$ is harmonic if and only if $d''^{*}\omega = 0$, thus $\mathcal{H}^{p,1}(\Gamma) = \ker d''^{*}$ and $i : \mathcal{H}^{p,1}(\Gamma) \to H^1(\mathcal{L}^{p,*}(\Gamma), d'')$ is an isomorphism. Also we proved that

$$\mathcal{L}^{p,1}(\Gamma) = \mathcal{H}^{p,1}(\Gamma) \oplus \text{im } d''. \tag{7}$$

□

Theorem 3.19. *The Laplace-Beltrami operator is a self-adjoint operator.*

Proof. By the dimensional reasons the operator Δ is equal to either $\Delta = d''d''^{*}$ or $\Delta = d''^{*}d''$. By von Neumann theorem for any closed densely operator A the operator $A^{*}A$ is a self-adjoint operator [1, Theorem 7.3]. Thus by Proposition 3.14 both $d''d''^{*}$ and $d''^{*}d''$ are self-adjoint. □

Remark 3.20. Let us describe the operator Δ in terms of local coordinates. Let x be a local coordinate on an edge of Γ and the Kähler form g is locally given by the equation $g = g(x)d'x \wedge d''x$. Then by straightforward computation we obtain

$$\Delta(f(x)) = -\frac{1}{g(x)}\frac{\partial^2 f(x)}{\partial x^2},$$

$$\Delta(f(x)d'x) = -\frac{1}{g(x)}\frac{\partial^2 f(x)}{\partial x^2} + \frac{1}{g^2(x)}\frac{\partial f(x)}{\partial x}\frac{\partial g(x)}{\partial x}d'x,$$

$$\Delta(f(x)d''x) = -\frac{1}{g(x)}\frac{\partial^2 f(x)}{\partial x^2} + \frac{1}{g^2(x)}\frac{\partial f(x)}{\partial x}\frac{\partial g(x)}{\partial x}d''x,$$

$$\Delta(f(x)d'x\wedge d''x) = -\frac{1}{g(x)}\frac{\partial^2 f(x)}{\partial x^2}+\frac{2}{g^2(x)}\frac{\partial f(x)}{\partial x}\frac{\partial g(x)}{\partial x}+\frac{2}{g^2(x)}\frac{\partial^2 g(x)}{\partial x^2}f(x)-\frac{2}{g^3(x)}(\frac{\partial g(x)}{\partial x})^2 f(x)d'x\wedge d''x.$$

Proposition 3.21. *The Hodge star operator commutes with the Laplace-Beltrami operator, i.e.,* $* \Delta = \Delta *$ *and* $\varphi \in \mathcal{D}(\Delta : \mathcal{L}^{p,q}(\Gamma) \to \mathcal{L}^{p,q}(\Gamma))$ *if and only if* $* \varphi \in \mathcal{D}(\Delta : \mathcal{L}^{1-p,1-q}(\Gamma) \to \mathcal{L}^{1-p,1-q}(\Gamma))$.

Proof. Firstly, let us check that $\varphi \in \mathcal{D}(\Delta : \mathcal{L}^{p,q}(\Gamma) \to \mathcal{L}^{p,q}(\Gamma))$ if and only if $* \varphi \in \mathcal{D}(\Delta : \mathcal{L}^{1-p,1-q}(\Gamma) \to \mathcal{L}^{1-p,1-q}(\Gamma))$.

The domain of Δ equals either

$$\mathcal{D}(\Delta : \mathcal{L}^{p,0}(\Gamma) \to \mathcal{L}^{p,0}(\Gamma)) = \{\omega : \omega \in \mathcal{D}(d''), d''\omega \in \mathcal{D}(d''^{*})\},$$

or

$$\mathcal{D}(\Delta : \mathcal{L}^{p,1}(\Gamma) \to \mathcal{L}^{p,1}(\Gamma)) = \{\omega : \omega \in \mathcal{D}(d''^*), d''^*\omega \in \mathcal{D}(d'')\}.$$

By Proposition 3.14 $\omega \in \mathcal{D}(d'')$ if and only if $* \omega \in \mathcal{D}(d''^*)$, and $d''^* = - * d'' *$. Thus, $d''\omega \in \mathcal{D}(d''^*)$ if and only if $* d''\omega \in \mathcal{D}(d'')$. Using the equality

$$* d'' = \pm * d'' ** = \pm d''^* *$$

and the previous statement we get $d''\omega \in \mathcal{D}(d''^*)$ if and only if $d''^* * \omega \in \mathcal{D}(d'')$. Thus $\omega \in \mathcal{D}(d'')$ and $d''\omega \in \mathcal{D}(d''^*)$ if and only if $* \omega \in \mathcal{D}(d''^*)$ and $d''^* * \omega \in \mathcal{D}(d'')$ which is equivalent to

$$\omega \in \mathcal{D}(\Delta) \Longleftrightarrow * \omega \in \mathcal{D}(\Delta).$$

Now, let us check the commutativity $* \Delta = \Delta *$. The equality $* \Delta = \Delta *$ can be written as

$$* \Delta = - * d'' * d'' * - ** d'' * d'' = -d'' * d'' ** - * d'' * d'' * = \Delta * .$$

Since $** = (-1)^{p+q}\mathrm{Id}$, as an operator on $\mathcal{L}^{p,q}(\Gamma)$, we get

$$- * d'' * d'' * + (-1)^{p+q+1} d'' * d'' = (-1)^{p+q+1} d'' * d'' - * d'' * d'' * .$$

Thus the equality $* \Delta = \Delta *$ holds. □

Proposition 3.22. *There are the following decompositions:*

$$\mathcal{L}^{p,1}(\Gamma) = \mathcal{H}^{p,1}(\Gamma) \oplus \mathrm{im}\ d'',$$

$$\mathcal{L}^{0,q}(\Gamma) = \mathcal{H}^{0,q}(\Gamma) \oplus \mathrm{im}\ d''^*.$$

Proof. The first decomposition was already proved (7). Since the Hodge star is an isometry we get

$$\mathcal{L}^{0,q}(\Gamma) = * \mathcal{L}^{1-q,1}(\Gamma) = * \mathcal{H}^{1-q,1}(\Gamma) \oplus * \mathrm{im}\ d''$$

By Proposition 3.21 we have $\mathcal{H}^{0,q}(\Gamma) = * \mathcal{H}^{1-q,1}(\Gamma)$. By Proposition 3.14 we have $d''^* = - * d'' *$, thus $* \mathrm{im}\ d'' = \mathrm{im}\ d''^*$. Combining these facts we obtain the second decomposition. □

3.10 The main result.

The main result of this paper is the following

Theorem 3.23. *Let* Γ *be a tropical curve of genus* n*. The Hodge star operator maps harmonic superform to harmonic superform and the map* $*\colon \mathcal{H}^{p,q}(\Gamma) \to \mathcal{H}^{1-p,1-q}(\Gamma)$ *is an isomorphism, and, consequently,* $H^{p,q}(\Gamma) \simeq H^{1-p,1-q}(\Gamma)$*. In particular,*

$$H^{1,1}(\Gamma) \simeq H^{0,0}(\Gamma) \simeq H^0(\Gamma, \mathbb{R}) \cong \mathbb{R}$$

and

$$H^{1,0}(\Gamma) \simeq H^{0,1}(\Gamma) \simeq H^1(\Gamma, \mathbb{R}) \cong \mathbb{R}^n.$$

Proof. By Proposition 3.21 if ω is an element of $\mathcal{H}^{p,q}(\Gamma)$, then $*\,\omega$ is also a harmonic form, $*\,\omega \in \mathcal{H}^{1-p,1-q}(\Gamma)$. Since $** = \pm\mathrm{Id}$, we get that $*$ is an isomorphism between $\mathcal{H}^{p,q}(\Gamma)$ and $\mathcal{H}^{1-p,1-q}(\Gamma)$. Thus, by Proposition 3.17 we get $H^{p,q}(\Gamma) \simeq H^{1-p,1-q}(\Gamma)$. Since by definition $H^{0,q}(\Gamma)$ is the cohomology group of the sheaf $\mathbb{R}_\Gamma$ of locally constant functions, $H^{0,q}(\Gamma)$ is isomorphic to the usual topological cohomology group $H^q(\Gamma, \mathbb{R})$. □

The space $H^{0,0}(\Gamma)$ is generated by a nonzero constant function. Since the Hodge star of a constant function is proportional to the Kähler form g, the class of g is a generator of $H^{1,1}(\Gamma)$. The space $\mathcal{H}^{1,0}(\Gamma)$ is the space of differential forms with coefficients constant on edges and satisfying the Kirchhoff's law and the Regularity at infinity conditions. Since $*\,d'x = d''x$, the group $\mathcal{H}^{0,1}(\Gamma) \cong H^{0,1}(\Gamma)$ is generated by essentially the same differential forms which are considered as $(0,1)$–forms.

Remark 3.24. Theorem 3.23 is a tropical analog of the Hodge theory on a compact Riemann surface. We proved this theorem using methods of the Hodge theory, but one can prove that there is an isomorphism $H^{p,q}(\Gamma) \simeq H^{1-p,1-q}(\Gamma)$ using quite simple combinatorial methods. For example, since, by definition, $\mathcal{H}^{0,p}(\Gamma)$ coincides with usual topological cohomologies and can be easily computed, one can apply the tropical Poincaré duality duality [8, Theorem 4.33] and get $\mathcal{H}^{0,p}(\Gamma) \simeq \mathcal{H}^{1,1-p}(\Gamma)$. Also in the case of tropical curves in a bit more general situation, which is a tropical analog of curve with punctures, the cohomologies were computed in [6, Proposition 2.4.2.] using quite simple methods. So for the purpose of this result our paper is overcomplicated. One can consider this paper as a proof of concept for the Hodge theory on higher dimensional tropical varieties.

3.11 Relation between cycles and harmonic tropical superforms.

Suppose some orientation of edges of a metric graph Γ is chosen. We can consider Γ as a CW-complex, where edges of the graph are $1-$cells and vertices are $0-$cells. Consider the group of $1-$chains $C_1(\Gamma, \mathbb{R})$ of Γ with coefficients in $\mathbb{R}$. An element β of this group is a linear combination of edges

$$\beta = \sum_{e \in E} \beta_e e,\ \ \beta_e \in \mathbb{R}.$$

There is the boundary operator ∂ on edges, $\partial e = v_1 - v_0$, where v_1, v_0 are, consequently, inward and outward vertices of the edge e. Denote by $Z_1(\Gamma, \mathbb{R})$ the groups of cycles of Γ with coefficients $\mathbb{R}$. By definition, a cycle β is a chain such that its boundary is equal to zero, i.e., $\partial\beta = 0$. Obviously, $Z_1(\Gamma, \mathbb{R})$ is isomorphic to homology group $H_1(\Gamma, \mathbb{R})$.

Let us define the linear map

$$\varphi : C_1(\Gamma, \mathbb{R}) \to \widetilde{\mathcal{E}}^{0,1}(\Gamma)$$

as follows, an image $\varphi(e)$ of an edge $e \in E$ is a $(0,1)-$tropical form such that its restriction to any edge $e' \neq e$ equals 0 and its restriction to the edge $e \cong [-l(e), 0]$ equals $d''x$.

Proposition 3.25. *The map φ is an isomorphism between $Z_1(\Gamma, \mathbb{R})$ and $\mathcal{H}^{0,1}(\Gamma)$. Moreover, suppose $\beta \in C_1(\Gamma, \mathbb{R})$ and $\omega \in \mathcal{E}^{1,0}(\Gamma)$, then*

$$\int_\beta \omega = \int_\Gamma \omega \wedge \varphi(\beta),$$

where the integral on the left hand side is the integral of a usual differential $1-$form over $1-$chain, we identify the tropical $(1,0)-$superform ω with the corresponding differential $1-$form, and on the right hand side there is the tropical integral of the tropical $(1,1)-$superform over the tropical curve Γ.

Proof. Recall that the Hodge star is an isomorphism between $\mathcal{H}^{1,0}(\Gamma)$ and $\mathcal{H}^{0,1}(\Gamma)$, and $\mathcal{H}^{1,0}(\Gamma)$ consists of closed regular forms. Therefore, these forms are forms with coefficients constants on each edge, regular at infinity, in this case it means equal to zero at each infinite length edge, and satisfying the Kirchhoff's law at each vertex.

In the other hand, the image $\varphi(\beta)$ of a chain β is a form with coefficients constants on each edge. Suppose that β is a cycle. It can't have infinite length edges because an infinite length edge is incident to a vertex of valence 1 and this vertex will appear as a nonvanishing term in the boundary, hence $\varphi(\beta)$ is regular at infinity. It is easy to understand that condition $\partial\beta = 0$ is equivalent to the Kirchhoff's law. Therefore, φ is an isomorphism between $Z_1(\Gamma, \mathbb{R})$ and $\mathcal{H}^{0,1}(\Gamma)$.

Consider a chain

$$\beta = \sum_{e\in E} \beta_e e.$$

On the edge e the form $\varphi(\beta)$ is equal to $\beta_e d''x$. Consider a $(1,0)-$tropical from ω. Its restriction to the edge $e \cong [-l(e), 0]$ is equal to $\omega_e(x)d'x$.

Let us show that

$$\int_\beta \omega = \int_\Gamma \omega \wedge \varphi(\beta).$$

Indeed,

$$\int_\beta \omega = \sum_{e\in E} \beta_e \int_{-l(e)}^{0} \omega_e(x)dx = \sum_{e\in E} \int_e \beta_e \omega_e(x) d'x \wedge d''x = \int_\Gamma \omega \wedge \varphi(\beta).$$

□

3.12 Pair of examples.

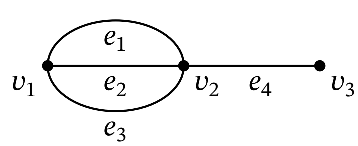

Figure 1:

Example 3.26. Consider the graph Γ with three vertices v_1, v_2, v_3 and four edges $e_1, \dots, e_4$, see Figure 1. The edges e_1, e_2, e_3 are incident to the vertices v_1 and v_2, and the edge e_4 is incident the vertices v_2 and v_3. Therefore, its genus equals 2. Edges are isometric to the intervals $e_i \simeq [-l_i, 0], 0 < l_i < +\infty, i = 1,2,3$ and, since e_4 is incident to degree 1 vertex, $e_4 \simeq [-\infty, 0]$. Let us assume that 0 of each interval corresponds to the vertex v_2. Let ω be a Kähler form on Γ, this form have to satisfy the conditions from Definition 2.17. Consider restriction of this form to edges $\omega|_{e_i} = \omega_i(x) d'x \wedge d''x$, then $\omega_i(x), i = 1,2,3$ have to be smooth positive function on the corresponding interval $e_i \simeq [-l_i, 0]$, and $\omega_4(x)$ is a smooth positive function on $(-\infty, 0]$ such that $\int_{-\infty}^{0} x^2 \omega_4(x) dx < +\infty$. In particular, we can choose $\omega_i(x) \equiv V_i > 0, i = 1,2,3$ to be any positive constants and $\omega_4(x) = 2\frac{e^{2x}}{(1+e^{2x})^2}$, this $\omega_4(x)$ is the analog of Fubini-Study form from Example 2.22. One can check that

$$\int_\Gamma \omega = V_1 l_1 + V_2 l_2 + V_3 l_3 + \frac{1}{2}.$$

The space $\mathcal{H}^{0,0}(\Gamma)$ is the space of constant functions on Γ, and $\mathcal{H}^{1,1}(\Gamma)$ is a linear span of the Kähler form ω. The space $\mathcal{H}^{1,0}(\Gamma)$ consists of forms ψ such that

$$\psi|_{e_i} = C_i d'x,\ C_i \in \mathbb{R},\ i = 1,2,3,\ C_1 + C_2 + C_3 = 0,\ \psi|_{e_4} = 0.$$

Indeed, because $(1,0)$–harmonic forms are closed they must have constant coefficients on each edge. These forms have to satisfy conditions of Definition 2.8. Since they have to be regular at infinity we get $\psi|_{e_4} = 0$, and the Kirchhoff's law condition at the vertex v_2 or, equivalently, v_1 give us $C_1 + C_2 + C_3 = 0$. Since the Hodge star is an isomorphism between $\mathcal{H}^{1,0}(\Gamma)$ and $\mathcal{H}^{0,1}(\Gamma)$, the space $\mathcal{H}^{0,1}(\Gamma)$ is defined by the same condition but with $d''x$ instead of $d'x$.

Consider the basis ψ_1 and ψ_2 of $\mathcal{H}^{1,0}(\Gamma)$, where

$$\begin{aligned} &\psi_1|_{e_1} = d'x, \quad \psi_1|_{e_2} = -d'x, \quad \psi_1|_{e_3} = 0, \qquad \psi_1|_{e_4} = 0;\\ &\psi_2|_{e_1} = 0, \qquad \psi_2|_{e_2} = \quad d'x, \quad \psi_2|_{e_3} = -d'x, \quad \psi_2|_{e_4} = 0, \end{aligned}$$

then

$$\int_\Gamma \psi_1 \wedge * \psi_1 = l_1 + l_2, \int_\Gamma \psi_2 \wedge * \psi_2 = l_2 + l_3, \int_\Gamma \psi_1 \wedge * \psi_2 = \int_\Gamma \psi_2 \wedge * \psi_1 = -l_2.$$

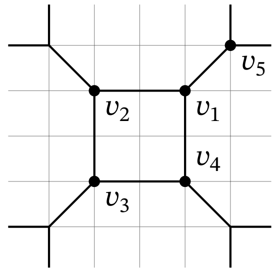

Figure 2:

Example 3.27. Now we are going to consider an example of a tropical curve defined by a tropical polynomial, it is more standard approach in the tropical geometry, for details see [3, Section 2]. We can think that $\mathbb{R}^2$ is the tropical analog of $(\mathbb{C}\setminus\{0\})^2$, and its compactification

$$\mathbb{R}^2 \subset [-\infty,+\infty]\times[-\infty,+\infty] = \mathbb{TP}^1\times\mathbb{TP}^1$$

is the analog of $\mathbb{CP}^1\times\mathbb{CP}^1$. We can take the sum of Fubini-Study forms from Example 2.22

$$\omega = 2\frac{e^{2x}}{(1+e^{2x})^2}d'x\wedge d''x + 2\frac{e^{2y}}{(1+e^{2y})^2}d'y\wedge d''y$$

as an $(1,1)$–form Kähler on $\mathbb{TP}^1\times\mathbb{TP}^1$.

Consider the tropical polynomial

$$P(x,y) = \varepsilon 3 + 2\cdot(x+y+x^{-1}+y^{-1}) + 0\cdot(xy+xy^{-1}+x^{-1}y+x^{-1}y^{-1})\varepsilon,$$

this notation means that

$$P(x,y) = \max(3, 2+x, 2+y, 2-x, 2-y, x+y, x-y, -x+y, -x-y),$$

it is a convex piecewise linear function. This tropical polynomial defines a tropical curve Γ of genus 1 in $\mathbb{TP}^1\times\mathbb{TP}^1$, this curve is the corner locus of the function $P(x,y)$, see Figure 2.

The length of an edge of Γ should be measured with respect to the minimal integer tangent vector to this edge. For example, the length of the edge (v_1,v_5) equals 1 since the minimal integer tangent vector is $(1, 1)$. For the edge (v_1, v_2) the length is equal to 2 and the minimal integer tangent vector is $(-1, 0)$.

The restriction of ω to Γ is a Kähler form on Γ and its linear span is the space of harmonic $(1, 1)$–forms $\mathcal{H}^{1,1}(\Gamma)$.

Suppose edges $e_1 = (v_1, v_2), e_2 = (v_2, v_3), e_3 = (v_3, v_4), e_4 = (v_4, v_1)$ are orientated in the cyclic order, these edges are isometric to the interval $[-2, 0]$. Let ψ be a $(1, 0)$–form equal to $d'x$ on these edges and to 0 at all other edges. Since this form is closed and regular it is harmonic and it is a basis of $\mathcal{H}^{1,0}(\Gamma)$. Its Hodge star $* \psi$, which is equal to $d''x$ on $e_1, \ldots, e_4$, is a basis of $\mathcal{H}^{0,1}(\Gamma)$.

Since

$$\omega = \frac{1}{2} d'd''(\log(1 + e^{2x}) + \log(1 + e^{2y}))$$

using Stokes' theorem argument one can show that $\int_\Gamma \omega = 4$. Also one can show that $\int_\Gamma \psi \wedge * \psi = 8$.

3.13 Final remarks.

Remark 3.28. Now we would like to discuss the relation of our paper to the quantum graphs.

The research in quantum graphs is mostly devoted to the study of the Schrödinger equation on metric graphs. This study usually based on study of stationary states, i.e., eigenfunctions of the Laplace operator. For the Laplace operator to be self-adjoint some boundary conditions at the vertexes of a graph are needed. There are a variety of such boundary conditions, some of them resembles our boundary conditions.

There is a difference between out approach and the standard quantum graph theory. Usually only functions are considered, but we also consider differential forms and tensor fields. We use harmonic forms as a tool to compute some cohomologies and the Laplace-Beltrami operator arise from the chain complex. Usually quantum graphs are not related to the study of cohomologies and chain complexes. In the quantum graphs setting the whole spectrum of the Laplace operator is studied, but we are only working with harmonic functions and forms, i.e., with zero-eigenvectors.

In our case there is a Riemannian metric g on Γ which we consider as an analog of a Kähler form. This metric is unrelated to the metric structure on Γ, i.e., to the length of edges, but if we take g to be trivial

$$g = d'x \wedge d''x \simeq dx \otimes dx,$$

then the length $l(e)$ of an edge e, the function $l(e)$ is a part of initial data for the metric graph Γ, coincides with the length with respect to the Riemannian metric g. In this case our theory is practically identical to the standard quantum graph theory, at least if we work with functions only.

Also, in our case we require that all infinite-length edges should have finite length with respect to the Riemannian metric g, otherwise we would get a different behavior of harmonic forms. For example, constant functions are harmonic on Γ, but if there is an infinite-length edge and g is the trivial metric, a non-zero constant function does not belong L^2 since it has infinite norm. So if there are infinite-length edges, than the trivial Riemannian metric is not a viable option for our purpose.

Remark 3.29. It would be interesting to study spectral properties (like asymptotics of eigenvalues, Weyl law, and so on) of the Laplace-Beltrami operator and compare them with spectral properties of complex curves and quantum graphs.

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Gravitational Billiard - Bouncing in a Paraboloid Cavity

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Abstract: In this work the confined domains for a point-like particle propagating within the boundary of an ideally reflecting paraboloid mirror are derived. Thereby it is proven that all consecutive flight parabola foci points lie on the surface of a common sphere of radius R. The main results are illustrated in various limiting cases and are compared to its one-dimensional counterpart. In the maximum angular momentum configuration we explicitly state the coordinates of the particle at any time t within the cavity.

AMS Classification: 35A01, 65L10, 65L12, 65L20, 65L70

Key words and phrases: billiards, gravity, foci, paraboloid mirror, confined trajectories

1 Introduction

Over the last decades, the dynamics of a point-like particle confined to some domain under the influence of a constant gravitational force, shortly called gravitational billiards,

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has been studied from various perspectives. Starting from [LM86] who first performed a numerical study of the simplest imaginable system, the wedge, showing that the system can be integrable for certain angles of the wedge. Further study of the system have e.g. been performed in [And22, KJ99]. Extensions to other one dimensional boundaries (circular, elliptic, oval) or potentials have been performed in e.g. [dCDL15, KL91], who showed that for the quadratic and Coulomb potential the system becomes integrable. Generalizations to the motion in general high dimensional quadrics under the influence of a harmonic, isotropic potential have been performed by Fedorov in [Fed01]. Thereby it has been shown that the trajectories again are tangent to confocal quadrics. This result is consistent with our results derived in Section 4 in presence of a homogeneous gravitational field. Note that similar results (see e.g. [Ves88]) already were obtained for the force free system of a particle bouncing in a higher-dimensional ellipsoid. Also for these systems the trajectories are tangent to confocal quartics. The explicit theta-function solutions for these problems were given e.g in [Ves88], who studied the system using a discrete Lagrangian approach.

In 2015, the first study of the dynamics in a two-dimensional cone were performed by [LM15, Lan15] showing that certain quantities of the one-dimensional framework map one-to-one to the embedded surface in $\mathbb{R}^3$.

In general, the motion of the particle is highly non-trivial and a neat expression for the trajectory at each time is not accessible. For this reason, following [Mas14, Mas20], the confined domains for a point like particle bouncing in a parabolic, one-dimensional cavity under the influence of a homogeneous gravitational force were derived through a geometric-analytic approach. In particular the author showed in [Mas20] that the geometric interpretation of the integrable system is that the foci of consecutive flight parabolae all lie on a common circle of fixed radius with center being the focus of the parabolic mirror.

Recently associated foci curves and confined domains for a particle ideally bouncing inside general one-dimensional boundaries were obtained in [Jau23a]. As a generalization

Galavotti and Jauslin [GJ20] considered the so called Boltzman system (a particle under the influence of a Keplerian potential being reflected along a straight line within $\mathbb{R}^2$). They showed that this system, in contrast to the belief of Boltzman himself, is integrable and that the second foci of the associated Kepler ellipses lie on a common circle with center being the mirror point of the center of the gravitational force with respect to the hard wall line. Following this work Felder [Fel21] proved that for this system one can associated an elliptic curve for the family of consecutive flight ellipses. Gasiorek and Radnoviȁ used the result of Felder in order to derive periodicity conditions for the orbits, generalizing Poncelet Porims to the case of a Keplerian force [GR23] with a straight line boundary. In a recent study [JZ24] it has been shown that a particle under the influence of a Keplerian force centered at one focus of a conic section in two dimensions is integrable and that the second foci of the flight trajectories also lie on a common circle centered at the second focus of the conic boundary. In particular the authors derived via a conformal transformation (Kepler-Hooke-Duality) the corresponding "foci curve" (as they denote it) in the case of a Hooke potential with conic section boundaries. These curves, on which all consecutive foci lie, correspond to Cassini ovals.

In the work of Borisov et al. [ABM19] the frictionless motion of a point mass on an elliptic (or hyperbolic) paraboloid $U = \{z = \frac{x^2}{a} + \frac{y^2}{b}\}$ in presence of a constant gravitational force parallel to the $z-$axis was considered. Already in [Cha33] the authors showed that the system is separable in parabolic coordinates and therefore integrable. The geometric interpretation of integrability is that the particle trajectory is tangent to two other confocal paraboloids. When one of the parameters, a or b, tends to zero, one recovers the situation studied by [Mas14].

In the following the confined domains for a particle bouncing inside a rotational symmetric paraboloid under the influence of a constant gravitational force parallel to the axis of symmetry is studied. Our analysis will show that some one-dimensional features obtained e.g. in [Mas20, Jau23a, Jau23b] will carry over (in some cases) to the two-dimensional scenario. Due to the additional rotational movement associated to conserved angular momentum along the $z-$direction further restrictions compared to the one-dimensional boundary case will emerge.

The structure of this work presents as follows: in Section 2 we will briefly introduce all necessary assumptions and general ideas that we will benefit from in our later analysis. Before diving into a general analysis, Section 3 will show that under certain restrictions the two-dimensional particle motion can be reduced to the one-dimensional force free case within a circle. For this case, corresponding to maximal angular momentum, we give a neat expression for the particles position within the cavity at any time t. In particular we again obtain the same result of the trajectories lying along a common confocal paraboloid resembling similar results as obtained in [Ves88, Fed01] for the free billiard and the billiard with harmonic potential. In Section 4, we will first show that all consecutive flight parabola foci points lie on a sphere of radius R. This result can be seen as a limiting case of [Fed01] and as the generalization of [Mas20] to the next dimension. With this result, we derive general formulas for the confined regions depending on the system parameters. For a deeper understanding of the general results and related physics, Section 5 will derive the associated envelope curves and therefore two-dimensional sections of the rotational confined regions for different values of the sphere radius R as well as (reduced) angular momentum l_z. Finally a conclusion and outlook on possible future research topics related to this work is made in Section 6.

2 Generalities for the paraboloid billiard

Here some general results for the motion of an particle under the influence of a constant gravitational force within a cavity are stated. All obtained results are direct generalizations from the one-dimensional case already discussed in e.g. [Mas20, Jau23a].

We are considering the movement for a particle of mass m propagating inside a paraboloid mirror under the influence of the constant gravitational force $\vec{F} = -mg\vec{e}_z$ parallel to the $z-$axis. The equation for the boundary of the paraboloid in Cartesian coordinates (x, y, z) reads

$$M(x, y, z) = z - \frac{x^2 + y^2}{4f_M} + f_M = 0. \tag{1}$$

The focus of the paraboloid is centered at the origin of the coordinate system and f_M denotes the focal length of this ideally reflecting mirror (see Figure 1).

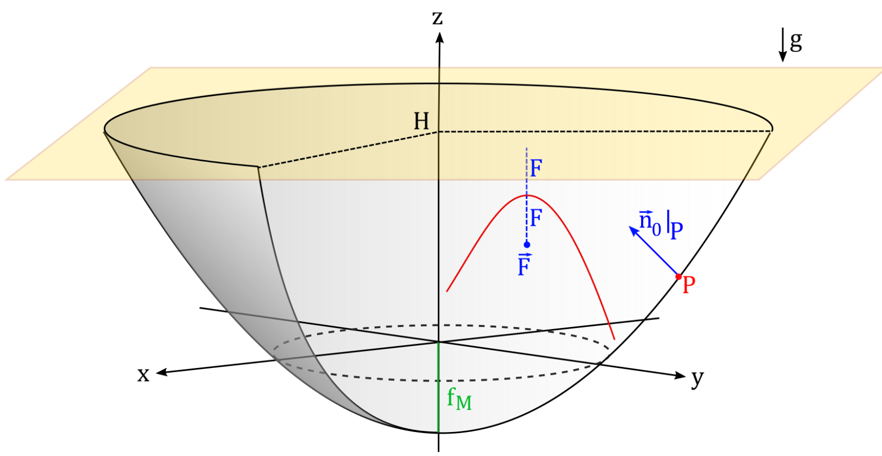

Figure 1: Visualization of main quantities for the paraboloid gravitational billiard.

For a general point $P(x, y, z)$ along the mirror boundary, the associated normalized normal-vector pointing inside the mirror domain is given by

$$\vec{n}_0|_P = \frac{1}{|\vec{\nabla}M|}\vec{\nabla}M|_P = \frac{1}{\sqrt{1 + \frac{x^2+y^2}{4f_M^2}}}\begin{pmatrix} -\frac{x}{2f_M} \\ -\frac{y}{2f_M} \\ 1 \end{pmatrix}. \tag{2}$$

As usual, the trajectory of one specific flight parabola can be written as a function of time t via

$$\vec{r}(t) = -\frac{1}{2}gt^2\vec{e}_z + \vec{v}t + \vec{r}_0, \tag{3}$$

where $\vec{v}^T = (v_x, v_y, v_z)$. All flight parabolas posses a focal length F associated with the velocity components in $x-$ and $y-$direction by

$$F = \frac{v_x^2 + v_y^2}{2g}. \tag{4}$$

Note that for a given flight parabola v_x, v_y are constant along the motion and thus F is well defined. Conservation of energy $E = \frac{m}{2}\vec{v}^2 + mgz$ yields a maximal reachable height H for all flight parabolas within the paraboloid. Considering the velocities $\vec{v}_S^T = (v_x, v_y, 0)$ and associated heights z_S at the vertex of a flight parabola we find

$$H = \frac{E}{mg} = \frac{\vec{v}_S^2}{2g} + z_S = const.. \tag{5}$$

In analogy to the one-dimensional case (see [Mas20, Jau23a]) H refers to the flight parabola vertex plane (see Figure 1). As a direct consequence, the $z-$coordinate of the flight parabola focus point $\vec{F}$ fulfills $F_z = H - 2F$. In Section 4 we will make use of this relation.

As a last component we state the law of reflection in vector form whenever the particle hits the boundary of the mirror at a point P and gets ideally reflected. For the velocities $\vec{v}$ before and $\vec{v}'$ after the reflection holds

$$\vec{v}' = \vec{v} - 2(\vec{n}_0 \circ \vec{v}) \cdot \vec{n}_0|_P = \vec{v} - \frac{2}{|\vec{\nabla} M|^2}(\vec{v} \circ \vec{\nabla} M) \cdot \vec{\nabla} M|_P. \tag{6}$$

A direct consequence of the law of reflection is stated in the following Lemma which in Section 4 will be used in order to reduce the number free parameters of our system.

Proposition 1. *Angular momentum per unit mass about the $Oz-$axis, i.e. $l_z = L_z/m$, in the further course to be called reduced angular momentum, is a conserved quantity in particular at any point P of reflection.*

Proof. It is sufficient to proof the statement at some general point of reflection P associated with the vector $\vec{r}$. Using the law of reflection (6) for reduced angular momentum in $z-$direction results in:

$$l_z' = (\vec{r} \times \vec{v}')_z = (\vec{r} \times \vec{v})_z - \frac{2}{|\vec{\nabla} M|^2}(\vec{v} \circ \vec{\nabla} M) \cdot (\vec{r} \times \vec{\nabla} M)_z = (\vec{r} \times \vec{v})_z = l_z.$$

Here we used in the last step the fact, that $(\vec{r} \times \vec{\nabla} M)_z = 0$ in P. Therefore $l_z' = x_0 v_y - y_0 v_x = x_0 v_y' - y_0 v_x' = l_z$ is conserved. Note that l_z conservation along the flight parabola is a direct consequence by the properties of the cross product. □

3 Special orbits with reflections along the same circle

In this section first we want to discuss the simplified case in which all consecutive points of reflection P_i lie on a common circle of radius r_0 (and consequently height $z_0 = \frac{1}{4f_M} r_0^2 - f_M$) with respect to the $z-$axis. Naturally for this section polar coordinates are chosen to describe the dynamics. When viewed from above, the system can uniquely be described by the angle ϑ enclosed by two consecutive points of reflection and the 'origin' at height z_0 (see Figure 2). Without loss of generality our starting position may be chosen in polar coordinates at $P_0(r_0, 0, z_0)$ and consequently $P_i(r_0, i \cdot \vartheta, z_0)$. We choose our particle, when viewed from above, traveling in counter-clockwise direction. The velocity values for the new flight parabola at the point of reflections are given by $(v_{r,i}, v_{\varphi,i}, v_{z,i})$. We will use in later sections the index i to label the $i-$th consecutive flight parabola.

From our setup it is clear that the allowed values for $(v_{r,i}, v_{\varphi,i}, v_{z,i})$ are restricted by the condition that all point of reflection P_i have to lie on the same circle. In particular, such kind of behavior can only exist if the associated flight parabolas are each a copy of the same fundamental, symmetric, parabola up to an rotation by ϑ spanned by the two initial reflection points P_0 and P_1.

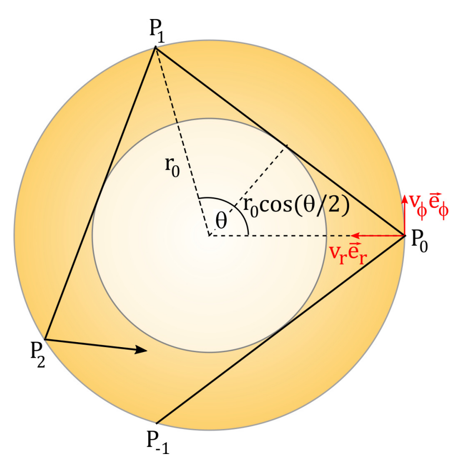

Figure 2: Projection to the parallel $x-y-$plane at height z_0 with relevant system parameter ϑ associated to force free billiards inside the circle.

Due to rotational symmetry it thus is sufficient to determine the restrictions on $(v_{r,0}, v_{\varphi,0}, v_{z,0}) =: (v_r, v_\varphi, v_z)$. Applying the law of reflection (6) at P_0 yields expressions (v_r', v_φ', v_z') right before the reflection. Those velocity values correspond to the flight parabola starting at P_{-1} propagating to P_0. For all flight parabolas being the same copy one thus obtains the restriction

$$|v_r| = |v_r'| \quad \text{and} \quad |v_z| = |v_z'|. \tag{7}$$

Both conditions are fulfilled if

$$v_r \vec{e}_r + v_z \vec{e}_z \parallel \vec{\nabla} M|_{P_0}, \tag{8}$$

i.e. the (r, z)–components of the velocity vector stand perpendicular on the tangent plane at the point of reflection. The flight time $t = \frac{2v_z}{g}$ for reaching the initial height z_0 again is uniquely determined by the motion in z–direction. Within this time the particle starting at P_0 has to reach P_1. In the $x - y$–plane, the angle ϑ between two consecutive points of reflection (see Figure 2) is related to the velocity values (v_r, v_φ) via

$$\vartheta = \pi - 2\arctan\left(\left|\frac{v_\varphi}{v_r}\right|\right). \tag{9}$$

Demanding for the reflection points all to lie on the same circle of radius r_0 gives a further restriction on the system within the given flight time t from P_i to P_{i+1}. Direct calculation shows that the allowed velocity components are completely determined by the angle ϑ, the radius r_0 of the common reflection points circle as well as the focal length f_M of the paraboloid mirror:

$$v_r = -\frac{r_0}{2f_M} \cdot \sqrt{g f_M \cdot [1 - \cos(\vartheta)]}, \tag{10}$$

$$v_\varphi = \frac{r_0}{2f_M} \cdot \sqrt{g f_M \cdot [1 + \cos(\vartheta)]}, \tag{11}$$

$$v_z = \sqrt{g f_M \cdot [1 - \cos(\vartheta)]}. \tag{12}$$

Depending on the values for ϑ (see. e.g. [Roz18, Jau23b]) we obtain periodic or non-periodic orbits, where all flight parabolas lie on a common rotational surface around the $z-$axis (see Figure 3) whose radial function is purely determined by the mirror parameters

$$g(r) = \frac{r_0^2}{4f_M} - \frac{f_M \cdot r^2}{r_0^2}, \quad \text{with} \quad r \in [r_0 \cos(\vartheta/2); r_0]. \tag{13}$$

As mentioned before in the introduction, this result is consistent with former studies, e.g. [Fed01, Ves88], where the trajectories inside a general quadric are tangent to other confocal quadrics.

Considering the flight parabolas dividing the rotational flight surface $g(r)$ consecutively into smaller sub regions, one can map this to the circle case as shown in [Jau23b] that for specific values of ϑ the surface division sequence is given by an integer series.

As a remark for $\vartheta = \pi$ the $\varphi-$velocity component equals zero, i.e. $v_\varphi = 0$. For this case there is no rotational motion (angular momentum being zero) and the particle bounces along $g(r)$ forming a two periodic orbit reproducing one-dimensional results obtained in e.g. [KL91, Jau23a]. Again this can be interpreted as the limiting case of [Fed01], where one axis tends to infinity and the harmonic potential, in this limit, resembling a constant force leading to two confocal, parabolic envelop curves for the motion.

The trajectory of the particle in our szenario now is completely determined by the parameters (r_0, ϑ). In particular the explicit coordinates at a given time read

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} v_r \cdot (t - \lfloor \frac{t}{T} \rfloor T) \cdot \cos\left(\lfloor \frac{t}{T} \rfloor \vartheta\right) - v_\varphi \cdot (t - \lfloor \frac{t}{T} \rfloor T) \cdot \sin\left(\lfloor \frac{t}{T} \rfloor \vartheta\right) \\ v_r \cdot (t - \lfloor \frac{t}{T} \rfloor T) \cdot \sin\left(\lfloor \frac{t}{T} \rfloor \vartheta\right) + v_\varphi \cdot (t - \lfloor \frac{t}{T} \rfloor T) \cdot \cos\left(\lfloor \frac{t}{T} \rfloor \vartheta\right) \\ -\frac{1}{2} g \cdot \left(t - \lfloor \frac{t}{T} \rfloor T\right)^2 + v_z \cdot \left(t - \lfloor \frac{t}{T} \rfloor T\right) \end{pmatrix}, \tag{14}$$

where the particle at time $t = 0$ starts from $(r_0, 0, \frac{r_0^2}{4f_M} - f_M)$ and

$$T = \sqrt{\frac{8f_M}{g}} \sin(\vartheta/2), \tag{15}$$

corresponds to the propagation time between two consecutive point of reflections.

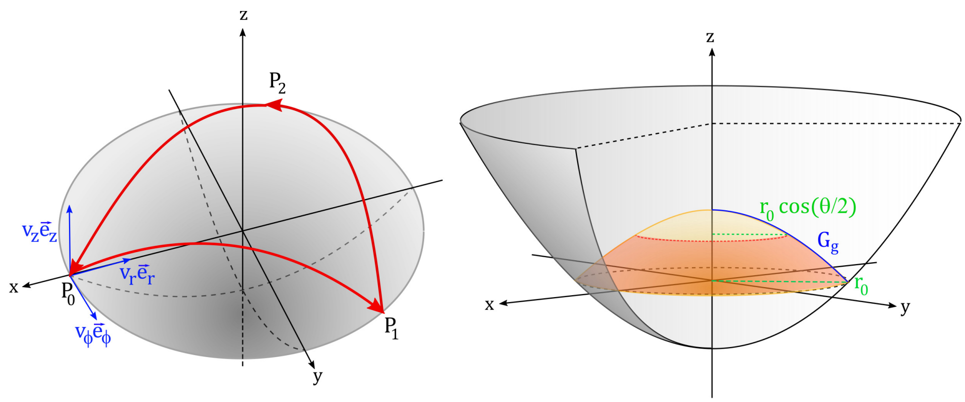

Figure 3: *left*: Example for 3-periodic orbit ($\vartheta = \frac{2\pi}{3}$) along the same circle. *right*: Swept out flight parabola surface (light red) for non-periodic case and $\vartheta \neq \pi$.

4 General flight parabola domain

In this section now we want to derive the confined domains which the particle at a given initial condition can not leave during its motion. We will use the same notations as introduced in Section 2. It is clear that for certain choices of initial conditions the actual flight orbits will not fill out the entire confined domains. In particular we are considering non periodic orbits in which the swept out region becomes dense. A main component for obtaining expressions for the confined domains is stated in the following theorem.

Theorem 2. *For each initial conditions, the foci $\vec{F}_i$ of all flight parabolas of the billiard trajectory lie on the same sphere with radius R centered at the origin O, i.e. the focus of the paraboloid.*

Proof. Without loss of generality consider let the point of reflection being located in Cartesian coordinates at $P(r_0, 0, z_0)$ and the associated velocity vector right before the reflection take the form $\vec{v} = (v_x, v_y, v_z)^T$. The flight parabola focus then is given by

$$\vec{F} = \begin{pmatrix} r_0 \\ 0 \\ 2z_0 - H \end{pmatrix} + \frac{v_z}{g}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}. \quad (16)$$

Applying the law of reflection in P yields the velocity vector $\vec{v}'$ after the reflection and consequently $\vec{F}'$. A direct but lengthy calculation shows that

$$|\vec{F}|^2 = |\vec{F}'|^2 = R^2 = const.,$$

i.e. both foci lie on a common sphere of radius R. Due to rotational symmetry of the system, conservation of energy as well as reduced angular momentum conservation in $z-$direction, all consecutive foci lie on the same sphere of radius R. Since $\vec{r}_0$ and the initial velocity $\vec{v}_0$ have been chosen arbitrarily (under assumption of same total energy) all consecutive foci have to saturate this equality. Note that this is a direct generalization of the one-dimensional case. As a reminder we state that similar one-dimensional results recently have been obtained for other mirror geometries [Jau23a, GJ20, Fel21, GR23, JZ24]. □

Now we are in the position to reduce the six-dimensional phase space with the knowledge of Theorem 2, conservation of energy and reduced angular momentum as well as rotational invariance, to two free parameters corresponding to confined domains which the particle at given values (H, l_z, R) cannot leave.

The vertex $\vec{S}$ of each flight parabola in spherical coordinates thus can be written as

$$\vec{S}(R,\varphi,\vartheta) = \vec{F}(R,\varphi,\vartheta) + F\vec{e}_z = R\begin{pmatrix}\cos(\varphi)\sin(\vartheta)\\ \sin(\varphi)\sin(\vartheta)\\ \cos(\vartheta)\end{pmatrix} + F\vec{e}_z = \begin{pmatrix}R\cos(\varphi)\sin(\vartheta)\\ R\sin(\varphi)\sin(\vartheta)\\ \frac{H+R\cos(\vartheta)}{2}\end{pmatrix}. \tag{17}$$

Thereby we used that for the focal length of Section 2 holds $F = \frac{H-R\cos(\vartheta)}{2}$. Note that ϑ in this case corresponds to the polar angle (compare Figure 4) in contrast to the definition for ϑ of Section 3. Energy conservation yields an expression for the absolute value of the velocity $|\vec{v}_S|$ at the vertex

$$H = \frac{\vec{v}_S^2}{2g} + z_S \quad \leftrightarrow \quad v_S = |\vec{v}_S| = \sqrt{2g(H - z_S)} = \sqrt{g(H - R\cos(\vartheta)}, \tag{18}$$

where H is the height of the directrix plane and z_S is the associated vertex height (consider Figure 4). We choose v_S to be positive; negative values simply correspond to a time inverted system. Since the orientation of $\vec{v}_S$ is not fixed by the equation above we may take a general ansatz $\vec{v}_S = v_S(\cos(\varphi'), \sin(\varphi'), 0)^T$. Reduced angular momentum conservation along the $z-$axis (compare Lemma 1)

$$l_z = (\vec{S} \times \vec{v}_S)_z = R\sqrt{g(H - R\cos(\vartheta))} \cdot \sin(\vartheta) \cdot \sin(\varphi' - \varphi) = const., \tag{19}$$

restricts the allowed values for the orientation of $\vec{v}_S$ related to the rotation angle φ' as follows

$$\varphi' = \varphi + \arcsin\left(\frac{l_z}{R\sin(\vartheta) \cdot \sqrt{g(H - R\cos(\vartheta))}}\right). \tag{20}$$

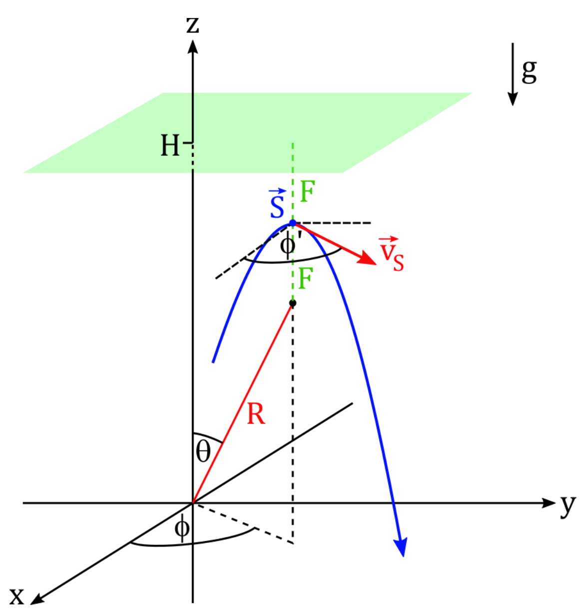

Figure 4: Flight parabola setup.

Due to rotational symmetry it is sufficient to consider the case $\varphi = 0$ from here on. The corresponding allowed flight parabolas

$$\vec{r}(t, \vartheta) = \begin{pmatrix} R\sin(\vartheta) + t \cdot \sqrt{g(H - R\cos(\vartheta)) - \frac{l_z^2}{R^2 \sin^2(\vartheta)}} \\ t \cdot \frac{l_z}{R\sin(\vartheta)} \\ -\frac{1}{2}gt^2 + \frac{H + R\cos(\vartheta)}{2} \end{pmatrix}, \tag{21}$$

at fixed (H, l_z, R) form a one-parameter family of curves in $\mathbb{R}^3$. Note that the $x-$component velocity term restricts the allowed values for ϑ at a given value of l_z according to (compare Figure 5)

$$J(H, R, \vartheta) = gR^2 \sin^2(\vartheta) \cdot (H - R\cos(\vartheta)) \geq l_z^2 \quad \leftrightarrow \quad \vartheta \in [\vartheta_0; \vartheta_1]. \tag{22}$$

Clearly l_z is the main limiting factor to ϑ with large l_z associated to a motion farther away from the $z-$axis as in the case for l_z being small, resulting in the possibility of approaching the $z-$axis. Further l_z is bound from above by the maximum of $J(H, R, l_z)$ which is saturated for

$$\cos(\vartheta_{max}) = \frac{H - \sqrt{H^2 + 3R^2}}{3R}, \tag{23}$$

and therefore takes the value

$$J(H, R, \vartheta_{max}) = \frac{2}{27} g \left(\sqrt{H^2 + 3R^2} + 2H\right) \left(H \left(\sqrt{H^2 + 3R^2} - H\right) + 3R^2\right). \tag{24}$$

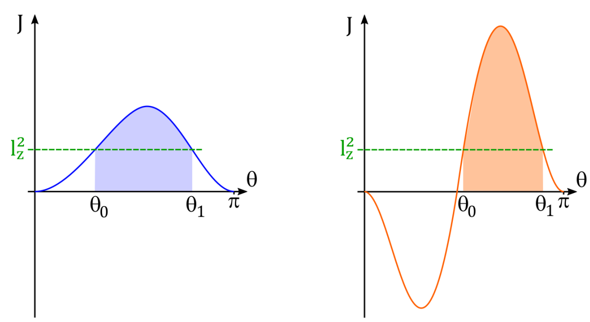

Figure 5: Qualitative restriction of allowed ϑ values at given l_z^2. *left* for $H > R > 0$ and *right* for $0 < H < R$.

Theorem 3. *The rotational symmetric allowed propagation heights* h *of the particle at a given radial distance* r *and angle* ϑ *are given by*

$$h_\pm(r, \vartheta) = \frac{H + R\cos(\vartheta)}{2} - \frac{g}{2} \left(\frac{\sqrt{r^2 \cdot g(H - R\cos(\vartheta)) - l_z^2} \pm \sqrt{R^2 \sin^2(\vartheta) \cdot g(H - R\cos(\vartheta)) - l_z^2}}{g(H - R\cos(\vartheta))} \right)^2,$$

with the restriction on the radius $r \geq \frac{l_z}{\sqrt{g(H - R\cos(\vartheta))}}$.

Proof. Equation (21) defines possible trajectories at given (H, R, l_z). Considering the associated radial distance $r^2 = x(t,\vartheta)^2 + y(t,\vartheta)^2$ one can express the t variable in terms of r and ϑ. Inserting this expression into the $z-$component of (21) yields the expression for the allowed heights $h_\pm$, where the two different solutions correspond to the left and right parabola arc measured from the minimal distance $r_{min} = l_z/\sqrt{g(H - R\cos(\vartheta))}$. □

In order to obtain expressions for the associated envelope curves we define a new quantity

$$K(z, r, \vartheta, H, R, l_z) := z - h_\pm(r, \vartheta). \tag{25}$$

The envelope curves restricting the confined domains then are obtained eliminating ϑ by solving the following system of equations (see [BG92])

$$K(z, r, \vartheta, H, R, l_z) = 0 \qquad \text{and} \qquad \frac{\partial K}{\partial \vartheta} = 0. \tag{26}$$

A computer animated picture of allowed flight parabolas is shown in Figure 6. In the next section our obtained results will be illustrated in various extremal limits.

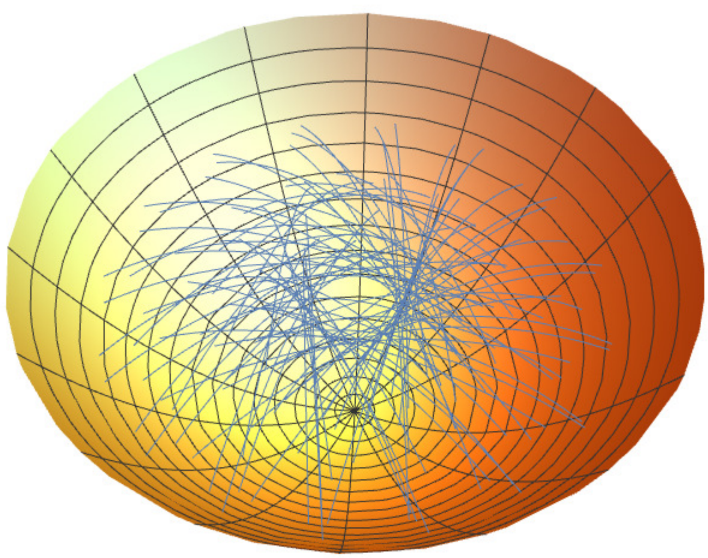

Figure 6: Computer animated flight trajectories.

5 Discussion of limiting cases

In this section four limiting cases in terms of the reduced angular momentum l_z are discussed. For all cases we determine the associated height function from Theorem 3 and calculate, if possible, the corresponding envelope curves restricting the motion of the particle, in general, to a rotational symmetric region. All results of this section are displayed in Figure 7 for illustrative purposes.

5.1 The $l_z = 0$ case

In the simplest case of no reduced angular momentum ($l_z = 0$) the height functions of Theorem 3 significantly simplify to

$$h_\pm(r, \vartheta) = \frac{H + R\cos(\vartheta)}{2} - \frac{(r \pm R\sin(\vartheta))^2}{2(H - R\cos(\vartheta))}. \tag{27}$$

Solving the system (26) yields the envelope curves denoted by $c_\pm$

$$c_\pm(r) = \frac{H \pm R}{2} - \frac{r^2}{2(H \pm R)}. \tag{28}$$

This reproduces the results obtained geometrically in [Mas20] and analytically in [Jau23a]. Since the motion lies in a common plane containing the $z-$axis it is clear that r can take values in $\mathbb{R}$.

5.2 The small l_z case

For l_z small the deviation from the $l_z = 0$ case is marginal. Thus one can conclude that in first approximation one obtains the same envelope curves $c_\pm(r)$ as before. An additional restriction comes from the fact that the allowed values for r are bound from below by $r \geq \frac{l_z}{\sqrt{g(H-R\cos(\vartheta))}}$. If this inequality is saturated, i.e. we consider the case of minimal radial distance in terms of ϑ, one can solve $r = \frac{l_z}{\sqrt{g(H-R\cos(\vartheta))}}$ for $\cos(\vartheta)$ and insert this expression into the height functions of Theorem 3 yielding one additional (approximate) envelope curve associated with the angular momentum barrier as

$$c_0(r) = \frac{(H^2 - R^2)g}{2l_z^2} \cdot r^2 + \frac{gr^4}{2l_z^4}. \tag{29}$$

This envelope curve is reminiscent of the Higgs-potential in particle physics, in which in the cases $R > H$ one obtains the well known Mexican-hat like function.

5.3 The large l_z case

In the large l_z limit the second square root appearing in the height functions of Theorem 3 in lowest order can be neglected since $l_z^2 \approx J(H, R, \vartheta_{max})$. The associated envelope curves thus approximately resemble the height functions for small variations of ϑ

$$\tilde{c}_\pm = \frac{H + R\cos(\vartheta)}{2} - \frac{r^2 - R^2\sin^2(\vartheta_{max})}{2(H - R\cos(\vartheta))}, \tag{30}$$

where $\vartheta \in [\vartheta_{max} - \delta; \vartheta_{max} + \delta]$ for δ small.

5.4 The maximal l_z case

The maximal value for l_z follows from (24) and is given by

$$l_z = \sqrt{J(H, R, \vartheta_{max})}. \tag{31}$$

In these cases, the second square root for the height function of Theorem 3 vanishes, resulting in a single height function for $\vartheta = \vartheta_{max}$ as

$$d(r) = \frac{H + R\cos(\vartheta_{max})}{2} - \frac{r^2 - R^2\sin^2(\vartheta_{max})}{2(H - R\cos(\vartheta_{max}))}, \tag{32}$$

with $r \geq R\sin(\vartheta_{max})$. Note that for $R < H$ this reproduces the results of Section 3. For $R > H$ it depends on the mirror boundary if the condition $r \geq R\sin(\vartheta_{max})$ can be saturated, cases exist in which the maximal l_z–value is not accessible due to the mirror boundary.

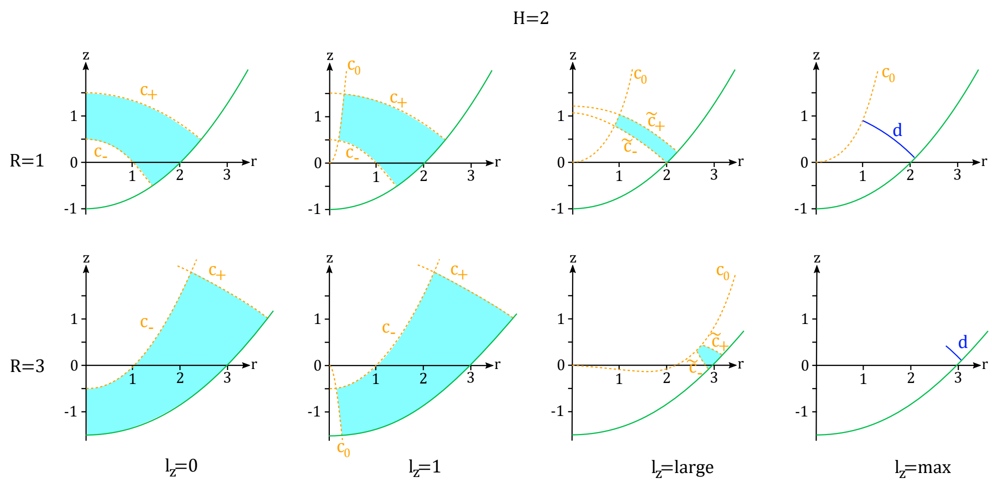

Figure 7: Two-dimensional section of confined domains associated to the four discussed limiting cases. The three-dimensional confined regions are obtained by rotation around the $z-$axis.

6 Conclusion and Outlook

In this work the rotational symmetric confined domains of a point-like particle bouncing inside a paraboloid cavity under the influence of a homogeneous gravitational field in terms of the directrix height H, reduced angular momentum l_z and foci sphere radius R were derived. It has been shown that some one-dimensional results map one to one to the 3D case. In addition, reduced angular momentum conservation (absent in 2D) yields some additional physics in 3D.

For future works it would be interesting to generalize our results to other rotational symmetric domains. Also, the motion in a non-constant, e.g. Keplerian-field, would be of interest, generalizing e.g. the results obtained in [Fed01, JZ24].

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Declarations

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Contact geometry of Hill's approximation in a spatial restricted four-body problem

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Abstract: It is well-known that the planar and spatial circular restricted three-body problem (CR3BP) is of contact type for all energy values below the first critical value. Burgos-Garcia and Gidea extended Hill's approach in the CR3BP to the spatial equilateral CR4BP, which can be used to approximate the dynamics of a small body near a Trojan asteroid of a Sun–planet system. Our main result in this paper is that this Hill four-body system also has the contact property. In other words, we can "contact" the Trojan. Such a result enables to use holomorphic curve techniques and Floer theoretical tools in this dynamical system in the energy range where the contact property holds.

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1 Introduction

Astronomical significance. One of the first triumphs in celestial mechanics was the Lagrange central configuration, one of the first explicit solutions to the three-body problem discovered by Lagrange [29] in 1772. It consists of three bodies, not necessarily of equal masses, forming the vertices of an equilateral triangle, each moving on a specific Kepler orbit. The triangular configuration of the bodies is maintained throughout the entire motion. A special type of Lagrange's solution is the rigid circular motion of the three bodies around their center of mass. It is common to use the term "Trojan" to describe a small body, an asteroid or a natural satellite, that lies in such equilateral triangular configuration together with the Sun and a planet, or with a planet and a moon. In other words, such small bodies remain near triangular points 60° ahead of or behind the orbit of a planet or a moon. Such triangular points correspond to the two equilateral Lagrange points, L_4 (leading) and L_5 (trailing), of a Sun–planet or a planet–moon system. Since the discovery of the first Trojan asteroid, 588 Achilles, near Jupiter's Lagrange point L_4 by Max Wolf of the Heidelberg Observatory in 1906 (see [37]), such configurations have not only deserved attention in theory, but have also gained tremendous astronomical significance. By now many other examples of Trojan-like asteroids in our solar system have become known. Jupiter has thousands of Trojans [44]; Mars [17] and Neptune [2] also have some; only two Earth Trojans have been discovered so far [41]. Meanwhile it is known [36] that the Saturn–Tethys system has two Trojans, Telesto (L_4-Trojan) and Calypso (L_5-Trojan), and the Saturn–Dione system has two as well, Helene (L_4-Trojan) and Polydeuces (L_5-Trojan). A twelve-year space probe to several Jupiter Trojans is currently being operated by NASA's Lucy mission, which was launched on 16 October 2021 as the first mission to the Jupiter Trojans (see e.g., [38] for a recent research result). Outside the solar system there exists also the possibility of a Trojan planet associated to extrasolar systems, formed by a star with similar mass as the Sun and a giant gas planet. Although such Trojan planets only play a fictitious role at the moment, their dynamics are already being analyzed theoretically [43].

In order to describe conveniently the dynamics of small bodies attracted by the gravitational field of three bodies in such a triangular central configuration, a restricted four-body problem (R4BP) becomes necessary. There are plenty of results on various models of the R4BP, such as [3], [8], [9], [13, 11, 12], [19], [22], [28], [31], [35], [42], [45]. Relevant for this work is the spatial equilateral circular one, in which three primaries circle around their common center of mass and forming an equilateral triangular configuration. In view of astronomical data associated to such configurations in the solar system, the mass of one of the primaries (the Trojan) is much smaller than the other two primaries. If one equates the mass of the Trojan to zero, the system represents the circular restricted three-body problem (CR3BP). Therefore, to study the dynamics in the vicinity of the Trojan, a practical and intelligent concept is to perform a Hill's approximation in the equilateral circular R4BP.

Hill's approximation. In 1878 Hill [24] introduced a limiting case of the CR3BP, as an approach to solve the motion of the Moon in the Sun–Earth problem. As a first approximation, the infinitesimal body (Moon) moves in the vicinity of the smaller primary (Earth) and, by a symplectic rescaling of coordinates, the remaining primary (Sun) is pushed infinitely far away in a way that it acts as a velocity independent gravitational perturbation of the rotating Kepler problem formed by the Earth and the Moon.

Extending Hill's concept to the equilateral circular R4BP was performed by Burgos-García and Gidea [12], which is the central system in this paper. This problem studies the dynamics near the Trojan and pushes the two remaining primaries (e.g., Sun and Jupiter) to infinity, and depends on two parameters, the energy of the system and the mass ratio $\mu \in [0, \frac{1}{2}]$ of the two primaries at infinity (system is symmetric with respect to $\mu = \frac{1}{2}$).

The case $\mu = 0$ corresponds to the classical Hill 3BP, therefore this Hill four-body model generalizes the classical Hill's approach. It is worth noting that this system is different as the one introduced by Scheeres [42], in which the motion of a spacecraft in the Sun perturbed Earth–Moon system is studied. Moreover, this Hill four-body system was extended in [11] as a problem with oblate bodies modeling the Sun–Jupiter–Hektor–Skamandrios system.

Why we care about contact property. One of Hill's main contributions was the discovery of one periodic solution with a period of one synodic month of the Moon. Hill's lunar theory was, as Wintner said [47, p. 1], *"considered by Poincaré as representing a turning point in the history of celestial mechanics"*. Poincaré sought to make periodic solutions central in the study of the global dynamics, a focus that has persisted since his pioneering work [39]. Inspired by Poincaré's concept of using global surface of sections for proving existence results of periodic orbits in the CR3BP [40], Birkhoff conjectured [10] that retrograde periodic orbits in the CR3BP bound a disk-like global surface of section (retrograde means that the motion of the orbit is in opposite direction as the coordinate system is rotating; direct is the one that rotates in the same direction). Due to preservation of an area form with finite total area, one can apply Brouwer's translation theorem to the Poincaré return map associated to the disk-like global surface of section and find at least one fixed point that should correspond to a direct periodic orbit. The direct orbit is astronomically more significant, since our Moon moves in a direct motion around the Earth, whose existence is based so far on numerical computations, as Hill's lunar orbit. Such fixed point approaches related to existence results of periodic orbits are sources of inspiration that have laid the fruitful fundamental principles of powerful abstract methods and important breakthroughs in symplectic and contact geometry, such as the work by Floer [20] on the Arnold conjecture, by Hofer [25] and Taubes [46] on the Weinstein conjecture, and by Hofer–Wysocki–Zehnder [26] on the construction of disk-like global surface of sections with the help of holomorphic curves. The assumption that energy level sets are of contact type enable to use holomorphic curve and Floer theoretical techniques in the energy range where the contact property holds. Especially, many recent deep results associated with the dynamics of low-dimensional contact manifolds have been proved: In the 3-dimensional case, [16] proved the existence of supporting broken book decompositions, [15] showed how to use these broken book decompositions to construct Birkhoff sections or global surfaces of section, [18] gave a detailed description of the dynamics when there are exactly two simple Reeb orbits, and [27] described an abstract framework for proving strong closing properties based on the smooth closing lemma for Reeb flows; in the 5-dimensional case, [33] showed the relation between the spatial dynamics of the CR3BP and iterated open book decompositions. We also refer to [21] for a profound introduction to holomorphic techniques and their use in celestial mechanics, particularly focused on the CR3BP and the above Birkhoff's conjecture. Another dynamical consequence of the contact property of energy level sets, discussed in the latter reference, is that blue sky catastrophes cannot occur. On a practical level, Floer theoretic bifurcation tools have recently been applied to numerical investigations of periodic orbits [4], [6], [32].

Main result. For the planar CR3BP it is well-known that below the first critical value, the two bounded components of the energy level sets, after Moser regularization, are of contact type [1]. Each component corresponds to the unit cotangent bundle of S^2 with the standard contact structure, meaning that each contact manifold corresponds to (S^*S^2, ξ_{st}). The same result for the spatial case was shown in [14], where each contact manifold corresponds to (S^*S^3, ξ_{st}). We note that [21, Chapter 6.1] proved the same result for the classical planar Hill 3BP.

The Hill four-body system we consider has four Lagrange points, where L_1 is symmetric to L_2 (lying on the x-axis), and L_3 is symmetric to L_4 (lying on the y-axis). If the energy value c is below the first critical value $H(L_{1/2})$, then the energy level set has one bounded component (where the origin is contained), which we denote by Σ_c^b. This component is non-compact because of a singularity at the origin corresponding to collision. After performing Moser regularization, we obtain a compact 5-dimensional manifold, which we denote by $\widetilde{\Sigma}_c^b$. The spatial system is invariant under a symplectic involution σ which is induced by the reflection at the ecliptic. The restriction of the spatial problem to the fixed point set $\mathrm{Fix}(\sigma)$ corresponds to the planar problem. In fact, we can restrict the whole procedure to $\mathrm{Fix}(\sigma)$ and obtain a compact 3-dimensional manifold, which we denote by $\widetilde{\Sigma}_c^b|_{\mathrm{Fix}(\sigma)}$. Our main result in this paper is the following theorem.

Theorem 1.1. *For any given $\mu \in [0, \frac{1}{2}]$ it holds that*

$$\widetilde{\Sigma}_c^b \cong (S^*S^3, \xi_{st}), \quad \text{if } c < H(L_{1/2}),$$
$$\widetilde{\Sigma}_c^b|_{\mathrm{Fix}(\sigma)} \cong (S^*S^2, \xi_{st}), \quad \text{if } c < H(L_{1/2}).$$

Our method to prove Theorem 1.1 is the same as in [1], [14], namely we find a Liouville vector field on the cotangent bundle which is transverse to $\widetilde{\Sigma}_c^b$ whenever $c < H(L_{1/2})$. This transversality result implies the contact property. The Liouville vector field we use is inspired by Moser regularization, which first interchanges the roles of position and momenta, and then uses the stereographic projection. In this setting, the Liouville vector field is the natural one (i.e., the radial vector field in fiber direction) on the new cotangent bundle structure after switching position and momenta. Therefore, our transversality result implies in particular fiberwise starshapedness.

Organization of the paper. In Section 2 we discuss the Hamiltonian, its linear symmetries, Lagrange points and Hill's regions. The goal of Section 3 is to prove Theorem 1.1. We first recall some basic definitions and notations from contact geometry, and then show transversality in the non-regularized case. After this, we perform Moser regularization and prove therein the transversality property.

2 Hill's approximation in the spatial equilateral circular R4BP

2.1 Hamiltonian

We consider three point masses (primaries), B_1, B_2 and B_3, moving in circular periodic orbits in the same plane with constant angular velocity around their common center of gravity fixed at the origin, while forming an equilateral triangle configuration (see Figure 1). A fourth body B_4 is significantly smaller than the other three and thus a negligible effect on their motion. We set B_1 on the negative x-axis at the origin of time and assume that the corresponding three masses are $m_1 \geq m_2 \geq m_3$. It is convenient to choose the units of mass, distance and time such that the gravitational constant is 1, and the period of the circular orbits is 2π. In these units the side length of the equilateral triangle configuration is normalized to be one, and $m_1 + m_2 + m_3 = 1$. Moreover, it is convenient to use a rotating frame of reference that rotates with an angular velocity of the orbital angular rate of the primaries. Then, the dynamics of the infinitesimal body B_4 is described by the Hamiltonian

$$H(x, y, z, p_x, p_y, p_z) = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) - \frac{m_1}{r_1} - \frac{m_2}{r_2} - \frac{m_3}{r_3} + p_x y - p_y x,$$

which is a first integral of the system. An equivalent first integral is the Jacobi integral C defined by $C = -2H$. Notice that r_i indicates the corresponding distance from B_4 to i-th primary, for $i = 1, 2, 3$. The general expressions of the position coordinates $(x_i, y_i, 0)$ can be seen in [9]. If $m_3 = 0$ and $m_2 = \mu$, then one recovers the constellation of the CR3BP associated to B_1 and B_2, where B_3 is located at the equilateral Lagrange point L_4. Moreover, the phase space is the trivial cotangent bundle $T^*\left(\mathbb{R}^3 \setminus \{B_1, B_2, B_3\}\right) = \left(\mathbb{R}^3 \setminus \{B_1, B_2, B_3\}\right) \times \mathbb{R}^3$, endowed with the standard symplectic form $\omega = \sum dp_k \wedge dk$ $(k = x, y, z)$. The flow of the Hamiltonian vector field X_H, defined by $dH(\cdot) = \omega(\cdot, X_H)$, is equivalent to the equations of motion, $\left\{\dot{k} = \frac{\partial H}{\partial p_k}, \dot{p}_k = -\frac{\partial H}{\partial k}\right\}$ $(k = x, y, z)$.

We now briefly recall the fundamental steps of Hill's approximation, as performed in [12] where the details can be seen. Let B_3 be the primary (the Trojan), whose mass is much smaller than the other two primaries. The first step is to set the Trojan to the origin.

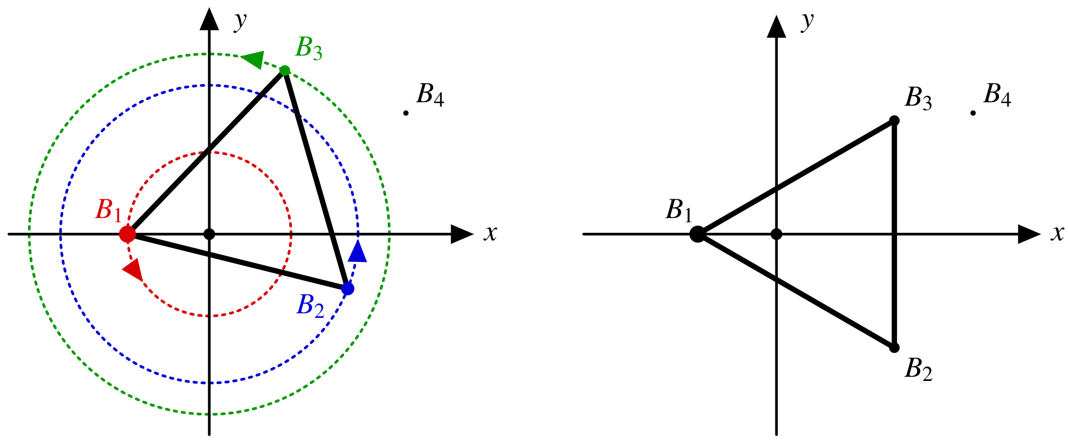

Figure 1: Equilateral circular restricted four-body problem. Left: Case of $m_1 > m_2 > m_3$. Right: Case of $m_2 = m_3$ in a rotating frame of reference; B_2 and B_3 are located symmetrically with respect to B_1.

The second step rescales symplectically the coordinates depending on $m_3^{1/3}$. The third step makes use of a Taylor expansion of the gravitational potential of the Hamiltonian in powers of $m_3^{1/3}$. Finally, the limiting case for $m_3 \to 0$ yields the Hamiltonian

$$H(x, y, z, p_x, p_y, p_z) = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + p_x y - p_y x - \frac{1}{r} + \frac{1}{8}x^2 - \frac{3\sqrt{3}}{4}(1-2\mu)xy - \frac{5}{8}y^2 + \frac{1}{2}z^2,$$

where $r = \left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}$, $m_1 = 1 - \mu$ and $m_2 = \mu$. Notice that if one expands the Hamiltonian of the CR3BP centered at the equilateral Lagrange point L_4, then the quadratic part corresponds to $H + 1/r$.

Furthermore, after applying a rotation in the xy-plane, the system is equivalent with the system characterized by the Hamiltonian

$$H(x, y, z, p_x, p_y, p_z) = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + p_x y - p_y x - \frac{1}{r} + ax^2 + by^2 + \frac{1}{2}z^2, \tag{1}$$

where

$$a = \frac{1}{2}(1-\lambda_2), \quad b = \frac{1}{2}(1-\lambda_1), \quad \lambda_1 = \frac{3}{2}(1-d), \quad \lambda_2 = \frac{3}{2}(1+d), \quad d = \sqrt{1-3\mu+3\mu^2}.$$

Since $d(1-\mu) = d(\mu)$, we can assume that $\mu \in [0, \frac{1}{2}]$. Notice that λ_1 and λ_2 are the eigenvalues corresponding to the rotation transformation in the xy-plane. The quantities $a, b, \lambda_1, \lambda_2$

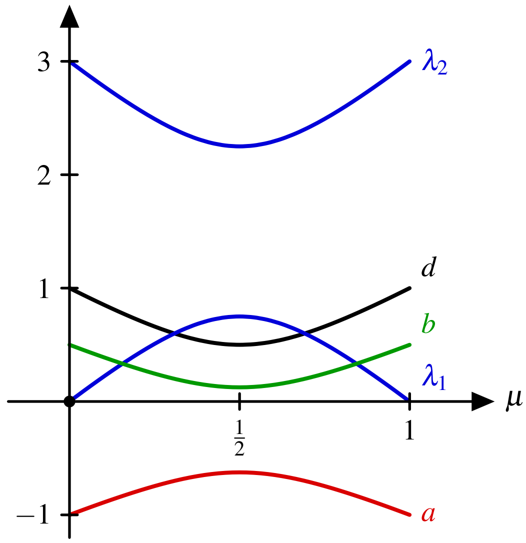

Figure 2: The quantities a (red), b (green), λ_1, λ_2 (both blue) and d (black).

and d are plotted in Figure 2. The Hamiltonian (1) consists of the rotating Kepler problem (formed by the Trojan and the infinitesimal body) with a velocity independent gravitational perturbation produced by the two remaining massive primaries (the degree 2 term $ax^2 + by^2 + \frac{1}{2}z^2$) which are sent at infinite distance. By introducing the effective potential

$$U: \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}, \quad (x, y, z) \longmapsto -\frac{1}{r} - \frac{1}{2}\left(\lambda_2 x^2 + \lambda_1 y^2 - z^2\right), \tag{2}$$

the Hamiltonian (1) can be written as

$$H(x, y, z, p_x, p_y, p_z) = \frac{1}{2}\left((p_x + y)^2 + (p_y - x)^2 + p_z^2\right) + U(x, y, z), \tag{3}$$

and the equations of motion are given by

$$\begin{aligned} \ddot{x} - 2\dot{y} &= -\frac{\partial U}{\partial x} = \left(\lambda_2 - \frac{1}{r^3}\right)x \\ \ddot{y} + 2\dot{x} &= -\frac{\partial U}{\partial y} = \left(\lambda_1 - \frac{1}{r^3}\right)y \\ \ddot{z} &= -\frac{\partial U}{\partial z} = -\left(1 + \frac{1}{r^3}\right)z. \end{aligned} \tag{4}$$

In particular, the case $\mu = 0$ recovers the classical Hill 3BP. While the Hill 3BP depends only on the energy of the orbit, this systems depends on two parameters, the mass ratio

μ and the energy of the system. Specific μ-values of practical interest are for example $\mu = 0.00095$, which approximates the Sun–Jupiter mass ratio, and $\mu = 0.00547$, which corresponds to the extrasolar system associated to the Sun-like star HD 28185 and its Jupiter-like exoplanet HD 28185 b.

2.2 Linear symmetries

A "symmetry" σ is, by definition, a symplectic or anti-symplectic involution of the phase space which leaves the Hamiltonian invariant, i.e.,

$$H \circ \sigma = H, \quad \sigma^2 = \mathrm{id}, \quad \sigma^* \omega = \pm \omega. \tag{5}$$

Anti-symplectic symmetries denote time-reversal symmetries in the Hamiltonian context, see e.g., [30]. A periodic solution $\mathbf{x} \equiv (x, y, z, p_x, p_y, p_z)$ is symmetric with respect to an anti-symplectic symmetry ρ if $\mathbf{x}(t) = \rho(\mathbf{x}(-t))$ for all t, and symmetric with respect to a symplectic one σ if $\mathbf{x}(t) = \sigma(\mathbf{x}(t))$ for all t.

The reflection at the ecliptic $\{z = 0\}$ gives rise to a linear symplectic symmetry of (1), denoted by

$$\sigma(x, y, z, p_x, p_y, p_z) = (x, y, -z, p_x, p_y, -p_z), \tag{6}$$

whose fixed point set $\mathrm{Fix}(\sigma) = \{(x, y, 0, p_x, p_y, 0)\}$ corresponds to the planar problem. Other linear symplectic symmetries are $-\sigma$ and $\pm\mathrm{id}$, where $-\sigma$ corresponds to the π-rotation around the z-axis, hence the z-axis is invariant under $-\sigma$. Linear anti-symplectic symmetries are determined by

- $\rho_1(x, y, z, p_x, p_y, p_z) = (x, -y, -z, -p_x, p_y, p_z)$ (π-rotation around the x-axis),
- $\rho_2(x, y, z, p_x, p_y, p_z) = (x, -y, z, -p_x, p_y, -p_z)$ (reflection at the xz-plane),
- $\rho_3(x, y, z, p_x, p_y, p_z) = (-x, y, -z, p_x, -p_y, p_z)$ (π-rotation around the y-axis),
- $\rho_4(x, y, z, p_x, p_y, p_z) = (-x, y, z, p_x, -p_y, -p_z)$ (reflection at the yz-plane).

Together with the previous linear symplectic symmetries, they form the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. If one restrict the system to $\mathrm{Fix}(\sigma)$, linear anti-symplectic symmetries for the planar problem are given by

- $\rho_x(x, y, 0, p_x, p_y, 0) = (x, -y, 0, -p_x, p_y, 0)$ (reflection at the x-axis),
- $\rho_y(x, y, 0, p_x, p_y, 0) = (-x, y, 0, p_x, -p_y, 0)$ (reflection at the y-axis),

that together with the linear symplectic ones $\{\pm \mathrm{id}\}$ form a Klein-four group $\mathbb{Z}_2 \times \mathbb{Z}_2$. These symmetries show that it is not possible to say which of the two primaries at infinity we are moving to or away from.

Remark 2.1. In [7] it shown that the Hill 3BP ($\mu = 0$) has two special properties.

i) *The spatial linear symmetries already determine the planar ones.* The same phenomenon is also true for all $\mu \in [0, \frac{1}{2}]$. To see this, let us denote by Σ_s and Σ_p each set of spatial and planar linear symmetries. Consider the projection map given by the restriction to $\mathrm{Fix}(\sigma)$,

$$\pi : \Sigma_s \to \Sigma_p, \quad \rho \longmapsto \rho|_{\mathrm{Fix}(\sigma)}.$$

If $\rho \in \Sigma_s$, then $\rho|_{\mathrm{Fix}(\sigma)} \in \Sigma_p$ with the corresponding (anti-)symplectic property. While π is not injective (since $\pi(\rho_1) = \pi(\rho_2)$), it is surjective. If $\rho \in \Sigma_p$ is symplectic (or anti-symplectic), then a symplectic (or anti-symplectic) extension is given by $z \longmapsto z$ and $p_z \longmapsto p_z$ (or $z \longmapsto -z$ and $p_z \longmapsto p_z$).

ii) *There are no other linear symmetries.* This statement also holds for all $\mu \in [0, \frac{1}{2}]$. Its proof uses the equations (5) and the properties of linear symplectic and anti-symplectic involutions. Since the exact same computations work for (1) for all $\mu \in [0, \frac{1}{2}]$, we forgo its proof in this paper.

2.3 Lagrange points and Hill's region

From the third equation in (4) it is obvious that all Lagrange points are located at the ecliptic $\{z = 0\}$. Using the projection onto the configuration space given by

$$\pi : \mathbb{R}^3 \setminus \{0\} \times \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{0\}, \quad (x, y, z, p_x, p_y, p_z) \longmapsto (x, y, z), \tag{7}$$

there is a one-to-one correspondence between critical points of the Hamiltonian (3) and the effective potential (2), determined by $\left(\pi|_{\mathrm{crit}(H)}\right)^{-1}(x, y, 0) = (x, y, 0, -y, x, 0)$. In [12] it is shown that (2) has four critical points, whose coordinates are given explicitly in terms of μ,

$$L_1 = \left(\frac{1}{\sqrt[3]{\lambda_2}}, 0, 0\right), \quad L_2 = \left(-\frac{1}{\sqrt[3]{\lambda_2}}, 0, 0\right), \quad L_3 = \left(0, \frac{1}{\sqrt[3]{\lambda_1}}, 0\right), \quad L_4 = \left(0, -\frac{1}{\sqrt[3]{\lambda_1}}, 0\right).$$

Note that $L_{1/2}$ are related to each other by ρ_y (reflection at the y-axis), and $L_{3/4}$ are related to each other by ρ_x (reflection at the x-axis). The classical Hill 3BP ($\mu = 0$) only has $L_{1/2}$, and especially, if $\mu \to 0$ then $\lambda_1 \to 0$, which means that L_3 and L_4 are sent to infinity. Therefore, the presence of a second primary at infinity for $\mu \in (0, \frac{1}{2}]$ produces the two additional Lagrange points $L_{3/4}$. Since $\lambda_2 > \lambda_1$, we have for the critical values

$$H(L_{1/2}) = -\frac{3}{2}\sqrt[3]{\lambda_2} < -\frac{3}{2}\sqrt[3]{\lambda_1} = H(L_{3/4}), \quad \text{for all } \mu \in (0, \frac{1}{2}].$$

We now consider the energy level set $\Sigma_c := H^{-1}(c)$, for $c \in \mathbb{R}$. In view of the footpoint projection (7), the "Hill's region" of Σ_c is defined as

$$\mathcal{K}_c := \pi(\Sigma_c) \subset \mathbb{R}^3 \setminus \{0\},$$

which means that the Hill's region of the energy level set is its shadow under the footpoint projection. Since the first three terms in (3) are quadratic and hence non-negative, we can obtain the Hill's region by

$$\mathcal{K}_c = \left\{(x, y, z) \in \mathbb{R}^3 \setminus \{0\} \mid U(x, y, z) \leq c\right\}.$$

The topology of the Hill's region depends on the energy level. If $c < H(L_{1/2})$, then the Hill's region has two connected components, one bounded and one unbounded (see Figure 3). We denote the bounded component by $\mathcal{K}_c^b$ and abbreviate by

$$\Sigma_c^b := \pi^{-1}(\mathcal{K}_c^b) \cap \Sigma_c \tag{8}$$

the corresponding connected component of Σ_c.

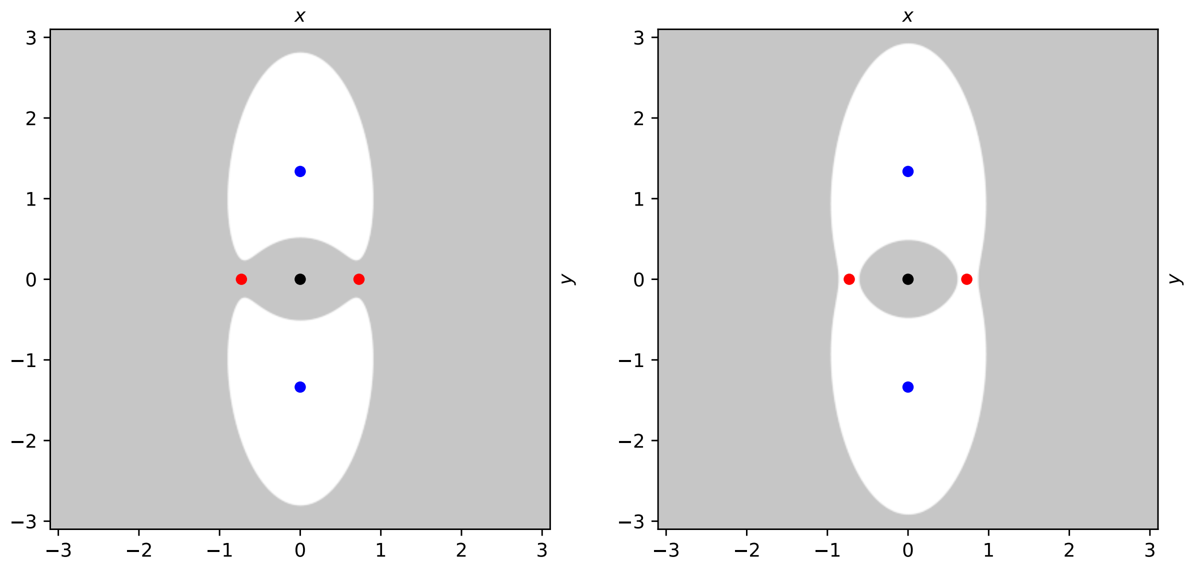

Figure 3: Hill's region (gray shaded domains) for planar problem $\{z = 0\}$ for $\mu = 0.2$. White domains correspond to forbidden regions. Red dots indicate $L_{1/2}$; blue dots indicate $L_{3/4}$. Right: For $c < H(L_{1/2})$. Left: For $H(L_{1/2}) < c < H(L_{3/4})$. In the Hill 3BP ($\mu = 0$), when $L_{3/4}$ are sent to infinity, below the critical value the Hill's region consists of one bounded component and two unbounded components.

3 Contact property - Proof of Theorem 1.1

3.1 Basic notations

We now recall some basic definitions and notations from contact geometry, and refer for details to [23].

Definition 3.1. Let M be a smooth manifold of odd dimension $2n+1$. A "contact form" on M is a 1-form $\alpha \in \Omega^1(M)$ such that $\alpha \wedge (d\alpha)^{\wedge n} \neq 0$. Given a contact form α, the hyperplane field $\xi = \ker\alpha \subset TM$ is oriented by $d\alpha$, and this oriented codimension-1 field is called the "contact structure". The pair (M, ξ) is called "contact manifold". The "Reeb vector field" R_α is the unique vector field defined by the equations $d\alpha(R_\alpha, \cdot) = 0$ and $\alpha(R) = 1$, whose flow is called "Reeb flow".

Definition 3.2. A "Liouville vector field" X on a symplectic manifold (M, ω) is a vector field satisfying $\mathscr{L}_X \omega = \omega$, where $\mathscr{L}$ denotes the Lie derivative, i.e., the Lie derivative along X preserves ω.

By Cartan's formula and the closedness of the symplectic form ω, we have $\mathscr{L}_X \omega = d(\iota_X \omega) + \iota_X d\omega = d(\iota_X \omega)$ and therefore, we can write the Liouville condition as $d(\iota_X \omega) = \omega$, where $\iota_X \omega(\cdot) = \omega(X, \cdot)$.

Example 3.3. The cotangent bundle T^*Q of a smooth manifold Q of dimension n is endowed with the so-called "Liouville one-form". In local coordinates $(q_1, ..., q_n)$ on Q and dual coordinates $(p_1, ..., p_n)$ on the fibers of T^*Q, the Liouville one-form is defined by $\lambda_{can} = \sum_{i=1}^n p_i dq_i$. Since the standard symplectic form is characterized by $\omega_{can} = d\lambda_{can} = \sum_{i=1}^n dp_i \wedge dq_i$, the "natural Liouville vector field" X on T^*Q associated to λ_{can} is defined by $\iota_X \omega_{can} = \lambda_{can}$. In local coordinates,

$$X = \sum_{i=1}^n p_i \frac{\partial}{\partial p_i},$$

that is, the radial vector field in fiber direction.

Hypersurfaces of contact type. Let X be a Liouville vector field on a $2n+2$ dimensional symplectic manifold (M, ω). Then $\alpha := \iota_X \omega|_\Sigma$ is a contact form on any hypersurface $\Sigma \subset M$ transverse to X (i.e., with X nowhere tangent to Σ). Such hypersurfaces are said to be of "contact type". To see this, let $x \in \Sigma$ and let $\{v_1, ..., v_{2n+1}\}$ be a basis of $T_x\Sigma$. By using the Liouville condition we have,

$$\alpha \wedge (d\alpha)^{\wedge n}(v_1, ..., v_{2n+1}) = \iota_X \omega \wedge \omega^{\wedge n}(v_1, ..., v_{2n+1}) = \frac{1}{n} \omega^{\wedge(n+1)}(X, v_1, ..., v_{2n+1}). \tag{9}$$

Since $\{X, v_1, ..., v_{2n+1}\}$ is a basis of T_xM (due to transversality) and $\omega^{\wedge(n+1)}$ is a volume form on M, we obtain that (9) is non-zero, i.e., the contact condition is satisfied.

Any hypersurface $\Sigma \subset M$ has a characteristic foliation L which is a rank 1 foliation with $L_x = \ker(\omega|_{T_x\Sigma})$, for $x \in \Sigma$. If Σ is a energy level set of a Hamiltonian $H: M \to \mathbb{R}$, then for $x \in \Sigma$ we have that $X_H(x) \in L_x$. If Σ is of contact type, then $R_\alpha(x) \in L_x$, i.e., the Reeb flow of α is a reparametrization of the Hamiltonian flow. In the case of $M = T^*Q$, if the contact form on $\Sigma \subset T^*Q$ is induced by the transversality of the natural Liouville vector field X on T^*Q, then the contact structure is called the “standard contact structure” determined by

$$\xi_{st} = \ker \alpha_{can}, \quad \alpha_{can} := \iota_X \omega_{can}|_\Sigma = \lambda_{can}|_\Sigma.$$

Moreover, in this case the energy hypersurface $\Sigma \subset T^*Q$ is “fiberwise starshaped”, i.e., for each point $q \in Q$ the intersection $\Sigma \cap T_q^*Q$ bounds a starshaped domain in the linear space T_q^*Q, which means that the natural Liouville vector field is transverse to each $\Sigma \cap T_q^*Q$.

3.2 Proof of transversality in non-regularized case

We now consider the Liouville vector field on $T^*\mathbb{R}^3$ given by

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}. \tag{10}$$

Proposition 3.4. *For any given $\mu \in [0, \frac{1}{2}]$ assume that $c < H(L_{1/2}) = -\frac{3}{2}\sqrt[3]{\lambda_2}$. Then the bounded component Σ_c^b of the energy level set, as defined by (8), is transverse to X.*

As a consequence of Proposition 3.4, $\iota_X \omega|_{\Sigma_c^b}$ defines a contact form on Σ_c^b. In order to prove Proposition 3.4, we need some properties of the effective potential (2), which we formulate in three lemmas and discuss in spherical coordinates,

$$x = \rho \cos\theta \sin\varphi$$

$$y = \rho \sin\theta \sin\varphi$$

$$z = \rho \cos\varphi$$

where $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \pi$. Since we consider energy level sets below the first critical value, the radius ρ is always smaller than the distance from $L_{1/2}$ to the origin, which is $1/\sqrt[3]{\lambda_2}$ and always less than 1. Therefore, we assume that the radius ρ is smaller than 1. Now the effective potential (2) reads

$$U(\rho, \theta, \varphi) = -\frac{1}{\rho} - \frac{1}{2}\rho^2(\lambda_2 \cos^2\theta \sin^2\varphi + \lambda_1 \sin^2\theta \sin^2\varphi - \cos^2\varphi),$$

which is π-periodic in the variables θ and φ.

Lemma 3.5. *For fixed radius $\rho \in (0,1)$ the function $U_\rho := U(\rho,\cdot,\cdot)$ has its minimum at $(\theta,\varphi) = (0,\frac{\pi}{2})$.*

Proof. The differential is given by

$$dU_\rho(\theta,\varphi) = \rho^2(\lambda_2-\lambda_1)\cos\theta\sin\theta\sin^2\varphi d\theta + \rho^2\sin\varphi\cos\varphi(\lambda_2\cos^2\theta + \lambda_1\sin^2\theta + 1)d\varphi.$$

Since $\lambda_2 > \lambda_1$, and the term $\lambda_2\cos^2\theta + \lambda_1\sin^2\theta + 1$ is strictly positive, we find four critical points at $(0,0)$, $(0,\frac{\pi}{2})$, $(\frac{\pi}{2})$ and $(\frac{\pi}{2},\frac{\pi}{2})$. The corresponding Hessians are given by

$$H_{U_\rho}(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -\rho^2(\lambda_2+1) \end{pmatrix}, \quad H_{U_\rho}(0,\frac{\pi}{2}) = \begin{pmatrix} \rho^2(\lambda_2-\lambda_1) & 0 \\ 0 & \rho^2(\lambda_2+1) \end{pmatrix}$$
$$H_{U_\rho}(\frac{\pi}{2},0) = \begin{pmatrix} 0 & 0 \\ 0 & -\rho^2(\lambda_1+1) \end{pmatrix}, \quad H_{U_\rho}(\frac{\pi}{2},\frac{\pi}{2}) = \begin{pmatrix} -\rho^2(\lambda_2-\lambda_1) & 0 \\ 0 & \rho^2(\lambda_1+1) \end{pmatrix}.$$

Therefore, the function U_ρ attains its minimum at $(\theta,\varphi) = (0,\frac{\pi}{2})$. □

We denote by $r := 1/\sqrt[3]{\lambda_2}$ the distance from $L_{1/2}$ to the origin and introduce

$$B_r(0) := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq r^2\}$$

the ball of radius r centered at the origin.

Corollary 3.6. *The bounded part of Hill's region, $\mathcal{K}_c^b$, is contained in $B_r(0)$.*

Proof. Let $(\rho,\theta,\varphi) \in \partial B_r(0)$, i.e., $\rho = r = 1/\sqrt[3]{\lambda_2}$. Then, by Lemma 3.5,

$$U(r,\theta,\varphi) \geq U(r,0,\frac{\pi}{2}) = -\frac{1}{r} - \frac{1}{2}r^2\lambda_2 = -\frac{3}{2}\sqrt[3]{\lambda_2} = H(L_{1/2}) > c. \tag{11}$$

Therefore, (r,θ,φ) does not lie in $\mathcal{K}_c^b$, and hence, $\partial B_r(0) \cap \mathcal{K}_c^b = \emptyset$. Since $\mathcal{K}_c^b$ is connected and contains the origin in its closure, $\mathcal{K}_c^b$ is contained in $B_r(0)$. □

Lemma 3.7. *For every $(\rho,\theta,\varphi) \in B_r(0)$ with $\rho \in (0,r)$ it holds that $\frac{\partial U}{\partial \rho}(\rho,\theta,\varphi) > 0$.*

Proof. Let $(\rho, \theta, \varphi) \in B_r(0)$ with $\rho \in (0, r)$. Since $\lambda_2 > \lambda_1$ we have the following equivalences

$$(\lambda_1 - \lambda_2)\sin^2\theta \leq 0 \quad \Leftrightarrow \quad \lambda_2(\cos^2\theta - 1) + \lambda_1\sin^2\theta \leq 0 \quad \Leftrightarrow \quad \lambda_2\cos^2\theta + \lambda_1\sin^2\theta \leq \lambda_2. \quad (12)$$

By using (12), we estimate

$$\frac{\partial U}{\partial \rho} = \frac{1}{\rho^2} - \rho\left(\lambda_2\cos^2\theta\sin^2\varphi + \lambda_1\sin^2\theta\sin^2\varphi - \cos^2\varphi\right) \geq \frac{1}{\rho^2} - \lambda_2\rho > 0. \quad (13)$$

The last strict inequality holds since the function $f\colon (0, r) \to \mathbb{R}$, $x \longmapsto \frac{1}{x^2} - \lambda_2 x$ is strictly positive on its domain. □

Lemma 3.8. *For every* $(\rho, \theta, \varphi) \in B_r(0)$ *with* $\rho > 0$ *it holds that* $\frac{\partial^2 U}{\partial \rho^2}(\rho, \theta, \varphi) \leq -\sin^2\varphi$.

Proof. Let $(\rho, \theta, \varphi) \in B_r(0)$ with $\rho > 0$. Since the function $f\colon (0, r] \to \mathbb{R}$, $x \longmapsto -\frac{1}{x^3}$ takes the maximal value at $x = r$, and because $\lambda_2 \geq 2$, we estimate

$$\frac{\partial^2 U}{\partial \rho^2} = -\frac{2}{\rho^3} + \cos^2\varphi - \sin^2\varphi\left(\lambda_2\cos^2\theta + \lambda_1\sin^2\theta\right) \leq -\frac{2}{r^3} + 1 = -2\lambda_2 + 1 \leq -3 \leq -\sin^2\varphi. \quad □$$

Proof of Proposition 3.4. We show that

$$dH(X)|_{\Sigma_c^b} > 0. \quad (14)$$

The differential of the Hamiltonian (1) is given by

$$\begin{aligned} dH =& p_x dp_x + p_y dp_y + p_z dp_z + p_x dy + y dp_x - p_y dx - x dp_y \\ & + 2ax dx + 2by dy + z dz + \frac{x}{r^3}dx + \frac{y}{r^3}dy + \frac{z}{r^3}dz. \end{aligned} \quad (15)$$

By inserting the Liouville vector field (10) into (15) we obtain

$$dH(X) = p_x y - p_y x + 2ax^2 + 2by^2 + z^2 + \frac{1}{r}. \quad (16)$$

Recall that $a = \frac{1}{2}(1 - \lambda_2)$ and $b = \frac{1}{2}(1 - \lambda_1)$. In spherical coordinates the Liouville vector field (10) becomes

$$X = \rho\frac{\partial}{\partial \rho},$$

and (16) reads

$$dH(X) = p_x\rho\sin\theta\sin\varphi - p_y\rho\cos\theta\sin\varphi + (1+\lambda_2)\rho^2\cos^2\theta\sin^2\varphi + (1-\lambda_1)\rho^2\sin^2\theta\sin^2\varphi + \rho^2\cos^2\varphi + \frac{1}{\rho}. \tag{17}$$

In view of $\frac{\partial U}{\partial\rho}$ from (13), we write (17) in the form

$$dH(X) = \rho\sin\theta\sin\varphi(p_x + \rho\sin\theta\sin\varphi) - \rho\cos\theta\sin\varphi(p_y - \rho\cos\theta\sin\varphi) + \rho\frac{\partial U}{\partial\rho},$$

which we estimate by using the Cauchy–Schwarz inequality,

$$\begin{aligned} dH(X) &\geq \rho\frac{\partial U}{\partial\rho} - \rho\sin\varphi\sqrt{(p_x + \rho\sin\theta\sin\varphi)^2 + (p_y - \rho\cos\theta\sin\varphi)^2} \\ &= \rho\frac{\partial U}{\partial\rho} - \rho\sin\varphi\sqrt{2(H-U) - p_z^2} \\ &\geq \rho\frac{\partial U}{\partial\rho} - \rho\sin\varphi\sqrt{2(H-U)}. \end{aligned}$$

Therefore, we have

$$dH(X)|_{\Sigma_c^b} \geq \rho\left(\frac{\partial U}{\partial\rho} - \sin\varphi\sqrt{2(c-U)}\right).$$

Since the right hand side is independent of the momentum coordinates, to prove (14) it is suffices to show that

$$\left.\left(\frac{\partial U}{\partial\rho} - \sin\varphi\sqrt{2(c-U)}\right)\right|_{\mathscr{K}_c^b} > 0. \tag{18}$$

Let $(\rho,\theta,\varphi) \in \mathscr{K}_c^b$. In particular, $U(\rho,\theta,\varphi) \leq c$. By Corollary 3.6, we have $\rho < r$, and by (11) it holds that $U(r,\theta,\varphi) > c$. Therefore, it exists $\tau \in [0, r-\rho)$ such that

$$U(\rho+\tau,\theta,\varphi) = c.$$

By using Lemma 3.7 and Lemma 3.8 we obtain

$$\begin{aligned}\left(\frac{\partial U}{\partial \rho}(\rho,\theta,\varphi)\right)^2 &= \left(\frac{\partial U}{\partial \rho}(\rho+\tau,\theta,\varphi)\right)^2 - \int_0^\tau \frac{d}{dt}\left(\frac{\partial U}{\partial \rho}(\rho+t,\theta,\varphi)\right)^2 dt \\ &> -2\int_0^\tau \frac{\partial U}{\partial \rho}(\rho+t,\theta,\varphi)\frac{\partial^2 U}{\partial \rho^2}(\rho+t,\theta,\varphi)dt \\ &\geq 2\sin^2\varphi \int_0^\tau \frac{\partial U}{\partial \rho}(\rho+t,\theta,\varphi)dt \\ &= 2\sin^2\varphi\,(U(\rho+\tau,\theta,\varphi)-U(\rho,\theta,\varphi)) \\ &= 2\sin^2\varphi\,(c-U(\rho,\theta,\varphi)).\end{aligned}$$

Therefore, by using Lemma 3.7 once more, we imply

$$\frac{\partial U}{\partial \rho}(\rho,\theta,\varphi) > \sin\varphi\sqrt{2\,(c-U(\rho,\theta,\varphi))},$$

which shows (18) and thereby the proposition. □

3.3 Moser-regularized energy level set and proof of transversality near the origin

The Hamiltonian (1) has a singularity at the origin corresponding to collisions, thus the bounded component Σ_c^b of the energy level set is non-compact. Moser [34] observed that the regularized Kepler problem coincides with the geodesic flow on the sphere endowed with its standard metric by interchanging the roles of position and momenta. To remove the singularity in our problem, we use the same concept as introduced by Moser.

We abbreviate by $\mathbf{X} = (x,y,z)$ and $\mathbf{P} = (p_x,p_y,p_z)$ the corresponding position and momentum coordinates. We use a new time parameter s and define for an energy value $c < H(L_{1/2}) = -\frac{3}{2}\sqrt[3]{\lambda_2}$ a new Hamiltonian by

$$s = \int \frac{dt}{|\mathbf{X}|}, \quad K_c(\mathbf{X},\mathbf{P}) := |\mathbf{X}|\,(H(\mathbf{X},\mathbf{P})-c)\,,$$

Notice that the flow of H at energy level c corresponds to the flow of K_c at energy level 0. Now we interchange the roles of position and momenta by the symplectic transformation mapping $(\mathbf{X}, \mathbf{P})$ to $(-\mathbf{P}, \mathbf{X})$. For simplicity of notation, we replace the new coordinates $\mathbf{X}' = -\mathbf{P}$ and $\mathbf{P}' = \mathbf{X}$ by $\mathbf{X}$ and $\mathbf{P}$. Then, the new transformed Hamiltonian $\widetilde{K}_c(\mathbf{X}, \mathbf{P}) = K_c(-\mathbf{P}, \mathbf{X})$ is explicitly given by

$$\begin{aligned}\widetilde{K}_c(\mathbf{X}, \mathbf{P}) &= \frac{1}{2}|\mathbf{X}|^2|\mathbf{P}| + |\mathbf{P}|(p_x y - p_y x) - 1 + |\mathbf{P}|(ap_x^2 + bp_y^2 + \frac{1}{2}p_z^2) - |\mathbf{P}|c \\ &= \frac{1}{2}\left(|\mathbf{X}|^2 + 1\right)|\mathbf{P}| + (p_x y - p_y x)|\mathbf{P}| - 1 + (ap_x^2 + bp_y^2 + \frac{1}{2}p_z^2)|\mathbf{P}| - (c + \frac{1}{2})|\mathbf{P}|.\end{aligned} \tag{19}$$

The next step is to use the stereographic projection which induces a symplectic transformation between $T^*\mathbb{R}^3$ and T^*S^3 that extends $\widetilde{K}_c$ to a Hamiltonian on T^*S^3. Let $\xi = (\xi_0, \xi_1, \xi_2, \xi_3) \in \mathbb{R}^4$ with norm 1. We write a tangent vector $\eta \in T_\xi S^3$ as $\eta = (\eta_0, \eta_1, \eta_2, \eta_3)$, with inner product $(\xi, \eta) = 0$. We identify TS^3 with $T^*S^3 \subset T^*\mathbb{R}^4$ by using the standard metric on S^3. Then, the symplectic transformation is given by

$$\begin{gathered}x = \frac{\xi_1}{1-\xi_0}, \quad y = \frac{\xi_2}{1-\xi_0}, \quad z = \frac{\xi_3}{1-\xi_0}, \\ p_x = \eta_1(1-\xi_0) + \xi_1\eta_0, \quad p_y = \eta_2(1-\xi_0) + \xi_2\eta_0, \quad p_z = \eta_3(1-\xi_0) + \xi_3\eta_0.\end{gathered} \tag{20}$$

Notice that here (x, y, z) represents the momentum and (p_x, p_y, p_z) the position compared to the original picture before switching their roles. After this transformation, going to the North pole (where the momentum becomes infinite) corresponds to collision in the original picture (where the position becomes zero). Dynamically, at collision (going through the North pole) it bounces back. Therefore, Moser regularization is characterized by adding the fiber over the North pole. Moreover, the inverse transformation is given by

$$\begin{gathered}\xi_0 = \frac{|\mathbf{X}|^2 - 1}{|\mathbf{X}|^2 + 1}, \quad \xi_1 = \frac{2x}{|\mathbf{X}|^2 + 1}, \quad \xi_2 = \frac{2y}{|\mathbf{X}|^2 + 1}, \quad \xi_3 = \frac{2z}{|\mathbf{X}|^2 + 1}, \\ \eta_0 = \langle \mathbf{X}, \mathbf{P}\rangle, \quad \eta_1 = \frac{|\mathbf{X}|^2 + 1}{2}p_x - \langle \mathbf{X}, \mathbf{P}\rangle x, \quad \eta_2 = \frac{|\mathbf{X}|^2 + 1}{2}p_y - \langle \mathbf{X}, \mathbf{P}\rangle y, \quad \eta_3 = \frac{|\mathbf{X}|^2 + 1}{2}p_z - \langle \mathbf{X}, \mathbf{P}\rangle z,\end{gathered}$$

and, in addition, we have the relation

$$|\eta| = \frac{1}{2}(|\mathbf{X}|^2 + 1)|\mathbf{P}| = \frac{|\mathbf{P}|}{1-\xi_0}. \tag{21}$$

By inserting (20) and (21) into (19), the transformed Hamiltonian on T^*S^3, which we denote by the same letter, is given by

$$\widetilde{K}_c(\xi, \eta) = |\eta| f(\xi, \eta) - 1, \tag{22}$$

where

$$f(\xi,\eta) := 1 + (\eta_1\xi_2 - \eta_2\xi_1)(1-\xi_0) + (ag_1^2 + bg_2^2 + \frac{1}{2}g_3^2)(1-\xi_0) - (c + \frac{1}{2})(1-\xi_0),$$
$$g_k := g_k(\xi,\eta) := \eta_k(1-\xi_0) + \xi_k\eta_0, \quad k = 1,2,3.$$

By shifting and squaring the Hamiltonian (22) we obtain the new smooth Hamiltonian $Q(\xi,\eta)$ on a subset of T^*S^3,

$$Q(\xi,\eta) = \frac{1}{2}|\eta|^2 f(\xi,\eta)^2. \tag{23}$$

The level set $H^{-1}(c) = K_c^{-1}(0)$ is compactified to the level set $Q^{-1}(\frac{1}{2})$. Since Q is smooth near this level set, we consider $Q^{-1}(\frac{1}{2})$ as the regularized problem. Since the only problem in compactness of Σ_c^b comes from collisions with the origin, we consider points near the origin, i.e., in view of (21), points (ξ,η) satisfying

$$|\mathbf{P}| = |\eta|(1-\xi_0) < \varepsilon. \tag{24}$$

Proposition 3.9. *For $\varepsilon > 0$ small enough, the natural Liouville vector field on T^*S^3 given by*

$$X = \sum_{i=0}^{3} \eta_i \frac{\partial}{\partial \eta_i}, \tag{25}$$

is transverse to $Q^{-1}(\frac{1}{2})$ over points (ξ,η) satisfying (24).

Notice that the Liouville vector field (10) on $T^*\mathbb{R}^3$ that we used for transversality in the unregularized case is mapped, via the composition of the symplectic transformation (20) with the symplectic switch map, to the natural Liouville vector field (25) on T^*S^3.

Proof of Proposition 3.9. We show that for $\varepsilon > 0$ small enough it holds that

$$dQ(X)|_{Q^{-1}(\frac{1}{2})} > 0. \tag{26}$$

The computation of $dQ(X)$, in view of (23) and (25), yields

$$\begin{aligned} dQ(X) &= |\eta|^2 f(\xi,\eta)^2 + |\eta|^2 f(\xi,\eta) \sum_{i=0}^{3} \frac{\partial f}{\partial \eta_i}(\xi,\eta)\eta_i \\ &= 2Q + |\eta|^2 f(\xi,\eta)(1-\xi_0)(\eta_1\xi_2 - \eta_2\xi_1 + 2ag_1^2 + 2bg_2^2 + g_3^2). \end{aligned}$$

In order to prove (26), we first show that we can choose $\varepsilon > 0$ so small such that

$$|f(\xi,\eta)| \geq \frac{1}{2}. \tag{27}$$

Since the energy value $c < H(L_{1/2}) = -\frac{3}{2}\sqrt[3]{\lambda_2}$ is negative, and in fact less then $-\frac{3}{2}$, the quantity $c + \frac{1}{2}$ is negative as well. Notice from Figure 2 that $a < 0$, $|a| \leq 1$ and $b > 0$. Therefore, $bg_2^2 + \frac{1}{2}g_3^2 - (c + \frac{1}{2})$ is positive. By using these, we estimate

$$\begin{aligned} |f(\xi,\eta)| &= \left|1 + (\eta_1\xi_2 - \eta_2\xi_1)(1-\xi_0) + (ag_1^2 + bg_2^2 + \frac{1}{2}g_3^2)(1-\xi_0) - (c+\frac{1}{2})(1-\xi_0)\right| \\ &= \left|1 + (bg_2^2 + \frac{1}{2}g_3^2 - (c+\frac{1}{2}))(1-\xi_0) + (\eta_1\xi_2 - \eta_2\xi_1)(1-\xi_0) + ag_1^2(1-\xi_0)\right| \\ &\geq 1 - |\eta_1\xi_2 - \eta_2\xi_1|(1-\xi_0) - |a|g_1^2(1-\xi_0) \\ &\geq 1 - |\eta_1\xi_2 - \eta_2\xi_1|(1-\xi_0) - g_1^2(1-\xi_0). \end{aligned}$$

Furthermore, $|\eta_1\xi_2 - \eta_2\xi_1| \leq |\eta||\xi|$, and because $|\xi| = 1$, we have in view of (24),

$$|\eta_1\xi_2 - \eta_2\xi_1|(1-\xi_0) \leq |\eta|(1-\xi_0) < \varepsilon. \tag{28}$$

This implies,

$$|f(\xi,\eta)| \geq 1 - \varepsilon - g_1^2(1-\xi_0).$$

If ε approaches 0, then $\xi_0 \to 1$, which means that we can choose ε so small such that (27) holds. By using the level set condition $Q^{-1}(\frac{1}{2})$ together with the lower bound (27) for $|f(\xi,\eta)|$, we find

$$\frac{1}{2} = Q(\xi,\eta) = \frac{1}{2}|\eta|^2 f(\xi,\eta)^2 \geq \frac{1}{2}|\eta|^2\frac{1}{2},$$

which gives an upper bound for $|\eta|$, i.e.,

$$|\eta| \leq 2. \tag{29}$$

We may write

$$dQ(X) \geq 2Q - |\eta|^2\,|f(\xi,\eta)|\left|(1-\xi_0)\left(\eta_1\xi_2 - \eta_2\xi_1 + 2ag_1^2 + 2bg_2^2 + g_3^2\right)\right|.$$

Notice that by (29) we obtain

$$|\eta||\eta||f(\xi,\eta)| \leq 2\sqrt{2Q(\xi,\eta)} = 2\sqrt{2\frac{1}{2}} = 2,$$

which implies, together with (28),

$$\begin{aligned} dQ(X) &\geq 1 - 2\left(\left|(1-\xi_0)(\eta_1\xi_2 - \eta_2\xi_1)\right| + \left|(1-\xi_0)(2ag_1^2 + 2bg_2^2 + g_3^2)\right|\right) \\ &\geq 1 - 2\varepsilon\left(1 + |2ag_1^2 + 2bg_2^2 + g_3^2|\right). \end{aligned}$$

Since the latter term can be bounded by some constant A on a compact set away from the origin, we obtain

$$dQ(X) \geq 1 - 2\varepsilon(1 + A).$$

Now we choose ε sufficiently small such that $dQ(X) > 0$, which proves (26). □

We have seen that for $c < H(L_{1/2})$ the bounded component Σ_c^b of the energy level set can be Moser-regularized to form a compact 5-dimensional manifold $\widetilde{\Sigma}_c^b \subset T^*S^3$ which is diffeomorphic to S^*S^3. Since the Liouville vector field (10) on $T^*\mathbb{R}^3$ and the natural one (25) on T^*S^3 coincide after Moser regularization, we obtain a Liouville vector field that is defined near the whole regularized level set, and in fact, it is the natural one. By the transversality results from Proposition 3.4 and Proposition 3.9 we obtain that the natural Liouville vector field on T^*S^3 is transverse to $\widetilde{\Sigma}_c^b$, which means that $\widetilde{\Sigma}_c^b$ is fiberwise starshaped, and moreover, $\widetilde{\Sigma}_c^b \cong (S^*S^3, \xi_{st})$.

For the planar problem, one can of course perform the same computation to obtain the same result. But since the planar problem corresponds to the restriction of the spatial system to the fixed point set of the symplectic symmetry σ from (6), the transversality result in the planar case follows immediately. This consequence is based on a general construction from [5]. Namely, if a energy level set Σ is of contact type and the entire system has a symplectic symmetry σ, such as (6), then the restriction of the contact form on Σ to $\Sigma|_{\mathrm{Fix}(\sigma|_\Sigma)}$ is a contact form on $\Sigma|_{\mathrm{Fix}(\sigma|_\Sigma)}$. Therefore, we have the same result in the planar problem, in which we denote the Moser-regularized compact 3-dimensional manifold by $\widetilde{\Sigma}_c^b|_{\mathrm{Fix}(\sigma)} \cong S^*S^2 \subset T^*S^2$. This completes the proof of Theorem 1.1.

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Classification of NMS-flows with unique twisted saddle orbit on orientable 4-manifolds

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Abstract: Topological equivalence of Morse-Smale flows without fixed points (NMS-flows) under assumptions of different generalities was studied in a number of papers. In some cases when the number of periodic orbits is small, it is possible to give exhaustive classification, namely to provide the list of all manifolds that admit flows of considered class, find complete invariant for topological equivalence and introduce each equivalence class with some representative flow. This work continues the series of such articles. We consider the class of NMS-flows with unique saddle orbit, under the assumption that it is twisted, on closed orientable 4-manifolds and prove that the only 4-manifold admitting the considered flows is the manifold $\mathbb{S}^3 \times \mathbb{S}^1$. Also, it is established that such flows are split into exactly eight equivalence classes and construction of a representative for each equivalence class is provided.

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1 Introduction and main results

In the present paper we consider *NMS-flows* f^t, namely *non-singular* (without fixed points) Morse-Smale flows which are defined on orientable-manifold M^4. Non-wandering set of such flow consist of a finite number of hyperbolic periodic orbits. Asimov proved [Asi75] that ambient manifold of such flow is a union of round handles. However, if the number of orbits is small, topology of the ambient manifold can be specified. For instance, in dimension 3 only lens spaces admit NMS-flows with two periodic orbits. Moreover, it was shown in [PS22a] that each lens space (closed orientable manifold obtained by gluing two solid tori along boundaries) admit exactly two classes of topological equivalence except for 3-sphere $\mathbb{S}^3$ and projective space $\mathbb{R}\mathrm{P}^3$ which both admit the unique equivalence class. Moreover only two 4-manifolds $\mathbb{S}^3 \times \mathbb{S}^1$, $\mathbb{S}^3 \tilde{\times} \mathbb{S}^1$ admit such flows and each admits exactly two topological equivalence classes.

Campos et al. [CCMAV04] argued that lens spaces are the only prime (a manifold that cannot be expressed as a non-trivial connected sum of two manifolds) 3-manifolds that are ambient for NMS-flow with unique saddle periodic orbit, but this is not so. There exists infinite series of mapping tori non-homeomorphic to lens spaces which admit such flows [Shu21]. Moreover, necessary and sufficient conditions for topological equivalence of such flows were obtained in [PS22b]. Finally, any 3-manifold admitting such flows is a lens space or connected sum[1] of two lens spaces or a small Seifert fibered[2] 3-manifold [PS22c]. The invariants constructed in these works are different from known ones, for example *scheme of the flow* constructed by Umanskii [Uma90] for Morse-Smale

[1]A *connected sum of two n-dimensional manifolds* is a manifold formed by deleting a ball inside each manifold and gluing together the resulting boundary spheres.

[2]A *Seifert fibered space* is a closed orientable 3-manifold that can be ecomposed into a disjoint union of circles (*fibers*) such that each fiber has a neigbourhood that is fiber-wise homeomorphic to standart fibered torus.

flows with finite number of singular trajectories (closed orbits, fixed points and heteroclinic orbits).

In the present paper we establish the topology of orientable 4-manifolds that are ambient for NMS-flows with exactly one saddle orbit assuming that it is *twisted* (its invariant manifolds are non-orientable). Remarkably, all such flows are suspensions over Morse-Smale diffeomorphisms on 3-manifolds, which are classified in [BGP19]. 3-diffeomorphisms are known to posess wild separatrices [Pix77], which complicates their classification. Pixton constructed an example of 3-diffeomorphism with one saddle orbit having wild unstable separatrice. Bonatti and Grines classified the class of sphere diffeomorphisms that have non-wandering set consisting of four fixed points: two sinks, a source, and a saddle [BG00]. They showed that the Pixton class contains a countable set of pairwise topologically non-conjugate diffeomorphisms.

As was shown in [PS20b] suspensions over Pixton diffeomorphisms also have wild unstable separatrices and such class contains a countable family of pairwise non-equivalent flows. However, in this case the saddle orbit of the flow is non-twisted. Surprisingly, there are no flows with unique saddle orbit that is twisted and having wild separatrix. Besides, the number of equivalence classes of such flows appear to equal 8.

Let us proceed to the formulation of the results.

Let M^4 be connected closed orientable 4-manifold. Flow $f^t : M^4 \to M^4$ is called *Morse-Smale flow* if (a) its chain-recurrent set[3] consist of finite number of periodic orbits and fixed points and (b) the unstable manifold of each chain-recurrent set component has transversal intersection with the stable manifold of any other chain-recurrent set component. Let f^t be NMS-flow (Morse-Smale flow without fixed points) and $\mathcal{O}$ be its periodic orbit. There exists tubular neighborhood $V_\mathcal{O}$ homeomorphic to $\mathbb{D}^3 \times \mathbb{S}^1$ such that the flow is topologically equivalent to the suspension over a linear diffeomorphism of the plane defined by the matrix which determinant is positive and eigenvalues are different

[3]Point $x \in M$ is called *chain-recurrent* for the flow f^t if for any $T,\ \varepsilon > 0$ there exist points $x_1, \dots x_n \in M$ and real numbers $t_0, \dots t_n > T$ such that $x = x_0 = x_n$ and $d(f^{t_i}(x_i), x_{i+1}) < \varepsilon$, where d is a metric on M

from ± 1 (see. Proposition 1). If absolute values of both eigenvalues are greater (less) than one, the corresponding periodic orbit is *attracting (repelling)*, otherwise it is *saddle*. The saddle orbit is called *twisted* if both eigenvalues are negative and *non-twisted* otherwise.

Consider the class $G_3^-(M^4)$ of NMS-flows $f^t : M^4 \to M^4$ with unique saddle orbit which is twisted. Since the ambient manifold M^4 is the union of stable (unstable) manifolds of its periodic orbits, the flow $f^t \in G_3^-(M^4)$ has at least one attracting and at least one repelling orbit. In Section 2 the following fact is established.

Lemma 1. *The non-wandering set of any flow $f^t \in G_3^-(M^4)$ consists of exactly three periodic orbits S, A, R, saddle, attracting and repelling, respectively.*

Unstable manifold os saddle orbit S of the flow $f^t \in G_3^-(M^4)$ can be either 3- or 2-dimensional; let $G_3^{-1}(M^4)$, $G_3^{-2}(M^4)$ denote corresponding subclasses of $G_3^-(M^4)$. Obviously, since dimension of unstable manifold is invariant under an equivalence homeomorphism no flow in $G_3^{-1}(M^4)$ is topologically equivalent to any flow in $G_3^{-2}(M^4)$. Furthermore, $G_3^{-2}(M^4) = \{f^{-t} : f^t \in G_3^{-1}(M^4)\}$ and the flows f^t, f'^t are topologically equivalent if and only if f^{-t}, f'^{-t} are topologically equivalent. This immediately implies that classification in the class $G_3^-(M^4)$ reduces to classification in the subclass $G_3^{-1}(M^4)$.

Let $f^t \in G_3^{-1}(M^4)$. Since the flow f^t in some tubular neighborhood of is topologically equivalent to the suspension over linear diffeomorphism of the plane, the topology of periodic orbits A, S, R stable and unstable manifolds is:

- $W^u_S \cong \mathbb{R}^2 \tilde{\times} \mathbb{S}^1$ (open solid Klein bottle);
- $W^s_S \cong \mathbb{R} \tilde{\times} \mathbb{S}^1$ (open Möbius band);
- $W^s_A \cong W^u_R \cong \mathbb{R}^3 \times \mathbb{S}^1$ (open solid torus);
- $W^u_A \cong W^s_R \cong \mathbb{S}^1$ (circle).

Let $\mathcal{O} \in \{S, A, R\}$. Choose the generator $\mathcal{G}_{\mathcal{O}}$ of boundary $T_{\mathcal{O}} = \partial V_{\mathcal{O}} \cong \mathbb{S}^2 \times \mathbb{S}^1$ fundamental group which is homologous to $\mathcal{O}$ in $V_{\mathcal{O}} \cong \mathbb{D}^3 \times \mathbb{S}^1$. By definition the manifold T_S is secant for all flow trajectories except the periodic ones. Since the flow in some tubular neighborhood of S is topologically equivalent to suspension, the set $K_S = W^u_S \cap T_S$ is homeomorphic to the Klein bottle. Let λ_S, μ_S be the knots (simple closed curves), which are generators of the fundamental group $\pi_1(K_S)$ with relation $[\lambda_S * \mu_S] = [\mu_S^{-1} * \lambda_S]$. We will call the curve μ_S *meridian* and the curve λ_S *longitude*. By virtue of Proposition 3 the Klein bottle longitude embedded in $\mathbb{S}^2 \times \mathbb{S}^1$ is a generator of fundamental group $\pi_1(\mathbb{S}^2 \times \mathbb{S}^1)$. Consider the longitude λ_S be oriented in such way that its homotopy type $\langle \lambda_S \rangle$ in T_S coincide with type $\langle \mathcal{G}_S \rangle$. So, the set $K_A = W^u_S \cap T_A$ is the Klein bottle with longitude λ_A which is pointwise transferred along the flow f^t orbits from λ_S.

Since the flow in some tubular neighborhood of S is topologically equivalent to suspension the set $\gamma_S = W^s_S \cap T_S$ is a knot in T_S, wrapping around $\mathcal{G}_S$ twice. We will assume that the knot γ_S is oriented in such way that its homotopy type $\langle \gamma_S \rangle$ on T_S coincides with homotopy type of $2\langle \mathcal{G}_S \rangle$. So the set $\gamma_R = W^s_S \cap T_R$ is a knot in T_R and its orientation is induced by the flow f^t from γ_S.

Lemma 2. *Let $f^t \in G_3^{-1}(M^4)$ then the following conditions hold:*

1. $\langle \lambda_A \rangle = \delta_A \langle \mathcal{G}_A \rangle$, $\delta_A \in \{-1, +1\}$ *in* T_A*;*
2. $\langle \gamma_R \rangle = \delta_R \langle \mathcal{G}_R \rangle$, $\delta_R \in \{-1, +1\}$ *in* T_R*.*

Let

$$C_{f^t} = (\delta_A, \delta_R).$$

Theorem 1. *Flows f^t, $f'^t \in G_3^{-1}(M^4)$ are topologically equivalent if and only if $C_{f^t} = C_{f'^t}$.*

Theorem 2. *For any element $C \in \mathbb{S}^0 \times \mathbb{S}^0$ there exists a flow $f^t \in G_3^{-1}(M^4)$ such that $C = C_{f^t}$.*

Theorem 3. *The only 4-manifold that is ambient for a flow of the class $G_3^-(M^4)$ is $\mathbb{S}^3 \times \mathbb{S}^1$. Moreover $G_3^-(\mathbb{S}^3 \times \mathbb{S}^1)$ consists of eight classes of topological equivalence.*

Note that weakening the saddle orbit twistedness condition fundamentally changes the picture. For example, in [PS20a] non-singular flows that are suspensions over Pixton diffeomorphisms on a three-dimensional sphere are considered. It is proved that in the class under consideration there exist flows with wildly embedded invariant manifolds of the saddle orbit. Moreover, there are an infinite number of topological equivalence classes for such flows.

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2 Flows of the class $G_3^{-1}(M^4)$

2.1 Structure of periodic orbits

This section is devoted to proof of Lemma 1: non-wandering set of any flow $f^t \in G_3^{-1}(M^4)$ consists of three periodic orbits S, A, R, saddle, attracting and repelling respectively.

Proof. The proof is based on the following representation of the ambient manifold M^4 of the NMS-flow f^t with the set of periodic orbits Per_{f^t} (see, for example, [Sma67])

$$M^4 = \bigcup_{\mathcal{O}\in Per_{f^t}} W^u_{\mathcal{O}} = \bigcup_{\mathcal{O}\in Per_{f^t}} W^s_{\mathcal{O}}, \tag{1}$$

as well as the asymptotic behavior of invariant manifolds

$$\mathrm{cl}(W^u_{\mathcal{O}}) \setminus W^u_{\mathcal{O}} = \bigcup_{\tilde{\mathcal{O}}\in Per_{f^t}\,:\, W^u_{\tilde{\mathcal{O}}}\cap W^s_{\mathcal{O}}\neq\varnothing} W^u_{\tilde{\mathcal{O}}}, \tag{2}$$

$$\mathrm{cl}(W^s_{\mathcal{O}}) \setminus W^s_{\mathcal{O}} = \bigcup_{\tilde{\mathcal{O}}\in Per_{f^t}\,:\, W^s_{\tilde{\mathcal{O}}}\cap W^u_{\mathcal{O}}\neq\varnothing} W^s_{\tilde{\mathcal{O}}}. \tag{3}$$

In particular, it follows from eq. (1) that any NMS-flow has at least one attracting orbit and at least one repulsive one. Moreover, if an NMS-flow has a saddle periodic orbit, then the basin of any attracting orbit has a non-empty intersection with an unstable manifold of at least one saddle orbit (see Proposition 2.1.3 [GMP16]) and a similar situation with the basin of a repulsive orbits.

Now let $f^t \in G_3^{-1}(M^3)$ and S be its only saddle orbit. It follows from eq. (2) that $W_S^u \setminus S$ intersects only basins of attracting orbits. Since the set $W_S^u \setminus S$ is connected and the basins of attracting orbits are open, then W_S^u intersects exactly one such basin. Denote by A the corresponding attracting orbit. Since there is only one saddle orbit, there is only one attracting orbit. Similar reasoning for W_S^s leads to the existence of a unique repulsive orbit R. □

2.2 Canonical neighborhoods of periodic orbits

Recall the definition of a suspension. Let $\varphi : M^3 \to M^3$ be a diffeomorphism of a 3-manifold. We define the diffeomorphism $g_\varphi : M^3 \times \mathbb{R}^1 \to M^3 \times \mathbb{R}^1$ by the formula

$$g_\varphi(x_1, x_2, x_3, x_4) = (\varphi(x_1, x_2, x_3), x_4 - 1).$$

Then the group $\{g_\varphi^n\} \cong \mathbb{Z}$ acts freely and discontinuously on $M^3 \times \mathbb{R}^1$, whence the orbit space $\Pi_\varphi = M^3 \times \mathbb{R}^1 g_\varphi$ is a smooth 4-manifold, and the natural projection $v_\varphi : M^3 \times \mathbb{R}^1 \to \Pi_\varphi$ is a covering. At the same time, the flow $\xi^t : M^3 \times \mathbb{R}^1 \to M^3 \times \mathbb{R}^1$, given by the formula

$$\xi^t(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4 + t),$$

induces the flow $[\varphi]^t = v_\varphi \xi^t v_\varphi^{-1} : \Pi_\varphi \to \Pi_\varphi$. The flow $[\varphi]^t$ is called the *suspension of the diffeomorphism* φ.

We define the diffeomorphisms $a_0, a_1, a_2, a_3 : \mathbb{R}^3 \to \mathbb{R}^3$ by the formulas

$$a_3(x_1, x_2, x_3) = (2x_1, 2x_2, 2x_3),\ a_0 = a_3^{-1},$$

$$a_{\pm 1}(x_1, x_2, x_3) = (\pm 2x_1, \pm 1/2x_2, 1/2x_3),\ a_{\pm 2} = a_{\pm 1}^{-1}.$$

Let

$$V_0 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 |\ 4^{x_4}x_1^2 + 4^{x_4}x_2^2 + 4^{x_4}x_3^2 \leqslant 1\},$$

$$V_{\pm 1} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 |\ 4^{-x_4}x_1^2 + 4^{x_4}x_2^2 + 4^{x_4}x_3^2 \leqslant 1\},$$

$$V_{\pm 2} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 |\ 4^{-x_4}x_1^2 + 4^{-x_4}x_2^2 + 4^{x_4}x_3^2 \leqslant 1\},$$

$$V_3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 |\ 4^{-x_4}x_1^2 + 4^{-x_4}x_2^2 + 4^{-x_4}x_3^2 \leqslant 1\}.$$

For $i \in \{0, \pm1, \pm2, 3\}$ we set $v_i = v_{a_i}$, $T_i = \partial V_i$ and $\mathbb{V}_i = v_i(V_i)$, $\mathbb{T}_i = v_i(T_i)$.

The following statement, proved by M. Irwin [Irw70], describes the behavior of flows in a neighborhood of hyperbolic periodic orbits.

Proposition 1 (M. Irwin [Irw70]). *If $\mathcal{O}$ is a hyperbolic orbit of a flow $f^t : M^4 \to M^4$ defined on an orientable 4-manifold M^4, then there exists a tubular neighborhood $V_\mathcal{O}$ of the orbit $\mathcal{O}$ such that the flow $f^t\big|_{V_\mathcal{O}}$ is topologically equivalent, by means of some homeomorphism $H_\mathcal{O}$, to one of the following streams:*

- $[a_0]^t|_{\mathbb{V}_0}$ *if $\mathcal{O}$ is an attracting orbit;*
- $[a_1]^t|_{\mathbb{V}_1}$ *if $\mathcal{O}$ is a non-twisted saddle orbit with a two-dimensional unstable manifold $W^u_\mathcal{O}$;*
- $[a_{-1}]^t|_{\mathbb{V}_{-1}}$ *if $\mathcal{O}$ is a twisted saddle orbit with a two-dimensional unstable manifold $W^u_\mathcal{O}$;*
- $[a_2]^t|_{\mathbb{V}_2}$ *if $\mathcal{O}$ is a non-twisted saddle orbit with an unstable 3-manifold $W^u_\mathcal{O}$;*
- $[a_{-2}]^t|_{\mathbb{V}_{-2}}$ *if $\mathcal{O}$ is a twisted saddle orbit with an unstable 3-manifold $W^u_\mathcal{O}$;*
- $[a_3]^t|_{\mathbb{V}_3}$ *if $\mathcal{O}$ is a repelling orbit.*

The neighborhood $V_\mathcal{O} = H_\mathcal{O}(\mathbb{V}_i)$, $i \in \{0, \pm1, \pm2, 3\}$ described in Proposition 1 is called *canonical neighborhood* of the periodic orbit $\mathcal{O}$.

When proving topological equivalence, we will use the following fact, which follows from the proof of Theorem 4 and Lemma 4 in [PS22a], and can also be found in [Uma90] (Theorem 1.1).

Proposition 2. *A homeomorphism $h_i : \partial\mathbb{V}_i \to \partial\mathbb{V}_i$ for $i \in \{0, 3\}$ extends to a homeomorphism $H_i : \mathbb{V}_i \to \mathbb{V}_i$ realizing the equivalence of the flows $[a_i]^t$ with itself if and only if the induced isomorphism $h_{i*} : \pi_1(\partial\mathbb{V}_i) \to \pi_1(\partial\mathbb{V}_i)$ is identical.*

2.3 Trajectory mappings

Consider a flow $f^t: M^4 \to M^4$ from the set $G_3^{-1}(M^3)$. Then $V_A = H_A(\mathbb{V}_0)$, $V_R = H_R(\mathbb{V}_3)$, $V_S = H_S(\mathbb{V}_{-2})$. Let $\Gamma = \{(x_1, x_2, x_3, x_4) \in T_{-2} |\ 4^{x_4} x_3^2 = 1/2\}$, $\Gamma^u = Ox_2x_3x_4 \cap T_{-2}$, $\Gamma^s = Ox_1x_4 \cap T_{-2}$. By construction, the set T_{-2} is homeomorphic to $\mathbb{S}^2 \times \mathbb{R}$, the set Γ consists of two surfaces, each of which is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$, dividing T_{-2} into three connected components, one of which N^u contains the cylinder $\Gamma^u \cong \mathbb{S}^1 \times \mathbb{R}$, and the union N^s the other two contains a pair of $\Gamma^s \cong \mathbb{S}^0 \times \mathbb{R}$ curves, one curve in each component. Then on T_S

- $K_S = H_S(v_{-2}(\Gamma^u)))$ is a Klein bottle;
- $\gamma_S = H_S(v_{-2}(\Gamma^s)))$ is a knot winding twice around the generator $\mathcal{G}_S$;
- $N_S^u = H_S(v_{-2}(\mathrm{cl}(N^u)))$ is a tubular neighborhood of K_S;
- $N_S^s = H_S(v_{-2}(\mathrm{cl}(N^s)))$ is a tubular neighborhood of γ_S;
- $\partial N_S^u = \partial N_S^s = H_S(v_{-2}((\Gamma)))$ is a two-dimensional torus.

Let

$$N_R^s = \left(\bigcup_{t>0,\, w \in N_S^s} f^{-t}(w) \right) \cap T_R, \ N_R^u = T_R \setminus N_R^s,$$

$$N_A^u = \left(\bigcup_{t>0,\, w \in N_S^u} f^{t}(w) \right) \cap T_A, \ N_A^s = T_A \setminus N_A^u$$

and introduce the following mappings:

- using Poincaré map between $N_R^u \subset T_R$ and N_S^s we define a continuous function $\tau_R: N_R^u \to \mathbb{R}^+$ so that $f^{\tau_R(r)}(r) \in N_S^s$ for $r \in N_R^s$. Next, we continuously extend it to $\tau_R: T_R \to \mathbb{R}^+$ and define the set $\mathcal{T} = \bigcup\limits_{r \in N_R^u} f^{\tau_R(r)}(r)$ which does not intersect the torus T_A if V_A is small enough. We set $\mathcal{T}_R = \mathcal{T} \cup N_S^s$ and define a homeomorphism $\psi_R: T_R \to \mathcal{T}_R$ by the formula $\psi_R(r) = f^{\tau_R(r)}(r)$, denote by $\mathcal{V}_R$ the closure of the connected component of the set $M^4 \setminus \mathcal{T}_R$, containing R;

- we set $\mathcal{T}_A = \mathcal{T} \cup N^u_S$. Since future orbits, that intersect $\mathcal{T}$ go towards A a continuous function $\tau_A : T_A \to \mathbb{R}^+$ such that $f^{-\tau_A(a)}(a) \in \mathcal{T}_A$ for $a \in T_A$ is uniquily defined; note that $\partial\mathcal{T} = \partial N^u_S = \partial N^s_S$. So, we define a homeomorphism $\psi_A : T_A \to \mathcal{T}_A$ by the formula $\psi_A(a) = f^{-\tau_A(a)}(a)$, denote by $\mathcal{V}_A$ the closure of the connected component of the set $M^4 \setminus \mathcal{T}_A$ containing A.

We will call the introduced homeomorphisms ψ_R, ψ_A *trajectory maps*. Note that the ambient manifold M^4 is represented as

$$M^4 = \mathcal{V}_A \cup V_S \cup \mathcal{V}_R.$$

Note that

$$\mathcal{V}_A \cap \mathcal{V}_R = \mathcal{T}, \quad \gamma_S \subset N^u_S = \mathcal{V}_A \cap V_S, \quad K_S \subset \mathcal{N}^s_S = V_R \cap V_S. \tag{4}$$

Moreover, in the manifolds $\mathcal{V}_A$ and $\mathcal{V}_R$ the flow f^t is topologically equivalent to the suspensions $[a_0]^t$ and $[a_3]^t$, respectively.

3 Homotopy types of knots λ_A, γ_R

In this section, we will prove Lemma 2. To do this, we first describe the properties of the embedding of the Klein bottle into the manifold $\mathbb{S}^2 \times \mathbb{S}^1$.

Recall that the Klein bottle $\mathbb{K}$ is the square $[0,1]\times[0,1]$ with sides glued by the relation

$$(x,0) \sim (x,1), \quad (0,y) \sim (0,1-y).$$

Let $\upsilon : [0,1]\times[0,1] \to \mathbb{K}$ be the natural projection, then the curves

$$\lambda = \upsilon([0,1]\times\{1/2\}),\ \mu = \upsilon(\{0\}\times[0,1])$$

are generators of the fundamental group $\pi_1(\mathbb{K})$ with relation

$$[\lambda * \mu] = [\mu^{-1} * \lambda],$$

where the curve λ is called *longitude* and the curve μ is called *meridian*.

It is well known that the Klein bottle does not embed into $\mathbb{R}^3$, however, it can be embeddable into $\mathbb{S}^2\times\mathbb{S}^1$, for example by defining the embedding $\tilde{e}_0 : [0,1]\times[0,1] \to \mathbb{S}^2\times\mathbb{S}^1$ by the formula

$$\tilde{e}_0(x,y) = \left(\sin \pi x \cos 2\pi y,\ \cos \pi x \cos 2\pi y,\ \sin 2\pi y, e^{2\pi i x}\right)$$

and noticing that $\tilde{e}_0(x,y) = \tilde{e}_0(x',y') \iff (x,y) \sim (x',y')$. Then (see, for example, [Kos80, Chapter 5])

$$e_0 = \tilde{e}_0 v^{-1} : \mathbb{K} \to \mathbb{S}^2 \times \mathbb{S}^1$$

is the desired embedding of the Klein bottle in $\mathbb{S}^2 \times \mathbb{S}^1$. Let

$$K_0 = e_0(\mathbb{K}).$$

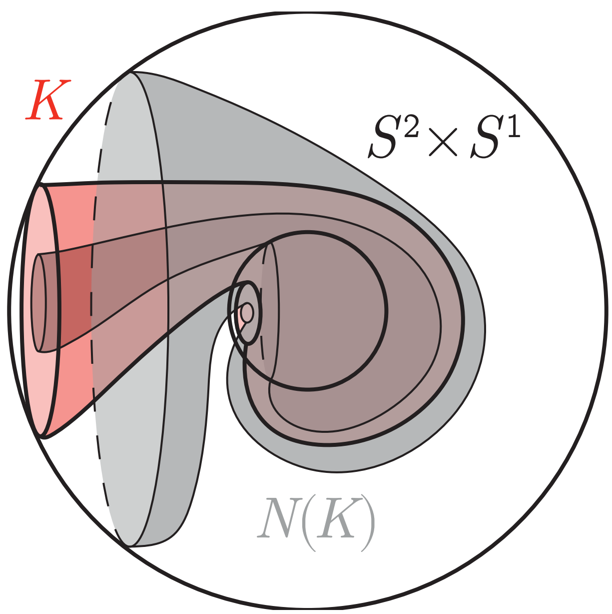

Figure 1: Bottle of Klein in $\mathbb{S}^2 \times \mathbb{S}^1$

Proposition 3 (Proposition 1.4, [BGP02]). *Let $e : \mathbb{K} \to \mathbb{S}^2\times\mathbb{S}^1$ be an embedding Klein bottles $\mathbb{K}$, $K = e(\mathbb{K})$, $N(K) \subset \mathbb{S}^2 \times \mathbb{S}^1$ be a tubular neighborhood of K and $V(K) = \mathbb{S}^2 \times \mathbb{S}^1 \setminus \operatorname{int} N(K)$ (see Figure 1). Then:*

1) *the curve* $e(\lambda)$ *is a generator of the fundamental group* $\pi_1(\mathbb{S}^2 \times \mathbb{S}^1)$*;*

2) *the set* $V(K)$ *is a solid torus whose meridian is homotopic to the curve* $e(\mu)$*;*

3) *there exists an orientation-preserving homeomorphism* $h: \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$ *such that* $h(K) = K_0$ *and* $h_* = id: \pi_1(\mathbb{S}^2 \times \mathbb{S}^1) \to \pi_1(\mathbb{S}^2 \times \mathbb{S}^1)$.

Proposition 4 (Proposition 4.2, [GMP16])**.** *A knot* γ *in manifold* $\mathbb{S}^2 \times \mathbb{S}^1$ *is trivial if and only if there exists tubular neighbourhood* $N(\gamma)$ *in* $\mathbb{S}^2 \times \mathbb{S}^1$ *such that the manifold* $(\mathbb{S}^2 \times \mathbb{S}^1) \setminus N(\gamma)$ *is homeomorphic to solid torus.*

It remains to prove Lemma 2. To do this, recall that we have chosen $T_\mathcal{O} = \partial V_\mathcal{O} \cong \mathbb{S}^2 \times \mathbb{S}^1$, $\mathcal{O} \in \{A, S, R\}$ generator $\mathcal{G}_\mathcal{O}$ of the fundamental group of $T_\mathcal{O}$, homologous in $V_\mathcal{O} \cong \mathbb{D}^3 \times \mathbb{S}^1$ orbit $\mathcal{O}$. Due to the fact that canonical neighborhoods of periodic orbits can be chosen so that $M^4 = \mathcal{V}_A \cup V_S \cup \mathcal{V}_R$ (see Section 2.3), everywhere below we assume that $V_A = \mathcal{V}_A$, $V_R = \mathcal{V}_R$.

We also established (see eq. (4)) that the set $K_S = W^u_S \cap T_S$ is a Klein bottle on $T_S \cong \mathbb{S}^2 \times \mathbb{S}^1$ and oriented its parallel λ_S so that $\langle \lambda_S \rangle = \langle \mathcal{G}_S \rangle$ on T_S. Since the set $K_A = W^u_S \cap T_A$ coincides with K_S, then $\lambda_A = \lambda_S$. We also established that the set $\gamma_S = W^s_S \cap T_S$ is a knot on T_S and oriented so that $\langle \gamma_S \rangle = 2\langle \mathcal{G}_S \rangle$ to T_S. Since the set $\gamma_R = W^s_S \cap T_R$ coincides with γ_S, then $\gamma_R = \gamma_S$.

Let us show that the knots λ_A, γ_R are generators in the fundamental groups of the manifolds T_A, T_R, respectively.

Proof. Since T_A is homeomorphic to the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ and λ_A is a parallel of the Klein bottle $K_A \subset T_A$, then item 1) of Proposition 3 implies that λ_A is a generator in the fundamental group of T_A, that is, on T_A

$$\langle \lambda_A \rangle = \delta_A \langle \mathcal{G}_A \rangle$$

for $\delta_A \in \{-1, +1\}$. The set $N(K_A) = N^u_S$ is a tubular neighborhood of the Klein bottle K_A in T_A. It follows from item 2) of Proposition 3 that the set $V(K_A) = T_A \setminus \operatorname{int} N(K_A)$ is a solid torus. The set $N(\gamma_R) = N^s_S$ is a tubular neighborhood of the knot γ_R in T_R. On the other hand $T_R \setminus \operatorname{int} N(\gamma_R) = V(K_A)$. Thus, the complement to the tubular neighborhood of γ_R in T_R is a solid torus. By Proposition 4, γ_R is a generator in T_R and, therefore,

$$\langle\gamma_R\rangle = \delta_R\langle\mathcal{G}_R\rangle$$

for $\delta_R \in \{-1, +1\}$. □

4 Classification of flows of the set $G_3^{-1}(M^4)$

In this section, we will prove Theorem 1.

Proof.

Necessity. Let flows f^t and f'^t have invariants $C_{f^t} = (\delta_A, \delta_R),\ C_{f'^t} = (\delta_{A'}, \delta_{R'})$ and are topologically equivalent by the homeomorphism $H:\ M^4 \to M^4$. Let us show that $C_{f^t} = C_{f'^t}$.

Let $h_A = H\big|_{T_A}$ and $T_{A'} = h(T_A)$. Then by Proposition 2

$$\langle\mathcal{G}_{A'}\rangle = h_{A*}\langle\mathcal{G}_A\rangle.$$

It follows from item 3) of Proposition 3 that $\lambda_{A'} = h_A(\lambda_A)$ is the parallel of the Klein bottle $K_{A'}$. Since the longitude λ_A of the Klein bottle is oriented consistent with the saddle orbit S and H transforms the orbit S into the orbit S' with orientation preservation, then $\lambda_{A'}$ is oriented consistent with the saddle orbit S'. On the other side, by Lemma 2,

$$\langle\lambda_A\rangle = \delta_A\langle\mathcal{G}_A\rangle,\ \langle\lambda_{A'}\rangle = \delta_{A'}\langle\mathcal{G}_{A'}\rangle,$$

whence, by virtue of a simple chain of equalities

$$\delta_{A'}\langle\mathcal{G}_{A'}\rangle = h_{A*}(\delta_A\langle\mathcal{G}_A\rangle) = \delta_A\langle\mathcal{G}_{A'}\rangle,$$

we get that $\delta_A = \delta_{A'}$. It is proved similarly that $\delta_R = \delta_{R'}$. Thus $C_{f^t} = C_{f'^t}$.

Sufficiency. Let the flows f^t and f'^t have equal invariants $C_{f^t} = (\delta_A, \delta_R),\ C_{f'^t} = (\delta_{A'}, \delta_{R'})$. Let us show that the flows f^t and f'^t are topologically equivalent.

Proposition 1 implies that the homeomorphism

$$H\big|_{V_S} = H_{S'}H_S^{-1} : V_S \to V_{S'}$$

is topological equivalence homeomorphism of the flows $f^t\big|_{V_S}$ and $f'^t\big|_{V_{S'}}$. It remains to extend this homeomorphism to $\mathcal{V}_A$ and $\mathcal{V}_R$.

The homeomorphism H is already defined on the set $\mathcal{T}_A \cap T_S$, which is a tubular neighborhood $N(K_A)$ of the Klein bottle K_A. By item 2) of Proposition 3 the set $V(K_A) = T_A \setminus \operatorname{int} N(K_A)$ is a solid torus whose meridian is homotopic to the meridian μ_A of the Klein bottle K_A to $N(K_A)$. It follows from the properties of the homeomorphism H that $K_{A'} = H(K_A)$ and $N(K_{A'}) = H(N(K_A))$ is a tubular neighborhood of the Klein bottle $K_{A'}$. By point 2) of Proposition 3 the set $V(K_{A'}) = T_{A'} \setminus \operatorname{int} N(K_{A'})$ is a solid torus whose meridian is homotopic to the meridian $\mu_{A'}$ of the Klein bottle $K_{A'}$ in $N(K_{A'})$. Since any homeomorphism of the Klein bottle does not change the homotopy class of the meridian (see, for example, [Lic63, Lemma 5]), the homeomorphism $H : \partial V(K_A) \to \partial V(K_{A'})$ extends to the homeomorphism $H : V(K_A) \to V(K_{A'})$ (see, for example, [Rol03, Exercise 2E5]). Thus H is defined on $\mathcal{T}_A$ and $\mathcal{T}_R$.

Since the parallel λ_A ($\lambda_{A'}$) of the Klein bottle is oriented consistent with the saddle orbit S (S') and H transforms the orbit S into the orbit S' orientation-preserving, then $H_*(\langle\lambda_A\rangle) = \langle\lambda_{A'}\rangle$. Since $\delta_A = \delta_{A'}$, then $H_*(\delta_A\langle\lambda_A\rangle) = \delta_{A'}\langle\lambda_{A'}\rangle$ and hence $H_*(\langle\mathcal{G}_A\rangle) = \langle\mathcal{G}_{A'}\rangle$. By Proposition 2 the homeomorphism $H\big|_{\mathcal{T}_A}$ extends to $\mathcal{V}_A$ by a homeomorphism realizing the equivalence of flows $f^t\big|_{\mathcal{V}_A}$ and $f'^t\big|_{\mathcal{V}_{A'}}$. Similarly, H can be extended to $\mathcal{V}_R$. Thus, the homeomorphism H is defined on the whole M^4 and realizes the equivalence of the flows f^t, f'^t. □

5 Realization of flows by admissible set

In this section, we will prove Theorem 2: for any element $C \in \mathbb{S}^0 \times \mathbb{S}^0$ there is a flow $f^t \in G_3^{-1}(M^4)$ such that $C = C_{f^t}$.

Proof. Let us construct the flow $f^t : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$ with the invariant $C_{f^t} = (+1, +1)$ as a suspension over the sphere diffeomorphism $\zeta : \mathbb{S}^2 \to \mathbb{S}^2$ with three periodic orbits. To do this, we describe the construction of the diffeomorphism ζ.

Let $\chi^t : \mathbb{R}^3 \to \mathbb{R}^3$ be the flow defined by system of equations:

$$\begin{cases} \dot{x} = -x \\ \dot{y} = -y \\ \dot{z} = -z(z-1)(z+1). \end{cases}$$

and the diffeomorphism $q : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by the formula:

$$q(x, y, z) = (x, -y, -z).$$

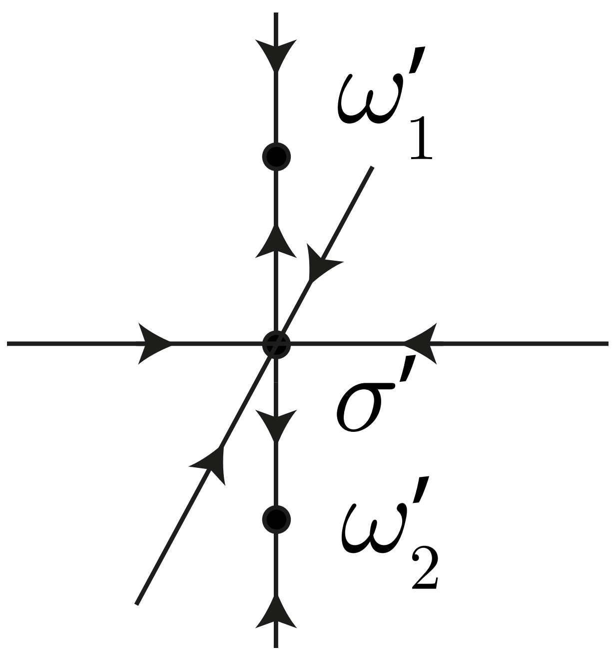

Figure 2: Flow χ^t phase portrait

Using stereographic projection (see Figure 3) $\vartheta : \mathbb{S}^3 \setminus \{N\} \to \mathbb{R}^3$ ($N = (0, 0, 0, 1), S =$

$(0, 0, 0, -1)$) by the given formula:

$$\vartheta(x_1, x_2, x_3, x_4) = \left(\frac{x_1}{1-x_4}, \frac{x_2}{1-x_4}, \frac{x_3}{1-x_4}\right).$$

project the diffeomorphism $q\chi^1$ onto $\mathbb{S}^3$:

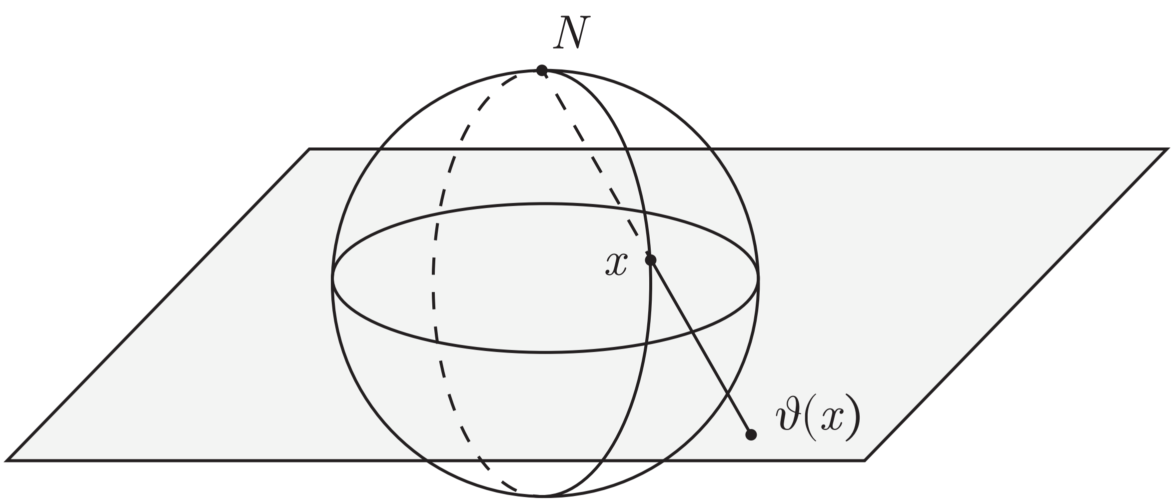

Figure 3: Stereographic projection

$$f(x) = \begin{cases} \vartheta^{-1} q \chi^1 \vartheta(x), & x \notin \{N, S\} \\ S, & x = N, \\ N, & x = S \end{cases}$$

Non-wandering set of diffeomorphism f consists of four periodic points:

- hyperbolic sink orbit ω_1, ω_2 of period 2: $\omega_1 = \vartheta^{-1}(0,\ 0,\ 1),\ \omega_2 = \vartheta^{-1}(0,\ 0,\ -1)$;
- hyperbolic saddle $\sigma = \vartheta^{-1}(0,\ 0,\ 0)$;
- hyperbolic sourse $\alpha = N$.

Then the flow $f^t = [f]^t$ belongs to the class $G_3^{-1}(M^3)$ and $C_{f^t} = (+1, +1)$. By construction, f is an orientation-preserving diffeomorphism of the 3-sphere, and hence the ambient manifold of the suspension $[f]^t$ is homeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$.

We construct the rest of the flows of class $G_3^{-1}(M^4)$ by modifying the constructed flow f^t in neighborhoods of attracting and repelling orbits such that its their stability does not change but the orbits go in opposite direction.

Let $\mathcal{O}$ be an attractive or repelling periodic orbit of f^t and $V_{\mathcal{O}} = V^0_{\mathcal{O}}$ be its canonical neighborhood, $V^t_{\mathcal{O}} = f^t(V^0_{\mathcal{O}})$. Without loss of generality, we assume that $V^{-1}_A \cap V^1_R = \varnothing$. Let $\vec{v}(x)$ denote the vector field induced by the flow f^t on $\mathbb{S}^3 \times \mathbb{S}^1$.

Recall that $V_{\mathcal{O}} \cong \mathbb{R}^3 \times \mathbb{S}^1$. For points x that belong to the basin of the orbit $\mathcal{O}$. Let $\vec{n}_{\mathcal{O}}(x)$ denote the field of unit outward normals to the hypersurfaces $\partial V^t_{\mathcal{O}} \cap \{(x,y) \in V_{\mathcal{O}} \mid y = const\}$ in $\{(x,y) \in V_{\mathcal{O}} \mid y = const\}$ and let $s_{\mathcal{O}}(x) \in \mathbb{R}$ be the time such that $f^{s_{\mathcal{O}}(x)}(x) \in \partial V^0_{\mathcal{O}}$. We define the vector field $\vec{v}'(x)$ on $\mathbb{S}^3 \times \mathbb{S}^1$ by the formulas

$$\vec{v}'(x) = \begin{cases} (1 - s^2_A(x))\vec{n}(x) + s^2_A(x)\vec{v}(x), & x \in V^{-1}_A \setminus V^1_A \\ (1 - s^2_R(x))\vec{n}(x) + s^2_R(x)\vec{v}(x), & x \in V^1_R \setminus V^{-1}_R \\ v(x), & \text{otherwise} \end{cases}$$

and denote by f'^t the flow it induces on $\mathbb{S}^3 \times \mathbb{S}^1$.

Recall, that flow $f'^t\big|_{V_{\mathcal{A}}}$ ($f'^t\big|_{V_{\mathcal{R}}}$) is congugated to $[a_0]^t\big|_{\mathbb{V}_0}$ ($[a_3]^t\big|_{\mathbb{V}_3}$) by a homeomorphism $h_A(h_R)$. For $\delta \in \{-1, +1\}$ we define the diffeomorphism $\bar{w}_\delta : \mathbb{R}^3 \to \mathbb{R}^3$ by the formula:

$$\bar{w}_\delta(x_1, x_2, x_3, x_4) = (4^{x_4}x_1, 4^{x_4}x_2, 4^{x_4}x_3, -x_4)\,.$$

Note, that $\bar{w}_\delta$ preserve V_0 and V_3. Next, define diffeomorphisms

$$w^A_\delta = h_A \bar{w}_\delta h^{-1}_A, \quad w^R_\delta = h_R \bar{w}_\delta h^{-1}_R$$

Note, that $\vec{v}'$ in invariant under w^A_δ, w^R_δ .

For $C = (\delta_A, \delta_R) \in \mathbb{S}^0 \times \mathbb{S}^0$ we induce the flow f^t_C on $\mathbb{S}^3 \times \mathbb{S}^1$ by vector field

$$\vec{v}_C(x) = \begin{cases} (dw^A_{\delta_A})\vec{v}' w^A_{\delta_A}(x), & x \in V_A \\ (dw^R_{\delta_R})\vec{v}' w^R_{\delta_R}(x), & x \in V_R \\ \vec{v}'(x), & \text{otherwise.} \end{cases}$$

It is easy to see that the flow's f^t_C invariant is C and its ambient manifold is $\mathbb{S}^3 \times \mathbb{S}^1$. $\square$

6 Ambient manifolds of the flow of class $G_3^-(M^4)$

In this section, we prove Theorem 3: the only 4-manifold admitting $G_3^-(M^4)$ flows is $\mathbb{S}^3 \times \mathbb{S}^1$. Moreover, the set $G_3^-(\mathbb{S}^3 \times \mathbb{S}^1)$ consists of exactly eight equivalence classes of the considered flows.

Proof. Assume that $f^t \in G_3^-(M^4)$ then by Theorem 1 f^t is topologically equivalent to either the flow f_C^t or flow f_C^{-t} for some $C \in \mathbb{S}^0 \times \mathbb{S}^0$. And since the topological equivalence of flows implies the homeomorphism of their ambient manifolds, the supporting manifold of the flow f^t is homeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$.

Since the elements of the set $C \in \mathbb{S}^0 \times \mathbb{S}^0$ correspond one-to-one to the equivalence classes of flows from $G_3^{-1}(M^4)$, the family $G_3^-(M^4) = G_3^{-2}(M^4) \sqcup G_3^{-1}(M^4)$ contains 8 topological equivalence classes. □

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