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RESEARCH CONTRIBUTION

Orbifold Jacobian Algebras for Exceptional Unimodal Singularities

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Abstract This note shows that the orbifold Jacobian algebra associated to each invertible polynomial defining an exceptional unimodal singularity is isomorphic to the (usual) Jacobian algebra of the Berglund–Hübsch transform of an invertible polynomial defining the strange dual singularity in the sense of Arnold.

Keywords Singularity theory · Arnold's strange duality · Berglund–Hübsch transform

1 Introduction

Exceptional unimodal singularities consist of 14 isolated hypersurface singularities—Q_{10}, Q_{11}, Q_{12}, S_{11}, S_{12}, U_{12}, Z_{11}, Z_{12}, Z_{13}, W_{12}, W_{13}, E_{12}, E_{13} and E_{14} in Arnold's notation (see Arnold et al. 2012). Arnold observed a "strange duality" in this class of singularities, the Dolgachev numbers (a triple of algebraically defined positive

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integers) of one singularity are equal to the Gabrielov numbers (a triple of positive integers associated to a Coxeter–Dynkin diagram) of another one and vice versa. It is now naturally understood as one of mirror symmetry phenomena (cf. Ebeling and Takahashi (2011) and references therein).

Let the polynomial $f \in \mathbb{C}[x_1, \ldots, x_N]$ be *invertible* (see Definition 1). For such polynomials f one can associate an a priori new polynomial $\widetilde{f} \in \mathbb{C}[x_1, \ldots, x_N]$, that is also invertible, called *Berglund–Hübsch transpose* of f (see Sect. 2 for details).

For any two exceptional unimodal singularities that are strangely dual in the sense of Arnold there is a particular choice of the polynomials f_1, f_2 representing them such that both f_1, f_2 are invertible and also Berglund–Hübsch transposes of each other. This was first observed in Kawai and Yang (1995), where the authors show the coincidence of the elliptic genera of dual pairs up to sign. This fact also plays an essential role in Ebeling and Takahashi (2011) for a precise formulation and generalization of Arnold's strange duality. However, the choice of an invertible polynomial, representing an exceptional unimodal singularity is not unique in general (we list all possible choices of an invertible polynomial, representing an exceptional unimodal singularity in Table 1).

For an invertible polynomial $f \in \mathbb{C}[x_1, \ldots, x_N]$ and its symmetry group $G^{\mathrm{SL}}(f)$ (see Sect. 2), let $\mathrm{Jac}(f, G^{\mathrm{SL}}(f))$ stand for the orbifold Jacobian algebra of the pair $(f, G^{\mathrm{SL}}(f))$ and $\mathrm{Jac}(f)$ be the "usual" Jacobian (or local) algebra. We prove the following theorem.

Theorem 1 *Let $f_1, f_2 \in \mathbb{C}[x_1, x_2, x_3]$ be invertible polynomials defining exceptional unimodal singularities (full list is given in Table 1). There exists a Frobenius algebra isomorphism*

$$\mathrm{Jac}(f_1) = \mathrm{Jac}(f_1, \{\mathrm{id}\}) \cong \mathrm{Jac}(\widetilde{f}_2, G^{\mathrm{SL}}(\widetilde{f}_2))$$

if and only if the associated singularities of f_1 and f_2 are strangely dual to each other in the sense of Arnold. Here $\widetilde{f}_2$ is the Berglund–Hübsch transpose of f_2.

For a fixed singularity and different choices of the invertible polynomial f_2 representing it, the function $\widetilde{f}_2$ can have different symmetry groups $G^{\mathrm{SL}}(\widetilde{f}_2)$ and even different Milnor numbers. In particular for U_{12} one will get the symmetry groups $\{\mathrm{id}\}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$ and Milnor numbers 12, 12, 15 by $\widetilde{f}_2$. The algebra $\mathrm{Jac}(f_1)$ in the theorem above will still be the same up to isomorphism. Hence Theorem 1 shows many non–trivial isomorphisms.

In order to visualize the statement of Theorem 1 consider the Fig. 1.

It is worth mentioning that Theorem 1 is compatible with mirror symmetry (using Frobenius structures). It was found in Krawitz et al. (2010) that for f_1, f_2 as above there is a Frobenius algebra isomorphism $\mathrm{Jac}(f_1, \{\mathrm{id}\}) \cong \mathcal{A}^{\mathrm{FJRW}}(f_2, \langle g_{f_2} \rangle)$, where the latter object stands for the so-called *FJRW ring*, the analogue of the quantum cohomology ring associated to the pair $(f_2, \langle g_{f_2} \rangle)$ and $\langle g_{f_2} \rangle$ is the certain symmetry group of f_2 (cf. loc. cit). As a corollary to Theorem 1, we get

$$\mathcal{A}^{\mathrm{FJRW}}(f_2, \langle g_{f_2} \rangle) \cong \mathrm{Jac}(\widetilde{f}_2, G^{\mathrm{SL}}(\widetilde{f}_2)),$$

which is the expected classical mirror symmetry isomorphism.

Table 1 This table shows all possible invertible polynomials representing an exceptional unimodal singularity of a given type

Type	f (v1)	f (v2)	f (v3)	Strange dual
Q_{10}	$x_1^4+x_2^3+x_1x_3^2$	–	–	E_{14}
Q_{11}	$x_1^3x_2+x_2^3+x_1x_3^2$	–	–	Z_{13}
Q_{12}	$x_1^3x_3+x_2^3+x_1x_3^2$	$x_1^5+x_2^3+x_1x_3^2$	–	Q_{12}
S_{11}	$x_1^4+x_2^2x_3+x_1x_3^2$	–	–	W_{13}
S_{12}	$x_1^3x_2+x_2^2x_3+x_1x_3^2$	–	–	S_{12}
U_{12}	$x_1^4+x_2^3+x_3^3$	$x_1^4+x_2^3+x_3^2x_2$	$x_1^4+x_2^2x_3+x_3^2x_2$	U_{12}
Z_{11}	$x_1^5+x_1x_2^3+x_3^2$	–	–	E_{13}
Z_{12}	$x_1^4x_2+x_1x_2^3+x_3^2$	–	–	Z_{12}
Z_{13}	$x_1^3x_3+x_1x_2^3+x_3^2$	$x_1^6+x_2^3x_1+x_3^2$	–	Q_{11}
W_{12}	$x_1^5+x_2^2x_3+x_3^2$	$x_1^5+x_2^4+x_3^2$	–	W_{12}
W_{13}	$x_1^4x_2+x_2^2x_3+x_3^2$	$x_1^4x_2+x_2^4+x_3^2$	–	S_{11}
E_{12}	$x_1^7+x_2^3+x_3^2$	–	–	E_{12}
E_{13}	$x_1^5x_2+x_2^3+x_3^2$	–	–	Z_{11}
E_{14}	$x_1^4x_3+x_2^3+x_3^2$	$x_1^8+x_2^3+x_3^2$	–	Q_{10}

Namely, the polynomials in each row are all right–equivalent to each other

It's important to note that similar results concerning the unimodal singularities are obtained in an apparently different context, in the study of matrix factorizations by the work of Carqueville et al. (2016) and Newton and Ros Camacho (2046). We expect that the Hochschild cohomology group of the category of G-equivariant matrix factorizations will naturally yield the relationship between theirs and ours. We hope to elaborate on this subject in the near future.

2 Orbifold Jacobian Algebra of an Invertible Polynomial

2.1 Invertible Polynomials

For a non-negative integer N and $f = f(x_1, \dots, x_N) \in \mathbb{C}[x_1, \dots, x_N]$ a polynomial, the *Jacobian algebra* $\mathrm{Jac}(f)$ of f is a $\mathbb{C}$-algebra defined as

$$\mathrm{Jac}(f) := \mathbb{C}[x_1, \dots, x_N] \Big/ \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_N} \right).$$

If $\mathrm{Jac}(f)$ is a finite–dimensional $\mathbb{C}$–algebra, f has at most an isolated critical point at the origin. Then set $\mu_f := \dim_{\mathbb{C}} \mathrm{Jac}(f)$ and call it the *Milnor number* of f. In particular, if $N = 0$ then $\mathrm{Jac}(f) = \mathbb{C}$ and $\mu_f = 1$.

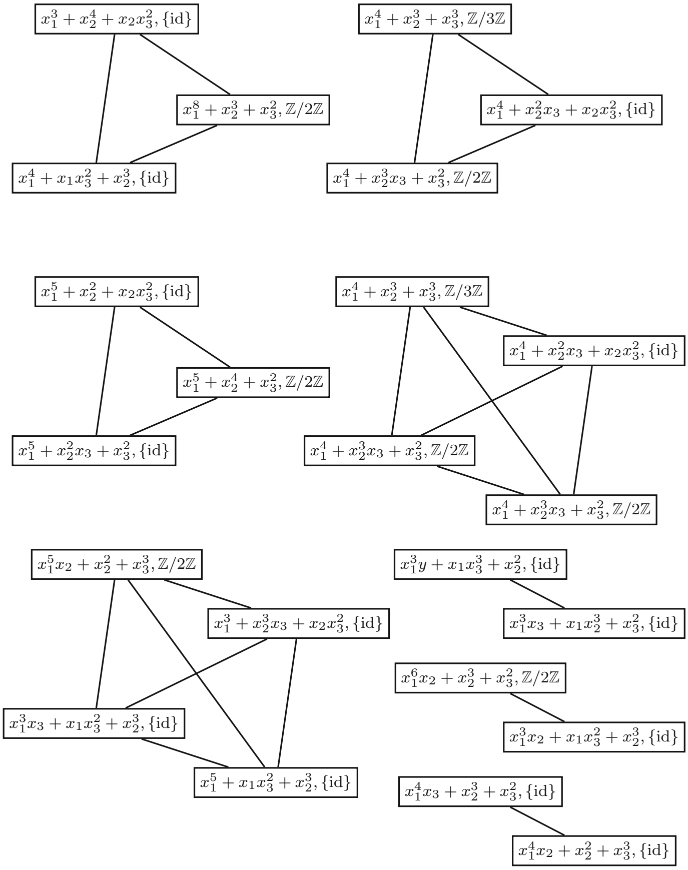

Fig. 1 This figure depicts the isomorphisms between the different orbifold Jacobian algebras. The nodes of this figure are the pairs (f, G) where f is an invertible polynomial and $G \subseteq G^{\mathrm{SL}}(f)$. All the pairs (f, G) considered are those from Table 2. The edge between two nodes labeled by (f_1, G_1) and (f_2, G_2) is drawn if and only if $\mathrm{Jac}(f_1, G_1) \cong \mathrm{Jac}(f_2, G_2)$

The *Hessian* of f is defined as

$$\mathrm{hess}(f) := \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=1,\ldots,N}.$$

In particular, if $N = 0$ then $\mathrm{hess}(f) = 1$.

A polynomial $f \in \mathbb{C}[x_1, \ldots, x_N]$ is called a *weighted homogeneous* polynomial if there are positive integers $w_1, \ldots, w_N$ and d such that

$$f(\lambda^{w_1}x_1, \ldots, \lambda^{w_N}x_N) = \lambda^d f(x_1, \ldots, x_N)$$

for all $\lambda \in \mathbb{C}^*$. A weighted homogeneous polynomial f is called *non-degenerate* if it has at most an isolated critical point at the origin in $\mathbb{C}^N$, equivalently, if the Jacobian algebra $\mathrm{Jac}(f)$ of f is finite-dimensional.

Definition 1 Let $f \in \mathbb{C}[x_1, \dots, x_N]$ be a non-degenerate weighted homogeneous polynomial. It's called *invertible* if the following conditions are satisfied.

– The number of variables ($= N$) coincides with the number of monomials in the polynomial f, namely,

$$f(x_1, \dots, x_N) = \sum_{i=1}^{N} c_i \prod_{j=1}^{N} x_j^{E_{ij}}$$

for some coefficients $c_i \in \mathbb{C}^*$ and non-negative integers E_{ij} for $i, j = 1, \dots, N$.
– The matrix $E := (E_{ij})$ is invertible over $\mathbb{Q}$.

Definition 2 For an invertible polynomial f we define the *Berglund–Hübsch transpose* $\widetilde{f}$ of f to be the following polynomial:

$$\widetilde{f}(x_1, \dots, x_N) := \sum_{i=1}^{N} c_i \prod_{j=1}^{N} x_j^{E_{ji}}$$

Definition 3 The *group of maximal diagonal symmetries* of an invertible polynomial $f(x_1, \dots, x_N)$ is defined as

$$G_f := \left\{ (\lambda_1, \dots, \lambda_N) \in (\mathbb{C}^*)^N \mid f(\lambda_1 x_1, \dots, \lambda_N x_N) = f(x_1, \dots, x_N) \right\}.$$

We shall always identify G_f with the subgroup of diagonal matrices of $\mathrm{GL}(N; \mathbb{C})$. Set also[1]

$$G^{\mathrm{SL}}(f) := G_f \cap \mathrm{SL}(N; \mathbb{C}).$$

Each element $g \in G_f$ has a unique expression of the form

$$g = \mathrm{diag}\left(\mathbf{e}\left[\frac{a_1}{r}\right], \dots, \mathbf{e}\left[\frac{a_N}{r}\right] \right) \quad \text{with } 0 \le a_i < r, \tag{1}$$

where $\mathbf{e}[\alpha] := \exp(2\pi\sqrt{-1}\alpha)$ and r is the order of g. We use the notation $(a_1/r, \dots, a_N/r)$ for the element g. The *age* of g is defined as the rational number

$$\mathrm{age}(g) := \frac{1}{r} \sum_{i=1}^{N} a_i.$$

Note that the $\mathrm{age}(g)$ is an integer if $g \in G^{\mathrm{SL}}(f)$.

[1] One often uses the notation G_f^{SL} rather than $G^{\mathrm{SL}}(f)$, however we need to change it a bit because of many embedded subscripts and superscripts in the text.

2.2 Orbifold Jacobian algebra

Let $f = f(x_1, \dots, x_N)$ be an invertible polynomial and G a subgroup of $G^{\mathrm{SL}}(f)$. *A G–twisted Jacobian algebra of f*, denoted $\mathrm{Jac}'(f, G)$, was introduced in Basalaev et al. (2016) axiomatically. It was also shown in Theorem 2 of loc. cit. that it exists and is uniquely defined up to an isomorphism when f is an invertible polynomial. The structure of $\mathrm{Jac}'(f, G)$ can be then given as follows (see Section 4 of loc.cit.).

As a $\mathbb{C}$-vector space, $\mathrm{Jac}'(f, G)$ is given by

$$\mathrm{Jac}'(f, G) = \bigoplus_{g \in G} \mathrm{Jac}(f^g) v_g, \tag{2}$$

where $f^g := f|_{\mathrm{Fix}(g)}$, $\mathrm{Fix}(g) \subseteq \mathbb{C}^N$ is the fixed locus of g and v_g is a generator (a formal letter) attached to each $g \in G$. Note that f^g is also an invertible polynomial and there is a surjective map $\mathrm{Jac}(f) \longrightarrow \mathrm{Jac}(f^g)$ (see Proposition 5 in Ebeling and Takahashi (2013) and Basalaev et al. (2016), Proposition 7). It is also important that $\mathrm{Jac}'(f, G)$ is equipped with a $\mathbb{Z}/2\mathbb{Z}$-grading according to the parity of $N - N_g$, the codimension of the fixed locus $\mathrm{Fix}(g)$, for each $g \in G$.

We are now ready to introduce the product structure on $\mathrm{Jac}'(f, G)$. For simplicity, we assume that G is a cyclic group whose order is a prime number. Denote also $I_g := \{i \mid g(x_i) = x_i\} \subseteq \{1, \dots, N\}$ for any $g \in G$.

For each pair (g, h) of elements in G and $\phi(\mathbf{x}), \psi(\mathbf{x}) \in \mathrm{Jac}(f)$, the product is defined as follows:

– Suppose that $I_g \cup I_h \cup I_{gh} = \{1, \dots, N\}$. Then

$$\begin{aligned} &[\phi(\mathbf{x})]v_g \circ [\psi(\mathbf{x})]v_h \\ &\quad := (-1)^{\frac{1}{2}(N-N_g)(N-N_g-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g)\right] \cdot [\phi(\mathbf{x})\psi(\mathbf{x})H_{g,h}]v_{gh}, \end{aligned} \tag{3}$$

  where $H_{g,h} \in \mathbb{C}[x_1, \dots, x_N]$ is defined by the following equation in $\mathrm{Jac}(f^{gh})$

$$\frac{1}{\mu_{f^{\langle g,h \rangle}}}[\mathrm{hess}(f^{\langle g,h \rangle})H_{g,h}] = \frac{1}{\mu_{f^{gh}}}[\mathrm{hess}(f^{gh})], \tag{4}$$

  and $f^{\langle g,h \rangle}$ is the invertible polynomial obtained by the restriction of f to the locus $\mathrm{Fix}(g) \cap \mathrm{Fix}(h)$.

– Suppose that $I_g \cup I_h \cup I_{gh} \neq \{1, \dots, N\}$. Then

$$[\phi(\mathbf{x})]v_g \circ [\psi(\mathbf{x})]v_h := 0. \tag{5}$$

This completes the definition of $\mathrm{Jac}'(f, G)$. It is easy to see that $v_{\mathrm{id}} = [1]v_{\mathrm{id}}$ is the identity of $\mathrm{Jac}'(f, G)$.

Note that we have a natural action of G on $\mathrm{Jac}(f^g)$ for any $g \in G$ and that the product structure is invariant under the G-action.

Definition 4 Let f and G be as above. The G-invariant $\mathbb{Z}/2\mathbb{Z}$-graded subalgebra $\mathrm{Jac}(f, G) := \left(\mathrm{Jac}'(f, G)\right)^G$ is called the *orbifold Jacobian algebra* of (f, G).

An important property of this algebra is the following

Proposition 1 (Basalaev et al. 2016) *The algebra* $\mathrm{Jac}(f, G)$ *is a* $\mathbb{Z}/2\mathbb{Z}$*-graded commutative Frobenius algebra. Namely, there is an even non-degenerate pairing* $\eta_{f,G}$ *such that*

$$\eta_{f,G}\,(X \circ Y, Z) = \eta_{f,G}\,(X, Y \circ Z)\,, \quad X, Y, Z \in \mathrm{Jac}(f, G), \tag{6}$$

$$\eta_{f,G}\,(v_{\mathrm{id}}, [\mathrm{hess}(f)]v_{\mathrm{id}}) = |G| \cdot \mu_f. \tag{7}$$

3 Proof of Theorem 1

The proof of Theorem 1 is done by direct calculation. In what follows let the notation be as in Theorem 1.

Skipping the trivial cases when $f_1 = \widetilde{f_2}$ and $G^{\mathrm{SL}}(\widetilde{f_2}) = \{\mathrm{id}\}$, to prove the theorem we only need to show that $\mathrm{Jac}(f_1) \cong \mathrm{Jac}(\widetilde{f_2}, G^{\mathrm{SL}}(\widetilde{f_2}))$ for each row of Table 2.

Further, note that if $G^{\mathrm{SL}}(\widetilde{f_2}) = \{1\}$ and f_1, $\widetilde{f_2}$ do not coincide but belong to the same right-equivalence class, the proof follows since the Jacobian algebra is an invariant of the right-equivalence class. Therefore, it is enough to show the statement for each row of Table 3.

3.1 Computations

From now on, we shall use the notation of Table 2. In order to check that two algebras are isomorphic, we represent $\mathrm{Jac}(\widetilde{f_2}, G^{\mathrm{SL}}(\widetilde{f_2}))$ as a quotient algebra of a polynomial ring in three variables. Namely, we will compute relations in $\mathrm{Jac}(\widetilde{f_2}, G^{\mathrm{SL}}(\widetilde{f_2}))$ and show the existence of a surjective algebra homomorphism from $\mathrm{Jac}(f_1)$, which turns out to be an isomorphism for dimension reasons.

3.1.1 Q_{10} and E_{14}

For $\widetilde{f_2} = x_1^8 + x_2^3 + x_3^2$, and $G^{\mathrm{SL}}(\widetilde{f_2}) = \langle g \rangle = \langle (\frac{1}{2}, 0, \frac{1}{2}) \rangle$, $\mathrm{Jac}(\widetilde{f_2}, G^{\mathrm{SL}}(\widetilde{f_2}))$ is a 10-dimensional $\mathbb{C}$-vector space, whose basis can be chosen as

$$v_{\mathrm{id}},\, [x_1^2]v_{\mathrm{id}},\, [x_1^4]v_{\mathrm{id}},\, [x_1^6]v_{\mathrm{id}},\, [x_2]v_{\mathrm{id}},\, [x_1^2x_2]v_{\mathrm{id}},\, [x_1^4x_2]v_{\mathrm{id}},\, [x_1^6x_2]v_{\mathrm{id}},$$
$$v_g,\, [x_2]v_g.$$

The only non–trivial[2] non-zero products in $\mathrm{Jac}(\widetilde{f_2}, G^{\mathrm{SL}}(\widetilde{f_2}))$, calculated by (3), are given by

[2] Here and henceforth we skip the products of the unit v_{id} with the other basis vectors.

Table 2 To prove Theorem 1, we need to show $\mathrm{Jac}(f_1) \cong \mathrm{Jac}(\widetilde{f}_2, G^{\mathrm{SL}}(\widetilde{f}_2))$ for every row of this table

Type of f_1	f_1	$\widetilde{f}_2$	$G^{\mathrm{SL}}(\widetilde{f}_2)$	Type of f_2
E_{14}	$x_1^8+x_2^3+x_3^2$	$x_1^4x_2+x_2^2+x_3^3$	$\{\mathrm{id}\}$	Q_{10}
Q_{10}	$x_1^4+x_2^3+x_1x_3^2$	$x_1^8+x_2^3+x_3^2$	$\langle(1/2,0,1/2)\rangle$	E_{14}
Q_{11}	$x_1^3x_2+x_2^3+x_1x_3^2$	$x_1^6x_2+x_2^3+x_3^2$	$\langle(1/2,0,1/2)\rangle$	Z_{13}
Q_{12}	$x_2^3+x_1^3x_3+x_1x_3^2$	$x_1^5x_2+x_2^2+x_3^3$	$\langle(1/2,1/2,0)\rangle$	Q_{12}
Q_{12}	$x_1^5+x_2^3+x_1x_3^2$	$x_1^3+x_2^3x_3+x_2x_3^2$	$\{\mathrm{id}\}$	Q_{12}
Q_{12}	$x_1^5+x_2^3+x_1x_3^2$	$x_1^5x_2+x_2^2+x_3^3$	$\langle(1/2,1/2,0)\rangle$	Q_{12}
S_{11}	$x_1^4+x_2^2x_3+x_1x_3^2$	$x_1^4+x_1x_2^4+x_3^2$	$\langle(0,1/2,1/2)\rangle$	W_{13}
U_{12}	$x_1^4+x_2^3+x_3^3$	$x_1^4+x_2^3+x_3^3$	$\langle(0,2/3,1/3)\rangle$	U_{12}
U_{12}	$x_1^4+x_2^3+x_3^3$	$x_1^4+x_2^3x_3+x_3^2$	$\langle(0,1/2,1/2)\rangle$	U_{12}
U_{12}	$x_1^4+x_2^3+x_3^3$	$x_1^4+x_2^2x_3+x_2x_3^2$	$\{\mathrm{id}\}$	U_{12}
U_{12}	$x_1^4+x_2^3+x_2x_3^2$	$x_1^4+x_2^3+x_3^3$	$\langle(0,2/3,1/3)\rangle$	U_{12}
U_{12}	$x_1^4+x_2^3+x_2x_3^2$	$x_1^4+x_2^3x_3+x_3^2$	$\langle(0,1/2,1/2)\rangle$	U_{12}
U_{12}	$x_1^4+x_2^3+x_2x_3^2$	$x_1^4+x_2^2x_3+x_2x_3^2$	$\{\mathrm{id}\}$	U_{12}
U_{12}	$x_1^4+x_2^2x_3+x_2x_3^2$	$x_1^4+x_2^3+x_3^3$	$\langle(0,2/3,1/3)\rangle$	U_{12}
U_{12}	$x_1^4+x_2^2x_3+x_2x_3^2$	$x_1^4+x_2^3x_3+x_3^2$	$\langle(0,1/2,1/2)\rangle$	U_{12}
W_{12}	$x_1^5+x_2^2x_3+x_3^2$	$x_1^5+x_2^4+x_3^2$	$\langle(0,1/2,1/2)\rangle$	W_{12}
W_{12}	$x_1^5+x_2^4+x_3^2$	$x_1^5+x_2^2+x_2x_3^2$	$\{\mathrm{id}\}$	W_{12}
W_{12}	$x_1^5+x_2^4+x_3^2$	$x_1^5+x_2^4+x_3^2$	$\langle(0,1/2,1/2)\rangle$	W_{12}
W_{13},	$x_1^4x_2+x_2^4+x_3^2$	$x_1^4x_2+x_2^2x_3+x_3^2$	$\{\mathrm{id}\}$	S_{11}
Z_{13},	$x_1^6+x_1x_2^3+x_3^2$	$x_1^3x_2+x_2^2+x_1x_3^3$	$\{\mathrm{id}\}$	Q_{11}

Table 3 It is enough to show $\mathrm{Jac}(f_1) \cong \mathrm{Jac}(\widetilde{f}_2, G^{\mathrm{SL}}(\widetilde{f}_2))$ for every row of this table to prove Theorem 1

Type of f_1	f_1	$\widetilde{f}_2$	$G^{\mathrm{SL}}(\widetilde{f}_2)$	Type of f_2
Q_{10}	$x_1^4+x_2^3+x_1x_3^2$	$x_1^8+x_2^3+x_3^2$	$\langle(1/2,0,1/2)\rangle$	E_{14}
Q_{11}	$x_1^3x_2+x_2^3+x_1x_3^2$	$x_1^6x_2+x_2^3+x_3^2$	$\langle(1/2,0,1/2)\rangle$	Z_{13}
Q_{12}	$x_1^5+x_2^3+x_1x_3^2$	$x_1^5x_2+x_2^2+x_3^3$	$\langle(1/2,1/2,0)\rangle$	Q_{12}
S_{11}	$x_1^4+x_2^2x_3+x_1x_3^2$	$x_1^4+x_1x_2^4+x_3^2$	$\langle(0,1/2,1/2)\rangle$	W_{13}
U_{12}	$x_1^4+x_2^3+x_3^3$	$x_1^4+x_2^3+x_3^3$	$\langle(0,2/3,1/3)\rangle$	U_{12}
U_{12}	$x_1^4+x_2^3+x_2x_3^2$	$x_1^4+x_2^3x_3+x_3^2$	$\langle(0,1/2,1/2)\rangle$	U_{12}
W_{12}	$x_1^5+x_2^4+x_3^2$	$x_1^5+x_2^4+x_3^2$	$\langle(0,1/2,1/2)\rangle$	W_{12}

$$[x_1^2]v_{\mathrm{id}} \circ [x_1^2]v_{\mathrm{id}} = [x_1^4]v_{\mathrm{id}}, \qquad [x_1^2]v_{\mathrm{id}} \circ [x_1^4]v_{\mathrm{id}} = [x_1^6]v_{\mathrm{id}},$$
$$[x_1^2]v_{\mathrm{id}} \circ [x_2]v_{\mathrm{id}} = [x_1^2x_2]v_{\mathrm{id}}, \qquad [x_1^4]v_{\mathrm{id}} \circ [x_2]v_{\mathrm{id}} = [x_1^4x_2]v_{\mathrm{id}},$$
$$[x_1^6]v_{\mathrm{id}} \circ [x_2]v_{\mathrm{id}} = [x_1^6x_2]v_{\mathrm{id}}, \qquad [x_2]v_{\mathrm{id}} \circ v_g = [x_2]v_g,$$
$$[x_1^2]v_{\mathrm{id}} \circ [x_1^2x_2]v_{\mathrm{id}} = [x_1^4x_2]v_{\mathrm{id}}, \qquad [x_1^4]v_{\mathrm{id}} \circ [x_1^2x_2]v_{\mathrm{id}} = [x_1^6x_2]v_{\mathrm{id}},$$

$$[x_1^2]v_{\rm id}\circ[x_1^4x_2]v_{\rm id}=[x_1^6x_2]v_{\rm id},\qquad v_g\circ v_g=16[x_1^6]v_{\rm id},$$
$$v_g\circ[x_2]v_g=16[x_1^6x_2]v_{\rm id}.$$

For example the product $v_g\circ[x_2]v_g$ is calculated as follows. We have $f^g=x_2^3=f^{\langle g,g\rangle}$ and

$$\frac{1}{\mu_{f^{\langle g,g\rangle}}}[\mathrm{hess}(f^{\langle g,g\rangle})H_{g,g}]=\frac{1}{2}[3\cdot 2x_2H_{g,g}],$$
$$\frac{1}{\mu_{f^{g^2}}}[\mathrm{hess}(f^{g^2})]=\frac{1}{\mu_f}[\mathrm{hess}(f)]=\frac{1}{14}[8\cdot 7x_1^6\cdot 3\cdot 2x_2\cdot 2].$$

So we see by equation (4), that $H_{g,g}=16x_1^6$. With this we get by equation (3)

$$v_g\circ[x_2]v_g=(-1)^{\frac{1}{2}(3-1)(3-1-1)}\cdot\mathbf{e}\left[-\frac{1}{2}\cdot 1\right]\cdot[x_2H_{g,g}]v_{g^2}=(-1)(-1)[x_2\cdot 16x_1^6]v_{\rm id}.$$

From the multiplication table above we see that $[x_1^2]v_{\rm id}$, $[x_2]v_{\rm id}$, v_g generate $\mathrm{Jac}(\widetilde{f}_2,G^{\rm SL}(\widetilde{f}_2))$ and are subject to the following relations

$$16([x_1^2]v_{\rm id})^{\circ 3}-v_g^{\circ 2}=0,\quad ([x_2]v_{\rm id})^{\circ 2}=0,\quad [x_1^2]v_{\rm id}\circ v_g=0.$$

On the other hand, the Jacobian algebra $\mathrm{Jac}(f_1)$ is given by

$$\mathrm{Jac}(f_1)=\mathbb{C}[y_1,y_2,y_3]\Big/\Big(4y_1^3+y_3^2,y_2^2,y_1y_3\Big).$$

Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(f_1)\xrightarrow{\cong}\mathrm{Jac}(\widetilde{f}_2,G^{\rm SL}(\widetilde{f}_2)),\quad y_1\mapsto[x_1^2]v_{\rm id},\ y_2\mapsto[x_2]v_{\rm id},\ y_3\mapsto\frac{1}{2}\sqrt{-1}v_g,$$

which is, moreover, an isomorphism of Frobenius algebras since by (7) we have

$$\eta_{\widetilde{f}_2,G^{\rm SL}(\widetilde{f}_2)}\Big(v_{\rm id},[x_1^6x_2]v_{\rm id}\Big)=\frac{1}{672}\cdot\eta_{\widetilde{f}_2,G^{\rm SL}(\widetilde{f}_2)}\big(v_{\rm id},[\mathrm{hess}(\widetilde{f}_2)]v_{\rm id}\big)=\frac{2\cdot 14}{672}=\frac{1}{24},$$
$$\eta_{f_1,\{\rm id\}}\Big(v_{\rm id},[y_1^3y_2]v_{\rm id}\Big)=\frac{1}{240}\cdot\eta_{f_1,\{\rm id\}}\left(v_{\rm id},[\mathrm{hess}(f_1)]v_{\rm id}\right)=\frac{1\cdot 10}{240}=\frac{1}{24}.$$

3.1.2 Q_{11} and Z_{13}

For $\widetilde{f}_2=x_1^6x_2+x_2^3+x_3^2$, and $G^{\rm SL}(\widetilde{f}_2)=\langle g\rangle=\langle(\frac{1}{2},0,\frac{1}{2})\rangle$, $\mathrm{Jac}(\widetilde{f}_2,G^{\rm SL}(\widetilde{f}_2))$ is a 11-dimensional $\mathbb{C}$-vector space, whose basis can be chosen as

$$v_{\rm id},[x_1^2]v_{\rm id},[x_1^4]v_{\rm id},[x_2]v_{\rm id},[x_1^2x_2]v_{\rm id},[x_1^4x_2]v_{\rm id},[x_2^2]v_{\rm id},$$
$$[x_1^2x_2^2]v_{\rm id},[x_1^4x_2^2]v_{\rm id},v_g,[x_2]v_g.$$

The only non-trivial non-zero products in $\mathrm{Jac}(\widetilde{f}_2, G^{\mathrm{SL}}(\widetilde{f}_2))$, calculated by (3), are given by

$$
\begin{aligned}
&[x_1^2]v_{\mathrm{id}} \circ [x_1^2]v_{\mathrm{id}} = [x_1^4]v_{\mathrm{id}}, && [x_1^2]v_{\mathrm{id}} \circ [x_1^4]v_{\mathrm{id}} = -3[x_2^2]v_{\mathrm{id}},\\
&[x_1^4]v_{\mathrm{id}} \circ [x_1^4]v_{\mathrm{id}} = -3[x_1^2x_2^2]v_{\mathrm{id}}, && [x_1^2]v_{\mathrm{id}} \circ [x_2]v_{\mathrm{id}} = [x_1^2x_2]v_{\mathrm{id}},\\
&[x_1^4]v_{\mathrm{id}} \circ [x_2]\widetilde{v}_{\mathrm{id}} = [x_1^4x_2]v_{\mathrm{id}}, && [x_2]v_{\mathrm{id}} \circ [x_2]v_{\mathrm{id}} = [x_2^2]v_{\mathrm{id}},\\
&[x_2]v_{\mathrm{id}} \circ v_g = [x_2]v_g, && [x_1^2]v_{\mathrm{id}} \circ [x_1^2x_2]\widetilde{v}_{\mathrm{id}} = [x_1^4x_2]v_{\mathrm{id}},\\
&[x_2]v_{\mathrm{id}} \circ [x_1^2x_2]\widetilde{v}_{\mathrm{id}} = [x_1^2x_2^2]v_{\mathrm{id}} && [x_1^2x_2]v_{\mathrm{id}} \circ [x_1^2x_2]v_{\mathrm{id}} = [x_1^4x_2^2]v_{\mathrm{id}},\\
&[x_2]\widetilde{v}_{\mathrm{id}} \circ [x_1^4x_2]\widetilde{v}_{\mathrm{id}} = [x_1^4x_2^2]v_{\mathrm{id}}, && [x_1^2]v_{\mathrm{id}} \circ [x_2^2]v_{\mathrm{id}} = [x_1^2x_2^2]v_{\mathrm{id}},\\
&[x_1^4]v_{\mathrm{id}} \circ [x_2^2]v_{\mathrm{id}} = [x_1^4x_2^2]v_{\mathrm{id}}, && [x_1^2]v_{\mathrm{id}} \circ [x_1^2x_2^2]v_{\mathrm{id}} = [x_1^4x_2^2]v_{\mathrm{id}},\\
&v_g \circ v_g = 12[x_1^4x_2]v_{\mathrm{id}}, && v_g \circ [x_2]v_g = 12[x_1^4x_2^2]v_{\mathrm{id}},
\end{aligned}
$$

which show that $[x_1^2]v_{\mathrm{id}}, [x_2]v_{\mathrm{id}}, v_g$ generate $\mathrm{Jac}(\widetilde{f}_2, G^{\mathrm{SL}}(\widetilde{f}_2))$ and are subject to the following relations

$$
\begin{aligned}
&12([x_1^2]v_{\mathrm{id}})^{\circ 2}[x_2]v_{\mathrm{id}} - (v_g)^{\circ 2} = 0, \ \ ([x_1^2]v_{\mathrm{id}})^{\circ 3} + 3([x_2]v_{\mathrm{id}})^{\circ 2} = 0,\\
&[x_1^2]v_{\mathrm{id}} \circ v_g = 0.
\end{aligned}
$$

On the other hand, the Jacobian algebra $\mathrm{Jac}(f_1)$ is given by

$$\mathrm{Jac}(f) = \mathbb{C}[y_1, y_2, y_3]\Big/\Big(3y_1^2y_2 + y_3^2, y_1^3 + 3y_2^2, 2y_1y_3\Big)$$

Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(f_1) \xrightarrow{\cong} \mathrm{Jac}(\widetilde{f}_2, G^{\mathrm{SL}}(\widetilde{f}_2)), \quad y_1 \mapsto [x_1^2]v_{\mathrm{id}}, \ y_2 \mapsto [x_2]v_{\mathrm{id}}, \ y_3 \mapsto \frac{1}{2}\sqrt{-1}v_g,$$

which is, moreover, an isomorphism of Frobenius algebras since by (7) we have

$$
\begin{aligned}
&\eta_{\widetilde{f}_2, G^{\mathrm{SL}}(\widetilde{f}_2)}\Big(v_{\mathrm{id}}, [x_1^4x_2^2]v_{\mathrm{id}}\Big) = \frac{1}{18},\\
&\eta_{f_1,\{\mathrm{id}\}}\Big(v_{\mathrm{id}}, [y_1^2y_2^2]v_{\mathrm{id}}\Big) = \frac{1}{198} \cdot \eta_{f_1,\{\mathrm{id}\}}\left(v_{\mathrm{id}}, [\mathrm{hess}(f_1)]v_{\mathrm{id}}\right) = \frac{1 \cdot 11}{198} = \frac{1}{18}.
\end{aligned}
$$

3.1.3 Q_{12} and Q_{12}

For $\widetilde{f}_2 = x_1^5x_2 + x_2^2 + x_3^3$, and $G^{\mathrm{SL}}(\widetilde{f}_2) = \langle g \rangle = \langle (\frac{1}{2}, \frac{1}{2}, 0) \rangle$, $\mathrm{Jac}(\widetilde{f}_2, G^{\mathrm{SL}}(\widetilde{f}_2))$ is a 12-dimensional $\mathbb{C}$-vector space, whose basis can be chosen as

$$
\begin{aligned}
&v_{\mathrm{id}}, [x_1^2]v_{\mathrm{id}}, [x_1^4]v_{\mathrm{id}}, [x_1x_2]v_{\mathrm{id}}, [x_1^3x_2]v_{\mathrm{id}}, [x_3]v_{\mathrm{id}}, [x_1^2x_3]v_{\mathrm{id}},\\
&[x_1^4x_3]v_{\mathrm{id}}, [x_1x_2x_3]v_{\mathrm{id}}, [x_1^3x_2x_3]v_{\mathrm{id}}, \ v_g, [x_3]v_g.
\end{aligned}
$$

The only non-trivial non-zero products in $\mathrm{Jac}(\widetilde{f}_2, G^{\mathrm{SL}}(\widetilde{f}_2))$, calculated by (3), are given by

$$
\begin{aligned}
&[x_1^2]v_{\mathrm{id}} \circ [x_1^2]v_{\mathrm{id}} = [x_1^4]v_{\mathrm{id}}, && [x_1^2]v_{\mathrm{id}} \circ [x_1^4]v_{\mathrm{id}} = -2[x_1x_2]v_{\mathrm{id}},\\
&[x_1^4]v_{\mathrm{id}} \circ [x_1^4]v_{\mathrm{id}} = -2[x_1^3x_2]v_{\mathrm{id}}, && [x_1^2]v_{\mathrm{id}} \circ [x_1x_2]v_{\mathrm{id}} = [x_1^3x_2]v_{\mathrm{id}},\\
&[x_1^2]v_{\mathrm{id}} \circ [x_3]v_{\mathrm{id}} = [x_1^2x_3]v_{\mathrm{id}}, && [x_1^4]v_{\mathrm{id}} \circ [x_3]v_{\mathrm{id}} = [x_1^4x_3]v_{\mathrm{id}},\\
&[x_1x_2]v_{\mathrm{id}} \circ [x_3]v_{\mathrm{id}} = [x_1x_2x_3]v_{\mathrm{id}}, && [x_1^3x_2]v_{\mathrm{id}} \circ [x_3]v_{\mathrm{id}} = [x_1^3x_2x_3]v_{\mathrm{id}},\\
&[x_3]v_{\mathrm{id}} \circ v_g = [x_3]v_g, && [x_1^2]v_{\mathrm{id}} \circ [x_1^2x_3]v_{\mathrm{id}} = [x_1^4x_3]v_{\mathrm{id}},\\
&[x_1^4]v_{\mathrm{id}} \circ [x_1^2x_3]v_{\mathrm{id}} = -2[x_1x_2x_3]v_{\mathrm{id}}, && [x_1x_2]v_{\mathrm{id}} \circ [x_1^2x_3]v_{\mathrm{id}} = [x_1^3x_2x_3]v_{\mathrm{id}},\\
&[x_1^2]v_{\mathrm{id}} \circ [x_1^4x_3]v_{\mathrm{id}} = -2[x_1x_2x_3]v_{\mathrm{id}}, && [x_1^4]v_{\mathrm{id}} \circ [x_1^4x_3]v_{\mathrm{id}} = -2[x_1^3x_2x_3]v_{\mathrm{id}},\\
&[x_1^2]v_{\mathrm{id}} \circ [x_1x_2x_3]v_{\mathrm{id}} = [x_1^3x_2x_3]v_{\mathrm{id}}, && v_g \circ v_g = 10[x_1^3x_2]v_{\mathrm{id}},\\
&v_g \circ [x_3]v_g = 10[x_1^3x_2x_3]v_{\mathrm{id}},
\end{aligned}
$$

which show that $[x_1^2]v_{\mathrm{id}}, [x_3]v_{\mathrm{id}}, v_g$ generate $\mathrm{Jac}(\widetilde{f}_2, G^{\mathrm{SL}}(\widetilde{f}_2))$ and are subject to the following relations

$$[x_1^2]v_{\mathrm{id}} \circ v_g = 0,\ (v_g)^{\circ 2} + 5([x_1^2]v_{\mathrm{id}})^{\circ 4} = 0,\ ([x_3]v_{\mathrm{id}})^{\circ 2} = 0.$$

On the other hand, the Jacobian algebra $\mathrm{Jac}(f_1)$ is given by

$$\mathrm{Jac}(f_1) = \mathbb{C}[y_1, y_2, y_3]\Big/\Big(5y_1^4 + y_3^2, 3y_2^2, 2y_1y_3\Big).$$

Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(f_1) \xrightarrow{\cong} \mathrm{Jac}(\widetilde{f}_2, G^{\mathrm{SL}}(\widetilde{f}_2)), \quad y_1 \mapsto [x_1^2]v_{\mathrm{id}},\ y_2 \mapsto -\frac{1}{4}[x_3]v_{\mathrm{id}},\ y_3 \mapsto v_g,$$

which is, moreover, an isomorphism of Frobenius algebras since by (7) we have

$$
\begin{aligned}
\eta_{\widetilde{f}_2, G^{\mathrm{SL}}(\widetilde{f}_2)}\Big(v_{\mathrm{id}}, [x_1^3x_2x_3]v_{\mathrm{id}}\Big) &= \frac{1}{15},\\
-\frac{4}{10} \cdot \eta_{f_1,\{\mathrm{id}\}}\Big(v_{\mathrm{id}}, [y_2y_3^2]v_{\mathrm{id}}\Big) &= \frac{4}{720} \cdot \eta_{f_1,\{\mathrm{id}\}}\left(v_{\mathrm{id}}, [\mathrm{hess}(f_1)]v_{\mathrm{id}}\right)\\
&= \frac{4 \cdot 12}{720} = \frac{1}{15}.
\end{aligned}
$$

3.1.4 S_{11} and W_{13}

For $\widetilde{f}_2 = x_1^4 + x_1x_2^4 + x_3^2$, and $G^{\mathrm{SL}}(\widetilde{f}_2) = \langle g \rangle = \langle (0, \frac{1}{2}, \frac{1}{2}) \rangle$, $\mathrm{Jac}(\widetilde{f}_2, G^{\mathrm{SL}}(\widetilde{f}_2))$ is a 11-dimensional $\mathbb{C}$-vector space, whose basis can be chosen as

$$v_{\rm id}, [x_1]v_{\rm id}, [x_1^2]v_{\rm id}, [x_1^3]v_{\rm id}, [x_2^2]v_{\rm id}, [x_1x_2^2]v_{\rm id}, [x_1^2x_2^2]v_{\rm id}, [x_1^3x_2^2]v_{\rm id},$$
$$v_g, [x_1]v_g, [x_1^2]v_g.$$

The only non-trivial non-zero products in $\mathrm{Jac}(\widetilde{f}_2, G^{\rm SL}(\widetilde{f}_2))$, calculated by (3), are given by

$$\begin{aligned}
&[x_1]v_{\rm id}\circ[x_1]v_{\rm id} = [x_1^2]v_{\rm id}, && [x_1]v_{\rm id}\circ v_g = [x_1]v_g,\\
&[x_1]v_{\rm id}\circ[x_1]v_g = [x_1^2]v_g, && [x_1]v_{\rm id}\circ[x_1^2]v_{\rm id} = [x_1^3]v_{\rm id},\\
&[x_1^2]v_{\rm id}\circ v_g = [x_1^2]v_g, && [x_1]v_{\rm id}\circ[x_2^2]v_{\rm id} = [x_1x_2^2]v_{\rm id},\\
&[x_1^2]v_{\rm id}\circ[x_2^2]v_{\rm id} = [x_1^2x_2^2]v_{\rm id}, && [x_1^3]v_{\rm id}\circ[x_2^2]v_{\rm id} = [x_1^3x_2^2]v_{\rm id},\\
&[x_2^2]v_{\rm id}\circ[x_2^2]v_{\rm id} = -4[x_1^3]v_{\rm id}, && [x_1]v_{\rm id}\circ[x_1x_2^2]v_{\rm id} = [x_1^2x_2^2]v_{\rm id},\\
&[x_1^2]v_{\rm id}\circ[x_1x_2^2]v_{\rm id} = [x_1^3x_2^2]v_{\rm id}, && [x_1]v_{\rm id}\circ[x_1^2x_2^2]v_{\rm id} = [x_1^3x_2^2]v_{\rm id},\\
&v_g\circ v_g = 8[x_1x_2^2]v_{\rm id}, && v_g\circ[x_1]v_g = 8[x_1^2x_2^2]v_{\rm id},\\
&[x_1]v_g\circ[x_1]v_g = 8[x_1^3x_2^2]v_{\rm id}, && v_g\circ[x_1^2]v_g = 8[x_1^3x_2^2]v_{\rm id},
\end{aligned}$$

which show that $[x_1]v_{\rm id}, [x_2^2]v_{\rm id}, v_g$ generate $\mathrm{Jac}(\widetilde{f}_2, G^{\rm SL}(\widetilde{f}_2))$ and are subject to the following relations

$$\begin{aligned}
&([x_2^2]v_{\rm id})^{\circ 2} + 4([x_1]v_{\rm id})^{\circ 3} = 0,\ [x_2^2]v_{\rm id}\circ v_g = 0,\\
&(v_g)^{\circ 2} - 8[x_1]v_{\rm id}\circ[x_2^2]v_{\rm id} = 0.
\end{aligned}$$

On the other hand, the Jacobian algebra $\mathrm{Jac}(f_1)$ is given by

$$\mathrm{Jac}(f_1) = \mathbb{C}[y_1, y_2, y_3]\Big/\Big(4y_1^3 + y_3^2, 2y_2y_3, y_2^2 + 2y_1y_3\Big).$$

Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(f_1) \xrightarrow{\cong} \mathrm{Jac}(\widetilde{f}_2, G^{\rm SL}(\widetilde{f}_2)),\quad y_1\mapsto[x_1]v_{\rm id},\ y_2\mapsto\frac{1}{2}\sqrt{-1}v_g,\ y_3\mapsto[x_2^2]v_{\rm id},$$

which is, moreover, an isomorphism of Frobenius algebras since by (7) we have

$$\begin{aligned}
&\eta_{\widetilde{f}_2, G^{\rm SL}(\widetilde{f}_2)}\Big(v_{\rm id}, [x_1^3x_2^2]v_{\rm id}\Big) = \frac{1}{16},\\
&\eta_{f_1,\{\rm id\}}\Big(v_{\rm id}, [y_1^3y_3]v_{\rm id}\Big) = \frac{1}{176}\cdot\eta_{f_1,\{\rm id\}}(v_{\rm id}, [\mathrm{hess}(f_1)]v_{\rm id}) = \frac{1\cdot 11}{176} = \frac{1}{16}.
\end{aligned}$$

3.1.5 U_{12} and U_{12}, part 1

For $\widetilde{f}_2 = x_1^4 + x_2^3 + x_3^3$, and $G^{\rm SL}(\widetilde{f}_2) = \langle g\rangle = \langle(0, \frac{2}{3}, \frac{1}{3})\rangle$, $\mathrm{Jac}(\widetilde{f}_2, G^{\rm SL}(\widetilde{f}_2))$ is a 12-dimensional $\mathbb{C}$-vector space, whose basis can be chosen as

$$v_{\mathrm{id}}, [x_1]v_{\mathrm{id}}, [x_1^2]v_{\mathrm{id}}, [x_2x_3]v_{\mathrm{id}}, [x_1x_2x_3]v_{\mathrm{id}}, [x_1^2x_2x_3]v_{\mathrm{id}},$$
$$v_{g^2}, [x_1]v_{g^2}, [x_1^2]v_{g^2},\ v_g, [x_1]v_g, [x_1^2]v_g.$$

The only non-trivial non-zero products in $\mathrm{Jac}(\widetilde{f_2}, G^{\mathrm{SL}}(\widetilde{f_2}))$, calculated by (3), are given by

$$\begin{aligned}
&[x_1]v_{\mathrm{id}} \circ [x_1]v_{\mathrm{id}} = [x_1^2]v_{\mathrm{id}}, && [x_1]v_{\mathrm{id}} \circ v_{g^2} = [x_1]v_{g^2},\\
&[x_1]v_{\mathrm{id}} \circ [x_1]v_{g^2} = [x_1^2]v_{g^2}, && [x_1]v_{\mathrm{id}} \circ v_g = [x_1]v_g,\\
&[x_1]v_{\mathrm{id}} \circ [x_1]v_g = [x_1^2]v_g, && [x_1^2]v_{\mathrm{id}} \circ v_{g^2} = [x_1^2]v_{g^2},\\
&[x_1^2 v_{\mathrm{id}}] \circ v_g = [x_1^2]v_g, && [x_1]v_{\mathrm{id}} \circ [x_2x_3]v_{\mathrm{id}}) = [x_1x_2x_3]v_{\mathrm{id}},\\
&[x_1^2]v_{\mathrm{id}} \circ [x_2x_3]v_{\mathrm{id}} = [x_1^2x_2x_3]v_{\mathrm{id}}, && [x_1]v_{\mathrm{id}} \circ [x_1x_2x_3]v_{\mathrm{id}}) = [x_1^2x_2x_3]v_{\mathrm{id}},\\
&v_g \circ v_{g^2} = 9[x_2x_3]v_{\mathrm{id}}, && v_g \circ [x_1]v_{g^2} = 9[x_1x_2x_3]v_{\mathrm{id}},\\
&v_g \circ [x_1^2]v_{g^2} = 9[x_1^2x_2x_3]v_{\mathrm{id}}, && [x_1]v_g \circ v_{g^2} = 9[x_1x_2x_3]v_{\mathrm{id}},\\
&[x_1]v_g \circ [x_1]v_{g^2} = 9[x_1^2x_2x_3]v_{\mathrm{id}}, && [x_1^2]v_g \circ v_{g^2} = 9[x_1^2x_2x_3]v_{\mathrm{id}},
\end{aligned}$$

which show that $[x_1]v_{\mathrm{id}},\ v_g,\ v_{g^2}$ generate $\mathrm{Jac}(\widetilde{f_2}, G^{\mathrm{SL}}(\widetilde{f_2}))$ and are subject to the following relations

$$([x_1]v_{\mathrm{id}})^{\circ 3} = 0,\ (v_g)^{\circ 2} = 0,\ (v_{g^2})^{\circ 2} = 0.$$

On the other hand, the Jacobian algebra $\mathrm{Jac}(f_1)$ is given by

$$\mathrm{Jac}(f_1) = \mathbb{C}[y_1, y_2, y_3]\Big/\left(4y_1^3, 3y_2^2, 3y_3^2\right).$$

Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(f_1) \xrightarrow{\cong} \mathrm{Jac}(\widetilde{f_2}, G^{\mathrm{SL}}(\widetilde{f_2})), \quad y_1 \mapsto \frac{1}{3}[x_1]v_{\mathrm{id}},\ y_2 \mapsto \frac{1}{\sqrt{3}}v_g,\ y_3 \mapsto \frac{1}{\sqrt{3}}v_{g^2},$$

which is, moreover, an isomorphism of Frobenius algebras since by (7) we have

$$\begin{aligned}
\eta_{\widetilde{f_2}, G^{\mathrm{SL}}(\widetilde{f_2})}\left(v_{\mathrm{id}}, [x_1^2x_2x_3]v_{\mathrm{id}}\right) &= \frac{1}{12},\\
3 \cdot \eta_{f_1,\{\mathrm{id}\}}\left(v_{\mathrm{id}}, [y_1^2y_2y_3]v_{\mathrm{id}}\right) &= \frac{3}{432} \cdot \eta_{f_1,\{\mathrm{id}\}}\left(v_{\mathrm{id}}, [\mathrm{hess}(f_1)]v_{\mathrm{id}}\right)\\
&= \frac{3 \cdot 12}{432} = \frac{1}{12}.
\end{aligned}$$

3.1.6 U_{12} and U_{12}, part 2

For $\widetilde{f}_2 = x_1^4 + x_2^3 x_3 + x_3^2$, and $G^{\mathrm{SL}}(\widetilde{f}_2) = \langle g \rangle = \langle (0, \frac{1}{2}, \frac{1}{2}) \rangle$, $\mathrm{Jac}(\widetilde{f}_2, G^{\mathrm{SL}}(\widetilde{f}_2))$ is a 12-dimensional $\mathbb{C}$-vector space, whose basis can be chosen as

$$v_{\mathrm{id}}, [x_1]v_{\mathrm{id}}, [x_1^2]v_{\mathrm{id}}, [x_2^2]v_{\mathrm{id}}, [x_1x_2^2]v_{\mathrm{id}}, [x_1^2x_2^2]v_{\mathrm{id}}, [x_2x_3]v_{\mathrm{id}},$$
$$[x_1x_2x_3]v_{\mathrm{id}}, [x_1^2x_2x_3]v_{\mathrm{id}}, v_g, [x_1]v_g, [x_1^2]v_g.$$

The only non-trivial non-zero products in $\mathrm{Jac}(\widetilde{f}_2, G^{\mathrm{SL}}(\widetilde{f}_2))$, calculated by (3), are given by

$$\begin{aligned}
&[x_1]v_{\mathrm{id}} \circ [x_1]v_{\mathrm{id}} = [x_1^2]v_{\mathrm{id}}, && [x_1]v_{\mathrm{id}} \circ v_g = [x_1]v_g,\\
&[x_1]v_{\mathrm{id}} \circ [x_1]v_g = [x_1^2]v_g, && [x_1^2]v_{\mathrm{id}} \circ v_g = [x_1^2]v_g,\\
&[x_1]v_{\mathrm{id}} \circ [x_2^2]v_{\mathrm{id}} = [x_1x_2^2]v_{\mathrm{id}}, && [x_1^2]v_{\mathrm{id}} \circ [x_2^2]v_{\mathrm{id}} = [x_1^2x_2^2]v_{\mathrm{id}},\\
&[x_2^2]v_{\mathrm{id}} \circ [x_2^2]v_{\mathrm{id}} = -2[x_2x_3]v_{\mathrm{id}}, && [x_1]v_{\mathrm{id}} \circ [x_1x_2^2]v_{\mathrm{id}} = [x_1^2x_2^2]v_{\mathrm{id}},\\
&[x_2^2]v_{\mathrm{id}} \circ [x_1x_2^2]v_{\mathrm{id}} = -2[x_1x_2x_3]v_{\mathrm{id}}, && [x_1x_2^2]v_{\mathrm{id}} \circ [x_1x_2^2]v_{\mathrm{id}} = -2[x_1^2x_2x_3]v_{\mathrm{id}},\\
&[x_2^2]v_{\mathrm{id}} \circ [x_1^2x_2^2]v_{\mathrm{id}} = -2[x_1^2x_2x_3]v_{\mathrm{id}}, && [x_1]v_{\mathrm{id}} \circ [x_2x_3]v_{id} = [x_1x_2x_3]v_{\mathrm{id}},\\
&[x_1^2]v_{\mathrm{id}} \circ [x_2x_3]v_{\mathrm{id}} = [x_1^2x_2x_3]v_{\mathrm{id}}, && [x_1]v_{\mathrm{id}} \circ [x_1x_2x_3]v_{\mathrm{id}} = [x_1^2x_2x_3]v_{\mathrm{id}},\\
&v_g \circ v_g = 6[x_2x_3]v_{\mathrm{id}}, && v_g \circ [x_1]v_g = 6[x_1x_2x_3]v_{\mathrm{id}},\\
&[x_1]v_g \circ [x_1]v_g = 6[x_1^2x_2x_3]v_{\mathrm{id}}, && v_g \circ [x_1^2]v_g = 6[x_1^2x_2x_3]v_{\mathrm{id}},
\end{aligned}$$

which show that $[x_1]v_{\mathrm{id}}, [x_2^2]v_{\mathrm{id}}, v_g$ generate $\mathrm{Jac}(\widetilde{f}_2, G^{\mathrm{SL}}(\widetilde{f}_2))$ and are subject to the following relations

$$([x_1]v_{\mathrm{id}})^{\circ 3} = 0, \ (v_g)^{\circ 2} + 3([x_2^2]v_{\mathrm{id}})^{\circ 2}, \ [x_2^2]v_{\mathrm{id}} \circ v_g = 0.$$

On the other hand, the Jacobian algebra $\mathrm{Jac}(f_1)$ is given by

$$\mathrm{Jac}(f_1) = \mathbb{C}[y_1, y_2, y_3] \Big/ \Big(4y_1^3, 3y_2^2 + y_3^2, 2y_2y_3\Big).$$

Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(f_1) \xrightarrow{\cong} \mathrm{Jac}(\widetilde{f}_2, G^{\mathrm{SL}}(\widetilde{f}_2)), \quad y_1 \mapsto \frac{1}{\sqrt{-6}}[x_1]v_{\mathrm{id}}, \ y_2 \mapsto [x_2^2]v_{\mathrm{id}}, \ y_3 \mapsto v_g,$$

which is, moreover, an isomorphism of Frobenius algebras since by (7) we have

$$\begin{aligned}
\eta_{\widetilde{f}_2, G^{\mathrm{SL}}(\widetilde{f}_2)}\left(v_{\mathrm{id}}, [x_1^2x_2x_3]v_{\mathrm{id}}\right) &= \frac{1}{12},\\
\frac{6}{9} \cdot \eta_{f_1,\{\mathrm{id}\}}\left(v_{\mathrm{id}}, [y_1^2y_3^2]v_{\mathrm{id}}\right) &= \frac{6}{9 \cdot 96} \cdot \eta_{f_1,\{\mathrm{id}\}}\left(v_{\mathrm{id}}, [\mathrm{hess}(f_1)]v_{\mathrm{id}}\right)\\
&= \frac{6 \cdot 12}{9 \cdot 96} = \frac{1}{12}.
\end{aligned}$$

3.1.7 W_{12} and W_{12}

For $\widetilde{f_2} = x_1^5 + x_2^4 + x_3^2$, and $G^{\rm SL}(\widetilde{f_2}) = \langle g \rangle = \langle (0, \frac{1}{2}, \frac{1}{2}) \rangle$, $\mathrm{Jac}(\widetilde{f_2}, G^{\rm SL}(\widetilde{f_2}))$ is a 12-dimensional $\mathbb{C}$-vector space, whose basis can be chosen as

$$v_{\rm id}, [x_1]v_{\rm id}, [x_1^2]v_{\rm id}, [x_1^3]v_{\rm id}, [x_2^2]v_{\rm id}, [x_1x_2^2]v_{\rm id}, [x_1^2x_2^2]v_{\rm id}, [x_1^3x_2^2]v_{\rm id},$$
$$v_g, [x_1]v_g, [x_1^2]v_g, [x_1^3]v_g.$$

The only non-trivial non-zero products in $\mathrm{Jac}(\widetilde{f_2}, G^{\rm SL}(\widetilde{f_2}))$, calculated by (3), are given by

$$[x_1]v_{\rm id} \circ [x_1]v_{\rm id} = [x_1^2]v_{\rm id}, \qquad [x_1]v_{\rm id} \circ v_g = [x_1]v_g,$$
$$[x_1]v_{\rm id} \circ [x_1]v_g = [x_1^2]v_g, \qquad [x_1]v_{\rm id} \circ [x_1^2]v_g = [x_1^3]v_g,$$
$$[x_1]v_{\rm id} \circ [x_1^2]v_{\rm id} = [x_1^3]v_{\rm id}, \qquad [x_1^2]v_{\rm id} \circ v_g = [x_1^2]v_g,$$
$$[x_1^2]v_{\rm id} \circ [x_1]v_g = [x_1^3]v_g, \qquad [x_1^3]v_{\rm id} \circ v_g = [x_1^3]v_g,$$
$$[x_1]v_{\rm id} \circ [x_2^2]v_{\rm id} = [x_1x_2^2]v_{\rm id}, \qquad [x_1^2]v_{\rm id} \circ [x_2^2]v_{\rm id} = [x_1^2x_2^2]v_{\rm id},$$
$$[x_1^3]v_{\rm id} \circ [x_2^2]v_{\rm id} = [x_1^3x_2^2]v_{\rm id}, \qquad [x_1]v_{\rm id} \circ [x_1x_2^2]v_{\rm id} = [x_1^2x_2^2]v_{\rm id},$$
$$[x_1^2]v_{\rm id} \circ [x_1x_2^2]v_{\rm id} = [x_1^3x_2^2]v_{\rm id}, \qquad [x_1]v_{\rm id} \circ [x_1^2x_2^2]v_{\rm id} = [x_1^3x_2^2]v_{\rm id},$$
$$v_g \circ v_g = 8[x_2^2]v_{\rm id}, \qquad v_g \circ [x_1]v_g = 8[x_1x_2^2]v_{\rm id},$$
$$[x_1]v_g \circ [x_1]v_g = 8[x_1^2x_2^2]v_{\rm id}, \qquad v_g \circ [x_1^2]v_g = 8[x_1^2x_2^2]v_{\rm id},$$
$$[x_1]v_g \circ [x_1^2]v_g = 8[x_1^3x_2^2]v_{\rm id}, \qquad v_g \circ [x_1^3]v_g = 8[x_1^3x_2^2]v_{\rm id},$$

which show that $[x_1]v_{\rm id}$ and v_g generate $\mathrm{Jac}(\widetilde{f_2}, G^{\rm SL}(\widetilde{f_2}))$ and are subject to the following relations

$$([x_1]v_{\rm id})^{\circ 4} = 0, \;\; (v_g)^{\circ 3} = 0.$$

On the other hand, the Jacobian algebra $\mathrm{Jac}(f_1)$ is given by

$$\mathrm{Jac}(f_1) = \mathbb{C}[y_1, y_2, y_3] \Big/ \Big(5y_1^4, 4y_2^3, y_3\Big).$$

Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(f_1) \xrightarrow{\cong} \mathrm{Jac}(\widetilde{f_2}, G^{\rm SL}(\widetilde{f_2})), \quad y_1 \mapsto \frac{1}{2}[x_1]v_{\rm id}, \;\; y_2 \mapsto \frac{1}{\sqrt{2}} v_g,$$

which is, moreover, an isomorphism of Frobenius algebras since by (7) we have

$$\eta_{\widetilde{f_2}, G^{\rm SL}(\widetilde{f_2})} \Big(v_{\rm id}, [x_1^3x_2^2]v_{\rm id}\Big) = \frac{1}{20},$$
$$\frac{16}{8} \cdot \eta_{f_1, \{\rm id\}} \Big(v_{\rm id}, [y_1^3y_2^2]v_{\rm id}\Big) = \frac{16}{8 \cdot 480} \cdot \eta_{f_1, \{\rm id\}} (v_{\rm id}, [\mathrm{hess}(f_1)]v_{\rm id})$$
$$= \frac{16 \cdot 12}{480} = \frac{1}{20}.$$

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