

# Semiconjugate Rational Functions: A Dynamical Approach

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**Abstract** Using dynamical methods we give a new proof of the theorem saying that if  $A, B, X$  are rational functions of complex variable  $z$  of degree at least two such that  $A \circ X = X \circ B$  and  $\mathbb{C}(B, X) = \mathbb{C}(z)$ , then the Galois closure of the field extension  $\mathbb{C}(z)/\mathbb{C}(X)$  has genus zero or one.

**Keywords** Semiconjugate rational functions · Poincaré functions · Invariant curves · Galois closure · Orbifolds

## 1 Introduction

Let  $A$  and  $B$  be rational functions of degree at least two on the Riemann sphere. The function  $B$  is said to be semiconjugate to the function  $A$  if there exists a non-constant rational function  $X$  such that

$$A \circ X = X \circ B. \quad (1)$$

Notice that for  $\deg X = 1$  condition (1) reduces to the usual conjugacy condition while for  $B = A$  it reduces to the commutativity condition

$$A \circ X = X \circ A.$$

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A solution of Eq. (1) is called primitive if the functions  $X$  and  $B$  generate the whole field of rational functions  $\mathbb{C}(z)$ . Up to a certain degree, the description of solutions of (1) reduces to the description of primitive solutions. Indeed, by the Lüroth theorem, the field  $\mathbb{C}(X, B)$  is generated by some rational function  $W$ . Therefore, if  $\mathbb{C}(X, B) \neq \mathbb{C}(z)$ , then there exists a rational function  $W$  of degree greater than one such that

$$B = \tilde{B} \circ W, \quad X = \tilde{X} \circ W \quad (2)$$

for some rational functions  $\tilde{X}$  and  $\tilde{B}$ . Substituting now (2) in (1) we see that the triple  $A, \tilde{X}, W \circ \tilde{B}$  is another solution of (1). This new solution is not necessary primitive, however  $\deg \tilde{X} < \deg X$ . Therefore, after a finite number of similar transformations we will arrive to a primitive solution.

Semiconjugate rational functions were investigated at length in the series of papers (Pakovich 2016a, b, 2017b, 2018b). In particular, it was shown in Pakovich (2016a) that all primitive solutions of (1) are related to discrete automorphism groups of  $\mathbb{C}$  and  $\mathbb{C}P^1$ , implying that corresponding functions  $X$  have a very restricted form. Recall that for a rational function  $X$  its normalization  $\tilde{X}$  is defined as a holomorphic function of the lowest possible degree between compact Riemann surfaces  $\tilde{X} : \tilde{S}_X \rightarrow \mathbb{C}P^1$  such that  $\tilde{X}$  is a Galois covering and

$$\tilde{X} = X \circ H$$

for some holomorphic map  $H : \tilde{S}_X \rightarrow \mathbb{C}P^1$ . From the algebraic point of view the passage from  $X$  to  $\tilde{X}$  corresponds to the passage from the field extension  $\mathbb{C}(z)/\mathbb{C}(X)$  to its Galois closure. In these terms, the main result of Pakovich (2016a) about primitive solutions of (1) may be formulated as follows.

**Theorem 1.1** *Let  $A, B, X$  be rational functions of degree at least two such that  $A \circ X = X \circ B$  and  $\mathbb{C}(B, X) = \mathbb{C}(z)$ . Then the Galois closure of the field extension  $\mathbb{C}(z)/\mathbb{C}(X)$  has genus zero or one.*

Observe a similarity between this result and the Ritt theorem (Ritt 1923) saying that if two rational functions  $A$  and  $X$  commute and have no iterate in common, then  $A$  and  $X$  either are Lattès maps, or are conjugate to powers or Chebyshev polynomials. Indeed, powers and Chebyshev polynomials are the simplest examples of rational functions such that  $g(\tilde{S}_X) = 0$ . On the other hand, Lattès maps are examples of rational functions with  $g(\tilde{S}_X) = 1$ . Rational functions  $X$  with  $g(\tilde{S}_X) = 0$  can be listed explicitly, while functions with  $g(\tilde{S}_X) = 1$  admit a simple geometric description (see Pakovich 2018a). Notice that rational functions with  $g(\tilde{S}_X) \leq 1$  can be described through their ramification, implying that Theorem 1.1 is equivalent to Theorem 6.1 of Pakovich (2016a) (see Sect. 5 below).

The problem of describing commuting and semiconjugate rational functions naturally belongs to dynamics (see e.g. the papers Buff and Epstein 2007; Eremenko 2012, 1989; Fatou 1923; Julia 1922; Medvedev and Scanlon 2014; Pakovich 2017a). In particular, in the papers of Fatou (1923) and Julia (1922) commuting rational functions

were investigated by dynamical methods, requiring however an assumption that the Julia sets of considered functions do not coincide with the whole Riemann sphere. On the other hand, the Ritt theorem about commuting rational functions cited above was proved by non-dynamical methods. In his paper, Ritt remarked that “it would be interesting to know whether a proof can also be effected by the use of Poincaré functions employed by Julia”. Sixty-six years later such a proof was given by Eremenko (1989). Notice that the Ritt theorem also follows from the results of Pakovich (2016b) about solutions of Eq. (1) with fixed  $B$ .

Similarly to the paper Ritt (1923), the paper Pakovich (2016a) does not use any dynamical methods, but relies on a study of maps between two-dimensional orbifolds associated with rational functions. At the same time, it is interesting to find approaches to Eq. (1) involving ideas from dynamics, and the goal of this paper is to provide a “dynamical” proof of Theorem 1.1. In fact, we give *two* such proofs. The first one exploits a link between Eq. (1) and Poincaré functions. The second one is based on the interpretation of  $\tilde{S}_X$  as an invariant curve for the dynamical system

$$(x_1, x_2, \dots, x_n) \rightarrow (A(x_1), A(x_2), \dots, A(x_n)) \tag{3}$$

on  $(\mathbb{C}\mathbb{P}^1)^n$ . The last proof is inspired by the recent paper Medvedev and Scanlon (2014) describing invariant varieties for dynamical systems of the form

$$(x_1, x_2, \dots, x_n) \rightarrow (C_1(x_1), C_2(x_2), \dots, C_n(x_n)),$$

where  $C_1, C_2, \dots, C_n$  are *polynomials*, and relating such varieties with polynomial solutions of (1). The analysis of Eq. (1) in the paper Medvedev and Scanlon (2014), based on the Ritt theory of polynomial decompositions (Ritt 1922), does not extend to arbitrary rational functions. Nevertheless, the relation between the semiconjugacy condition and invariant varieties established in Medvedev and Scanlon (2014) suggests that there should be some interpretation of the results of Pakovich (2016a) in terms of dynamical systems of form (3), and we show that this is indeed the case.

The paper is organized as follows. In the second section we recall the description of  $\tilde{S}_X$  in terms of algebraic equations, and give a criterion for a rational function  $X$  to satisfy the condition  $g(\tilde{S}_X) \leq 1$ . In the third and the fourth sections we provide two proofs of Theorem 1.1 using two approaches described above. Finally, in the fifth section we show that Theorem 1.1 is equivalent to Theorem 6.1 of Pakovich (2016a) which describes primitive solutions of (1) in terms of orbifolds.

## 2 Meromorphic Parametrizations and Normalizations

Let  $\mathcal{C}$  be an irreducible algebraic curve in  $\mathbb{C}^n$ . Recall that a meromorphic parametrization of  $\mathcal{C}$  on  $\mathbb{C}$  is a collection of functions  $\psi_1, \psi_2, \dots, \psi_n$  such that

- $\psi_1, \psi_2, \dots, \psi_n$  are non-constant and meromorphic on  $\mathbb{C}$ ,
- $(\psi_1(z), \psi_2(z), \dots, \psi_n(z)) \in \mathcal{C}$  whenever  $\psi_i(z) \neq \infty, 1 \leq i \leq n$ ,
- with finitely many exceptions, every point of  $\mathcal{C}$  is of the form  $(\psi_1(z), \psi_2(z), \dots, \psi_n(z))$  for some  $z \in \mathbb{C}$ .

Notice that the last condition in fact is a corollary of the first two.

By the classical theorem of Picard (1887), a plane algebraic curve  $\mathcal{C}$  which can be parametrized by functions meromorphic on  $\mathbb{C}$  has genus zero or one (see e.g. Beardon and Ng 2006). We will use the following slightly more general version of this theorem which can be proved in the same way.

**Theorem 2.1** *If an irreducible algebraic curve  $\mathcal{C}$  in  $\mathbb{C}^n$  has a meromorphic parametrization on  $\mathbb{C}$ , then  $\mathcal{C}$  has genus zero or one.*  $\square$

Let  $X : \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$  be a rational function of degree  $d$ . The normalization  $\tilde{X} : \tilde{S}_X \rightarrow \mathbb{C}\mathbb{P}^1$  can be described by the following construction (see Fried 1995, §I.G). Consider the fiber product of the cover  $X : \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$  with itself  $d$  times, that is a subset  $\mathcal{L}$  of  $(\mathbb{C}\mathbb{P}^1)^d$  consisting of  $d$ -tuples with a common image under  $X$ . Clearly,  $\mathcal{L}$  is an algebraic variety of dimension one defined by the algebraic equations

$$X(z_i) - X(z_j) = 0, \quad 1 \leq i, j \leq d, \quad i \neq j. \tag{4}$$

Let  $\mathcal{L}_0$  be a variety obtained from  $\mathcal{L}$  by removing the components where two or more coordinates coincide,  $\mathcal{N}$  an irreducible component of  $\mathcal{L}_0$ , and  $\mathcal{N}' \xrightarrow{\pi'} \mathcal{N}$  the desingularization map. In this notation the following statement holds.

**Theorem 2.2** *The map  $\psi : \mathcal{N}' \rightarrow \mathbb{C}\mathbb{P}^1$  given by the composition*

$$\mathcal{N}' \xrightarrow{\pi'} \mathcal{N} \xrightarrow{\pi_i} \mathbb{C}\mathbb{P}^1 \xrightarrow{X} \mathbb{C}\mathbb{P}^1, \tag{5}$$

where  $\mathcal{N}$  is any irreducible component of  $\mathcal{L}_0$  and  $\pi_i$  is the projection to any coordinate, is the normalization of  $X$ .  $\square$

Combining Theorems 2.1 and 2.2 we obtain the following characterization of rational functions  $X$  with  $g(\tilde{S}_X) \leq 1$ .

**Theorem 2.3** *Let  $X$  be a rational function of degree  $d$ . Then  $g(\tilde{S}_X) \leq 1$  if and only if there exist  $d$  distinct functions  $\psi_1, \psi_2, \dots, \psi_d$  meromorphic on  $\mathbb{C}$  such that*

$$X(\psi_i) - X(\psi_j) = 0, \quad 1 \leq i, j \leq d, \quad i \neq j. \tag{6}$$

*Proof* Equalities (6) imply that some irreducible component  $\mathcal{N}$  of  $\mathcal{L}_0$  admits a meromorphic parametrization. Since  $\mathcal{N}' = \tilde{S}_X$  by Theorem 2.2, it follows from Theorem 2.1 that  $g(\tilde{S}_X) \leq 1$ .

In the other direction, if  $g(\tilde{S}_X) \leq 1$ , then taking different coordinate projections in (5) we obtain  $d$  distinct functions

$$\theta_i = \pi_i \circ \pi', \quad 1 \leq i \leq d,$$

from  $\tilde{S}_X$  to  $\mathbb{C}\mathbb{P}^1$  such that

$$X(\theta_i) - X(\theta_j) = 0, \quad 1 \leq i, j \leq d, \quad i \neq j.$$

If  $g(\tilde{S}_X) = 0$ , these functions are rational and therefore meromorphic on  $\mathbb{C}$ . On the other hand, if  $g(\tilde{S}_X) = 1$ , we obtain meromorphic functions satisfying (6) setting

$$\psi_i = \theta_i \circ \tau, \quad 1 \leq i \leq d,$$

where  $\tau : \mathbb{C} \rightarrow \tilde{S}_X$  is the universal covering of  $\tilde{S}_X$ . □

### 3 Semiconjugate Functions and Poincaré Functions

Let  $A$  be a rational function and  $z_0$  its repelling fixed point. Recall that the Poincaré function  $\mathcal{P}_{A,z_0}$  associated with  $z_0$  is a function meromorphic on  $\mathbb{C}$  such that  $\mathcal{P}_{A,z_0}(0) = z_0$ ,  $\mathcal{P}'_{A,z_0}(0) = 1$ , and the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\lambda z} & \mathbb{C} \\ \downarrow \mathcal{P}_{A,z_0} & & \downarrow \mathcal{P}_{A,z_0} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array}$$

commutes. The Poincaré function always exists and is unique (see e.g. Milnor 2006).

**Lemma 3.1** *Let  $X$  and  $B$  be rational functions such that  $\mathbb{C}(X, B) = \mathbb{C}(z)$ . Then for all but finitely many  $z_0 \in \mathbb{C}$  the set  $B(X^{-1}\{z_0\})$  contains  $\deg X$  distinct points.*

*Proof* Since  $\mathbb{C}(X, B) = \mathbb{C}(z)$ , there exist  $U, V \in \mathbb{C}[x, y]$  such that

$$z = \frac{U(X, B)}{V(X, B)}.$$

This implies that for  $z_1 \neq z_2$  such that  $X(z_1) = X(z_2)$  the inequality  $B(z_1) \neq B(z_2)$  holds, unless  $z_1$  or  $z_2$  is a zero of the polynomial  $V(X, B)$ . Therefore, if  $z_0$  is neither a critical value of  $X$  nor an  $X$ -image of a zero of  $V(X, B)$ , the set  $B(X^{-1}\{z_0\})$  contains  $\deg X$  distinct points, since  $X^{-1}\{z_0\}$  contains  $\deg X$  distinct points and their  $B$ -images are distinct. □

Combining the uniqueness of the Poincaré function with Theorem 2.3 we can prove Theorem 1.1 as follows. Let  $A, B$ , and  $X$  be rational functions of degree at least two such that the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ \downarrow X & & \downarrow X \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \tag{7}$$

commutes and  $\mathbb{C}(X, B) = \mathbb{C}(z)$ . Since the number of repelling periodic points of  $A$  is infinite, it follows from Lemma 3.1 that we can find a repelling periodic point  $z_0 \in \mathbb{C}$

such that for any point  $z$  in the forward  $A$ -orbit of  $z_0$  the set  $B(X^{-1}\{z\})$  contains  $\deg X$  distinct points. Since (7) implies that

$$B(X^{-1}\{z_0\}) \subseteq X^{-1}\{A(z_0)\},$$

this yields that

$$B(X^{-1}\{z_0\}) = X^{-1}\{A(z_0)\},$$

and, inductively, that

$$B^{ok}(X^{-1}\{z_0\}) = X^{-1}\{A^{ok}(z_0)\}, \quad k \geq 1.$$

In particular, for  $k$  equal to the period of  $z_0$  we have:

$$B^{ok}(X^{-1}\{z_0\}) = X^{-1}\{z_0\}.$$

Therefore, the restriction of the rational function  $B^{ok}$  on the set  $X^{-1}\{z_0\}$  is a permutation of its elements, and hence for certain  $l \geq 1$  all the points of  $X^{-1}\{z_0\}$  are fixed points of  $B^{okl}$ . Thus, considering instead of  $A$  and  $B$  their iterates we can assume that  $z_0$  is a fixed point of  $A$  and the set  $X^{-1}\{z_0\}$  consists of  $d = \deg X$  distinct fixed points  $z_1, z_2, \dots, z_d$  of  $B$ .

Since the points  $z_1, z_2, \dots, z_d$  are not critical points of  $X$ , the map  $X$  is invertible near each of them implying that the multipliers of  $B$  at  $z_1, z_2, \dots, z_d$  are all equal to the multiplier  $\lambda$  of  $A$  at  $z_0$ , so that  $z_1, z_2, \dots, z_d$  are repelling fixed points of  $B$ . Clearly, for each  $i, 1 \leq i \leq d$ , we can complete commutative diagram (7) to the commutative diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\lambda z} & \mathbb{C} \\ \downarrow \mathcal{P}_{B,z_i} & & \downarrow \mathcal{P}_{B,z_i} \\ \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ \downarrow X & & \downarrow X \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1, \end{array}$$

where  $\mathcal{P}_{B,z_i}, 1 \leq i \leq d$ , is the corresponding Poincaré function for  $B$ . Since the functions  $X \circ \mathcal{P}_{B,z_i}, 1 \leq i \leq d$ , are meromorphic, it follows now from the uniqueness of the Poincaré function that there exist  $\alpha_1, \alpha_2, \dots, \alpha_d \in \mathbb{C} \setminus \{0\}$  such that

$$\mathcal{P}_{A,z_0}(z) = X \circ \mathcal{P}_{B,z_1}(\alpha_1 z) = X \circ \mathcal{P}_{B,z_2}(\alpha_2 z) = \dots = X \circ \mathcal{P}_{B,z_d}(\alpha_d z). \quad (8)$$

Moreover, the functions  $\mathcal{P}_{B,z_i}(\alpha_i z), 1 \leq i \leq d$ , are distinct since the points

$$z_i = \mathcal{P}_{B,z_i}(0), \quad 1 \leq i \leq d,$$

are distinct. Applying now Theorem 2.3 to equality (8), we see that  $g(\tilde{S}_X) \leq 1$ .

### 4 Semiconjugate Functions and Invariant Curves

Let  $R_1, R_2, \dots, R_d$  be rational functions, and let  $\mathcal{R} : (\mathbb{C}\mathbb{P}^1)^d \rightarrow (\mathbb{C}\mathbb{P}^1)^d$  be the map

$$(x_1, x_2, \dots, x_d) \rightarrow (R_1(x_1), R_2(x_2), \dots, R_d(x_d)).$$

Say that an algebraic curve  $\mathcal{C}$  in  $(\mathbb{C}\mathbb{P}^1)^d$  is  $\mathcal{R}$ -invariant if  $\mathcal{R}(\mathcal{C}) = \mathcal{C}$ . Invariant curves possess the following property (cf. Medvedev and Scanlon 2014, Proposition 2.34).

**Theorem 4.1** *Let  $R_1, R_2, \dots, R_d$  be rational functions of degree at least two and  $\mathcal{C}$  an irreducible  $\mathcal{R}$ -invariant curve in  $(\mathbb{C}\mathbb{P}^1)^d$ . Then  $g(\mathcal{C}) \leq 1$ .*

*Proof* Since  $\mathcal{C}$  is  $\mathcal{R}$ -invariant, the map  $\mathcal{R}$  lifts to a holomorphic map

$$\mathcal{R}' : \mathcal{C}' \rightarrow \mathcal{C}',$$

where  $\mathcal{C}'$  is a desingularization of  $\mathcal{C}$ . Applying now the Riemann-Hurwitz formula

$$2g(\mathcal{C}') - 2 = (2g(\mathcal{C}') - 2)\deg \mathcal{R}' + \sum_{P \in \mathcal{C}'} (e_P - 1),$$

we see that  $g(\mathcal{C}') \leq 1$ , unless  $\deg \mathcal{R}' = 1$ .

Furthermore, if  $\deg \mathcal{R}' = 1$  the inequality  $g(\mathcal{C}') \leq 1$  still holds. Indeed, since the automorphism group of a Riemann surface of genus greater than one is finite, if  $g(\mathcal{C}') \geq 2$ , then for some  $k \geq 1$  the map  $(\mathcal{R}')^{ok}$  is the identical automorphism of  $\mathcal{C}'$ , implying that the maps

$$(z_1, z_2, \dots, z_d) \rightarrow R_i^{ok}(z_i), \quad 1 \leq i \leq d,$$

are identical on  $\mathcal{C}$ . However, since each  $R_i, 1 \leq i \leq d$ , has degree at least two, in this case for every point of  $\mathcal{C}$  its  $i$ th coordinate belongs to a finite subset of  $\mathbb{C}\mathbb{P}^1$  consisting of fixed point of  $R_i^{ok}$ , implying that  $\mathcal{C}$  is a finite set. □

Using Theorems 4.1 and 2.2 we obtain a proof of Theorem 1.1 as follows. Define the maps  $\mathcal{A}, \mathcal{B}$ , and  $\mathcal{X}$  from  $(\mathbb{C}\mathbb{P}^1)^d$  to  $(\mathbb{C}\mathbb{P}^1)^d$  by the formulas

$$\begin{aligned} \mathcal{A} &: (x_1, x_2, \dots, x_d) \rightarrow (A(x_1), A(x_2), \dots, A(x_d)), \\ \mathcal{B} &: (x_1, x_2, \dots, x_d) \rightarrow (B(x_1), B(x_2), \dots, B(x_d)), \\ \mathcal{X} &: (x_1, x_2, \dots, x_d) \rightarrow (X(x_1), X(x_2), \dots, X(x_d)). \end{aligned}$$

Clearly, equality (1) implies that the diagram

$$\begin{CD}
 (\mathbb{CP}^1)^d @>{\mathcal{B}}>> (\mathbb{CP}^1)^d \\
 @V{\mathcal{X}}VV @VV{\mathcal{X}}V \\
 (\mathbb{CP}^1)^d @>{\mathcal{A}}>> (\mathbb{CP}^1)^d
 \end{CD} \tag{9}$$

commutes. By construction, the variety  $\mathcal{L}$  defined by Eq. (4) is the preimage of the diagonal  $\Delta$  in  $(\mathbb{CP}^1)^d$  under the map  $\mathcal{X} : (\mathbb{CP}^1)^d \rightarrow (\mathbb{CP}^1)^d$ . Therefore, since  $\mathcal{A}(\Delta) = \Delta$ , it follows from (9) that  $\mathcal{B}(\mathcal{L}) \subseteq \mathcal{L}$ . Moreover, Lemma 3.1 implies that  $\mathcal{B}(\mathcal{L}_0) \subseteq \mathcal{L}_0$ . Since  $\mathcal{L}_0$  has a finite number of irreducible components, this implies that there exists an irreducible component  $\mathcal{N}_0$  of  $\mathcal{L}_0$  such that  $\mathcal{B}^{\circ k}(\mathcal{N}_0) = \mathcal{N}_0$  for some  $k \geq 1$ . Since by Theorem 2.2 the equality  $g(\mathcal{N}_0) = g(\tilde{\mathcal{S}}_X)$  holds, it follows now from Theorem 4.1 that  $g(\tilde{\mathcal{S}}_X) \leq 1$ .

### 5 Semiconjugate Functions and Orbifolds

Recall that an orbifold  $\mathcal{O}$  on  $\mathbb{CP}^1$  is a ramification function  $\nu : \mathbb{CP}^1 \rightarrow \mathbb{N}$  which takes the value  $\nu(z) = 1$  except at finite number of points. The Euler characteristic of an orbifold  $\mathcal{O}$  is defined by the formula

$$\chi(\mathcal{O}) = 2 + \sum_{z \in \mathbb{CP}^1} \left( \frac{1}{\nu(z)} - 1 \right).$$

A rational function  $f$  is called a covering map  $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  between orbifolds  $\mathcal{O}_1$  and  $\mathcal{O}_2$  if for any  $z \in \mathbb{CP}^1$  the equality

$$\nu_2(f(z)) = \nu_1(z) \deg_z f \tag{10}$$

holds, where  $\deg_z f$  denotes the local degree of  $f$  at the point  $z$ . If  $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  is a covering map, then the Riemann-Hurwitz formula implies that

$$\chi(\mathcal{O}_1) = \deg f \cdot \chi(\mathcal{O}_2). \tag{11}$$

In case if a weaker than (10) condition

$$\nu_2(f(z)) = \nu_1(z) \text{GCD}(\deg_z f, \nu_2(f(z)))$$

holds,  $f$  is called a minimal holomorphic map between orbifolds  $\mathcal{O}_1$  and  $\mathcal{O}_2$ .

With each rational function  $f$  one can associate in a natural way two orbifolds  $\mathcal{O}_1^f$  and  $\mathcal{O}_2^f$ , setting  $\nu_2^f(z)$  equal to the least common multiple of the local degrees of  $f$  at the points of the preimage  $f^{-1}\{z\}$ , and

$$\nu_1^f(z) = \nu_2^f(f(z)) / \deg_z f.$$



By construction,  $\mathcal{O}_1^f \rightarrow \mathcal{O}_2^f$  is a covering map between orbifolds. The following statement expresses the condition  $g(\tilde{S}_f) \leq 1$  in terms of the Euler characteristic of  $\mathcal{O}_2^f$  (see Pakovich 2018a, Lemma 2.1).

**Lemma 5.1** *Let  $f$  be a rational function. Then  $g(\tilde{S}_f) = 0$  if and only if  $\chi(\mathcal{O}_2^f) > 0$ , and  $g(\tilde{S}_f) = 1$  if and only if  $\chi(\mathcal{O}_2^f) = 0$ .  $\square$*

Using Lemma 5.1 one can show that Theorem 1.1 is equivalent to the following statement proved in the paper Pakovich (2016a, Theorem 6.1).

**Theorem 5.1** *Let  $A, B, X$  be rational functions of degree at least two such that  $A \circ X = X \circ B$  and  $\mathbb{C}(B, X) = \mathbb{C}(z)$ . Then  $\chi(\mathcal{O}_1^X) \geq 0$ ,  $\chi(\mathcal{O}_2^X) \geq 0$ , and the commutative diagram*

$$\begin{array}{ccc} \mathcal{O}_1^X & \xrightarrow{B} & \mathcal{O}_1^X \\ \downarrow X & & \downarrow X \\ \mathcal{O}_2^X & \xrightarrow{A} & \mathcal{O}_2^X \end{array}$$

*consists of minimal holomorphic maps between orbifolds.*

Indeed, a direct calculation shows that if  $A, B, X$  is a primitive solution of (1), then  $A : \mathcal{O}_1^X \rightarrow \mathcal{O}_1^X$  and  $B : \mathcal{O}_2^X \rightarrow \mathcal{O}_2^X$  are minimal holomorphic maps between orbifolds (see Pakovich 2016a, Theorem 4.2). If Theorem 1.1 is true, then Lemma 5.1 implies that  $\chi(\mathcal{O}_2^X) \geq 0$ . Furthermore,  $\chi(\mathcal{O}_1^X) \geq 0$ , by (11). In turn, Theorem 5.1 implies Theorem 1.1, since  $\chi(\mathcal{O}_2^X) \geq 0$  implies  $g(\tilde{S}_X) \leq 1$ .

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