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RESEARCH EXPOSITION

Two-Valued Groups, Kummer Varieties, and Integrable Billiards

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Abstract A natural and important question of study two-valued groups associated with hyperelliptic Jacobians and their relationship with integrable systems is motivated by seminal examples of relationship between algebraic two-valued groups related to elliptic curves and integrable systems such as elliptic billiards and celebrated Kowalevski top. The present paper is devoted to the case of genus 2, to the investigation of algebraic two-valued group structures on Kummer varieties. One of our approaches is based on the theory of σ-functions. It enables us to study the dependence of parameters of the curves, including rational limits. Following this line, we are introducing a notion of n-groupoid as natural multivalued analogue of the notion of topological groupoid. Our second approach is geometric. It is based on a geometric approach to addition laws on hyperelliptic Jacobians and on a recent notion of billiard algebra. Especially important is connection with integrable billiard systems within confocal quadrics. The third approach is based on the realization of the Kummer variety in the framework of moduli of semi-stable bundles, after Narasimhan and Ramanan. This construction of the two-valued structure is remarkably similar to the historically first example of topological formal two-valued group from 1971, with a significant difference: the resulting bundles in the 1971 case were "virtual", while in the present case the resulting bundles are effectively realizable.

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Contents

1 Introduction

The deep connection between two-valued groups and integrable systems is well-known. In Buchstaber and Veselov (1996), a two-valued action of two-valued group Z_+ has been related to the Euler-Chasles correspondence, as a basis of billiard dynamics within conics, as an example of integrable discrete dynamics.

The concept of integrability in context of discrete dynamical systems goes back to Veselov (1992), Moser and Veselov (1991). They established integrability of billiard dynamics generated by pencils of quadrics and related it in case of two-dimensional quadrics of three-dimensional space with Jacobians of hyperelliptic curves of genus 2.

The aim of this paper is to develop algebraic two-valued group structure on the Kummer varieties and to study its relationship with integrable billiard systems within pencils of quadrics. The Kummer variety is a classical and well-studied object of the algebraic geometry. It appears as a variety of orbits of a Jacobian of a curve of genus 2, factorized by the hyperelliptic involution. Thus, there are various algebro-geometric and analytical structures on it, related to the moduli of curves of genus 2. We want to understand and present the structure of an algebraic two-valued group associated to different appearances of the Kummer varieties. Let us mention that from the point of view of differential geometry and the theory of abelian functions of genus 2, Kummer varieties have been studied by Baker.

One side of our approach is based on the theory of σ-functions. The choice of σ functions is motivated by our wish to study dependence of parameters of the curves.

As we know from the theory of integrable systems, (see Dubrovin and Novikov 1975), it is important to study not only a single curve and its Jacobian, but rather a whole class of curves and their Jacobians. Following this line, let B denote a set parameterizing nonsingular curves of genus 2 and let U be the analytic bundle over B with a genus 2 curve as a fibre. We denote by $J(U)$ the associated bundle over B with a Jacobian of a genus 2 curve as a fiber. Then, $J(U)$ has a natural structure of a topological groupoid, see Buchstaber and Leykin (2005). The description of the corresponding associated bundle over B with a Kummer variety as a fiber was done in Buchstaber (2016).

Higher genera Poncelet type problems, integrable billiards and so-called billiard algebra associated with pencils of quadrics in arbitrary dimensions have been studied in Dragović and Radnović (2006, 2008), where the relationship with hyperelliptic curves and their Jacobians has been considered (see also Dragović and Radnović 2011).

In the next section, after recalling the definition of the n valued group, we are introducing a notion of n-groupoid, which enables us to consider not only a single structure of a two-valued group, but also dependance of the parameters of the curve. Thus the notion of n-groupoid is a natural analogue of the notion of topological groupoid.

Another important reason of using σ-functions, lies in the fact that addition relations for them are well adjusted to two-valued structure, as one can easily see from the genus one case formula:

$$\sigma(u+v)\sigma(u-v) = \sigma(u)^2\sigma(v)^2\big(\wp(v) - \wp(u)\big),$$

The program of construction of σ-functions of higher genera had been proposed by Klein. In genus 2 case, the program got rather advanced development by Baker (see Baker 1898), although in his last review of the program in 1923, Klein noted that program for genus $g > 2$ was still far from being completed. We base our first approach on the results from Buchstaber and Leykin (2005) and references therein. For some recent applications to the theory of integrable systems, see Buchstaber (2016) and Buchstaber and Mikhailov (2017).

Our second approach is geometric. Following Donagi (1980), Knörrer (1980), Griffiths and Harris (1994), Audin (1994) a geometric approach to addition laws on hyperelliptic Jacobians has been developed further in Dragović and Radnović (2008) (see also Dragović and Radnović 2011). It has been based on the notion of billiard algebra and it has been connected to integrable billiard systems within confocal quadrics.

Our first approach is developed in Sect. 4, and the second one in Sect. 6. The study of the rational limit of a Kummer surface and related two-valued group law is performed in Sect. 3.

In Sect. 7, the geometric approach from the Sect. 6, develops further toward integrable systems, through the integrable billiards and the billiard algebra and is motivated by Audin (1994), Dragović and Radnović (2008) .

The final Sect. 8 gives yet another construction of the two-valued group law on the Kummer variety, based on the realization of the Kummer variety in the framework of moduli of semi-stable bundles. This framework has been developed by Narasimhan and Ramanan, see Narasimhan and Ramanan (1969). This last construction of the two-valued structure is remarkably similar to the historically first example of topological two-valued group from 1971 (see Buchstaber and Novikov 1971), with a significant difference. The resulting bundles in the 1971 case of Buchstaber and Novikov (1971) are "virtual", while in the present case the resulting bundles are effectively realizable.

2 From *n*-Valued Groups to *n*-Groupoids

The study of structure of multivalued groups has been started in 1971 [see Buchstaber and Novikov (1971)] within the study of characteristic classes of vector bundles, at that time as the structure of formal (local) multivalued groups. In 1990, in Buchstaber (1990) has been introduced algebraic two-valued group structure on $\mathbb{C}P^1$ by using addition theorems on elliptic curves. The structure of algebraic multivalued groups has been studied since then extensively [see Buchstaber (2006) and references therein] in different contexts.

However, it has got a new impetus with a recent paper Dragović (2010), where a connection between two-valued groups on $\mathbb{C}P^1$ and the celebrated Kowalevski top has been discovered. There, it was shown that the famous, and still not fully understood Kowalevski change of variables [see, for example, Audin (1999)] corresponds to the operation in two-valued group W/τ, where W is an elliptic curve, in the standard Weierstrass model of elliptic curve, with τ as the canonical involution. More detailed description of this two-valued group is presented in Sect. 2.1, see Example 3. This deep connection of the Kowalevski integration procedure with a structure of elliptic curve, on the first glance, may be in surprising contrast to a well known fact that the integration of the Kowalevski top is performed on Jacobian of a genus two curve. Moreover, as it has been shown in Dragović (2010), the Kowalevski integration procedure can be interpreted as a certain deformation of the two-valued group p_2 (see Example 2) toward the structure on W/τ. The structure of p_2 is the rational limit of the one on W/τ. In addition, in Dragović (2010) it has been proven the equivalence between associativity of the two-valued group on W/τ with the Poncelet theorem of pencils of conics, in the case of triangles.

2.1 Defining Notions and Basic Examples of *n*-Valued Groups

We follow Buchstaber (2006) and define an n-valued group on X as a map:

$$m : X \times X \to (X)^n$$
$$m(x, y) = x * y = [z_1, \ldots, z_n].$$

Here $(X)^n$ denotes the symmetric n-th power of X and z_i coordinates therein.

Associativity is a generalization of the usual condition of associativity and is expressed as equality of two n^2-sets

$$[x * (y * z)_1, \ldots, x * (y * z)_n]$$
$$[(x * y)_1 * z, \ldots, (x * y)_n * z]$$

for all triplets $(x, y, z) \in X^3$. An element $e \in X$ is *a unit* if

$$e * x = x * e = [x, \dots, x],$$

for all $x \in X$. Similarly, a map $\mathrm{inv} : X \to X$ is *an inverse* if it satisfies

$$e \in \mathrm{inv}(x) * x, \quad e \in x * \mathrm{inv}(x),$$

for all $x \in X$.

As in Buchstaber (2006), m determines *an n-valued group structure* (X, m, e, inv) if it is associative, with a unit and with an inverse. Naturally, an n-valued group X acts on the set Y if there is a mapping

$$\begin{aligned} \phi : \ X \times Y &\to (Y)^n \\ \phi(x, y) &= x \circ y, \end{aligned}$$

such that the two n^2-multisubsets of Y

$$x_1 \circ (x_2 \circ y) \quad (x_1 * x_2) \circ y$$

are equal for all $x_1, x_2 \in X$, $y \in Y$. It is additionally required that

$$e \circ y = [y, \dots, y]$$

for all $y \in Y$.

Example 1 [A two-valued group structure on $\mathbb{Z}_+$, Buchstaber and Veselov (1996)] Let us consider the set of nonnegative integers $\mathbb{Z}_+$ and define a mapping

$$\begin{aligned} m : \ \mathbb{Z}_+ \times \mathbb{Z}_+ &\to (\mathbb{Z}_+)^2, \\ m(x, y) &= [x + y, |x - y|]. \end{aligned}$$

This mapping provides a structure of a two-valued group on $\mathbb{Z}_+$ with the unit $e = 0$ and the inverse equal to the identity $\mathrm{inv}(x) = x$.

In Buchstaber and Veselov (1996) sequence of two-valued mappings associated with the Poncelet porism was identified as the algebraic representation of this 2-valued group. Moreover, the algebraic action of this group on $\mathbb{C}P^1$ was studied and it was shown that in the irreducible case all such actions are generated by Euler-Chasles correspondences.

Example 2 [2-valued group on $(\mathbb{C}, +)$] Among the basic examples of multivalued groups, there are n-valued additive group structures on $\mathbb{C}$. For $n = 2$, this is a two-valued group p_2 defined by the relation

$$\begin{aligned} m_2 : \ \mathbb{C} \times \mathbb{C} &\to (\mathbb{C})^2 \\ x *_2 y &= [(\sqrt{x} + \sqrt{y})^2, (\sqrt{x} - \sqrt{y})^2] \end{aligned} \tag{1}$$

The product $x *_2 y$ corresponds to the roots in z of the polynomial equation

$$p_2(z, x, y) = 0,$$

where

$$p_2(z, x, y) = (x + y + z)^2 - 4(xy + yz + zx).$$

This two-valued group structure has been connected with degenerations of the Kowalevski top in Dragović (2010). Similar integrable systems were studied by Appel'rot, Delone, Mlodzeevskii [see Golubev 1960 and references therein.]

As it has been observed in Dragović (2010), the general Kowalevski case is connected with p_2 together with its deformation on $\mathbb{C}P^1$ as a factor of an elliptic curve, see the next example.

Example 3 [2-valued group on $S^2 = \hat{\mathbb{C}}$, associated with an elliptic curve] Suppose that a cubic W is given in the standard form

$$W : t^2 = J(s) = 4s^3 - g_2 s - g_3.$$

Consider the mapping $W \to S^2 = \hat{\mathbb{C}} : (s, t) \mapsto s$, where $\hat{\mathbb{C}}$ represents a complex line extended by ∞.

The curve W as a cubic curve has the group structure. Together with its canonical involution $\tau : (s, t) \mapsto (s, -t)$, it defines the standard two-valued group structure of coset type (see Buchstaber and Rees 2002; Buchstaber 2006) on S^2 with the unit at infinity in S^2. The product is defined by the formula:

$$[s_1] *_c [s_2] = \left[\left[-s_1 - s_2 + \left(\frac{t_1 - t_2}{2(s_1 - s_2)}\right)^2\right], \left[-s_1 - s_2 + \left(\frac{t_1 + t_2}{2(s_1 - s_2)}\right)^2\right]\right], \tag{2}$$

where $t_i = J(s_i)$, $i = 1, 2$, and

$$[s_i] = \{(s_i, t_i), (s_i, -t_i)\}, \quad s_i = \wp(u_i), t_i = \wp'(u_i),$$

by using addition theorem for the Weierstrass function $\wp(u)$:

$$\wp(u_1 + u_2) = -\wp(u_1) - \wp(u_2) + \left(\frac{\wp'(u_1) - \wp'(u_2)}{2(\wp(u_1) - \wp(u_2))}\right)^2.$$

The Kowalevski integration procedure was explained in Dragović (2010) as certain deformation of p_2 to (W, τ) [see also Dragović and Kukić (2014a, b, c), Dragović and Gajić (2016) for further developments].

The 2-group structure (W, τ) is also connected with the Poncelet and the Darboux theorem [see Darboux 1917; Dragović 2009; Dragović and Radnović 2008].

In this example, g_2, g_3 are parameters of the curve, and they lead to a rational limit, in the standard limit procedure.

Example 4 Let X be a topological space. Denote by $W(X)$ the set of all 2-dimensional complex vector bundles $\zeta = \eta + \bar{\eta}$ over X, where η is a linear complex vector bundle over X. Then the formula

$$\zeta_1 \otimes \zeta_2 = (\eta_1 \otimes \eta_2 + \bar{\eta}_1 \otimes \bar{\eta}_2) + (\bar{\eta}_1 \otimes \eta_2 + \eta_1 \otimes \bar{\eta}_2)$$

gives the structure of a 2-valued group on $W(X)$ defined by the 2-valued multiplication

$$\zeta_1 \star \zeta_2 = [(\eta_1 \otimes \eta_2 + \bar{\eta}_1 \otimes \bar{\eta}_2), (\bar{\eta}_1 \otimes \eta_2 + \eta_1 \otimes \bar{\eta}_2)].$$

2.2 Topological *n*-Groupoids

Following Buchstaber and Leykin (2005), let us fix a topological space Y. A space X with a map $p_X\colon X \to Y$ is called a space over Y, and the map p_X is called anchor. For Y, the anchor p_Y is the identity map.

A map $f\colon X_1 \to X_2$ of two spaces over Y is called a map over Y if $p_{X_2} \circ f = p_{X_1}$. For two spaces X_1, X_2 over Y, their direct product over Y is defined as

$$X_1 \times_Y X_2 = \{(x_1, x_2) \in X_1 \times X_2 \mid p_{X_1}(x_1) = p_{X_2}(x_2)\}$$

with the map $p_{X_1 \times_Y X_2}(x_1, x_2) = p_{X_1}(x_1)$. Along this line, one may define a product over Y of n spaces $X_1, \ldots X_n$ over Y

$$\begin{aligned} X_1 \times_Y \cdots \times_Y X_n = \{(x_1, \ldots, x_n) \in X_1 \times X_2 \times \cdots \times X_n \mid p_{X_1}(x_1) \\ = p_{X_2}(x_2) = \cdots = p_{X_n}(x_n)\}. \end{aligned}$$

In a special case $X_1 = \cdots = X_n = X$, we define *n-th power of X over Y* and denote it as X^n_Y.

For a space X over Y we define its *n-th symmetric power over Y*, denoted as

$$(X)^n_Y$$

as the quotient of X^n_Y by the action of the permutation group.

We define also the diagonal map over Y

$$D : X \to (X)^n_Y, \quad x \mapsto (x, x, \ldots, x).$$

Definition 1 A space X with an anchor $p_X : X \to Y$ and structural maps over Y:

$$\mu : X \times_Y X \to (X)^n_Y, \quad inv : X \to X, \quad e : Y \to X$$

is called *an n-groupoid over Y* if the following conditions are satisfied:

(1) For x_1, x_2, x_3 such that $p_X(x_1) = p_X(x_2) = p_X(x_3)$: if

$$\mu(x_1, x_2) = [z_1, \dots, z_n], \quad \mu(x_2, x_3) = [w_1, \dots, w_n],$$

then

$$[\mu(x_1, w_1), \dots, \mu(x_1, w_n)] = [\mu(z_1, x_3), \dots, \mu(z_n, x_3)].$$

(2) For every $x \in X$ and $y = p_X(x) \in Y$:

$$\mu(e(y), x) = \mu(x, e(y)) = D(x).$$

(3) For every $x \in X$ and $y = p_X(x) \in Y$:

$$e(y) \in \mu(x, inv(x)), \quad e(y) \in \mu(inv(x), x).$$

Example 5 Let $X = \mathbb{C} \times \mathbb{C}$, $Y = \mathbb{C}$ and an anchor is defined as the projection to the second component

$$p_X := p_2 : X \to Y, \quad p_X(x, \lambda) := \lambda.$$

We define a 2-groupoid over Y starting with an operation over Y

$$\mathcal{A}((x_1, \lambda), (x_2, \lambda)) = (x_1 + x_2 - \lambda x_1 x_2, \lambda).$$

An involutive automorphism I over Y is defined by

$$I : X \to X, \quad (\bar{u}, \lambda) = \left(-\frac{u}{1 - \lambda u}, \lambda \right).$$

If we denote by $u\bar{u} = x,\ v\bar{v} = y$, then the 2-groupoid over Y is defined by

$$\mu((x, \lambda), (y, \lambda)) = [(z_1, \lambda), (z_2, \lambda)]$$

where $z_i,\ i = 1, 2$ are solutions of the following quadratic equation

$$Z^2 - \left(2(x + y) - \lambda^2 xy\right) Z + (x - y)^2 = 0.$$

Notice that we get p_2 for $\lambda = 0$. Thus the structure of the above 2-groupoid is certain deformation of p_2.

3 Rational Limit of a Kummer Surface and Two-Valued Addition Law

We start with a genus two curve V given by an affine equation

$$V = \{(t, s) \in \mathbb{C}^2 : t^2 = s^5 + \lambda_4 s^3 + \lambda_6 s^2 + \lambda_8 s + \lambda_{10}\}.$$

Corresponding Jacobian $\mathrm{Jac}(V)$ as two-dimensional complex torus is a factor of $\mathbb{C}^2$ with a lattice Γ. The lattice Γ is determined by the vector $(\lambda_4, \lambda_6, \lambda_8, \lambda_{10})$. Note that the indices (4, 6, 8, 10) of λ are chosen to fit with the graduation in sequel. The Kummer surface K is the factor of the Jacobian by the group of automorphisms of order 2:

$$K = \mathrm{Jac}(V)/\pm\,.$$

Locally, in a vicinity of $0 = (0, 0)$ the Kummer surface K is isomorphic to $\mathbb{C}^2/\pm$.

Moreover, all constructions allow the rational limit:$\lambda \to 0$.

Starting from the vector $(\lambda_4, \lambda_6, \lambda_8, \lambda_{10})$ and a vector $u = (u_1, u_3)$, a function $\sigma(u, \lambda)$ is constructed as an entire function in u and λ, such that all coefficients $c_{ij}(\lambda)$ in expansion in $u_1^i u_3^j$ are *polynomials* in λ.

3.1 Two-Dimensional Abelian Functions

Following Buchstaber et al. (1997), we introduce

$$\zeta_k = \frac{\partial}{\partial u_k} \ln \sigma(u, \lambda), \qquad k = 1, 3$$

$$\wp_{kl} = -\frac{\partial^2}{\partial u_k \partial u_l} \ln \sigma(u, \lambda), \qquad k, l = 1, 3.$$

The last, *abelian functions* are functions on the Jacobian $\mathrm{Jac}(V)$. All necessary details about these functions are presented in Buchstaber et al. (2012) and Buchstaber (2016).

We are going to use the standard sigma function, the one defined by *arf-invariants* (ℓ, ℓ') to be equal to

$$\ell = (1, 1), \quad \ell' = (2, 1).$$

The standard sigma function is odd in u,

$$\sigma(u, \lambda) = -\sigma(-u, \lambda),$$

while the $\wp$- functions are *even*:

$$\wp_{k,l}(u, \lambda) = \wp_{k,l}(-u, \lambda).$$

Thus, the $\wp$- functions $\wp_{k,l}(u,\lambda)$ generate well-defined functions on the Kummer surface K.

There is an embedding with each fixed λ:

$$\begin{aligned}\pi_\lambda : K &\to \mathbb{C}P^3\\ [u,-u] &\mapsto [\sigma^2(u), \sigma^2(u)\wp_{11}(u), \sigma^2(u)\wp_{13}(u), \sigma^2(u)\wp_{33}(u)].\end{aligned}$$

Note that $\sigma^2(u)\wp_{kl}(u)$ are entire functions. The embeddings π_λ serve to describe addition law on K in terms of coordinates on $\mathbb{C}P^3$.

One can easily compute the limit of the functions when λ tends to zero:

$$\begin{aligned}
&\lim_{\lambda\to 0}\sigma(u,\lambda) = \sigma_0(u) = u_3 - \frac{1}{3}u_1^3\\
&\zeta_1^{(0)} = \frac{\partial}{\partial u_1}\ln\sigma_0(u) = -\frac{u_1^2}{\sigma_0(u)}\\
&\zeta_3^{(0)} = \frac{\partial}{\partial u_3}\ln\sigma_0(u) = -\frac{1}{\sigma_0(u)}\\
&\wp_{11}^{(0)} = -\frac{\partial^2}{\partial u_1^2}\ln\sigma_0(u) = \frac{2u_1\sigma_0(u)+u_1^4}{\sigma_0^2(u)}\\
&\wp_{13}^{(0)} = -\frac{\partial}{\partial u_1}\zeta_3^{(0)}(u) == -\frac{u_1^2}{\sigma_0(u)^2}\\
&\wp_{33}^{(0)} = -\frac{\partial}{\partial u_3}\zeta_3^{(0)}(u) = \frac{1}{\sigma_0^2(u)}.
\end{aligned}$$

Thus,

$$\sigma_0^2\wp_{13}^{(0)} = -u_1^2, \quad \sigma_0^2\wp_{33}^{(0)} = 1.$$

3.2 Rational Limit Embedding and Two-Valued Group Law

Now, we are going to construct a new two-valued group law on $\mathbb{C}^2/\pm$ associated with an embedding π_K induced by the rational limit of a Kummer surface:

$$\begin{aligned}&\pi_K : \mathbb{C}^2/\pm \to \mathbb{C}^3\\ &\pi_K([u,-u]) = \left(\left(u_3 - \frac{1}{3}u_1^3\right)^2, 2u_1u_3 + \frac{1}{3}u_1^4, -u_1^2\right),\end{aligned}$$

where

$$u = (u_1, u_3).$$

One checks that an inverse image is a two element set if nonempty:

$$x_1 = \left(u_3 - \frac{1}{3}u_1^3\right)^2$$
$$x_2 = 2u_1u_3 + \frac{1}{3}u_1^4$$
$$x_3 = -u_1^2.$$

The multiplication μ given by the formula

$$\mu([u, -u], [v, -v]) = [(u_1 + v_1), -(u_1 + v_1), (u_3 + v_3), -(u_3 + v_3), (u_1 - v_1), -(u_1 - v_1), (u_3 - v_3), -(u_3 - v_3)]$$

after composition with the embedding $(\pi_K)^2$ leads to the formulae

$$\hat{X}_6 = \left((u_3 + v_3) - \frac{1}{3}(u_1 + v_1)^3\right)^2$$
$$\hat{X}_4 = 2(u_1 + v_1)(u_3 + v_3) + \frac{1}{3}(u_1 + v_1)^4$$
$$\hat{X}_2 = -(u_1 + v_1)^2$$

and

$$\hat{Y}_6 = \left((u_3 - v_3) - \frac{1}{3}(u_1 - v_1)^3\right)^2$$
$$\hat{Y}_4 = 2(u_1 - v_1)(u_3 - v_3) + \frac{1}{3}(u_1 - v_1)^4$$
$$\hat{Y}_2 = -(u_1 - v_1)^2.$$

The last formulae lead to the following change of variables:

$$\hat{X}_2 = -X_2$$
$$\hat{X}_4 = 2X_4 + \frac{1}{3}X_2^2$$
$$\hat{X}_6 = X_6 - \frac{2}{3}X_2X_4 + \frac{1}{9}X_2^3.$$

This is an *algebraic* change of variables and the inverse change is given by the formulae:

$$X_2 = -\hat{X}_2$$
$$X_4 = \frac{1}{2}\left(\hat{X}_4 - \frac{1}{3}\hat{X}_2^2\right)$$
$$X_6 = \hat{X}_6 - \frac{1}{3}\hat{X}_2\hat{X}_4 - \frac{2}{9}X_2^3$$

By applying the last algebraic change of variables on the equation of the quadric $Q : X_2X_6 = X_4^2$ we get the equation of the rational limit of the Kummer surface.

Proposition 1 *The rational limit of the Kummer surface is given by the surface in* $\mathbb{C}^3$ *by the equation*

$$-9\hat{X}_4^2 - 36\hat{X}_2\hat{X}_6 + 12\hat{X}_2^2\hat{X}_4 + 7\hat{X}_2^4 = 0. \tag{3}$$

3.3 Rational Kummer Two-Valued Group

Now, using the algebraic change of variables, we are going to construct a new two-valued group law, and we will call it *the rational Kummer two-valued group*.

First, we consider $(\hat{X}_2, \hat{Y}_2)$. We have

$$\hat{X}_2 = -x_3 - 2u_1v_1 - y_3$$
$$\hat{Y}_2 = -x_3 + 2u_1v_1 - y_3$$

therefore

$$\hat{X}_2 + \hat{Y}_2 = -2(x_3 + y_3)$$
$$\hat{X}_2\hat{Y}_2 = (x_3 - y_3)^2.$$

Thus we see that the pair $(\hat{X}_2, \hat{Y}_2)$ is the solution of the quadratic equation

$$\mathcal{Z}^2 + 2(x_3 + y_3)\mathcal{Z} + (x_3 - y_3)^2 = 0. \tag{4}$$

Now, we pass to the pair $(\hat{X}_4, \hat{Y}_4)$. They can be represented in the form

$$\hat{X}_4 = \hat{X}_4^+ + \hat{X}_4^-$$
$$\hat{Y}_4 = \hat{X}_4^+ - \hat{X}_4^-$$

where

$$\hat{X}_4^+ = 2(u_1u_3 + v_1v_3) + \frac{1}{3}(u_1^4 + 6u_1^2v_1^2 + v_1^4)$$
$$\hat{X}_4^- = 2(v_1u_3 + u_1v_3) + \frac{4}{3}u_1v_1(u_1^2 + v_1^2).$$

Then, we have

$$\hat{X}_4 + \hat{Y}_4 = 2\hat{X}_4^+$$
$$\hat{X}_4\hat{Y}_4 = (\hat{X}_4^+)^2 - (\hat{X}_4^-)^2.$$

Thus, $(\hat{X}_4, \hat{Y}_4)$ are the roots of the quadratic equation

$$\mathcal{Z}^2 - 2\hat{X}_4^+\mathcal{Z} + (\hat{X}_4^+)^2 - (\hat{X}_4^-)^2 = 0. \tag{5}$$

The last equation is equivalent to

$$(\mathcal{Z} - (\hat{X}_4^+))^2 = (\hat{X}_4^-)^2,$$

where $\hat{X}_4^+$, $\hat{X}_4^-$ can be rewritten in the form

$$\begin{aligned}
\hat{X}_4^+ &= 2(u_1u_3 + v_1v_3) + \frac{1}{3}(u_1^4 + 6u_1^2v_1^2 + v_1^4)\\
\hat{X}_4^- &= 2(v_1u_3 + u_1v_3) + \frac{4}{3}u_1v_1(u_1^2 + v_1^2).
\end{aligned}$$

We pass to the last pair $\hat{X}_6$, $\hat{Y}_6$:

$$\begin{aligned}
\hat{X}_6 &= \left((u_3 + v_3) - \frac{1}{3}(u_1 + v_1)^3\right)^2\\
\hat{Y}_6 &= \left((u_3 - v_3) - \frac{1}{3}(u_1 - v_1)^3\right)^2.
\end{aligned}$$

One can easily calculate

$$\begin{aligned}
\hat{X}_6 + \hat{Y}_6 =& 2\Big[u_3^2 + v_3^2 - \frac{2}{3}(u_1^3u_3 + 3u_1u_3v_1^2 + 3u_1^2v_1v_3 + v_1^4)\\
&+ \frac{1}{9}(u_1^6 + 15u_1^4v_1^2 + 15u_1^2v_1^4 + v_1^6)\Big]\\
\hat{X}_6\hat{Y}_6 =& \left((u_3^2 - v_3^2) + \frac{1}{9}(u_1^2 - v_1^2)^3 - \frac{2}{3}(u_3u_1^3 + 3u_1v_1^2u_3^2 - 3u_1^2v_1v_3 - v_1^3v_3)\right)^2,
\end{aligned}$$

or, in the old coordinates

$$\begin{aligned}
\hat{X}_6 + \hat{Y}_6 =& 2\Big(x_3 + y_3 - \frac{2}{3}(x_3x_2 + 3x_2y_1 + 3x_1y_2 + y_1^2)\\
&+\frac{1}{9}(x_1^3 + 15x_1^2y_1 + 15x_1y_1^2 + y_1^3)\Big)\\
\hat{X}_6\hat{Y}_6 =& \left((x_3 - y_3) + \frac{1}{9}(x_1 - y_1)^2 - \frac{2}{3}(x_2x_1 + 3x_2y_1 - 3x_1y_2 - y_1y_2)\right)^2.
\end{aligned}$$

In the new coordinates one may rewrite

$$\begin{aligned}
B_3 &:= \hat{X}_6 + \hat{Y}_6\\
C_3 &:= \hat{X}_6\hat{Y}_6
\end{aligned}$$

and to get finally

$$B_3 = 2\left(\left(\left(\hat{x}_6 - \frac{1}{3}\hat{x}_4\hat{x}_2 - \frac{2}{9}\hat{x}_2^3\right) + (\hat{y}_6 - \frac{1}{3}\hat{y}_4\hat{y}_2 - \frac{2}{9}\hat{y}_2^3\right)\right.$$
$$- \frac{1}{3}\left(-\hat{x}_2\hat{x}_4 + \frac{1}{3}\hat{x}_2^3 - 3\hat{x}_4\hat{y}_2 + \hat{x}_2^2\hat{y}_2 - 3\hat{x}_2\hat{y}_4 + \hat{x}_2\hat{y}_2^2 + 2\hat{y}_2^2\right)$$
$$\left. + \frac{1}{9}(-\hat{x}_2^3 + 15\hat{x}_2^2\hat{y}_2 + 15\hat{x}_2\hat{y}_2^2 - \hat{y}_2^3)\right)$$

and

$$C_3 = \left[\left(\hat{x}_6 - \frac{1}{3}\hat{x}_4\hat{x}_2 - \frac{2}{9}\hat{x}_2^3 - \hat{y}_6 + \frac{1}{3}\hat{y}_4\hat{y}_2 + \frac{2}{9}\hat{y}_2^3\right) + \frac{1}{9}(\hat{y}_2 - \hat{x}_2)^3\right.$$
$$- \frac{1}{9}\left(\frac{1}{3}\hat{x}_2^2 - \hat{x}_2\hat{x}_4 + \hat{x}_2^2\hat{y}_2 - 3\hat{x}_4\hat{y}_2 + 3\hat{y}_4\hat{x}_2 - \hat{y}_2^2\hat{x}_2\right.$$
$$\left.\left. + \hat{y}_2\hat{y}_4 - \frac{1}{3}\hat{y}_2^3\right)\right]^2.$$

Thus, we may conclude that the pair $(\hat{X}_6, \hat{Y}_6)$ is determined as the roots of the quadratic equation

$$\mathcal{Z}^2 - B_3\mathcal{Z} + C_3 = 0, \tag{6}$$

where B_3, C_3 are functions of the coordinates $(\hat{x}_2, \hat{x}_4, \hat{x}_6, \hat{y}_2, \hat{y}_4, \hat{y}_6)$ given above.

4 Two-Valued Group Structures on Kummer Varieties and Sigma-Functions

We start with the sigma-function $\sigma(u) = \sigma(u, \lambda)$,where $u^\top = (u_1, u_3)$, $\lambda^\top = (\lambda_4, \lambda_6, \lambda_8, \lambda_{10})$, associated with a curve

$$V = \{(t, s) \in \mathbb{C}^2 \ : \ t^2 = s^5 + \lambda_4 s^3 + \lambda_6 s^2 + \lambda_8 s + \lambda_{10}\}.$$

We assume that $\lambda \in \mathbb{C}^4$ is a non-discriminant point of the curve V and $u \in \mathbb{C}^2$, where $du_1 = \frac{sds}{t}$, $du_3 = \frac{ds}{t}$. Indexation of the coordinates of the vector of the parameter λ is chosen according to the graduation $\deg s = -2$, $\deg t = -5$, $\deg \lambda_{2i} = -2i$. Moreover, $\deg u_1 = 1$, $\deg u_3 = 3$, and the sigma-function

$$\sigma(u, \lambda) = u_3 - \frac{1}{3}u_1^3 + \frac{1}{6}\lambda_6 u_3^3 + (u^5)$$

is an entire and homogeneous of degree 3 in u and λ. Recurrent description of the series for $\sigma(u, \lambda)$ is given in Buchstaber and Leykin (2005).

We consider the Jacobian $J_2 = \mathrm{Jac}(V) = \mathbb{C}^2/\Gamma_2$ of the curve V and a vector-function is defined by the formulae

$$\wp(u) = \big(\wp_{33}(u), \wp_{13}(u), \wp_{11}(u)\big),$$

where $\wp_{kl}(u) = -\frac{\partial^2}{\partial u_k \partial u_l} \ln \sigma(u),\ k, l = 1, 3$. Since the function $\sigma(u)$ is odd, we get a mapping

$$i\colon J_2 \longrightarrow \mathbb{C}P^3 \ :\ i(u) = (x_0 : x_2 : x_4 : x_6),$$

where $x_0 = \sigma(u)^2\wp_{33}(u),\ x_2 = \sigma(u)^2\wp_{13}(u),\ x_4 = \sigma(u)^2\wp_{11}(u),\ x_6 = \sigma(u)^2$. The last mapping factorizes through the Kummer variety K together with an embedding

$$\widehat{i}\colon K = (J_2/\pm) \longrightarrow \mathbb{C}P^3.$$

Let us note, that the embedding $\widehat{i}$ is defined with entire homogeneous functions $x_{2k}(u),\ \deg x_{2k} = 2k,\ k = 0, \ldots, 3$.

We are going to use a ramified covering

$$\gamma\colon \left(\mathbb{C}P^3\right)^2 \longrightarrow \mathbb{C}P^6,$$

defined by the relation

$$[p(x, t), p(y, t)] \longrightarrow p(x, t)p(y, t),$$

where $p(x, t) = x_0 t_3^3 + x_2 t_3^2 t_1 + x_4 t_3 t_1^2 + x_6 t_1^3$. By putting $\deg t_k = k$, we get $p(x, t)$ as a homogeneous polynomial of degree 9.

Theorem 1 *Multiplication m in the two-valued group on K*

$$m\colon K \times K \longrightarrow (K)^2 \ :\ [u] * [u] = \big([u + v], [u - v]\big)$$

is defined through an algebraic mapping

$$\mu\colon \mathbb{C}P^3 \times \mathbb{C}P^3 \longrightarrow \mathbb{C}P^6 \ :$$

and it is uniquely defined by the commuting condition of the following diagram

$$\begin{array}{ccccc} K \times K & \xrightarrow{\ m\ } & (K)^2 & \xrightarrow{\ (\widehat{i})^2\ } & \left(\mathbb{C}P^3\right)^2 \\ \downarrow{\scriptstyle \widehat{i}\times\widehat{i}} & & & & \downarrow{\scriptstyle \gamma} \\ \mathbb{C}P^3 \times \mathbb{C}P^3 & & \xrightarrow{\qquad \mu \qquad} & & \mathbb{C}P^6 \end{array}$$

Proof Set

$$X(u) = \big(\wp_{33}(u), \wp_{31}(u), \wp_{11}(u), 1\big) \qquad \mathcal{X}(u) = \sigma(u)^2 X(u).$$

Consider the canonical projection

$$\pi : \mathbb{C}^7 \backslash 0 \longrightarrow \mathbb{C}P^6 \ :\ \pi(z) = (z_0 : z_2 : \ldots : z_{12}) = [z].$$

According to the construction, we have

$$\widehat{i}\,[u] = [\mathcal{X}(u)], \quad \gamma(\widehat{i}\,)^2\big([u] * [v]\big) = \gamma\big([\mathcal{X}(u+v), \mathcal{X}(u-v)]\big).$$

Thus, we have to show that each coordinate $z_{2k}(u, v)$, $k = 0, \ldots, 6$, of the point $\gamma\big([\mathcal{X}(u+v), \mathcal{X}(u-v)]\big)$ is a polynomial of the coordinates x_{2i}, y_{2i}, $i = 0, \ldots, 3$ of the points $\mathcal{X}(u)$ $\mathcal{X}(v)$.

Genus two sigma-function $\sigma(u)$ satisfies the following addition theorem (see Buchstaber et al. 1997; Buchstaber and Leykin 2005):

$$\sigma(u+v)\sigma(u-v) = \mathcal{X}(u)^\top \mathcal{J}\,\mathcal{X}(v).$$

where $\mathcal{J} = \begin{pmatrix} 0 & -\mathcal{E} \\ \mathcal{E} & 0 \end{pmatrix}$ and $\mathcal{E} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We have,

$$z_{12} = z_{12}(u, v) = \big(\mathcal{X}(u)^\top \mathcal{J}\,\mathcal{X}(v)\big)^2.$$

Thus, we get that in the mapping $\mu(x, y) = [z] = (z_0 : z_2 : \ldots : z_{12}) \in \mathbb{C}P^6$, the coordinate z_{12} is defined by the formula $z_{12} = (x^\top \mathcal{J} y)^2$, where $(x, y) \in \mathbb{C}P^3 \times \mathbb{C}P^3$.

Analogous result for the rest of the coordinates is based on deep facts about algebraic generators of the ring generated by logarithmic derivatives of order 2 and higher of the sigma-function $\sigma(u)$ (see Buchstaber et al. 1997).

Set

$$M(u, v) = X(u)^\top \mathcal{J}\,X(v).$$

We have

$$\ln\sigma(u+v) + \ln\sigma(u-v) = 2\big(\ln\sigma(u) + \ln\sigma(v)\big) + \ln M(u, v).$$

Apply the operators $\partial_k = \frac{\partial}{\partial u_k} + \frac{\partial}{\partial v_k}$, for $k = 1$ and 3. We get

$$2\zeta_k(u+v) = 2\big(\zeta_k(u) + \zeta_k(v)\big) + \frac{1}{M(u, v)}\Big(X_k(u)^\top \mathcal{J}\,X(v) + X(u)^\top \mathcal{J}\,X_k(v)\Big),$$

where $\zeta_k(u) = \frac{\partial}{\partial u_k}\ln\sigma(u)$ $X_k(u) = \frac{\partial}{\partial u_k}X(u)$. Now, we apply the operator $\frac{\partial}{\partial u_l}$, and we get

$$\wp_{kl}(u+v) = -\frac{\partial}{\partial u_l}\,\zeta_k(u+v) = \varphi_{kl}(u, v) - \psi_{kl}(u, v),$$

where

$$\varphi_{kl}(u, v) = \wp_{kl}(u) - \frac{1}{2M(u, v)^2}\Big\{\big(X_{kl}(u)^\top \mathcal{J}\,X(v)\big)M(u, v) - X_k(u)^\top B(v)X_l(u)\Big\},$$
$$\psi_{kl}(u, v) = \frac{1}{2M(u, v)^2}\big(X_l(u)^\top C(u, v)X_k(v)\big)$$

and $B(v) = \mathcal{J}\, X(v)X(v)^\top \mathcal{J}^\top, \quad C(u,v) = \big(M(u,v) - \mathcal{J}\, X(v)X(u)^\top\big)\mathcal{J}$.

Note $B(-v) = B(v)$ $C(-u,v) = C(u,-v) = C(u,v)$. Thus, $\varphi_{kl}(u,-v) = \varphi_{kl}(u,v)$ $\psi_{kl}(u,-v) = -\psi_{kl}(u,v)$. We get

$$\mathcal{X}(u+v) = \sigma(u+v)^2(X_1 - X_2) \qquad \mathcal{X}(u-v) = \sigma(u-v)^2(X_1 + X_2),$$

where $X_1 = (\varphi_{33}, \varphi_{13}, \varphi_{11}, 1)$ $X_2 = (\psi_{33}, \psi_{13}, \psi_{11}, 0)$. We obtain

$$p\big(\mathcal{X}(u+v), t\big)p\big(\mathcal{X}(u-v), t\big) = z_1 \cdot \big(p(X_1,t)^2 - p(X_2,t)^2\big)$$

where

$$\begin{aligned} p(X_1,t) &= \varphi_{33}t_3^3 + \varphi_{13}t_3^2 t_1 + \varphi_{11}t_3 t_1^2 + t_1^3, \\ p(X_2,t) &= t_3(\psi_{33}t_3^2 + \psi_{13}t_3 t_1 + \psi_{11}t_1^2). \end{aligned}$$

From the above formulae for $\varphi_{kl}(u,v)$ $\psi_{kl}(u,v)\psi_{pq}(u,v)$, it immediately follows that these functions are polynomials of $\wp_{ij}(u)$, $\wp_{ijk}(u)\wp_{i'j'k'}(u)$, $\wp_{ijpq}(u)$ $\wp_{ij}(v)$, $\wp_{ijk}(v)\wp_{i'j'k'}(v)$, $\wp_{ijpq}(v)$. The functions $\wp_{ijk}(u)\wp_{i'j'k'}(u)$ $\wp_{ijpq}(u)$, where $i, j, k, i', j', k', p, q$ take values 1 or 3 independently, are polynomials of $\wp_{ij}(u)$ (see Buchstaber et al. 1997). Consequently, using $(x,y)|_{\widehat{i}\times\widehat{i}} = \big(\mathcal{X}(u), \mathcal{X}(v)\big)$, we get coordinates z_{2k} of the vector $\mu(x,y)$ as polynomials of the coordinates of the vectors x and y. □

5 Homomorphism of Rings of Functions, Induced by Abel Mapping in Genus 2

Let $V = \big\{(s,\mu) \in \mathbb{C}^2 \;:\; \mu^2 = s^5 + \lambda_4 s^3 + \lambda_6 s^2 + \lambda_8 s + \lambda_{10}\big\}$ denotes a hyperelliptic curve.

Theorem 2 *The Abel mapping*

$$\mathcal{A}\colon (V)^2 \longrightarrow \mathrm{Jac}V$$

induces a homomorphism of rings of functions

$$\mathcal{A}^*\colon \mathcal{F}(\mathrm{Jac}V) \longrightarrow \mathcal{F}\big((V)^2\big)$$

such that

$$\mathcal{A}^*(\wp_{11}(u)) = s_1 + s_2, \quad \mathcal{A}^*(\wp_{13}(u)) = -s_1 s_2, \quad \mathcal{A}^*(\wp_{33}(u)) = \frac{F(s_1,s_2) - 2\mu_1\mu_2}{(s_1 - s_2)^2},$$

where

$$
\begin{aligned}
F(s_1, s_2) &= 2\lambda_{10} + \lambda_8(s_1 + s_2) + s_1 s_2\big(2\lambda_6 + \lambda_4(s_1 + s_2)\big),\\
\wp_{111}(u) &= 2\frac{\mu_1 - \mu_2}{s_1 - s_2}, \quad \wp_{113}(u) = 2\frac{s_1\mu_2 - s_2\mu_1}{s_1 - s_2}\\
\wp_{331}(u) &= -2\frac{s_1^2\mu_2 - s_2^2\mu_1}{s_1 - s_2}, \quad \wp_{333}(u) = 2\frac{\psi(s_1, s_2)\mu_2 - \psi(s_2, s_1)\mu_1}{(s_1 - s_2)^3},
\end{aligned}
$$

and

$$\psi(s_1, s_2) = 4\lambda_{10} + \lambda_8(3s_1 + s_2) + 2\lambda_6 s_1(s_1 + s_2) + \lambda_4 s_1^2(s_1 + 3s_2) + s_1^3 s_2(3s_1 + s_2).$$

The proof of the Theorem can be found in Buchstaber et al. (1997). Let us note that we use the indexation here different from Buchstaber et al. (1997), in correspondence with the graduation:

$$\deg s = -2;\ \deg \mu = -5;\ \deg \lambda_{2k} = -2k;\ k = 2, 3, 4, 5;\ \deg u_i = i,\ i = 1, 3.$$

This provides additional opportunity to check the formulae. Observe that $\deg \wp_{kl}(u) = -(k + l),\ \deg \wp_{klp}(u) = -(k + l + p)$.

Any Abelian function on Jac$V$ represents a linear function of $\wp_{111}(u)$ with coefficients which are rational functions of $\wp_{11}(u)$ and $\wp_{13}(u)$. On the other hand, from the theory of polysymmetric functions (see Gelfand et al. 1994 and Buchstaber and Rees 2002), it is known that the field of rational functions on $(\mathbb{C}^2)^2$ in the coordinates $[(s_1, \mu_1), (s_2, \mu_2)]$ is generated by polysymmetric functions

$$e_{10} = s_1 + s_2,\ e_{20} = s_1 s_2,\ e_{01} = \mu_1 + \mu_2,\ e_{02} = \mu_1\mu_2,\ e_{11} = s_1\mu_2 + s_2\mu_1,$$

which are related by unique (for $(\mathbb{C}^2)^2$) relation

$$\left(e_{10}^2 - 4e_{20}\right)\left(e_{01}^2 - 4e_{02}\right) = (e_{10}e_{01} - 2e_{11}).$$

The mapping $\mathcal{A}$ is a birational equivalence, thus $\mathcal{A}^*$ induces isomorphism of the field of Abelian functions $\mathcal{F}(\mathrm{Jac}V)$ on Jac$V$ with the field of rational functions $\mathcal{F}\left((V)^2\right)$ on $(V)^2$. We have:

$$\mathcal{A}^*(\wp_{11}(u)) = e_{10}, \quad \mathcal{A}^*(\wp_{13}(u)) = -e_{20}, \quad \mathcal{A}^*(\wp_{111}(u)) = \frac{e_{10}e_{01} - 2e_{11}}{e_{10}^2 - 4e_{20}}.$$

In this way, the Theorem 2 completely describes the isomorphism $\mathcal{A}^*$ and gives an opportunity to express explicitly, for example, the even functions $\wp_{klp}(u)\wp_{k'l'p'}(u)$ as polynomials of $\wp_{kl}(u)$. The explicit formulae for those polynomials are given in the book Buchstaber et al. (1997).

The hyperelliptic involution acts on $(V)^2$ according to the formula

$$[(s_1, \mu_1), (s_2, \mu_2)] \longrightarrow [(s_1, -\mu_1), (s_2, -\mu_2)] .$$

The Abel mapping is invariant with respect to this involution on $(V)^2$ and the involution $u \to -u$ on $\operatorname{Jac} V$.

From the above formulae one can see that the images of the functions $\wp_{kl}(u)$ and $\wp_{klp}(u)\wp_{k'l'p'}(u)$ are even, while the images of the functions $\wp_{klp}(u)$ are odd. Thus, the mapping

$$\widehat{\mathcal{A}}\colon (V)^2/_{\pm} \longrightarrow K = (\operatorname{Jac} V)/_{\pm},$$

is defined and it induces a homomorphism between rings of functions.

There is an addition law on $(V)^2$, with the Abel mapping $\mathcal{A}$ as a homomorphism. Explicit form of this operation in the coordinates $[(s_1, \mu_1), (s_2, \mu_2)]$ has been described in Buchstaber and Leykin (2005).

Thus, on $(V)^2/_{\pm}$ there is corresponding two-valued addition, such that $\widehat{\mathcal{A}}$ is a homomorphism with respect to the two-valued group structure on the Kummer variety K, defined above.

6 Geometric Two-Valued Group Laws and Kummer Varieties

6.1 A Quadric in $\mathbb{C}P^5$ and A Line Complex in $\mathbb{C}P^3$

Following classics, see Donagi (1980) and references therein, let us consider a three-dimensional projective space $\mathbb{C}P^3 = P(\mathbb{C}^4)$ and corresponding Grassmannian $Gr(2, 4)$ of all lines in $\mathbb{C}P^3$. By Plücker embedding, the Grassmannian $Gr(2, 4)$ can be realized as a quadric G in $\mathbb{C}P^5 = P(\wedge^2 V)$, where $V = \mathbb{C}^4$:

$$\begin{gathered} Gr(2,4) \hookrightarrow \mathbb{C}P^5 \\ \ell = \ell < v_1, v_2 > \mapsto v_1 \wedge v_2, \quad v_1, v_2 \in V^4. \end{gathered}$$

The quadric G parameterizes all decomposable elements $w = v_1 \wedge v_2$ of $P(\wedge^2 V)$ and the quadric is described by the Plücker quadratic relation:

$$G : w \wedge w = 0.$$

For a given element $x \in G$, denote by ℓ_x the line in $\mathbb{C}P^3$ which maps to x by the above embedding.

We consider so-called Schubert cycles:

$$\begin{aligned} \sigma_1(\ell) &= \{x \in G \mid \ell_x \cap \ell \neq \emptyset\} \\ \sigma_2(p) &= \{x \in G \mid \ell_x \ni p\} \\ \sigma_{1,1}(h) &= \{x \in G \mid \ell_x \subset h\} \\ \sigma_{2,1}(p, h) &= \{x \in G \mid p \in \ell_x \subset h\}. \end{aligned}$$

One can easily see that every cycle $\sigma_1(\ell)$ is a hyperplane section of the quadric G. If the line $\ell \in \mathbb{C}P^3$ is determined by vectors $v_1, v_2 \in V^4$ then the hyperplane of intersection is of the form $H_{v_1\wedge v_2} = \{w \mid w \wedge v_1 \wedge v_2 = 0\}$.

Every cycle $\sigma_{2,1}(p, h)$ is a line in $\mathbb{C}P^5$. Every line $L \subset G$ is of the form $L = \sigma_{2,1}(p, h)$.

A line $L \subset G$ is a pencil of lines in $\mathbb{C}P^3$. This is a confocal pencil with the common point, *the focus* $p \in \mathbb{C}P^3$. At the same time this pencil is coplanar with the common plane $h \in \mathbb{C}P^{3*}$.

Every cycle of the form $\sigma_2(p)$ or of the form $\sigma_{1,1}(h)$ is a two-plane in $\mathbb{C}P^5$. Conversely, every two-plane in G is of the form $\sigma_2(p)$ or of the form $\sigma_{1,1}(h)$.

Let us recall some general properties of a quadric Q in $\mathbb{C}P^m$. The rank of the quadric is equal to the rank of any of its symmetric $(m + 1) \times (m + 1)$ matrices. The quadric is smooth if its rank is maximal, i.e. if rank of Q is $m + 1$. If the rank of Q is r then it is a cone over a smooth quadric in $\mathbb{C}P^{r-1}$ with a vertex $\mathbb{C}P^{m-r}$. Quadrics of rank $m + 1$ and m are called *general*.

Now, we give a number of well-known facts for the case of a smooth quadric in $\mathbb{C}P^5$, see Tyurin (1975), Donagi (1980) and references therein.

Proposition 2 *Let G be a smooth quadric in $\mathbb{C}P^5$. Then:*

(a) There exists a four-dimensional vector space V^4 such that G is the Plücker embedding of the Grassmannian of lines in $P(V^4)$.
(b) The maximal linear subspaces of G are two-dimensional and they are of the form $\sigma_2(p)$ or of the form $\sigma_{1,1}(h)$, where $p \in \mathbb{C}P^3 = P(V^4)$ and $h \in \mathbb{C}P^{3*}$.
(c) The variety $C(Q)$ of all two-dimensional subspaces of G is three-dimensional and it has two irreducible components, A and B:

$$A = \{\sigma_2(p) \mid p \in \mathbb{C}P^3 = P(V^4)\}$$
$$B = \{\sigma_{1,1}(h) \mid h \in \mathbb{C}P^{3*}\}.$$

6.2 Smooth Intersection of Two Quadrics and Abelian Varieties

Now we consider the intersection X of two quadrics G and F in $\mathbb{C}P^5$. Such a set is classically called a quadratic complex of lines, if G is understood as a Grassmannian of lines in some $\mathbb{C}P^3$. Together with two quadrics G and F, one may consider the whole *pencil of quadrics*:

$$F_\lambda := F + \lambda G,$$

and X is the base set for the pencil, the common intersection of quadrics from the pencil, see Tyurin (1975), Donagi (1980) and references therein.

A pencil of quadrics is *generic* if associated pencil of 6×6 symmetric matrices contains six different singular matrices. For a generic pencil F_λ denote by $\lambda_1, \dots, \lambda_6$ the corresponding values of the pencil parameter associated with the singular matrices.

The condition that X is smooth is equivalent to the condition that the pencil F_λ is generic. Smoothness of X is also equivalent to the fact that all quadrics F_λ are general and that exactly six of them, $F_{\lambda_1}, \dots, F_{\lambda_6}$ are singular.

Proposition 3 *Suppose X is a smooth intersection of two quadrics $X = G \cap F$ in $\mathbb{C}P^5$. Then:*

(a) The maximal linear subspaces of X are one-dimensional.
(b) There is a maximal linear subspace through each point of X.
(c) There are four maximal linear subspaces passing through a generic point of X.

Given a smooth intersection of quadrics $X = G \cap F$ in $\mathbb{C}P^5$, following Narasimhan, Ramanan, Reid and Donagi, see Donagi (1980) and references there in, in particular M. Reid's unpublished Cambridge 1972 thesis, let us consider the set of one-dimensional linear subspaces

$$\mathcal{A}(X) = \{L \mid L \subset X, L \in Gr(2, 6)\}.$$

Suppose G is realized as a Grassmannian of lines in some $\mathbb{C}P^3$, and denote as before the two components of $C(G)$ of two-dimensional linear subspaces of G as

$$A = \{\sigma_2(p) \mid p \in \mathbb{C}P^3 = P(V^4)\}$$
$$B = \{\sigma_{1,1}(h) \mid h \in \mathbb{P}^{3*}\}.$$

Lemma 1 *Given $L \in \mathcal{A}(X)$. Then:*

(a) There is a unique two-dimensional linear subspace of G, $\sigma_2(p) \in A$ such that $L \subset \sigma_2(p)$. There is a unique one-dimensional linear subspace of X

$$L_1 \in \mathcal{A}(X),$$

such that

$$\sigma_2(p) \cap F = L \cup L_1.$$

(b) There is a unique two-dimensional linear subspace of G, $\sigma_{1,1}(h) \in A$ such that $L \subset \sigma_{1,1}(h)$. There is a unique one-dimensional linear subspace of X

$$L_2 \in \mathcal{A}(X),$$

such that

$$\sigma_{1,1}(h) \cap F = L \cup L_2.$$

The last Lemma introduces two involutions

$$i_1 : \mathcal{A}(X) \to \mathcal{A}(X), \quad i_1 : \quad L \mapsto L_1$$
$$i_2 : \mathcal{A}(X) \to \mathcal{A}(X), \quad i_2 : \quad L \mapsto L_2.$$

As pencils of lines in $\mathbb{C}P^3$ the subspaces L and L_1 are confocal, they have the same focus p. In the same manner, the subspaces L and L_2 are coplanar, they have the same plane h.

Moreover, there are two mappings

$$k_1 : \mathcal{A}(X) \to \mathbb{C}P^3, \quad k_1 : \quad L \mapsto p$$
$$k_2 : \mathcal{A}(X) \to \mathbb{C}P^{3*}, \quad k_2 : \quad L \mapsto h.$$

The mapping k_1 maps a pencil L to its focus in $\mathbb{C}P^3$ while k_2 maps a pencil to its plane in $\mathbb{C}P^{3*}$.

Denote by $K \subset \mathbb{C}P^3$ the image of $\mathcal{A}(X)$ by k_1. We see that k_1 is a double covering of $\mathcal{A}(X)$ over K and that the involution i_1 interchanges the leaves of the covering. We are going to call K *the Kummer variety* of $\mathcal{A}(X)$. It is associated to the choice of a quadric G from the pencil and to the choice of a connected component A of $C(G)$.

Similarly, denote by $K^* \subset \mathbb{C}P^{3*}$ the image of $\mathcal{A}(X)$ by k_2. The mapping k_2 is a double covering of $\mathcal{A}(X)$ over K and the involution i_2 interchanges the leaves of the covering. We are going to call K^* *the dual Kummer variety* of $\mathcal{A}(X)$. It is associated to the choice of a quadric G from the pencil and to the choice of a connected component B of $C(G)$.

It can be shown that K^* is dual to K, which means that every plane from K^* is tangent to K. From the previous considerations it can also be seen that the degree of K is equal to 4.

For a general point $p \in \mathbb{C}P^3$ the plane $\sigma_2(p) \subset G$ intersects F along a smooth conic. The Kummer variety K coincides with the Kummer variety associated with the quadratic line complex $X = G \cup F$, as defined in Griffiths and Harris (1994), Ch. 6.2: the set of the points $p \in \mathbb{C}P^3$ for which this intersection is a conic which is not smooth. For a general point of K this intersection is a degenerate conic which is a union of two different lines L and $i_1(L)$. But, there is a subset $R \subset K$ of sixteen points, for which this intersection is a degenerate conic of double line. These sixteen points from R correspond to the fixed points of the involution i_1 and to the ramification points of the double covering. We also denote $R^* \subset K^*$ the set of points that correspond to the fixed points of the involution i_2.

6.3 Pencils of Quadrics, Hyperelliptic Curves and Geometric Group Laws

With a generic pencil of quadrics F_λ in $\mathbb{C}P^5$ with six singular quadrics $F_{\lambda_1}, \dots, F_{\lambda_6}$ one may associate a genus two curve

$$\Gamma : y^2 = \prod_{i=1}^{6} (x - \lambda_i).$$

As it was shown by Reid and Donagi, this correspondence between a generic pencil of quadrics and a hyperelliptic curve is not just formal. After Donagi, denote by E the family of all connected components of $C(F_\lambda)$ of all quadrics from the pencil together with a projection

$$p : E \to \mathbb{C}P^1$$

which maps a given irreducible family of two-subspaces of a quadric F_μ to the value μ of the pencil parameter. The projection p is obviously double covering. It is ramified over the points λ_i, $i = 1, \ldots, 6$ since the singular quadrics are the only one with unique component of maximal linear subspaces. Thus, Donagi showed the isomorphism between E and Γ.

But, there is yet another natural realization of the hyperelliptic curve Γ in the context of the pencil F_λ, as it was demonstrated by Reid.

For $L \in \mathcal{A}(X)$ denote by $\mathcal{A}_L(X)$ the closure of the set $\{L' \in \mathcal{A}(X) \mid L \cap L' \neq \emptyset\}$. There is a natural projection

$$q : \mathcal{A}_L(X) \backslash \{L\} \to \mathbb{C}P^1$$

which maps L' to the parameter μ of a quadric F_μ if the space $< L, L' >$ belongs to $C(F_\mu)$. The mapping q is double covering ramified over the six points λ_i, $i = 1, \ldots, 6$ and $\mathcal{A}_L(X)$ is isomorphic to the hyperelliptic curve Γ. The natural involution on $\mathcal{A}_L(X)$ which interchanges the folds of q will be denoted by τ_L.

Moreover, as it was shown by Reid and Donagi, $\mathcal{A}(X)$ is an Abelian variety, isomorphic to the Jacobian of the curve Γ.

We will refer to the curves $\mathcal{A}_L(X)$ and E as *the Donagi-Reid-Knörrer curves* (DRK) associated with the intersection of quadrics X of a generic pencil.

It can easily be shown that for any hyperelliptic curve Γ, there exists a pencil of quadrics with the base set X such that $\mathcal{A}(X)$ is isomorphic to the Jacobian of Γ.

The addition laws on the Abelian varieties $\mathcal{A}(X)$ are such that

$$L_1 + L_2 = M_1 + M_2$$

if there exists μ such that

$$< L_1, L_2 >, < M_1, M_2 > \in C(F_\mu)$$

and if the two two-dimensional spaces $< L_1, L_2 >, < M_1, M_2 >$ belong to the same component of $C(F_\mu)$.

Proposition 4 *Every point $e \in E$ determines its Kummer variety involution i_e and its Kummer variety K_e. The hyperelliptic involution on E, τ interchanges a Kummer variety and its dual:*

$$K_{\tau(e)} = K_e^*.$$

We will fix a point $e_0 \in E$ and a line $L_0 \in \mathcal{A}(X)$ as the origin of a group structure on $\mathcal{A}(X)$ such that $L_0 + \hat{L}_0 = 0$, whenever $< L_0, \hat{L}_0 > \in e_0$.

As above, denote by τ_{L_0} the natural involution on $\mathcal{A}_{L_0}(X)$. Then we have

Lemma 2 *The involutions τ_{L_0} and i_{e_0} are related according to the formula*

$$i_{e_0}|_{\mathcal{A}_{L_0}(X)} = \tau_{L_0}.$$

Now, we are ready to define a two-valued group structure on the Kummer variety K_{e_0}. We will define a mapping

$$\star : K_{e_0} \times K_{e_0} \longrightarrow (K_{e_0})^2$$

by the following procedure.

Take a pair

$$(p_1, p_2) \in K_{e_0} \times K_{e_0}.$$

Denote by

$$[L_1] = \{L_1, \hat{L}_1\}, \quad L_1, \hat{L}_1 \in \mathcal{A}(X)$$

the class of lines in $\mathcal{A}(X)$ which represents the two confocal pencils of lines in $\mathbb{C}P^3$ with the focal point p_1. Similarly, denote by

$$[L_2] = \{L_2, \hat{L}_2\}, \quad L_2, \hat{L}_2 \in \mathcal{A}(X)$$

the class of lines in $\mathcal{A}(X)$ which represents the two confocal pencils of lines in $\mathbb{C}P^3$ with the focal point p_2.

Assume that L_1, L_2 don't intersect L_0. Denote by N_1, N_2 the two lines in $\mathcal{A}_{L_0}(X)$ of intersection of the space $< L_0, L_1 >$ with X and by $e_1, e_2 \in E$ denote the classes determined by

$$< L_0, N_1 >\in e_1, \quad < L_0, N_2 >\in e_2.$$

Denote also by

$$\mu_1 = p(e_1), \quad \mu_2 = p(e_2),$$

and by

$$N_1', N_1'', N_2', N_2'' \in \mathcal{A}_{L_2}(X)$$

the lines which intersect L_2 and which are uniquely defined by the conditions

$$\begin{aligned} &< N_1', L_2 >\in e_1, \quad < N_1'', L_2 >\in \tau(e_1) \\ &< N_2', L_2 >\in e_1, \quad < N_2'', L_2 >\in \tau(e_2). \end{aligned}$$

In other words N_1', N_1'' belong to the two two-dimensional spaces of the different classes of $C(F_{\mu_1})$ which contain L_2; N_2', N_2'' belong to the two two-dimensional spaces of the different classes of $C(F_{\mu_2})$ which contain L_2.

Let M_1 be the fourth intersection line in the intersection of X with the space generated with L_2, N_1', N_2' and let M_2 be the fourth intersection line in the intersection of X with the space generated with L_2, N_1'', N_2''. The line M_1 represents a pencil of lines in $\mathbb{C}P^3$ with the focal point w_1 and M_2 represents a pencil of lines in $\mathbb{C}P^3$ with the focal point w_2.

If we repeat the above procedure with $\hat{L}_1, \hat{L}_2$ instead of L_1, L_2 we come to the lines $\hat{M}_1$ and $\hat{M}_2$ which are confocal with M_1 and M_2.

One can easily adjust previous construction to the case where L_1, L_2 intersect L_0: denote by M_1 the fourth line of the intersection of X with the space generated with $\hat{L}_1, \hat{L}_2, L_0$; denote by M_2 the fourth line of the intersection of X with the space generated with $L_1, \hat{L}_2, L_0$.

Thus we get

Theorem 3 *The mapping defined by the formulae*

$$\star : K_{e_0} \times K_{e_0} \longrightarrow (K_{e_0})^2$$
$$p_1 \star p_2 = (w_1, w_2)$$

defines a structure of two-valued group on the Kummer variety K_{e_0}.

Definition 2 The mapping $\star$ defined by previous construction defines *the geometric two-valued group law* on the Kummer variety K_{e_0}.

7 Integrable Billiards and Two-Valued Laws

7.1 Pencils of Quadrics and Billiards

We begin this Section by recalling the basic definitions related to billiard systems of confocal quadrics from Dragović and Radnović (2006), Dragović and Radnović (2008). [For some most recent developments, see Dragović and Radnović (2011, 2014a, b, 2015a, b).] Here we assume that the dimension of the space is $d = 3$.

Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two quadrics. Denote by u the tangent plane to $\mathcal{Q}_1$ at point x and by z the pole of u with respect to $\mathcal{Q}_2$. Suppose lines ℓ_1 and ℓ_2 intersect at x, and the plane containing these two lines meet u along ℓ.

If lines ℓ_1, ℓ_2, xz, ℓ are coplanar and harmonically conjugated, we say that rays ℓ_1 and ℓ_2 *obey the reflection law* at the point x of the quadric $\mathcal{Q}_1$ with respect to the confocal system which contains $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

If we introduce a coordinate system in which quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are confocal in the usual sense, reflection defined in this way is same as the standard one.

The next assertions are crucial for applications to the billiard dynamics.

Theorem 4 (One Reflection Theorem) *Suppose rays ℓ_1 and ℓ_2 obey the reflection law at x of $\mathcal{Q}_1$ with respect to the confocal system determined by quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Let ℓ_1 intersects $\mathcal{Q}_2$ at y_1' and y_1, u is a tangent plane to $\mathcal{Q}_1$ at x, and z its pole with respect to $\mathcal{Q}_2$. Then lines $y_1'z$ and y_1z respectively contain intersecting points y_2' and y_2 of ray ℓ_2 with $\mathcal{Q}_2$. Converse is also true.*

Theorem 5 (Double Reflection Theorem) *Suppose that $\mathcal{Q}_1$, $\mathcal{Q}_2$ are given quadrics and ℓ_1 line intersecting $\mathcal{Q}_1$ at the point x_1 and $\mathcal{Q}_2$ at y_1. Let u_1, v_1 be tangent planes to $\mathcal{Q}_1$, $\mathcal{Q}_2$ at points x_1, y_1 respectively, and z_1, w_1 their poles with respect to $\mathcal{Q}_2$ and $\mathcal{Q}_1$. Denote by x_2 second intersecting point of the line w_1x_1 with $\mathcal{Q}_1$, by y_2 intersection of y_1z_1 with $\mathcal{Q}_2$ and by ℓ_2, ℓ_1', ℓ_2' lines x_1y_2, y_1x_2, x_2y_2. Then pairs ℓ_1, ℓ_2; ℓ_1, ℓ_1'; ℓ_2, ℓ_2'; ℓ_1', ℓ_2' obey the reflection law at points x_1 (of $\mathcal{Q}_1$), y_1 (of $\mathcal{Q}_2$), y_2 (of $\mathcal{Q}_2$), x_2 (of $\mathcal{Q}_1$) respectively.*

Corollary 1 *If the line ℓ_1 is tangent to a quadric $\mathcal{Q}$ confocal with $\mathcal{Q}_1$ and $\mathcal{Q}_2$, then rays ℓ_2, ℓ_1', ℓ_2' also touch $\mathcal{Q}$.*

The famous Chasles theorem (see Arnol'd 1978) states that any line is tangent to exactly two quadrics from a given confocal family:

$$\mathcal{Q}_\lambda : \; Q_\lambda(x) = 1. \tag{7}$$

where we denote:

$$Q_\lambda(x) = \frac{x_1^2}{a_1 - \lambda} + \frac{x_2^2}{a_2 - \lambda} + \frac{x_3^2}{a_3 - \lambda}.$$

We assume that the family (7) is generic, i.e. the constants a_1, a_2, a_3 are all distinct.

Suppose a line ℓ is tangent to quadrics $\mathcal{Q}_{\alpha_1}$, $\mathcal{Q}_{\alpha_2}$ from the given confocal family. Denote by $\mathcal{A}_\ell$ the family of all lines which are tangent to the same two quadrics. The set $\mathcal{A}_\ell$ is invariant to the billiard reflection off any of the confocal quadrics, according to the previous Corollary.

Following Knörrer (1980), together with two affine confocal quadrics $\mathcal{Q}_{\alpha_1}$, $\mathcal{Q}_{\alpha_2}$, one can consider their projective closures $\mathcal{Q}^p_{\alpha_1}$, $\mathcal{Q}^p_{\alpha_2}$ and the intersection X of two quadrics in $\mathbb{C}P^5$:

$$x_1^2 + x_2^2 + x_3^2 - y_1^2 - y_2^2 = 0, \tag{8}$$
$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 - \alpha_1 y_1^2 - \alpha_2 y_2^2 = x_0^2. \tag{9}$$

Denote by $\mathcal{A}(X)$ the set of all 1-dimensional linear subspaces of X. For a given $L \in \mathcal{A}$, denote by $\mathcal{A}_L(X)$ the closure in $\mathcal{A}(X)$ of the set $\{ L' \in \mathcal{A} \mid \dim L \cap L' = 0 \}$. It was shown in the Reid's thesis, see Donagi (1980), Knörrer (1980), that $\mathcal{A}_L$ is a nonsingular hyperelliptic curve of genus 2.

The projection

$$\pi' \; : \; \mathbb{C}P^5 \backslash \{(x, y) | x = 0\} \to \mathbb{C}P^3, \quad \pi'(x, y) = x,$$

maps $L \in \mathcal{A}(X)$ to a subspace $\pi'(L) \subset \mathbb{CP}^3$ of the codimension 2. $\pi'(L)$ is tangent to the quadrics $\mathcal{Q}^{p*}_{\alpha_1}$, $\mathcal{Q}^{p*}_{\alpha_2}$ that are dual to $\mathcal{Q}^p_{\alpha_1}$, $\mathcal{Q}^p_{\alpha_2}$.

The projection π' commutes with involutions $\bar{j}_1$, $\bar{j}_2$ of $\mathbb{C}P^5$, where $\bar{j}_k$ flips the sign of y_k in $(x_0 : x_1 : x_2 : x_3 : y_1 : y_2)$. Denote by D the group generated by $\bar{j}_1$, $\bar{j}_2$ and by H its index two subgroup generated by $\bar{j}_1 \circ \bar{j}_2$.

Thus, the space dual to $\pi'(L)$, denoted by $\pi^*(L)$, is a line tangent to the quadrics $\mathcal{Q}^p_{\alpha_1}, \mathcal{Q}^p_{\alpha_2}$.

Recall the notion of the generalized Cayley's curve from Dragović and Radnović (2006, 2008).

Definition 3 The *generalized Cayley curve* $\mathcal{C}_\ell$ is the variety of hyperplanes tangent to quadrics of a given confocal family in $\mathbb{C}P^3$ at the points of a given line ℓ.

This curve is naturally embedded in the dual space $\mathbb{C}P^{3*}$.

Proposition 5 *The generalized Cayley curve in $\mathbb{C}P^3$, is a hyperelliptic curve of genus* 2. *Its natural realization in $\mathbb{C}P^{3*}$ is of degree* 5*:*

$$y^2 = (x - a_1)(x - a_2)(x - a_3)(x - \alpha_1)(x - \alpha_2).$$

The natural involution τ_ℓ on the generalized Cayley's curve $\mathcal{C}_\ell$ maps to each other the tangent planes at the points of intersection of ℓ with any quadric of the confocal family. Thus, the factor:

$$Y = \mathcal{C}_\ell/\tau_\ell, \tag{10}$$

is naturally seen as *the pencil of quadrics*; thus an isomorphism is defined $i_2 : Y \to \mathbb{C}P^1$ and each point in Y corresponds to a quadric of the pencil.

(It was observed in Dragović and Radnović (2006) that this curve is isomorphic to the Veselov-Moser isospectral curve.)

Now we are going to mention a connection obtained in Dragović and Radnović (2008), between generalized Cayley's curve defined above and the curves (see the previous Section) studied by Knörrer, Donagi, Reid. This connection traces out the relationship between billiard constructions and the algebraic structure of the corresponding Abelian varieties.

We can reinterpret the generalized Cayley's curve $\mathcal{C}_\ell$, which is a family of tangent hyperplanes, as a set of lines from $\mathcal{A}_\ell$ which intersect ℓ. Namely, for almost every tangent hyperplane there is a unique line ℓ', obtained from ℓ by the billiard reflection. Having this identification in mind, it is easy to prove the following

Corollary 2 *There is a birational morphism between the generalized Cayley's curve $\mathcal{C}_\ell$ and the Reid-Donagi-Knörrer's curve F_L, with $L = \pi^{*-1}(\ell)$, defined by*

$$j : \ell' \mapsto L', \quad L' = \pi^{*-1}(\ell'),$$

where ℓ' is a line obtained from ℓ by the billiard reflection off a confocal quadric.

Thus, there is a link between the dynamics of ellipsoidal billiards and algebraic structure of certain Abelian varieties.

Denote by K the Kummer variety of $\mathcal{A}(X)$. Denote by K_1 the Kummer variety of the Jacobian of the Cayley curve Jac$(\mathcal{C}_\ell$. Then, see Knörrer (1980), $K_1 = \mathcal{A}(X)/G = K/H$. Then the embedding $i : \mathcal{C}_\ell \to \mathrm{Jac}(\mathcal{C}_\ell)$ induces an embedding $\hat{i} : Y \to K_1$.

If we denote by $m : K \times K \to (K)^2$ the structure of algebraic two-valued group on the Kummer variety as defined above, then there is an induced two-valued group structure b on the Kummer variety K_1:

$$\begin{array}{ccc} K \times K & \xrightarrow{\quad m \quad} & (K)^2 \\ \Big\downarrow{\scriptstyle p\times p} & & \Big\downarrow{\scriptstyle (p)^2} \\ K_1 \times K_1 & \xrightarrow{\quad b \quad} & (K_1)^2, \end{array}$$

where p is the projection induced by the projection $\hat{p} : \mathcal{A}(X) \to \mathcal{A}(X)/H$.

If we specialize the above morphisms to Y defined by the Eq. (10), we get the following commutative diagram:

$$\begin{array}{ccccc} Y \times K_1 & \xrightarrow{\hat{i}\times id} & K_1 \times K_1 & \xrightarrow{m} & (K_1)^2 \\ \Big\downarrow{\scriptstyle i_2\times id} & & & & \Big\downarrow{\scriptstyle (id)^2} \\ \mathbb{C}P^1 \times K_1 & & \xrightarrow{\qquad \hat{b} \qquad} & & (K_1)^2. \end{array}$$

For a given $\lambda \in \mathbb{C}P^1$ and a line $\ell_1 \in K_1$, which is tangent to the both given confocal quadrics Q_{α_1} and Q_{α_2}, we have that

$$\hat{b}(\lambda, \ell_1) = (\ell_2, \ell_3) \in (K_1)^2.$$

Here ℓ_2, ℓ_3 are the two lines tangent to the both given confocal quadrics Q_{α_1} and Q_{α_2}, which are obtained from ℓ_1 after the billiard reflection off the quadric Q_λ.

8 Moduli of Semi-Stable Bundles and Two-Valued Group Structure on Kummer Varieties

As we have mentioned before, historically first examples of 2-valued groups appeared in topological context in the study of the characteristic classes of vector bundles, see Buchstaber and Novikov (1971). There one-dimensional symplectic bundles over $\mathbb{HP}^n$, and together with the canonical projection $\mathbb{C}P^{2n+1} \to \mathbb{H}P^n$, the associated two-dimensional complex vector bundles over $\mathbb{C}P^{2n+1}$ with $c_1 = 0$, were considered.

For a pair of such two-dimensional bundles,

$$\xi_1,\ \xi_2, \quad c_1(\xi_i) = 0,\ i = 1, 2$$

its tensor product

$$\xi_1 \otimes_C \xi_2$$

is a four-dimensional bundle, with the first two Potryagin classes $p_1(\xi_1 \otimes_C \xi_2)$ and $p_2(\xi_1 \otimes_C \xi_2)$. In Buchstaber and Novikov (1971), to the initial pair of bundles, a pair of *virtual* two-bundles, were associated according to the formula

$$Z^2 - p_1(\xi_1 \otimes_C \xi_2)Z + p_2(\xi_1 \otimes_C \xi_2) = 0.$$

The solutions $Z_{1,2}$ of the last quadratic equation play a role of the first Pontryagin classes of two virtual two-bundles. The last quadratic equation defines a two-valued group structure in $x = p_1(\xi_1)$ and $y = p_1(\xi_2)$, since $p_1(\xi_1 \otimes_C \xi_2) = \Theta_1(x, y)$ and $p_2(\xi_1 \otimes_C \xi_2) = \Theta_2(x, y)$, where Θ_i are certain series.

In this Section, we are going to construct the two-valued group structure on the Kummer variety, in the context of two-dimensional varieties, semi-stable in the sense of Mumford and Seshadri. The significance of the present situation, lies in the fact that obtained resulting two-bundles *are not virtual* - they are realized as a pair of two-bundles, semi-stable but not stable.

To get this, yet another interpretation of Kummer varieties, we are going back to Narasimhan and Ramanan (1969) Let X be a genus two curve. Following Mumford and Seshadri, the notions of stable and semi-stable vector bundles of rank n and degree d have been introduced respectively. For a holomorphic nonzero vector bundle W on X one introduces a rational-valued function $\mu(W) = \deg W/\mathrm{rank} W$. A vector bundle W is stable if for every proper subbundle V the condition

$$\mu(V) < \mu(W)$$

is satisfied. Similarly, a bundle is semi-stable if in the last inequality the sign $<$ is replaced by $\leq$. Any semi-stable bundle W has a strictly decreasing filtration

$$W = W_0 \supset W_1 \supset \cdots \supset W_n = \{0\}$$

such that W_{i-1}/W_i are stable and $\mu(W_{i-1}/W_i) = \mu(W)$. Denote by $Gr W = \oplus W_{i-1}/W_i$. As Seshadri defined, two semi-stable bundles W_1 and W_2 are S-equivalent if $Gr W_1 \approx Gr W_2$; a normal, projective (n^2+1)-dimensional variety of S-equivalence classes of semi-stable bundles of degree d and rank n denote $U(n, d)$.

Following Narasimhan and Ramanan (1969), for $U(2, 0)$, denote by S its three-dimensional sub-variety of bundles with trivial determinant. The non-stable bundles in S are of the form

$$j \oplus j^{-1},$$

with j a line bundle of degree 0. The Kummer surface K associated to the Jacobian of X is isomorphic to the set of all non-stable bundles in S.

Now, we define a structure of two-valued group on K. It is an important development of the Example 4 from Sect. 2. Denote $a, b \in K$, where

$$a = j \oplus j^{-1}, \quad b = l \oplus l^{-1},$$

where j, l are line bundles on X of degree 0. Then:

$$a \star_s b := (j \otimes l \oplus j^{-1} \otimes l^{-1}, j \otimes l^{-1} \oplus j^{-1} \otimes l). \tag{11}$$

Proposition 6 *The operation*

$$\star_s : K \times K \longrightarrow (K)^2$$

determined by the relation (11) defines a two-valued group structure on K.

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