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RESEARCH CONTRIBUTION

# Triangulated Endofunctors of the Derived Category of Coherent Sheaves Which Do Not Admit DG Liftings

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**Abstract**

In, Rizzardo and Van den Bergh (An example of a non-Fourier–Mukai functor between derived categories of coherent sheaves. arXiv:1410.4039, 2014) constructed an example of a triangulated functor between the derived categories of coherent sheaves on smooth projective varieties over a field $k$ of characteristic 0 which is not of the Fourier–Mukai type. The purpose of this note is to show that if char $k = p$ then there are very simple examples of such functors. Namely, for a smooth projective $Y$ over $\mathbb{Z}_p$ with the special fiber $i : X \hookrightarrow Y$, we consider the functor $Li^* \circ i_* : D^b(X) \to D^b(X)$ from the derived categories of coherent sheaves on $X$ to itself. We show that if $Y$ is a flag variety which is not isomorphic to $\mathbb{P}^1$ then $Li^* \circ i_*$ is not of the Fourier–Mukai type. Note that by a theorem of Toen (Invent Math 167:615–667, 2007, Theorem 8.15) the latter assertion is equivalent to saying that $Li^* \circ i_*$ does not admit a lifting to a $\mathbb{F}_p$-linear DG quasi-functor $D^b_{dg}(X) \to D^b_{dg}(X)$, where $D^b_{dg}(X)$ is a (unique) DG enhancement of $D^b(X)$. However, essentially by definition, $Li^* \circ i_*$ lifts to a $\mathbb{Z}_p$-linear DG quasi-functor.

Given smooth proper schemes $X_1, X_2$ over a field $k$ and an object $E \in D^b(X_1 \times_k X_2)$ of the bounded derived category of coherent sheaves on $X_1 \times_k X_2$ define a triangulated functor

$$\Phi_E : D^b(X_1) \to D^b(X_2) \tag{1}$$

sending a bounded complex $M$ of coherent sheaves on $X_1$ to $Rp_{2*}(E \overset{L}{\otimes} p_1^* M)$, where $p_i : X_1 \times_k X_2 \to X_i$ are the projections. Recall that a triangulated functor $D^b(X_1) \to D^b(X_2)$ is said to be of the Fourier–Mukai type if it is isomorphic to $\Phi_E$ for some $E$.

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Let $Y$ be a smooth projective scheme over $\operatorname{Spec} \mathbb{Z}_p$, and let $X$ be its special fiber, $i : X \hookrightarrow Y$ the closed embedding. Consider the triangulated functor $G : D^b(X) \to D^b(X)$ given by the formula

$$G = Li^* \circ i_*$$

We shall see that in general $G$ is not of the Fourier–Mukai type.

**Theorem** *Let $Z$ a smooth projective scheme over $\operatorname{Spec} \mathbb{Z}_p$, $Y = Z \times_{\mathbb{Z}_p} Z$, $X = Y \times_{\mathbb{Z}_p} \operatorname{Spec} \mathbb{F}_p$. Assume that*

*(1) The Frobenius morphism $Fr : \overline{Z} \to \overline{Z}$, where $\overline{Z} = Z \times_{\mathbb{Z}_p} \operatorname{Spec} \mathbb{F}_p$, does not lift modulo $p^2$.*
*(2) $H^1(X, T_X) = 0$, where $T_X$ is the tangent sheaf on $X$.*

*Then $G = Li^* \circ i_* : D^b(X) \to D^b(X)$ is not of the Fourier–Mukai type.*

For example, let $GL_n$ be the general linear group over $\operatorname{Spec} \mathbb{Z}_p$, $B \subset GL_n$ a Borel subgroup. Then, by Theorem 6 from Buch et al. (1997), for any $n > 2$, the flag variety $Z = GL_n/B$ satisfies the first assumption of the Theorem i.e., the Frobenius $Fr : \overline{Z} \to \overline{Z}$ does not lift on $Z \times_{\mathbb{Z}_p} \operatorname{Spec} \mathbb{Z}/p^2\mathbb{Z}$. By Kumar et al. (1999, Theorem 2), we have that $H^1(\overline{Z}, T_{\overline{Z}}) = H^1(\overline{Z}, \mathcal{O}_{\overline{Z}}) = 0$. It follows that $H^1(X, T_X) = 0$. Hence, by the Theorem, for $n > 2$, $G : D^b(X) \to D^b(X)$ is not of the Fourier–Mukai type.

***Proof*** Assume the contrary and let $E \in D^b(X \times_{\mathbb{F}_p} X)$ be a Fourier–Mukai kernel of $G$. By definition, for every $M \in D^b(X)$ we have a functorial isomorphism

$$G(M) \xrightarrow{\sim} Rp_{2*}(E \overset{L}{\otimes} p_1^* M). \tag{2}$$

By the projection formula (Hartshorne 1966, Chapter II, Prop. 5.6) we have that

$$i_* \circ Li^* \circ i_*(M) \xrightarrow{\sim} i_*(M) \overset{L}{\otimes} i_*(\mathcal{O}_X) \xrightarrow{\sim} i_*(M) \otimes (\mathcal{O}_Y \xrightarrow{p} \mathcal{O}_Y) \xrightarrow{\sim} i_*(M) \oplus i_*(M)[1]$$

In particular, if $M$ is a coherent sheaf then $\underline{H}^i(G(M)) \simeq M$ for $i = 0, -1$ and $\underline{H}^i(G(M)) = 0$ otherwise. Applying this observation and formula (2) to skyscraper sheaves, $M = \delta_x$, $x \in X(\overline{\mathbb{F}}_p)$, we conclude that the coherent sheaves $\underline{H}^i(E)$ are set theoretically supported on the diagonal $\Delta_X \subset X \times_{\mathbb{F}_p} X$. Applying the same formulas to $M = \mathcal{O}_X$ we see that $p_{2*}(\underline{H}^i(E)) = \mathcal{O}_X$ for $i = 0, -1$ and $p_{2*}(\underline{H}^i(E)) = 0$ otherwise. In fact, every coherent sheaf $F$ on $X \times_{\mathbb{F}_p} X$ which is set theoretically supported on the diagonal and such that $p_{2*}F = \mathcal{O}_X$ is isomorphic to $\mathcal{O}_{\Delta_X}$. It follows that $\underline{H}^0(E) = \underline{H}^{-1}(E) = \mathcal{O}_{\Delta_X}$. In the other words, $E$ fits into an exact triangle in $D^b(X \times X)$

$$\mathcal{O}_{\Delta_X}[1] \xrightarrow{\alpha} E \longrightarrow \mathcal{O}_{\Delta_X} \xrightarrow{\beta} \mathcal{O}_{\Delta_X}[2] \tag{3}$$

for some $\beta \in Ext^2_{\mathcal{O}_{X\times_{\mathbb{F}_p}X}}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X})$. We wish to show that the second assumption in the Theorem implies that $\beta = 0$, while the first one implies that $\beta \neq 0$. For every $M \in D^b(X)$, (3) gives rise to an exact triangle

$$M[1] \xrightarrow{\alpha_M} G(M) \longrightarrow M \xrightarrow{\beta_M} M[2] \tag{4}$$

Our main tool is the following result. □

**Lemma** *For a coherent sheaf $M$ on $X$ the following conditions are equivalent.*

(1) $\beta_M = 0$.
(2) $G(M) \xrightarrow{\sim} M \oplus M[1]$.
(3) *There exists a morphism $\lambda : G(M) \to M[1]$ such that $\lambda \circ \alpha_M$ is an isomorphism.*
(4) *$M$ admits a lift modulo $p^2$ i.e., there is a coherent sheaf $\tilde{M}$ on $Y$ flat over $\mathbb{Z}/p^2\mathbb{Z}$ such that $i^*\tilde{M} \simeq M$.*

***Proof*** The equivalence of (1), (2) and (3) is immediate. Let us check that (3) is equivalent to (4). By adjunction a morphism $\lambda : G(M) \to M[1]$ gives rise to a morphism $\gamma : i_*M \to i_*M[1]$. Note that $\tilde{M} := (\mathrm{cone}\,\gamma)[-1]$ is a coherent sheaf on $Y$ which is an extension of $i_*M$ by itself:

$$0 \longrightarrow i_*M \xrightarrow{v} \tilde{M} \xrightarrow{u} i_*M \longrightarrow 0. \tag{5}$$

It suffices to prove that $\lambda \circ \alpha_M : M[1] \to M[1]$ is an isomorphism if and only if $\tilde{M}$ is flat over $\mathbb{Z}/p^2\mathbb{Z}$.

The exact sequence (5) gives rise to an exact triangle

$$Li^*i_*M \to Li^*\tilde{M} \to Li^*i_*M \to Li^*i_*M[1].$$

This, in turn, yields a long exact sequence of the cohomology sheaves

$$0 = L_2i^*i_*M \to L_1i^*i_*M \xrightarrow{L_1i^*(v)} L_1i^*\tilde{M} \to M \xrightarrow{\lambda\circ\alpha_M[-1]} M \to i^*\tilde{M} \xrightarrow{L_1i^*(u)} M \to 0.$$

It follows that the morphism $\lambda \circ \alpha_M$ is an isomorphism if and only if the morphisms $v$ and $u$ from exact sequence (5) induce isomorphisms $i_*M \xrightarrow{\sim} \mathrm{Ker}(\tilde{M} \xrightarrow{p} \tilde{M})$, $\mathrm{Coker}(\tilde{M} \xrightarrow{p} \tilde{M}) \xrightarrow{\sim} i_*M$. The latter condition is equivalent to the flatness of $\tilde{M}$ over $\mathbb{Z}/p^2\mathbb{Z}$. □

We have a spectral sequence converging to $Ext^*_{\mathcal{O}_{X\times_{\mathbb{F}_p}X}}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X})$ whose second page is $H^*(X, \mathcal{E}xt^*_{\mathcal{O}_{X\times_{\mathbb{F}_p}X}}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}))$. In particular, we have a homomorphism

$$Ext^2_{\mathcal{O}_{X\times_{\mathbb{F}_p}X}}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}) \to H^0(X, \mathcal{E}xt^2_{\mathcal{O}_{X\times_{\mathbb{F}_p}X}}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X})) \xrightarrow{\sim} H^0(X, \wedge^2 T_X).$$

Let us check that the image $\mu$ of $\beta$ under this map is 0. To do this we apply the Lemma to skyscraper sheaves $\delta_x$, where $x$ runs over closed points of $X$. On the one hand, the evaluation of the bivector field $\mu$ at $x$ is equal to the class of $\beta_{\delta_x}$ in $Ext^2_{\mathcal{O}_X}(\delta_x, \delta_x) \xrightarrow{\sim} \wedge^2 T_{x,X}$. On the other hand, by the Lemma, $\beta_{\delta_x} = 0$ since $\delta_x$ is liftable modulo $p^2$. Next, the assumption that $H^1(X, T_X) = 0$ implies that $\beta$ lies in the image of the map

$$v : H^2(X, \mathcal{O}_X) \xrightarrow{\sim} H^2(X, \mathcal{E}xt^0_{\mathcal{O}_{X\times_{\mathbb{F}_p} X}}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X})) \to Ext^2_{\mathcal{O}_{X\times_{\mathbb{F}_p} X}}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}). \tag{6}$$

The map (6) has a left inverse $u : Ext^2_{\mathcal{O}_{X\times_{\mathbb{F}_p} X}}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}) \to H^2(X, \mathcal{O}_X)$ which takes $\beta$ to $\beta_{\mathcal{O}_X}$. But, by the Lemma, the later class is equal to 0 since $\mathcal{O}_X$ is liftable modulo $p^2$. It follows that $\beta$ is 0.

On the other hand, let $\Gamma \subset X = \overline{Z} \times_{\mathbb{F}_p} \overline{Z}$ be the graph of the Frobenius morphism $Fr : \overline{Z} \to \overline{Z}$ and $\mathcal{O}_\Gamma$ the structure sheaf of $\Gamma$ viewed as a coherent sheaf on $X$. Then, by our first assumption, the sheaf $\mathcal{O}_\Gamma$ is not liftable modulo $p^2$. Hence, by the Lemma, $\beta_{\mathcal{O}_\Gamma}$ is not 0. This contradiction completes the proof. □

***Remark*** Let $X$ be a smooth proper scheme over $\mathbb{F}_p$. The bounded derived category $D^b(X)$ of coherent sheaves on $X$ has a natural dg enhencement $L_{parf}(X)$ which is a dg category over $\mathbb{F}_p$ whose homotopy category $\mathrm{Ho}(L_{parf}(X))$ is equivalent to $D^b(X)$ [see, for example, (Toën 2007, §8.3)]. One has a functor

$$\mathrm{Ho}(R\underline{\mathrm{End}}_{\mathbb{F}_p}(L_{parf}(X))) \to \mathrm{End}(D^b(X)) \tag{7}$$

from the homotopy category of $\mathbb{F}_p$-linear dg quasi-endofunctors of $L_{parf}(X)$ to the category of triangulated endofunctors of $D^b(X)$. According to (Toën 2007, Theorem 8.15) the dg category $R\underline{\mathrm{Hom}}_{\mathbb{F}_p}(L_{parf}(X), L_{parf}(X))$ is homotopy equivalent to the dg category $L_{parf}(X \times_{\mathbb{F}_p} X)$, so that the essential image of (7) consists of triangulated endofunctors of the Fourier–Mukai type. On the other hand, any dg category over $\mathbb{F}_p$ can be viewed as a dg category over $\mathbb{Z}_p$. In particular, one can consider the dg category $R\underline{\mathrm{Hom}}_{\mathbb{Z}_p}(L_{parf}(X), L_{parf}(X))$ of $\mathbb{Z}_p$-linear dg quasi-endofunctors of $L_{parf}(X)$. Functor (7) factors as follows.

$$\mathrm{Ho}(R\underline{\mathrm{End}}_{\mathbb{F}_p}(L_{parf}(X))) \to \mathrm{Ho}(R\underline{\mathrm{End}}_{\mathbb{Z}_p}(L_{parf}(X))) \to \mathrm{End}(D^b(X)). \tag{8}$$

Given a flat lifting $Y$ of $X$ over $\mathbb{Z}_p$ one can view the functor $Li^*i_*$ as an object of the category $\mathrm{Ho}(R\underline{\mathrm{End}}_{\mathbb{Z}_p}(L_{parf}(X)))$. The construction from this paper is inspired by the simple observation that for any $X$ and $Y$ (for example, one can take $X = \mathrm{Spec}\,\mathbb{F}_p$, $Y = \mathrm{Spec}\,\mathbb{Z}_p$) the $\mathbb{Z}_p$-linear dg quasi-functor $Li^*i_*$ is not in the image of $\mathrm{Ho}(R\underline{\mathrm{End}}_{\mathbb{F}_p}(L_{parf}(X)))$.

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