



# Invariants of Graph Drawings in the Plane

A. Skopenkov<sup>1</sup>

Received: 9 July 2018 / Revised: 27 October 2019 / Accepted: 8 November 2019 /

Published online: 13 February 2020

© Institute for Mathematical Sciences (IMS), Stony Brook University, NY 2020

## Abstract

We present a simplified exposition of some classical and modern results on graph drawings in the plane. These results are chosen so that they illustrate some spectacular recent higher-dimensional results on the border of geometry, combinatorics and topology. We define a  $\mathbb{Z}_2$  valued *self-intersection invariant* (i.e. the van Kampen number) and its generalizations. We present elementary formulations and arguments accessible to mathematicians not specialized in any of the areas discussed. So most part of this survey could be studied before textbooks on algebraic topology, as an introduction to starting ideas of algebraic topology motivated by algorithmic, combinatorial and geometric problems.

## Contents

Introduction	22
1 Planarity of Graphs	25
1.1 Linear Realizations of Graphs	26
1.2 Algorithmic Results on Graph Planarity	27
1.3 Intersection Number for Polygonal Lines in the Plane	28
1.4 Self-Intersection Invariant for Graph Drawings	29
1.5 A Polynomial Algorithm for Recognizing Graph Planarity	30
1.5.1 Van Kampen–Hanani–Tutte Planarity Criterion	30
1.5.2 Intersection Cocycle	32
1.5.3 Intersection Cocycles of Different Maps	33
1.5.4 Intersections with Signs	34
2 Multiple Intersections in Combinatorial Geometry	35
2.1 Radon and Tverberg Theorems in the Plane	35
2.2 Topological Radon Theorem in the Plane	36
2.3 Topological Tverberg Theorem in the Plane	37
2.3.1 Statement	37

---

Supported in part by the Russian Foundation for Basic Research Grant No. 19-01-00169 and by Simons-IUM Fellowship.

---

✉ A. Skopenkov  
skopenko@mccme.ru  
<https://users.mccme.ru/skopenko/>

<sup>1</sup> Moscow Institute of Physics and Technology, Independent University of Moscow, Moscow, Russia

2.3.2 Ideas of Proofs . . . . .	39
2.3.3 Triple Self-Intersection Invariant for Graph Drawings . . . . .	40
2.3.4 An Approach Via Borsuk–Ulam Theorem . . . . .	41
2.4 Mapping Hypergraphs in the Plane and the Özaydin Theorem . . . . .	43
2.4.1 A Polynomial Algorithm for Recognizing Hypergraph Planarity . . . . .	43
2.4.2 Intersections with Signs for 2-Hypergraphs . . . . .	46
2.4.3 Elementary Formulation of the Özaydin Theorem . . . . .	47
3 Conclusion: Higher-Dimensional Generalizations . . . . .	48
3.1 Radon, Tverberg and van Kampen–Flores Theorems . . . . .	48
3.2 Realizability of Higher-Dimensional Hypergraphs . . . . .	50
3.3 Algorithmic Recognition of Almost Realizability of Hypergraphs . . . . .	52
References . . . . .	54

## Introduction

**Why this survey could be interesting.** In this survey we present a simplified exposition of some classical and modern results on graph drawings in the plane (Sects. 1, 2). These results are chosen so that they illustrate some spectacular recent higher-dimensional results on the border of geometry, combinatorics and topology (Sect. 3).

We exhibit a connection between non-planarity of the complete graph  $K_5$  on five vertices and results on intersections in the plane of *algebraic interiors* of curves (namely, the Topological Radon–Tverberg Theorems in the plane 2.2.2, 2.3.2). Recent resolution of the Topological Tverberg Conjecture 3.1.7 on multiple intersections for maps from simplex to Euclidean space used a higher-dimensional  $r$ -fold generalization of this connection (i.e. a connection between the  $r$ -fold van Kampen–Flores Conjecture 3.1.8 and Conjecture 3.1.7).

Recall that invariants of knots were initially defined using *presentations of the fundamental group* at the beginning of the twentieth century and, even in a less elementary way, at the end of the twentieth century. An elementary description of knot invariants via plane diagrams (initiated in J. Conway’s work of the second half of the twentieth century) increased interest in knot theory and made that part of topology a part of graph theory as well.

Analogously, we present elementary formulations and arguments that do not involve configuration spaces and cohomological obstructions. Nevertheless, the main contents of this survey is an *introduction to starting ideas of algebraic topology* (more precisely, to configuration spaces and cohomological obstructions) *motivated by algorithmic, combinatorial and geometric problems*. We believe that describing simple applications of topological methods in elementary language makes these methods more accessible (although this is called ‘detopologization’ in [Matoušek et al. 2012, Sect. 1]). Such an introduction is independent of textbooks on algebraic topology (if a reader is ready to accept without proof some results from Sect. 2.3.4). For textbooks written in the spirit of this article see e.g. [Skopenkov 2019, Skopenkov 2020].

More precisely, it is fruitful to invent or to interpret homotopy-theoretical arguments in terms of invariants defined via intersections or preimages.<sup>1</sup> In this survey we describe

<sup>1</sup> Examples are definition of the mapping degree [Matoušek 2008, Sect. 2.4], [Skopenkov 2020, Sect. 8] and definition of the Hopf invariant via linking, i.e., via intersection [Skopenkov 2020, Sect. 8]. Importantly, ‘secondary’ not only ‘primary’ invariants allow interpretations in terms of *framed* intersections; for a recent application see [Skopenkov 2017a].

in terms of double and multiple intersection numbers those arguments that are often exposed in a less elementary language of homotopy theory.

No knowledge of algebraic topology is required here. Important ideas are introduced in non-technical particular cases and then generalized.<sup>2</sup> So this survey is accessible to mathematicians not specialized in the area.

**Contents of this survey.** Both Sects. 1 and 2 bring the reader to the frontline of research.

In Sect. 1 we present a polynomial algorithm for recognizing graph planarity (Sect. 1.5), together with all the necessary definitions, some motivations and preliminary results (Sects. 1.1–1.4). This algorithm, the corresponding planarity criterion (Proposition 1.5.1) and an elementary proof of the non-planarity of  $K_5$  (Sect. 1.4) are interesting because they can be generalized to higher dimensions and higher multiplicity of intersections (Theorems 3.1.2, 3.1.6, 3.2.1, 3.2.3, 3.3.3 and 3.3.4, see also Conjectures 3.1.4 and 3.1.8).

In Sect. 2 we introduce in an elementary way results on multiple intersections in the plane of *algebraic interiors* of curves (namely, the topological Radon–Tverberg theorems 2.2.2, 2.3.2, and the topological Tverberg conjecture in the plane 2.3.3). An elementary generalization of the ideas from Sects. 1.4, 2.2, [Skopenkov and Tancer 2017, Lemmas 6 and 7] could give an elementary proof of the topological Tverberg theorem, and of its ‘quantitative’ version, at least for primes (Sect. 2.3.3). This is interesting in particular because the topological Tverberg conjecture in the plane 2.3.3 is still open. We also give an elementary formulation of the Özaydin Theorem in the plane 2.4.10 on cohomological obstructions for multiple intersections of algebraic interiors of curves. This formulation can perhaps be applied to obtain an elementary proof.

In Sect. 3 we indicate how elementary results of Sects. 1 and 2 illustrate some spectacular recent higher-dimensional results. Detailed description of those recent results is outside purposes of this survey. In Sect. 3.1 we state classical and modern results and conjectures on complete hypergraphs (since the results only concern *complete* hypergraphs, we present elementary statements not involving hypergraphs). These results generalize non-planarity of  $K_5$  (Proposition 1.1.1.a and Theorem 1.4.1) and the results on intersections of algebraic interiors of curves (linear and topological Radon and Tverberg theorems in the plane 2.1.1, 2.1.5, 2.2.2, 2.3.2). In Sect. 3.2 we state modern algorithmic results on realizability of arbitrary hypergraphs; they generalize Proposition 1.2.2.b. In Sect. 3.3 we do the same for *almost realizability*. This notion is defined there but implicitly appeared in Sects. 1.4, 2. We introduce Özaydin Theorem 3.3.6, which is a higher-dimensional version of the above-mentioned Özaydin Theorem in the plane 2.4.10, and which is an important ingredient in recent resolution of the topological Tverberg conjecture 3.1.7.

The main notion of this survey linking together Sects. 1 and 2 is a  $\mathbb{Z}_2$  valued ‘self-intersection’ invariant (i.e. the van Kampen and the Radon numbers defined in Sects. 1.4, 2.2). Its generalizations to  $\mathbb{Z}_r$  valued invariants and to cohomological obstructions are defined and used to obtain elementary formulations and proofs of

---

<sup>2</sup> The ‘minimal generality’ principle (to introduce important ideas in non-technical particular cases) was put forward by classical figures in mathematics and mathematical exposition, in particular by V. Arnold. Cf. ‘detopologization’ tradition described in [Matoušek et al. 2012, Historical notes in Sect. 1].

Sects. 1, 2 mentioned above. For applications of other generalizations see [Skopenkov 2018a, Sect. 4], [Skopenkov 2018b, Skopenkov and Tancer 2017]. For invariants of plane curves and caustics see [Arnold 1995] and the references therein.

**Remark 0.0.1** (generalizations in five different directions) The main results exposed in this survey can be obtained from the easiest of them (Linear van Kampen–Flores and Radon Theorems for the plane 1.1.1.a, 2.1.1) by generalizations in five different directions. Thus the results can naturally be numbered by a vector of length five.

First, a result can give intersection of simplices of some dimensions, or of the same dimension. This relates Sects. 1 to 2.

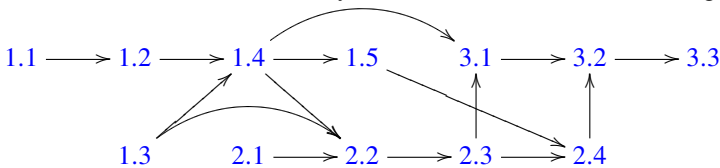
Second, a ‘qualitative’ result on the existence of intersection can be generalized to a ‘quantitative’ result on the algebraic number of intersections. (This relates Propositions 1.1.1.a to 1.1.1.b, Theorem 2.3.2 to Problem 2.3.7, etc.)

Third, a linear result can be generalized to a topological result, which is here equivalent to a piecewise linear result. (This relates Proposition 1.1.1.ab to Theorem 1.4.1 and Lemma 1.4.3, etc.)

Fourth, a result on double intersection can be generalized to multiple intersections. (This relates Proposition 1.1.1.a and Theorem 1.4.1 to Theorems 2.1.5 and 2.3.2, etc.; note that the  $r$ -tuple intersection version might not hold for  $r$  not a power of a prime.)

Fifth, a result in the plane can be generalized to higher dimensions. This relates Sects. 1 and 2 to 3.

**Structure of this survey.** Subsections of this survey can be read independently of each other, and so in any order. In one subsection we indicate relations to other subsections, but these indications can be ignored. If in one subsection we use a definition or a result from the other, then we only use a specific shortly stated definition or result. However, we recommend to read subsections in any order consistent with the following diagram.



Main statements are called theorems, important statements are lemmas or propositions, less important statements which are not referred to outside a subsection are assertions.

**Historical notes.** All the results of this survey are well-known.

For history, more motivation, more proofs, related problems and generalizations see surveys [Bárány et al. 2016, Ziegler 2011, Skopenkov 2018a, Blagojevič and Ziegler 2016, Shlosman 2018] (to Sects. 2 and 3.1) and [Skopenkov 2008, Skopenkov 2014], [Matoušek et al. 2011, Sect. 1], [Skopenkov 2019, Sect. 5 ‘Realizability of hypergraphs’] (to Sects. 1 and 3.2). Discussion of those related problems and generalizations is outside purposes of this survey.

Exposition of the polynomial algorithm for recognizing graph planarity (Sect. 1.5) is new. First, we give an elementary statement of the corresponding planarity criterion (Proposition 1.5.1). Second, we do not require knowledge of cohomology theory but

show how some notions of that theory naturally appear in studies planarity of graphs. Cf. [Fokkink 2004], [Matoušek et al. 2011, Appendix D].

Elementary formulation of the topological Radon theorem (Sect. 2.2) in the spirit of [Schöneborn 2004, Schöneborn and Ziegler 2005] is presumably folklore. The proof follows the idea of L. Lovasz and A. Schrijver [Lovasz and Schrijver 1998]. Elementary formulation of the topological Tverberg theorem and conjecture in the plane (Sect. 2.3.1) is due to T. Schöneborn and G. Ziegler [Schöneborn 2004, Schöneborn and Ziegler 2005]. An idea of an elementary proof of that result (Sect. 2.3.3) and elementary formulation of M. Özaydin's results (Sect. 2.4) are apparently new.

The paper [Enne et al. 2019] was used in preparation of the first version of this paper; most part of the first version of Sect. 2 is written jointly with A. Ryabichev. I am grateful to P. Blagojević, I. Bogdanov, G. Chelnokov, A. Enne, R. Fulek, R. Karasev, Yu. Makarychev, A. Ryabichev, M. Tancer, T. Zaitsev, R. Živaljević and anonymous referees for useful discussions.

**Conventions.** Unless the opposite is indicated, by  $k$  points in the plane we mean a  $k$ -element subset of the plane; so these  $k$  points are assumed to be pairwise distinct. We often denote points by numbers not by letters with subscript numbers. Denote  $[n] := \{1, 2, \dots, n\}$ .

## 1 Planarity of Graphs

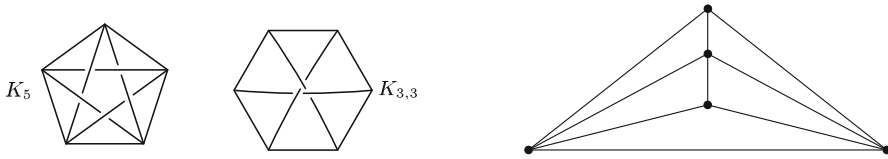
A (finite) **graph**  $(V, E)$  is a finite set  $V$  together with a collection  $E \subset \binom{V}{2}$  of two-element subsets of  $V$  (i.e. of non-ordered pairs of elements).<sup>3</sup> The elements of this finite set  $V$  are called *vertices*. Unless otherwise indicated, we assume that  $V = \{1, 2, \dots, |V|\}$ . The pairs of vertices from  $E$  are called *edges*. The edge joining vertices  $i$  and  $j$  is denoted by  $ij$  (not by  $(i, j)$  to avoid confusion with ordered pairs).

A *complete graph*  $K_n$  on  $n$  vertices is a graph in which every pair of vertices is connected by an edge, i.e.,  $E = \binom{V}{2}$ . A *complete bipartite graph*  $K_{m,n}$  is a graph whose vertices can be partitioned into two subsets of  $m$  elements and of  $n$  elements, so that

- every two vertices from different subsets are joined by an edge, and
- every edge connects vertices from different subsets.

In Sects. 1.1 and 1.2 we present two formalizations of realizability of graphs in the plane: the linear realizability and the planarity (i.e. piecewise linear realizability). The formalizations turn out to be equivalent by Fáry Theorem 1.2.1; their higher-dimensional generalizations are not equivalent, see [van Kampen 1941], [Matoušek et al. 2011, Sect. 2]. Both formalizations are important. These formalizations are presented independently of each other, so Sect. 1.1 is essentially not used below (except for Proposition 1.1.1.b making the proof of Lemma 1.4.3 easier, and footnote 7, which are trivial and not important). However, before more complicated study of planarity it could be helpful to study linear realizability. The tradition of studying both linear and piecewise linear problems is also important for Sect. 2, see Remark 0.0.1.

<sup>3</sup> The common term for this notion is a *graph without loops and multiple edges* or a *simple graph*.



**Fig. 1** (Left) Nonplanar graphs  $K_5$  and  $K_{3,3}$ . (Right) A planar drawing of  $K_5$  without one of the edges

## 1.1 Linear Realizations of Graphs

### Proposition 1.1.1<sup>4</sup>

- (a) (cf. Theorems 1.4.1 and 2.1.1) *From any 5 points in the plane one can choose two disjoint pairs such that the segment with the ends at the first pair intersects the segment with the ends at the second pair.*
- (b) (cf. Proposition 2.1.2 and Lemma 1.4.3) *If no 3 of 5 points in the plane belong to a line, then the number of intersection points of interiors of segments joining the 5 points is odd.*

Proposition 1.1.1 is easily proved by analyzing the *convex hull* of the points. See another proof in [Skopenkov 2018c, Sect. 1.6].

**Proposition 1.1.2** *Suppose<sup>5</sup> that no 3 of 5 points 1, 2, 3, 4, 5 in the plane belong to a line. If the segments*

- (a)  *$jk$ ,  $1 \leq j < k \leq 5$ ,  $(j, k) \neq (1, 2)$ , have disjoint interiors then the points 1 and 2 lie on different sides of the triangle 345, cf. Fig. 1, right;*
- (b)  *$jk$ ,  $1 \leq j < k \leq 5$ ,  $(j, k) \notin \{(1, 2), (1, 3)\}$ , have disjoint interiors then EITHER the points 1 and 2 lie on different sides of the triangle 345, OR the points 1 and 3 lie on different sides of the triangle 245.*
- (c)  *$jk$ ,  $1 \leq j < k \leq 5$ ,  $(j, k) \notin \{(1, 2), (1, 3), (1, 4)\}$ , have disjoint interiors then EITHER the points 1 and 2 lie on different sides of the triangle 345, OR the points 1 and 3 lie on different sides of the triangle 245, OR the points 1 and 4 lie on different sides of the triangle 235.*

Informally speaking, a graph is *linearly realizable* in the plane if the graph has a planar drawing without self-intersection and such that every edge is drawn as a line segment. Formally, a graph  $(V, E)$  is called **linearly realizable** in the plane if there exists  $|V|$  points in the plane corresponding to the vertices so that no segment joining a pair (of points) from  $E$  intersects the interior of any other such segment.<sup>6</sup>

The following results are classical:

<sup>4</sup> These are ‘linear’ versions of the nonplanarity of the graphs  $K_5$  and  $K_{3,3}$ . But they can be proved easier (because the Parity Lemma 1.3.2.b and [Skopenkov 2020, Intersection Lemma 1.4.4] are not required for the proof).

<sup>5</sup> See proof in [Skopenkov 2018c, Sect. 1.6]. Proposition 1.1.2 and [Skopenkov 2018c, 1.6.1] are not formally used in this paper. However, they illustrate by two-dimensional examples how boolean functions appear in the study of embeddings. This is one of the ideas behind recent higher-dimensional  $NP$ -hardness Theorem 3.2.3.b.

<sup>6</sup> We do not require that ‘no isolated vertex lies on any of the segments’ because this property can always be achieved.

- $K_4$  and  $K_5$  without one of the edges are linearly realizable in the plane (Fig. 1, right).
- neither  $K_5$  nor  $K_{3,3}$  is linearly realizable in the plane (Proposition 1.1.1.ac);
- every graph is linearly realizable in 3-space (linear realizability in 3-space is defined analogously to the plane).

A criterion for linear realizability of graphs in the plane follows from the Fáry Theorem 1.2.1 below and any planarity criterion (e.g. Kuratowski Theorem 1.2.3 below).

**Proposition 1.1.3** ([Tamassia 2019, Chapters 1 and 6]; cf. Sect. 3.2; see comments in [Skopenkov 2018c, Sect. 1.6]) *There is an algorithm for recognizing the linear realizability of graphs in the plane.*<sup>7</sup>

By the Fáry Theorem 1.2.1 and Proposition 1.2.2.bc polynomial and even linear algorithms exist.

## 1.2 Algorithmic Results on Graph Planarity

Informally speaking, a graph is planar if it can be drawn ‘without self-intersections’ in the plane. Formally, a graph  $(V, E)$  is called **planar** (or piecewise-linearly realizable in the plane) if in the plane there exist

- a set of  $|V|$  points corresponding to the vertices, and
- a set of non-self-intersecting polygonal lines joining pairs (of points) from  $E$

such that no of the polygonal lines intersects the interior of any other polygonal line.<sup>8</sup>

For example, the graphs  $K_5$  and  $K_{3,3}$  (Fig. 1) are not planar by Theorem 1.4.1 and its analogue for  $K_{3,3}$  [Skopenkov 2018c, Remark 1.4.4].

The following theorem shows that any planar graph can be drawn without self-intersections in the plane so that every edge is drawn as a segment.

**Theorem 1.2.1** (Fáry) *If a graph is planar (i.e. piecewise-linearly realizable in the plane), then it is linearly realizable in the plane.*

**Proposition 1.2.2** (a) *There is an algorithm for recognizing graph planarity.*

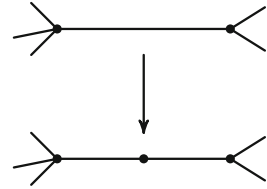
(b) (cf. Theorems 2.4.1 and 3.2.1.b) *There is an algorithm for recognizing graph planarity, which is polynomial in the number of vertices  $n$  in the graph (i.e. there are numbers  $C$  and  $k$  such that for each graph the number of steps in the algorithm does not exceed  $Cn^k$ ).*<sup>9</sup>

<sup>7</sup> Rigorous definition of the notion of algorithm is complicated, so we do not give it here. Intuitive understanding of algorithms is sufficient to read this text. To be more precise, the above statement means that there is an algorithm for calculating the function from the set of all graphs to  $\{0, 1\}$ , which maps graph to 1 if the graph is linearly realizable in the plane, and to 0 otherwise. All other statements on algorithms in this paper can be formalized analogously.

<sup>8</sup> Then any two of the polygonal lines either are disjoint or intersect by a common end vertex. We do not require that ‘no isolated vertex lies on any of the polygonal lines’ because this property can always be achieved. See an equivalent definition of planarity in the beginning of Sect. 1.4.

<sup>9</sup> Since for a planar graph with  $n$  vertices and  $e$  edges we have  $e \leq 3n - 6$  and since there are planar graphs with  $n$  vertices and  $e$  edges such that  $e = 3n - 6$ , the ‘complexity’ in the number of edges is ‘the same’ as the ‘complexity’ in the number of vertices.

Fig. 2 Subdivision of edge



- (c) *There is an algorithm for recognizing graph planarity, which is linear in the number of vertices  $n$  in the graph (linearity is defined as polynomiality with  $k = 1$ ).*

Part (a) follows from Proposition 1.1.3 and the Fáry Theorem 1.2.1. Part (a) can also be proved using Kuratowski Theorem 1.2.3 below (see for details [Tamassia 2019, Chapters 1 and 6]) or considering *thickenings* [Skopenkov 2020, Sect. 1]. However, the corresponding algorithms are slow, i.e. have more than  $2^n$  steps, if the graph has  $n$  vertices ('exponential complexity'). So other ways of recognizing planarity are interesting.

Part (b) is deduced from equivalence of planarity and solvability of certain system of linear equations with coefficients in  $\mathbb{Z}_2$  (see (i)  $\Leftrightarrow$  (iii) of Proposition 1.5.1 below). The deduction follows because there is a polynomial in  $N$  algorithm for recognizing the solvability of a system of  $\leq N$  linear equations with coefficients in  $\mathbb{Z}_2$  and with  $\leq N$  variables (this algorithm is constructed using *Gauss elimination of variables algorithm*).

Part (c) is proved in [Hopcroft and Tarjan 1974], see a short proof in [Boyer and Myrvold 2004]. The algorithm does not generalize to higher dimensions (as opposed to the algorithm of (b)).

The *subdivision of edge* operation for a graph is shown in Fig. 2. Two graphs are called *homeomorphic* if one can be obtained from the other by subdivisions of edges and inverse operations. This is equivalent to the existence of a graph that can be obtained from each of these graphs by subdivisions of edges. Some motivations for this definition are given in [Skopenkov 2020, Sect. 5.3].

**Theorem 1.2.3** (Kuratowski) *A graph is planar if and only if it has no subgraphs homeomorphic to  $K_5$  or  $K_{3,3}$  (Fig. 1).*

A simple proof of this theorem can be found e.g. in [Chernov et al. 2016, Sect. 2.9].

### 1.3 Intersection Number for Polygonal Lines in the Plane

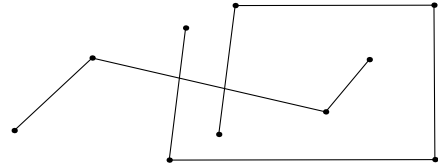
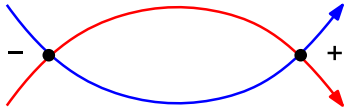
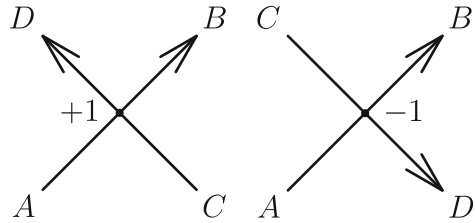
Before reading this section a reader might want to look at [Skopenkov 2018c, Assertion 1.3.4] and applications from [Skopenkov 2020, Sect. 1.4]. Comments and proofs are also presented in [Skopenkov 2018c, Sect. 1.3].

Some points in the plane **are in general position**, if no three of them lie in a line and no three segments joining them have a common interior point.

**Proposition 1.3.1** *Any two polygonal lines in the plane whose vertices are in general position intersect at a finite number of points.*



**Fig. 3** The sign of intersection point



**Fig. 4** Two curves intersecting at an even number of points the sum of whose signs is zero (left) or non-zero (right)

**Lemma 1.3.2** (Parity)

- (a) *If 6 vertices of two triangles in the plane are in general position, then the boundaries of the triangles intersect at an even number of points.*
- (b) *Any two closed polygonal lines in the plane whose vertices are in general position intersect at an even number of points.*

Let  $A, B, C, D$  be points in the plane, of which no three belong to a line. Define **the sign** of intersection point of oriented segments  $\overrightarrow{AB}$  and  $\overrightarrow{CD}$  as the number  $+1$  if  $ABC$  is oriented clockwise and the number  $-1$  otherwise (Figs. 3 and 4).

The following lemma is proved analogously to the Parity Lemma 1.3.2.

**Lemma 1.3.3** (Triviality) *For any two closed polygonal lines in the plane whose vertices are in general position the sum of signs of their intersection points is zero.*

**1.4 Self-Intersection Invariant for Graph Drawings**

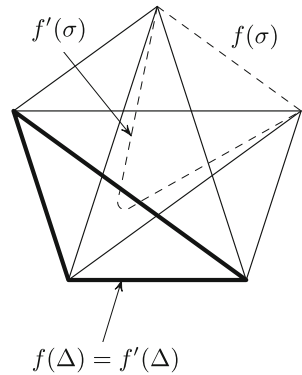
We shall consider plane drawings of a graph such that the edges are drawn as polygonal lines and intersections are allowed. Let us formalize this for graph  $K_n$  (formalization for arbitrary graphs is presented at the beginning of Sect. 1.5.2).

A **piecewise-linear (PL) map**  $f : K_n \rightarrow \mathbb{R}^2$  of the graph  $K_n$  to the plane is a collection of  $\binom{n}{2}$  (non-closed) polygonal lines pairwise joining some  $n$  points in the plane. **The image**  $f(\sigma)$  **of edge**  $\sigma$  **is the corresponding polygonal line. The image of a collection of edges** is the union of images of all the edges from the collection.

**Theorem 1.4.1** (Cf. Proposition 1.1.1.a and Theorems 2.2.2, 3.1.6) *For any PL (or continuous) map  $K_5 \rightarrow \mathbb{R}^2$  there are two non-adjacent edges whose images intersect.*

Theorem 1.4.1 is deduced from its ‘quantitative version’: for ‘almost every’ drawing of  $K_5$  in the plane the number of intersection points of non-adjacent edges is odd. The

**Fig. 5** The residue  $v(f)$  is independent of  $f$



words ‘almost every’ are formalized below in Lemma 1.4.3. Formally, Theorem 1.4.1 follows by Lemma 1.4.3 using a version of [Skopenkov 2020, Approximation Lemma 1.4.6], cf. Remark 3.3.1.c.

Let  $f : K_n \rightarrow \mathbb{R}^2$  be a PL map. It is called a **general position** PL map if all the vertices of the polygonal lines are in general position. Then by Proposition 1.3.1 the images of any two non-adjacent edges intersect by a finite number of points. Let **the van Kampen number** (or the self-intersection invariant)  $v(f) \in \mathbb{Z}_2$  be the parity of the number of all such intersection points, for all pairs of non-adjacent edges.

- Example 1.4.2** (a) A convex pentagon with the diagonals forms a general position PL map  $f : K_5 \rightarrow \mathbb{R}^2$  such that  $v(f) = 1$ .
- (b) A convex quadrilateral with the diagonals forms a general position PL map  $f : K_4 \rightarrow \mathbb{R}^2$  such that  $v(f) = 1$ . A triangle and a point inside forms a general position PL map  $f : K_4 \rightarrow \mathbb{R}^2$  such that  $v(f) = 0$ . Cf. Sects. 2.1 and 2.2.

**Lemma 1.4.3** (Cf. Proposition 1.1.1.b and Lemma 2.2.3) *For any general position PL map  $f : K_5 \rightarrow \mathbb{R}^2$  the van Kampen number  $v(f)$  is odd.*

**Proof** By Proposition 1.1.1.b it suffices to prove that  $v(f) = v(f')$  for each two general position PL maps  $f, f' : K_5 \rightarrow \mathbb{R}^2$  coinciding on every edge except an edge  $\sigma$ , and such that  $f|_\sigma$  is linear (Fig. 5). The edges of  $K_5$  non-adjacent to  $\sigma$  form a cycle (this very property of  $K_5$  is necessary for the proof). Denote this cycle by  $\Delta$ . Then

$$v(f) - v(f') = |(f\sigma \cup f'\sigma) \cap f\Delta| \pmod 2 = 0.$$

Here the second equality follows by the Parity Lemma 1.3.2.b. □

## 1.5 A Polynomial Algorithm for Recognizing Graph Planarity

### 1.5.1 Van Kampen–Hanani–Tutte Planarity Criterion

A polynomial algorithm for recognizing graph planarity is obtained using the van Kampen–Hanani–Tutte planarity criterion (Proposition 1.5.1 below). In the following

subsections we show how to invent and prove that criterion. We consider a natural object (intersection cocycle) for any general position PL map from a graph to the plane (Sect. 1.5.2). Then we investigate how this object depends on the map (Proposition 1.5.6.b below). So we derive from this object an obstruction to planarity which is independent of the map. Combinatorial and linear algebraic (=cohomological) interpretation of this obstruction gives the required planarity criterion.

**Proposition 1.5.1** *Take any ordering of the vertices of a graph. Then the following conditions are equivalent.*

- (i) *The graph is planar.*
- (ii) *There are vertices  $V_1, \dots, V_s$  and edges  $e_1, \dots, e_s$  such that  $V_i \notin e_i$  for any  $i = 1, \dots, s$ , and for any non-adjacent edges  $\sigma, \tau$  the following numbers have the same parity:*
  - *the number of endpoints of  $\sigma$  that lie between endpoints of  $\tau$  (for the above ordering; the parity of this number is one if the endpoints of edges are ‘intertwined’ and is zero otherwise).*
  - *the number of those  $i = 1, \dots, s$  for which either  $V_i \in \sigma$  and  $e_i = \tau$ , or  $V_i \in \tau$  and  $e_i = \sigma$ .*
- (iii) *The following system of linear equations over  $\mathbb{Z}_2$  is solvable. To every pair  $A, e$  of a vertex and an edge such that  $A \notin e$  assign a variable  $x_{A,e}$ . For every non-ordered pair of non-adjacent edges  $\sigma, \tau$  denote by  $b_{\sigma,\tau} \in \mathbb{Z}_2$  the number of endpoints of  $\sigma$  that lie between endpoints of  $\tau$ . For every such pairs  $(A, e)$  and  $\{\sigma, \tau\}$  let<sup>10</sup>*

$$a_{A,e,\sigma,\tau} = \begin{cases} 1 & \text{either } (A \in \sigma \text{ and } e = \tau) \text{ or } (A \in \tau \text{ and } e = \sigma) \\ 0 & \text{otherwise} \end{cases}.$$

For every such pair  $\{\sigma, \tau\}$  take the equation  $\sum_{A \notin e} a_{A,e,\sigma,\tau} x_{A,e} = b_{\sigma,\tau}$ .

The implication (ii)  $\Leftrightarrow$  (iii) is clear. The implication (ii)  $\Rightarrow$  (i) follows by the Kuratowski Theorem 1.2.3 and Assertion 1.5.2 below. The implication (i)  $\Rightarrow$  (iii) follows by the Hanani–Tutte Theorem 1.5.3, Example 1.5.4 and Proposition 1.5.9 below.

**Assertion 1.5.2** (see proof in [Skopenkov 2018c, Sect. 1.5.1]) The property (ii) above is not fulfilled for  $K_5$  and for  $K_{3,3}$ .

Let us present a direct reformulation for  $K_5$  (for  $K_{3,3}$  the reformulation and the proof are analogous).

Let  $A_1, \dots, A_5$  be five collections of 2-element subsets of  $\{1, 2, 3, 4, 5\}$  such that no  $j \in \{1, 2, 3, 4, 5\}$  is contained in any subset from  $A_j$ . Then for some four different elements  $i, j, k, l \in \{1, 2, 3, 4, 5\}$  the sum of the following three numbers is odd

- the number of elements  $s \in \{i, j\}$  lying between  $k$  and  $l$ ;
- the number of elements  $s \in \{i, j\}$  such that  $A_s \ni \{k, l\}$ ;
- the number of elements  $s \in \{k, l\}$  such that  $A_s \ni \{i, j\}$ .

<sup>10</sup> Example 1.5.4 and Proposition 1.5.6.b explain how  $b_{\sigma,\tau}$  and  $a_{A,e,\sigma,\tau}$  naturally appear in the proof.

### 1.5.2 Intersection Cocycle

A linear map  $f : K \rightarrow \mathbb{R}^2$  of a graph  $K = (V, E)$  to the plane is a map  $f : V \rightarrow \mathbb{R}^2$ . The image  $f(AB)$  of edge  $AB$  is the segment  $f(A)f(B)$ . A **piecewise-linear (PL) map**  $f : K \rightarrow \mathbb{R}^2$  of a graph  $K = (V, E)$  to the plane is a collection of (non-closed) polygonal lines corresponding to the edges of  $K$ , whose endpoints correspond to the vertices of  $K$ . (A PL map of a graph  $K$  to the plane is ‘the same’ as a linear map of some graph homeomorphic to  $K$ .) **The image** of an edge, or of a collection of edges, is defined analogously to the case of  $K_n$  (Sect. 1.4). So a graph is planar if there exists its PL map to the plane such that the images of vertices are distinct, the images of the edges do not have self-intersections, and no image of an edge intersects the interior of any other image of an edge.

A linear map of a graph to the plane is called a *general position linear map* if the images of all the vertices are in general position. A PL map  $f : K \rightarrow \mathbb{R}^2$  of a graph  $K$  is called a **general position PL map** if there exist a graph  $H$  homeomorphic to  $K$  and a general position linear map of  $H$  to the plane such that this map ‘corresponds’ to the map  $f$ .

A graph is called  $\mathbb{Z}_2$ -**planar** if there exists a general position PL map of this graph to the plane such that images of any two non-adjacent edges intersect at an even number of points.

By Lemma 1.4.3  $K_5$  is not  $\mathbb{Z}_2$ -planar. Analogously,  $K_{3,3}$  is not  $\mathbb{Z}_2$ -planar. Hence, if a graph  $K$  is homeomorphic to  $K_5$  or to  $K_{3,3}$ , then  $K$  is not  $\mathbb{Z}_2$ -planar (because any PL map  $K \rightarrow \mathbb{R}^2$  corresponds to some PL map  $K_5 \rightarrow \mathbb{R}^2$  or  $K_{3,3} \rightarrow \mathbb{R}^2$ ). Then using Kuratowski Theorem 1.2.3 one can obtain the following result.

**Theorem 1.5.3** (Hanani–Tutte; cf. Theorems 2.4.2 and 3.3.5) *A graph is planar if and only if it is  $\mathbb{Z}_2$ -planar.*

**Example 1.5.4** Suppose a graph and an arbitrary ordering of its vertices are given. Put the vertices on a circle, preserving the ordering. Take a chord for each edge. We obtain a general position linear map of the graph to the plane. For any pair of non-adjacent edges  $\sigma, \tau$  the number of intersection points of their images has the same parity as the number of endpoints of  $\sigma$  that lie between the endpoints of  $\tau$ .

Let  $f : K \rightarrow \mathbb{R}^2$  be a general position PL map of a graph  $K$ . Take any pair of non-adjacent edges  $\sigma, \tau$ . By Proposition 1.3.1 the intersection  $f\sigma \cap f\tau$  consists of a finite number of points. Assign to the pair  $\{\sigma, \tau\}$  the residue

$$|f\sigma \cap f\tau| \pmod{2}.$$

Denote by  $K^*$  the set of all unordered pairs of non-adjacent edges of the graph  $K$ . The obtained map  $K^* \rightarrow \mathbb{Z}_2$  is called **the intersection cocycle** (modulo 2) of  $f$  (we call it ‘cocycle’ instead of ‘map’ to avoid confusion with maps to the plane). In other words, we have obtained a subset of  $K^*$ , or a ‘partial matrix’, i.e., a symmetric arrangement of zeroes and ones in some cells of the  $e \times e$ -matrix corresponding to the pairs of non-adjacent edges, where  $e$  is the number of edges of  $K$ .

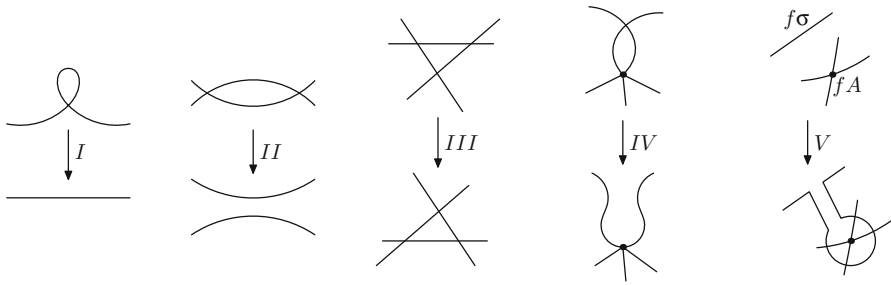


Fig. 6 The Reidemeister moves for graphs in the plane

**Remark 1.5.5** If a graph is  $\mathbb{Z}_2$ -planar, then the intersection cocycle is zero for some general position PL map of this graph to the plane.

**1.5.3 Intersection Cocycles of Different Maps**

Addition of maps  $K^* \rightarrow \mathbb{Z}_2$  is componentwise, i.e. is defined by adding modulo 2 numbers corresponding to the same pair (i.e. numbers in the same cell of a ‘partial matrix’).

**Proposition 1.5.6** (cf. Proposition 1.5.11)

- (a) *The intersection cocycle does not change under the first four Reidemeister moves in Fig. 6I–IV. (The graph drawing changes in the disk as in Fig. 6, while out of this disk the graph drawing remains unchanged. No other images of edges besides the pictured ones intersect the disk.)*
- (b) *Let  $K$  be a graph and  $A$  its vertex which is not the end of an edge  $\sigma$ . An elementary coboundary of the pair  $(A, \sigma)$  is the map  $\delta_K(A, \sigma) : K^* \rightarrow \mathbb{Z}_2$  that assigns 1 to any pair  $\{\sigma, \tau\}$  with  $\tau \ni A$ , and 0 to any other pair. Under the Reidemeister move in Fig. 6.V the intersection cocycle changes by adding  $\delta_K(A, \sigma)$ .*

**Example 1.5.7** The subset of  $\delta_K(A, \sigma)^{-1}(1) \subset K^*$  corresponding to the map  $\delta_K(A, \sigma)$  is also called elementary coboundary.

- (a) We have  $\{\{13, 24\}\} = \delta_{K_4}(1, 24) = \delta_{K_4}(2, 13) = \delta_{K_4}(3, 24) = \delta_{K_4}(4, 13)$ .
- (b) We have  $\delta_{K_5}(3, 12) = \{\{12, 34\}, \{12, 35\}\}$ .

Two maps  $\nu_1, \nu_2 : K^* \rightarrow \mathbb{Z}_2$  are called **cohomologous** if

$$\nu_1 - \nu_2 = \delta_K(A_1, \sigma_1) + \dots + \delta_K(A_k, \sigma_k)$$

for some vertices  $A_1, \dots, A_k$  and edges  $\sigma_1, \dots, \sigma_k$  (not necessarily distinct).

Proposition 1.5.6.b and the following Lemma 1.5.8 show that cohomology is the equivalence relation generated by changes of a graph drawing.

**Lemma 1.5.8** (cf. Lemmas 1.5.12 and 2.4.4) *The intersection cocycles of different general position PL maps of the same graph to the plane are cohomologous.*

The proof of Lemma 1.5.8 presented in [Skopenkov 2018c, Sect. 1.5] is a non-trivial generalization of the proof of Lemma 1.4.3. Lemma 1.5.8 and Proposition 1.5.6.b imply the following result.

**Proposition 1.5.9** (cf. Propositions 1.5.13 and 2.4.5) *A graph is  $\mathbb{Z}_2$ -planar if and only if the intersection cocycle of some (or, equivalently, of any) general position PL map of this graph to the plane is cohomologous to the zero map.*

### 1.5.4 Intersections with Signs

Here we generalize previous constructions from residues modulo 2 to integers. These generalizations are not formally used later. However, it is useful to make these simple generalizations (and possibly to make remark [Skopenkov 2018c, Remark 1.6.5]) before more complicated generalizations in Sects. 2.3.3, 2.4. Also, integer analogues are required for higher dimensions (namely, for proofs of Theorems 3.2.1.b, 3.3.4, cf. [Skopenkov 2018c, Remark 1.6.3]).

Suppose that  $P$  and  $Q$  are oriented polygonal lines in the plane whose vertices are in general position. Define the **algebraic intersection number**  $P \cdot Q$  of  $P$  and  $Q$  as the sum of the signs of the intersection points of  $P$  and  $Q$ . See Fig. 4.

**Assertion 1.5.10** (a) We have  $P \cdot Q = -Q \cdot P$ .

(b) If we change the orientation of  $P$ , then the sign of  $P \cdot Q$  will change.

(c) If we change the orientation of the plane, i.e. if we make axial symmetry, then the sign of  $P \cdot Q$  will change.

Let  $K$  be a graph and  $f : K \rightarrow \mathbb{R}^2$  a general position PL map. Orient the edges of  $K$ . Assign to every ordered pair  $(\sigma, \tau)$  of non-adjacent edges the algebraic intersection number  $f\sigma \cdot f\tau$ . Denote by  $\tilde{K}$  the set of all ordered pairs of non-adjacent edges of  $K$ . The obtained map  $\cdot : \tilde{K} \rightarrow \mathbb{Z}$  is called the **integral intersection cocycle** of  $f$  (for given orientations).

**Proposition 1.5.11** *Analogue of Proposition 1.5.6 is true for the integral intersection cocycle, with the following definition. Let  $K$  be an oriented graph and  $A$  a vertex which is not the end of an edge  $\sigma$ . An elementary skew-symmetric coboundary of the pair  $(A, \sigma)$  is the map  $\delta_{K, \mathbb{Z}}(A, \sigma) : \tilde{K} \rightarrow \mathbb{Z}$  that assigns*

- $+1$  to any pair  $(\sigma, \tau)$  with  $\tau$  issuing out of  $A$  and any pair  $(\tau, \sigma)$  with  $\tau$  going to  $A$ ,
- $-1$  to any pair  $(\sigma, \tau)$  with  $\tau$  going to  $A$  and any pair  $(\tau, \sigma)$  with  $\tau$  issuing out of  $A$ ,
- $0$  to any other pair.

Two maps  $N_1, N_2 : \tilde{K} \rightarrow \mathbb{Z}$  are called **skew-symmetrically cohomologous**, if

$$N_1 - N_2 = m_1 \delta_{K, \mathbb{Z}}(A_1, \sigma_1) + \cdots + m_k \delta_{K, \mathbb{Z}}(A_k, \sigma_k)$$

for some vertices  $A_1, \dots, A_k$ , edges  $\sigma_1, \dots, \sigma_k$  and integers  $m_1, \dots, m_k$  (not necessarily distinct).

The following integral analogue of Lemma 1.5.8 is proved analogously using the Triviality Lemma 1.3.3.

**Lemma 1.5.12** *The integer intersection cocycles of different maps of the same graph to the plane are skew-symmetrically cohomologous.*

**Proposition 1.5.13** (cf. Proposition 2.4.7) *Twice the integral intersection cocycle of any general position PL map of a graph in the plane is skew-symmetrically cohomologous to the zero map.*

This follows by Assertion 1.5.10.c and Lemma 1.5.12.

## 2 Multiple Intersections in Combinatorial Geometry

### 2.1 Radon and Tverberg Theorems in the Plane

The reader can find more complete exposition and illustrative examples e.g. in [Skopenkov 2018c, Sect. 2.1].

**Theorem 2.1.1** (Radon theorem in the plane) *For any 4 points in the plane either one of them belongs to the triangle with vertices at the others, or they can be decomposed into two pairs such that the segment joining the points of the first pair intersects the segment joining the points of the second pair.*

Cf. Proposition 1.1.1.a and Theorems 2.2.2, 3.1.1.

Radon theorem in the plane can be reformulated as follows: *any 4 points in the plane can be decomposed into two disjoint sets whose convex hulls intersect.* This reformulation has the following stronger ‘quantitative’ form.

**Proposition 2.1.2** (see proof in [Skopenkov 2018c, Sect. 2.5]; cf. Proposition 1.1.1.b and Lemma 2.2.3) *If no 3 of 4 points in the plane belong to a line, then there exists a unique partition of these 4 points into two sets whose convex hulls intersect.*

Now consider partitions of subsets of the plane into *more than two* disjoint sets.

**Example 2.1.3** In the plane take  $r - 1$  points at each vertex of a triangle (or a ‘similar’ set of distinct points). For any decomposition of these  $3r - 3$  points into  $r$  disjoint sets the convex hulls of these sets do not have a common point.

**Assertion 2.1.4** (see proof in [Skopenkov 2018c, Sect. 2.5]) For any  $r$  there exist  $N$  such that any  $N$  points in the plane can be decomposed into  $r$  disjoint sets whose convex hulls have a common point.

The following theorem shows that the minimal  $N$  is just one above the number of Example 2.1.3.

**Theorem 2.1.5** (Tverberg theorem in the plane, see proof in [Matoušek 2008], cf. Theorem 3.1.3) *For any  $r$  every  $3r - 2$  points in the plane can be decomposed into  $r$  disjoint sets whose convex hulls have a common point.*

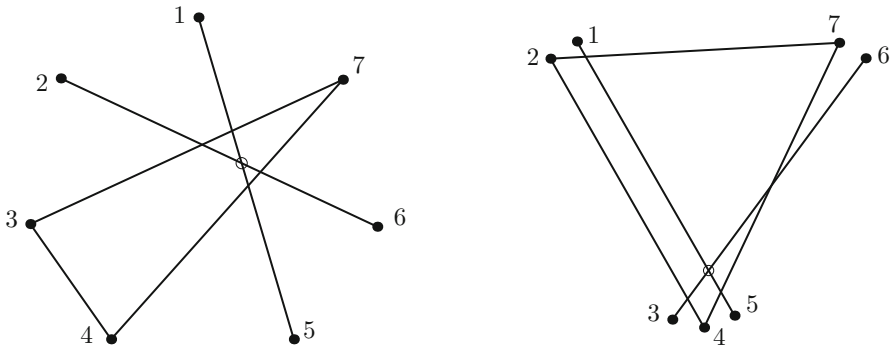


Fig. 7 A common point of convex hulls as in Theorem 2.1.5

**Example 2.1.6** (cf. Propositions 1.1.1 and 2.1.2) The sum

$$\sum_{\{R_1, R_2, R_3\} : M_i = R_1 \sqcup R_2 \sqcup R_3} |\langle R_1 \rangle \cap \langle R_2 \rangle \cap \langle R_3 \rangle|$$

has different parity for the two sets  $M_1, M_2$  of Fig. 7.

### 2.2 Topological Radon Theorem in the Plane

**Proposition 2.2.1** (see proof in [Skopenkov 2018c, Sect. 2.5]) *Take a closed polygonal line  $L$  in the plane whose vertices are in general position.*

- (a) *The complement to  $L$  has a chess-board coloring (so that the adjacent domains have different colors, see Fig. 8).*
- (b) (cf. Proposition 2.3.1.c) *The ends of a polygonal line  $P$  whose vertices together with the vertices of  $L$  are in general position have the same color if and only if  $|P \cap L|$  is even.*

The **modulo two interior** of a closed polygonal line in the plane whose vertices are in general position is the union of black domains for a chess-board coloring (provided the infinite domain is white).

Piecewise-linear (PL) and general position PL maps  $K_n \rightarrow \mathbb{R}^2$  are defined in Sect. 1.4.

**Theorem 2.2.2** (Topological Radon theorem in the plane [Bajmóczy and Bárány 1979], cf. Theorems 1.4.1, 2.1.1, 3.1.5)

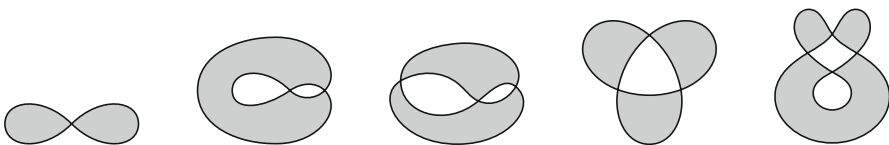


Fig. 8 The modulo two interiors of some closed polygonal lines



- (a) For any general position PL map  $f : K_4 \rightarrow \mathbb{R}^2$  either
- the images of some non-adjacent edges intersect, or
  - the image of some vertex belongs to the interior modulo 2 of the image of the cycle formed by those three edges that do not contain this vertex.
- (b) For any PL (or continuous) map of a tetrahedron to the plane either
- the images of some opposite edges intersect, or
  - the image of some vertex belongs to the image of the opposite face.

Part (a) follows from its ‘quantitative version’ Lemma 2.2.3 below using a version of [Skopenkov 2020, Approximation Lemma 1.4.6], cf. Remark 3.3.1.c.

Part (b) for PL general position maps follows from part (a) because the image  $f\Delta$  of a face  $\Delta$  contains the interior modulo 2 of the image of the boundary  $\partial\Delta$  of this face. (This fact follows because for a general position map  $f : \Delta \rightarrow \mathbb{R}^2$  a general position point from the interior modulo 2 of  $f(\partial\Delta)$  has an odd number of  $f$ -preimages.) Part (b) follows from part (b) for general position PL maps using a version of [Skopenkov 2020, Approximation Lemma 1.4.6], cf. Remark 3.3.1.c.

Also, the standard formulation (b) is equivalent to (a) by [Schöneborn 2004, Schöneborn and Ziegler 2005].

For any general position PL map  $f : K_4 \rightarrow \mathbb{R}^2$  let the **Radon number**  $\rho(f) \in \mathbb{Z}_2$  be the sum of the parities of

- the number of intersections points of the images of non-adjacent edges, and
- the number of vertices whose images belong to the interior modulo 2 of the image of the cycle formed by the three edges not containing the vertex.

**Lemma 2.2.3** (cf. Lemma 1.4.3, Proposition 2.1.2 and [Schöneborn 2004, Schöneborn and Ziegler 2005]) *For every general position PL map  $f : K_4 \rightarrow \mathbb{R}^2$  the Radon number  $\rho(f)$  is odd.*

**Proof** By Proposition 2.1.2 it suffices to prove that  $\rho(f) = \rho(f')$  for each two general position PL maps  $f, f' : K_4 \rightarrow \mathbb{R}^2$  coinciding on every edge except an edge  $\sigma$ , and such that  $f|_\sigma$  is linear. Denote by  $\tau$  the edge of  $K_4$  non-adjacent to  $\sigma$ , by  $S$  the modulo 2 interior of  $\partial S := f\sigma \cup f'\sigma$ . Then

$$\rho(f) - \rho(f') = (|\partial S \cap f\tau| + |S \cap f(\partial\tau)|) \pmod 2 = 0.$$

Here the second equality follows by Proposition 2.2.1.b. □

### 2.3 Topological Tverberg Theorem in the Plane

#### 2.3.1 Statement

The topological Tverberg theorem in the plane 2.3.2 generalizes both the Tverberg Theorem in the plane 2.1.5 and the Topological Radon Theorem in the plane 2.2.2. For statement we need a definition. **The winding number** of a closed oriented polygonal

line  $A_1 \dots A_n$  in the plane around a point  $O$  that does not belong to the polygonal line is the following sum of the oriented angles divided by  $2\pi$

$$A_1 \dots A_n \cdot O := (\angle A_1 O A_2 + \angle A_2 O A_3 + \dots + \angle A_{n-1} O A_n + \angle A_n O A_1) / 2\pi$$

**Proposition 2.3.1** (a) *The winding number of any polygon (without self-intersections and oriented counterclockwise) around any point in its exterior (interior) is 0 (respectively 1).*

(b) *The interior modulo 2 (Fig. 8) of any closed polygonal line is the set of points for which the winding number is odd.*

(c) (cf. Proposition 2.2.1.b) *Take a closed and a non-closed oriented polygonal lines  $L$  and  $P$  in the plane, all whose vertices are in general position. Let  $P_0$  and  $P_1$  be the starting point and the endpoint of  $P$ . Then  $L \cdot P = L \cdot \partial P := L \cdot P_1 - L \cdot P_0$ .<sup>11</sup>*

**Theorem 2.3.2** (Topological Tverberg theorem in the plane, [Bárány et al. 1981, Özaydin 2019, Volovikov 1996a], see Fig. 9) *If  $r$  is a power of a prime, then for any general position  $PL$  map  $f : K_{3r-2} \rightarrow \mathbb{R}^2$  either  $r - 1$  triangles wind around one vertex or  $r - 2$  triangles wind around the intersection of two edges, where the triangles, edges and vertices are disjoint. More precisely, the vertices can be numbered by  $1, \dots, 3r - 2$  so that either*

- *the winding number of each of the images  $f(3t - 1, 3t, 3t + 1)$ ,  $t = 2, 3, \dots, r - 1$ , around some point of  $f(12) \cap f(34)$  is nonzero, or*
- *the winding number of each of the images  $f(3t - 1, 3t, 3t + 1)$ ,  $t = 1, 2, 3, \dots, r - 1$ , around  $f(1)$  is nonzero.*

*(The condition ‘winding number is nonzero’ does not depend on orientation of  $f(ijk)$ .)*

Cf. Theorems 2.1.5, 2.2.2 and Conjecture 3.1.7.

By [Schöneborn 2004, Schöneborn and Ziegler 2005] Theorem 2.3.2 is equivalent to the following standard formulation: *If  $r$  is a power of a prime, then for every  $PL$  (or continuous) map of the  $(3r - 3)$ -simplex to the plane there exist  $r$  pairwise disjoint faces whose images have a common point.* Proofs of Theorem 2.3.2 can be found e.g. in Sects. 2.3.3, 2.3.4 for a prime  $r$ , and in the surveys cited in ‘historical notes’ of the Introduction for a prime power  $r$ .

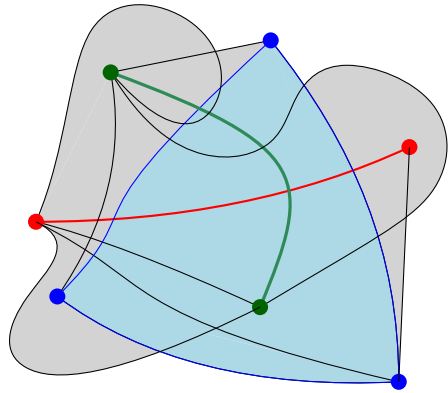
**Conjecture 2.3.3** (Topological Tverberg conjecture in the plane) *The analogue of the previous theorem remains correct if  $r$  is not a power of a prime.*

Let us state a refinement of Theorems 2.1.5 and 2.3.2.

<sup>11</sup> The number  $L \cdot P$  is defined in Sect. 1.5.4.

This version of the Stokes theorem shows that the complement to  $L$  has a *Möbius–Alexander numbering*, i.e. a ‘chess-board coloring by integers’ (so that the colors of the adjacent domains are different by  $\pm 1$  depending on the orientations; the ends of a polygonal line  $P$  have the same color if and only if  $L \cdot P = 0$ ). See more in [[https://en.wikipedia.org/wiki/Winding\\_number](https://en.wikipedia.org/wiki/Winding_number)].

**Fig. 9** Topological Tverberg theorem in the plane,  $r = 3$



An ordered partition  $(R_1, R_2, R_3)$  of  $M = R_1 \sqcup R_2 \sqcup R_3 \subset [7]$  into three sets (possibly empty) is called **spherical** if no set  $R_1, R_2, R_3$  contains any of the subsets  $\{1, 2\}, \{3, 4\}, \{5, 6\}$ . Generally, an ordered partition  $(R_1, \dots, R_r)$  of  $M = R_1 \sqcup \dots \sqcup R_r \subset [3r - 2]$  into  $r$  sets (possibly empty) is called **spherical** if for every  $j = 1, \dots, 3(r-1)/2$  if  $2j - 1 \in R_s$ , then  $2j \in R_{s-1} \cup R_{s+1}$ , where the  $r$  sets are numbered modulo  $r$ . Or, less formally, if consecutive odd and even integers are contained in consecutive sets. Spherical partitions appeared implicitly in [Vučić and Živaljević 1993] and in [Matoušek 2008, Sect. 6.5, pp. 166–167]. Cf. [Skopenkov 2018c, Remark 2.5.2].

- Example 2.3.4** (a) There are  $6^3 = 216$  spherical partitions of  $[6]$  into three sets. Indeed, each of the pairs  $\{1, 2\}, \{3, 4\}, \{5, 6\}$  can be distributed in 6 ways.  
 (b) A spherical partition  $(\{1\}, \{2, 4, 5\}, \{3\})$  of  $[5]$  extends to two spherical partitions  $(\{1, 6\}, \{2, 4, 5\}, \{3\})$  and  $(\{1\}, \{2, 4, 5\}, \{3, 6\})$  of  $[6]$ . The extension  $(\{1\}, \{2, 4, 5, 6\}, \{3\})$  is not spherical because  $\{2, 4, 5, 6\} \supset \{5, 6\}$ .

- Theorem 2.3.5** (a) For any prime  $r$  any  $3r - 2$  points  $1, \dots, 3r - 2$  in the plane can be spherically partitioned into  $r$  sets whose convex hulls have a common point.  
 (b) For any prime  $r$  and general position PL map  $f : K_{3r-2} \rightarrow \mathbb{R}^2$  either the images of  $r - 1$  triangles wind around the images of the remaining vertex, or  $r - 2$  triangles wind around the intersection of two edges. Moreover, the triangles and the vertex, or the triangles and the edges, respectively, form a spherical partition of  $[3r - 2] = V(K_{3r-2})$ .

Part (a) follows from (b). Part (b) is essentially proved in [Vučić and Živaljević 1993], [Matoušek 2008, Sect. 6.5]. This formulation is from [Schöneborn 2004, Theorem 3.3.1], [Schöneborn and Ziegler 2005, Theorem 5.8].

### 2.3.2 Ideas of Proofs

In the following subsections we present a standard proof, and an idea of an elementary proof, of the topological Tverberg Theorem for the plane 2.3.2 (in fact of Theorem 2.3.5).

The idea of an elementary proof presented in Sect. 2.3.3 generalizes the proofs of Lemmas 1.4.3, 1.5.8, 2.2.3 and [Skopenkov and Tancer 2017, Lemmas 6 and 7]. Instead of counting *double* intersection points we count *r-tuple* intersection points. Instead of counting points *modulo 2* we have to count points *with signs*, see Example 2.1.6. It is also more convenient (because of Lemma 2.3.12) instead of summing over *all* partitions to sum over *spherical* partitions. This is formalized by Problem 2.3.7 below which is a ‘quantitative version’ of Theorems 2.3.2 and 2.3.5. Formally, Theorems 2.3.2 and 2.3.5 for  $r = 3$  follow by resolution of Problem 2.3.7. Proofs of Theorems 2.3.2, 2.3.5 and of Conjecture 3.1.7 for (arbitrary  $d$  and) prime  $r$  would perhaps be analogous.

Similar proofs of Theorems 2.3.2, 2.3.5, and of Conjecture 3.1.7 for prime  $r$ , are given in [Blagojević et al. 2015, Matoušek et al. 2012], see [Skopenkov 2018c, Remark 2.5.2]. Those papers use more complicated language not necessary for these results (Sarkaria–Onn transform in [Matoušek et al. 2012], homology and equivariant maps between configuration spaces in [Blagojević et al. 2015]).

The standard proof of Theorem 2.3.2 is presented in Sect. 2.3.4 following [Vučić and Živaljević 1993], [Matoušek 2008, Sect. 6], cf. [Bárány et al. 1981], [Skopenkov 2018a, Sect. 2]. This proof also yields Theorem 2.3.5 and generalizes to Conjecture 3.1.7 for prime  $r$ . Theorem 2.3.2 is deduced from the  $r$ -fold Borsuk–Ulam Theorem 2.3.13 using Lemma 2.3.12, both below. Proof of the Borsuk–Ulam Theorem 2.3.8 via its ‘quantitative version’ Lemma 2.3.9 generalizes to a proof of the  $r$ -fold Borsuk–Ulam Theorem 2.3.13. So this deduction of Theorem 2.3.2 is not a proof essentially different from the idea of Sect. 2.3.3 but rather the same proof in a different language. (Therefore it is not quite correct that the main idea of the proof of the topological Tverberg Theorem is to apply the  $r$ -fold Borsuk–Ulam Theorem for configuration spaces.)

The case  $r = 3$  gives a non-trivial generalization of the case  $r = 2$ ; the generalization to arbitrary  $r$  (prime for some results below) is trivial.

### 2.3.3 Triple Self-Intersection Invariant for Graph Drawings

Suppose that every object of  $P_1, \dots, P_r$  is either a point, or an oriented non-closed polygonal line, or an oriented closed polygonal line, in the plane, all of whose vertices are in general position. Define **the  $r$ -tuple algebraic intersection number**  $P_1 \dots P_r$  to be

- (A)  $\sum_{X \in P_i \cap P_j} \operatorname{sgn} X \left( \prod_{s \neq i, j} (P_s \cdot X) \right)$ , if  $P_i, P_j$  are non-closed polygonal lines for some  $i < j$ , and the other  $P_s$  are closed polygonal lines;  
 (B)  $\prod_{s \neq i} (P_s \cdot P_i)$ , if  $P_i$  is a point and the other  $P_s$  are closed polygonal lines.

Here  $\operatorname{sgn} X$  and  $\cdot$  are defined in Sects. 1.3, 1.5.4 and 2.3.1; the number  $P_1 \dots P_r$  is only defined in cases (A) and (B).<sup>12</sup>

<sup>12</sup> This is an elementary interpretation in the spirit of [Schöneborn 2004, Schöneborn and Ziegler 2005] of the  $r$ -tuple algebraic intersection number  $f D^{n_1} \dots f D^{n_r}$  of a general position map  $f : D^{n_1} \sqcup \dots \sqcup D^{n_r} \rightarrow \mathbb{R}^2$ , where  $n_1, \dots, n_r \subset \{0, 1, 2\}$  and  $n_1 + \dots + n_r = 2r - 2$  [Mabillard and Wagner 2015, Sect. 2.2]. This agrees with [Mabillard and Wagner 2015, Sect. 2.2] by [Mabillard and Wagner 2015, Lemma 27.b]. For a degree interpretation see [Skopenkov 2018c, Assertion 2.5.4].

**Example 2.3.6** Assume that  $R_1, R_2, R_3$  are either

(A) two vectors and an oriented triangle, or (B) two oriented triangles and a point, in the plane. Assume that the vertices of  $R_1, R_2, R_3$  are pairwise disjoint subsets of the plane and their union is in general position. Then  $|R_1 \cdot R_2 \cdot R_3| = \langle R_1 \rangle \cap \langle R_2 \rangle \cap \langle R_3 \rangle$ .

When the intersection is non-empty,  $R_1 \cdot R_2 \cdot R_3 = +1$  if and only if up to a permutation of  $(R_1, R_2, R_3)$  not switching the order of vectors, the triangle  $R_1$  has the same orientation as the triangle

(A)  $A_2 B_2 B_3$ , where  $R_i = \overrightarrow{A_i B_i}$  for each  $i = 2, 3$ , (B)  $R_2$ .

**Problem 2.3.7** Let  $S$  be the set of all spherical partitions  $(T_1, T_2, T_3)$  of [7] such that  $7 \in T_3$ . Define a map  $\text{sgn} : S \rightarrow \{+1, -1\}$  so that for any general position PL map  $f : K_7 \rightarrow \mathbb{R}^2$  the following (‘triple van Kampen’) number is not divisible by 3:

$$V(f) := \sum_{T=(T_1, T_2, T_3) \in S} \text{sgn } T(fT_1 \cdot fT_2 \cdot fT_3).$$

Here  $fT_s$  is the  $f$ -image of either a vertex  $T_s$ , or an oriented edge  $T_s = ab, a < b$ , or an oriented cycle  $T_s = abc, a < b < c$ .

Analogously to Lemmas 1.4.3, 1.5.8, 2.2.3 and to [Skopenkov and Tancer 2017, Lemmas 6 and 7], the non-divisibility in Problem 2.3.7 could possibly be proved by calculating  $V(f)$  for a specific  $f$  and showing that  $V(f)$  modulo 3 is independent of  $f$ . This might be not so easy, cf. [Matoušek et al. 2012, second half of Sect. 8].

### 2.3.4 An Approach Via Borsuk–Ulam Theorem

A map  $f : S^n \rightarrow \mathbb{R}^m$  is called *odd*, or *equivariant*, or *antipodal* if  $f(-x) = -f(x)$  for any  $x \in S^n$ . We consider only continuous maps and omit ‘continuous’.

**Theorem 2.3.8** (Borsuk–Ulam)

- (a) For any map  $f : S^d \rightarrow \mathbb{R}^d$  there exists  $x \in S^d$  such that  $f(x) = f(-x)$ .
- (a’) For any equivariant maps  $f : S^d \rightarrow \mathbb{R}^d$  there exists  $x \in S^d$  such that  $f(x) = 0$ .
- (b) There are no equivariant maps  $S^d \rightarrow S^{d-1}$ .
- (b’) No equivariant map  $S^{d-1} \rightarrow S^{d-1}$  extends to  $D^d$ .
- (c) If  $S^d$  is the union of  $d + 1$  closed sets (or  $d + 1$  open sets), then one of the sets contains opposite points.

The equivalence of these assertions is simple. Part (a’) is deduced from its following ‘quantitative version’.

**Lemma 2.3.9** If  $0 \in \mathbb{R}^d$  is a regular point of a (PL or smooth) equivariant map  $f : S^d \rightarrow \mathbb{R}^d$ , then  $|f^{-1}(0)| \equiv 2 \pmod 4$ .

See the definition of a regular point e.g. in [Skopenkov 2020, Sect. 8.3]. Proof of Lemma 2.3.9 is analogous to Lemmas 1.4.3 and 1.5.8 (cf. Problem 2.3.7): calculate  $|f^{-1}(0)|$  for a specific  $f$  and prove that  $|f^{-1}(0)|$  modulo 4 is independent of  $f$ .

Realization of this simple idea is technical, see [Matoušek 2008, Sect. 2.2]. For other proofs of Theorem 2.3.8 see [Matoušek 2008] and the references therein.

Assume that  $U_1, \dots, U_r \subset \mathbb{R}^d$  are finite union of simplices. Identify them with subsets  $U'_1, \dots, U'_r \subset \mathbb{R}^N$  isometric to  $U_1, \dots, U_r$  and lying in pairwise skew affine subspaces. Define **the  $(r$ -tuple) join**  $U_1 * \dots * U_r$  to be

$$\left\{ t_1x_1 + \dots + t_r x_r \in \mathbb{R}^N : x_j \in U'_j, t_j \in [0, 1], t_1 + \dots + t_r = 1 \right\}.$$

Geometric, topological, and combinatorial join are introduced and discussed in [Matoušek 2008, Sect. 4.2].

- Assertion 2.3.10** (a) If  $U$  and  $V$  are unions of faces of some simplex  $\Delta_n$  and are disjoint, then  $U * V$  is a union of all faces of  $\Delta_n$  that correspond to subsets  $\sigma \sqcup \tau$ , where  $\sigma, \tau \subset [n + 1]$  correspond to faces of  $U, V$ , respectively.  
 (b) The join  $S^1 * S^1$  of two cycles is PL homeomorphic to  $S^3$ .  
 (c) The  $r$ -tuple join  $(S^1)^{*r} := S^1 * \dots * S^1$  of  $r$  cycles is PL homeomorphic to  $S^{2r-1}$ .

The proof is simple [Matoušek 2008, Sect. 4.2]. Part (a) implies that to every ordered partition of  $[3r - 3]$  into  $r$  sets there corresponds a  $(3r - 4)$ -simplex of  $\Delta_{3r-4}^{*r} := \Delta_{3r-4} * \dots * \Delta_{3r-4}$  ( $r$  ‘factors’). Denote by  $|S_r|$  the union of  $(3r - 4)$ -simplices of  $\Delta_{3r-4}^{*r}$  corresponding to spherical partitions of  $[3r - 3]$  into  $r$  sets.

**Assertion 2.3.11** The union  $|S_r|$  is PL homeomorphic to  $S^{3r-4}$ .

This assertion and the following lemma are easily deduced from Assertion 2.3.10.c, see details in [Matoušek 2008, pp. 166–167].

Denote by  $\Sigma_r$  the permutation group of  $r$  elements. The group  $\Sigma_r$  acts on the set of real  $3 \times r$ -matrices by permuting the columns. Denote by  $S_{\Sigma_r}^{3r-4}$  the set formed by all those of such matrices, for which the sum in every row is zero, and the sum of squares of the matrix elements is 1. This set is homeomorphic to the sphere of dimension  $3r - 4$ . Take a triangulation of this set given by some such homeomorphism. This set is invariant under the action of  $\Sigma_r$ . The cyclic permutation  $\omega : S_{\Sigma_r}^{3r-4} \rightarrow S_{\Sigma_r}^{3r-4}$  of the  $r$  columns has no fixed points and  $\omega^r = \text{id } S_{\Sigma_r}^{3r-4}$ .

**Lemma 2.3.12** *There is a PL homeomorphism  $h : |S_r| \rightarrow S_{\Sigma_r}^{3r-4}$  such that  $h(R_2, \dots, R_r, R_1)$  is obtained from  $h(R_1, R_2, \dots, R_r)$  by cyclic permutation of the  $r$  columns.*

**Theorem 2.3.13** ( $r$ -fold Borsuk–Ulam Theorem) *Let  $r$  be a prime and  $\omega : S^k \rightarrow S^k$  a PL map without fixed points such that  $\omega^r = \text{id } S^k$ . Then no map  $g : S^k \rightarrow S^k$  commuting with  $\omega$  (i.e. such that  $g \circ \omega = \omega \circ g$ ) extends to  $D^{k+1}$ .*

**Comments on the proof** Clearly, the theorem is equivalent to the following result.

*Extend  $\omega$  to  $S^k * \mathbb{Z}_3$  by  $\omega(ts \oplus (1 - t)m) := t\omega(s) \oplus (1 - t)(m + 1)$ . Let  $\omega_0 : \mathbb{R}^{k+1} \rightarrow \mathbb{R}^{k+1}$  be a map whose only fixed point is 0 and such that  $\omega_0^r = \text{id } \mathbb{R}^{k+1}$ . Then for any map  $g : S^k * \mathbb{Z}_3 \rightarrow \mathbb{R}^{k+1}$  commuting with  $\omega, \omega_0$  (i.e. such that  $g \circ \omega = \omega_0 \circ g$ ) there is  $x \in S^k * \mathbb{Z}_3$  such that  $g(x) = 0$ .*

This result is deduced from its ‘quantitative version’ analogous to Lemma 2.3.9.

For a standard proof see [Bárány et al. 1981], [Matoušek 2008, Sect. 6]. □

**Proof of Theorem 2.3.2** Consider the case  $r = 3$ , the general case is analogous. We use the standard formulation of Theorem 2.3.2 given after the statement. Suppose to the contrary that  $f : \Delta_6 \rightarrow \mathbb{R}^2$  is a continuous map and there are no 3 pairwise disjoint faces whose images have a common point.

For  $x \in \mathbb{R}^2$  let  $x^* := (1, x) \in \mathbb{R}^3$ . For  $x_1, x_2, x_3 \in \mathbb{R}^2$  and  $t_1, t_2, t_3 \in [0, 1]$  such that  $t_1 + t_2 + t_3 = 1$  and pairs  $(x_1, t_1), (x_2, t_2), (x_3, t_3)$  are not all equal define

$$S^* := t_1x_1^* + t_2x_2^* + t_3x_3^*,$$

$$\pi^{*'} := \left( t_1x_1^* - \frac{S^*}{3}, t_2x_2^* - \frac{S^*}{3}, t_3x_3^* - \frac{S^*}{3} \right) \quad \text{and} \quad \pi^* := \frac{\pi^{*'}}{|\pi^{*'}|}.$$

This defines a map

$$\pi^* : \mathbb{R}^2 * \mathbb{R}^2 * \mathbb{R}^2 - \text{diag}^* \rightarrow S_{\Sigma_3}^5, \quad \text{where} \quad \text{diag}^* := \left\{ \left( \frac{1}{3}x \oplus \frac{1}{3}x \oplus \frac{1}{3}x \right) \right\}.$$

So we obtain the map  $\pi^* \circ (f * f * f) : |S_3| \rightarrow S_{\Sigma_3}^5$ . This map extends to the union of 6-simplices of  $\Delta_6^{*3}$  corresponding to spherical partitions  $(T_1, T_2, T_3)$  of [7] into 3 sets such that  $7 \in T_3$ . The union is PL homeomorphic to  $\text{con}|S_3| \cong D^6$ . The map  $\pi^* \circ (f * f * f)$  commutes with the cyclic permutations of the three sets in  $|S_3|$  and of the three columns in  $S_{\Sigma_3}^5$ . Take any PL homeomorphism  $h$  of Lemma 2.3.12. The composition  $g := \pi^* \circ (f * f * f) \circ h^{-1} : S_{\Sigma_3}^5 \rightarrow S_{\Sigma_3}^5$  commutes with the cyclic permutation of the three columns and extends to  $D^6$ . A contradiction to Theorem 2.3.13. □

## 2.4 Mapping Hypergraphs in the Plane and the Özaydin Theorem

Formulation of the Özaydin Theorem 2.4.10 uses the definition of a *multiple (r-fold) intersection cocycle*. We preface the definition by simplified analogues. In Sect. 1.5 we have defined double (2-fold) intersection cocycle for graphs. In Sect. 2.4.1 we define double intersection cocycle modulo 2 for hypergraphs. In Sect. 2.4.2 we generalize that definition from residues modulo 2 to integers. In Sect. 2.4.3 we generalize definition of Sect. 2.4.2 from  $r = 2$  to arbitrary  $r$ .

### 2.4.1 A Polynomial Algorithm for Recognizing Hypergraph Planarity

A  **$k$ -hypergraph** (more precisely,  $k$ -dimensional, or  $(k + 1)$ -uniform, hypergraph)  $(V, F)$  is a finite set  $V$  together with a collection  $F \subset \binom{V}{k+1}$  of  $(k + 1)$ -element subsets of  $V$ . Elements of  $V$  and of  $F$  are called *vertices* and *faces*. An *edge* is a 2-element subsets of  $V$  contained in some face. The results of this paper for hypergraphs have straightforward generalization to simplicial complexes (see definition in [Matoušek 2008, Definition 1.5.1] or in [Skopenkov 2018a, Sect. 1]).

E.g.  $\Delta_n^2 := ([n + 1], \binom{[n+1]}{3})$  is called *the complete 2-hypergraph on  $n + 1$  vertices*.

Let  $K = (V, F)$  be 2-hypergraph. The graph  $(V, E)$  formed by vertices and edges of  $K$  is denoted by  $K^{(1)}$ . E.g.  $(\Delta_n^2)^{(1)} = K_{n+1}$ . For a PL map  $f : K^{(1)} \rightarrow \mathbb{R}^2$  and a face  $R = \{A, B, C\}$  the closed polygonal line  $f(AB) \cup f(BC) \cup f(CA)$  is denoted by  $f(\partial R)$ .

A 2-hypergraph  $K$  is called **planar** (or PL embeddable into the plane) if there exists a PL embedding  $f : K^{(1)} \rightarrow \mathbb{R}^2$  such that for any vertex  $A$  and face  $R \not\ni A$  the image  $fA$  does not lie inside the polygon  $f(\partial R)$ . Cf. [Schöneborn 2004, Definition 3.0.6 for  $q = 2$ ].

For illustration observe that the complete 2-hypergraph  $\Delta_3^2$  on 4 vertices (i.e. the boundary of a tetrahedron) is not planar (and not even  $\mathbb{Z}_2$ -planar, see below) by the Topological Radon Theorem in the plane 2.2.2.

**Theorem 2.4.1** ([Gross and Rosen 1979]; cf. Proposition 1.2.2 and Theorem 3.2.1) *There is a polynomial algorithm for recognition planarity of 2-hypergraphs.*

In [Matoušek et al. 2011, Appendix A] it is explained that this result (even with linear algorithm) follows from the Kuratowski-type Halin–Jung planarity criterion for 2-hypergraphs (stated there). We present a different proof similar to proof of Proposition 1.2.2.b (Sect. 1.5). This proof illustrates the idea required for elementary formulation of the Özaydin Theorem 2.4.10.

A 2-hypergraph  $K$  is called  $\mathbb{Z}_2$ -**planar** if there exists a general position PL map  $f : K^{(1)} \rightarrow \mathbb{R}^2$  such that the images of any two non-adjacent edges intersect at an even number of points and for any vertex  $A$  and face  $R \not\ni A$  the image  $fA$  does not lie in the interior modulo 2 of  $f(\partial R)$ .

**Theorem 2.4.2** (cf. Theorems 1.5.3 and 3.3.5) *A 2-hypergraph is planar if and only if it is  $\mathbb{Z}_2$ -planar.*

This is proved using the Halin–Jung criterion [Matoušek et al. 2011, Appendix A].

Let  $K = (V, F)$  be a 2-hypergraph and  $f : K^{(1)} \rightarrow \mathbb{R}^2$  a general position PL map. Assign to any pair  $\{\sigma, \tau\}$  of non-adjacent edges the residue

$$|f\sigma \cap f\tau| \pmod 2.$$

Assign to any pair of a vertex  $A$  and a face  $R \not\ni A$  the residue

$$|fA \cap \text{int}_2 f(\partial R)| \pmod 2,$$

where  $\text{int}_2 f(\partial R)$  is the interior modulo 2 of  $f(\partial R)$ .

Denote by  $K^*$  the set of unordered pairs  $\{R_1, R_2\}$  of disjoint subsets  $R_1, R_2 \in V \sqcup E \sqcup F$  such that  $|R_1| + |R_2| = 4$ , where  $E$  is the set of edges. Then either both  $R_1$  and  $R_2$  are edges, or one of  $R_1, R_2$  is a face and the other one is a vertex. The obtained map  $K^* \rightarrow \mathbb{Z}_2$  is called **the (double) intersection cocycle** (modulo 2) of  $f$  for  $K$ . Note that  $K^* \supset (K^{(1)})^*$  and the intersection cocycle of  $f$  for  $K^{(1)}$  is the restriction of the intersection cocycle of  $f$  for  $K$ . The intersection cocycle for  $K$  of the map  $f : K^{(1)} \rightarrow \mathbb{R}^2$  from Example 1.5.4 is the extension to  $K^*$  by zeroes of the intersection cocycle for  $K^{(1)}$  described there.



Comparing the definitions of the Radon number and the intersection cocycle we see that for every general position PL map  $f : K_4 = (\Delta_3^2)^{(1)} \rightarrow \mathbb{R}^2$  the Radon number  $\rho(f)$  equals to the sum of the values of the intersection cocycle for  $\Delta_3^2$ .

By Proposition 2.3.1.c for any disjoint edge  $\sigma$  and face  $R$  we have

$$\sum_{A \in \sigma} |fA \cap \text{int}_2 f(\partial R)| = \sum_{\tau \subset R} |f\tau \cap f\sigma|.$$

Analogue of Proposition 1.5.6 is true for the intersection cocycle for 2-hypergraph, with the following definition. Let  $K$  be a 2-hypergraph and  $A$  its vertex which is not the end of an edge  $\sigma$ . An **elementary coboundary** of the pair  $(A, \sigma)$  is the map  $\delta_K(A, \sigma) : K^* \rightarrow \mathbb{Z}_2$  that assigns 1 to the pair  $\{R_i, R_j\}$  if  $R_i \supset A$  and  $R_j \supset \sigma$  for some  $i \neq j$ , and 0 to any other pair.

The subset of  $\delta_K(A, \sigma)^{-1}(1) \subset K^*$  corresponding to the map  $\delta_K(A, \sigma)$  is also called elementary coboundary. So  $\delta_{\Delta_3^2}(1, 23) = \{\{14, 23\}, \{1, 234\}\}$ , cf. Example 1.5.7.a.

**Proposition 2.4.3** *Under the Reidemeister move in Fig. 6.V the intersection cocycle changes by adding  $\delta_K(A, \sigma)$ .*

Two maps  $\nu_1, \nu_2 : K^* \rightarrow \mathbb{Z}_2$  are called **cohomologous** if

$$\nu_1 - \nu_2 = \delta_K(A_1, \sigma_1) + \dots + \delta_K(A_k, \sigma_k)$$

for some vertices  $A_1, \dots, A_k$  and edges  $\sigma_1, \dots, \sigma_k$  (not necessarily distinct).

**Lemma 2.4.4** (cf. Lemmas 1.5.8 and 2.2.3) *For any 2-hypergraph  $K$  the intersection cocycles of different general position PL maps  $K^{(1)} \rightarrow \mathbb{R}^2$  are cohomologous.*

**Proposition 2.4.5** (cf. Proposition 1.5.9) *A 2-hypergraph  $K$  is  $\mathbb{Z}_2$ -planar if and only if the intersection cocycle modulo 2 of some (or, equivalently, of any) general position PL map  $K^{(1)} \rightarrow \mathbb{R}^2$  is cohomologous to the zero map.*

This proposition follows by Lemma 2.4.4 and Proposition 2.4.3.

**Proof of Theorem 2.4.1** Take a 2-hypergraph  $K$ . To every pair  $A, \sigma$  of a vertex and an edge such that  $A \notin \sigma$  assign a variable  $x_{A,\sigma}$ . For every  $\{R, R'\} \in K^*$  denote by  $b_{R,R'} \in \mathbb{Z}_2$  the value of the extension to  $K^*$  by zeroes of the intersection cocycle for  $K^{(1)}$  described in Example 1.5.4. For every such pairs  $(A, \sigma)$  and  $\{R, R'\}$  let

$$a_{A,\sigma,R,R'} = \begin{cases} 1 & \text{either } (R \ni A \text{ and } R' \supset \sigma) \text{ or } (R' \ni A \text{ and } R \supset \sigma) \\ 0 & \text{otherwise} \end{cases}.$$

For every pair  $\{R, R'\} \in K^*$  consider the linear equation  $\sum_{A \notin \sigma} a_{A,\sigma,R,R'} x_{A,\sigma} = b_{R,R'}$  over  $\mathbb{Z}_2$ . By Theorem 2.4.2 and Proposition 2.4.5 planarity of  $K$  is equivalent to solvability of this system of equations. This can be checked in polynomial time.  $\square$

### 2.4.2 Intersections with Signs for 2-Hypergraphs

Let  $K$  be a 2-hypergraph and  $f : K^{(1)} \rightarrow \mathbb{R}^2$  a general position PL map. Orient the edges and faces of  $K$ , i.e. choose some cyclic orderings on the subsets that are edges and faces. Assign to every ordered pair

- $(\sigma, \tau)$  of non-adjacent edges the algebraic intersection number  $f\sigma \cdot f\tau$  (defined in Sect. 1.5.4).
- $(A, R)$  or  $(R, A)$  of a vertex  $A$  and a face  $R \not\ni A$  minus the winding number  $-fA \cdot f(\partial R)$  of  $f(\partial R)$  around  $A$  (defined in Sect. 2.3.1).

Denote by  $\tilde{K}$  the set of ordered pairs  $(R_1, R_2)$  of disjoint subsets  $R_1, R_2 \in V \sqcup E \sqcup F$  such that  $|R_1| + |R_2| = 4$ . The obtained map  $\cdot : \tilde{K} \rightarrow \mathbb{Z}$  is called **the integral intersection cocycle** of  $f$  for  $K$  (and for given orientations).

For oriented 2-element set  $AB$  denote  $[AB : B] = 1$  and  $[AB : A] = -1$ . For oriented 3-element set  $ABC$  denote  $[ABC : BA] = [ABC : CB] = [ABC : AC] = 1$  and  $[ABC : AB] = [ABC : BC] = [ABC : CA] = -1$ . For other oriented sets  $R, R' \in V \sqcup E \sqcup F$  define  $[R : R'] = 0$ .

By Proposition 2.2.1.b for any disjoint edge  $\sigma$  and face  $R$  we have

$$\sum_{A \in \sigma} [\sigma : A](f(\partial R) \cdot fA) = \sum_{\tau \subset R} [R : \tau](f\sigma \cdot f\tau).$$

Analogue of Proposition 1.5.6 is true for the integral intersection cocycle for 2-hypergraph, with the following definition. Let  $K$  be a 2-hypergraph whose edges and faces are oriented, and  $A$  a vertex which is not the end of an edge  $\sigma$ . **An elementary super-symmetric coboundary** of the pair  $(A, \sigma)$  is the map  $\delta_K(A, \sigma) : \tilde{K} \rightarrow \mathbb{Z}$  that assigns

$$-[\tau : A] \text{ to } (\sigma, \tau), \quad [\tau : A] \text{ to } (\tau, \sigma) \quad \text{and} \quad [R : \sigma] \text{ both to } (A, R) \text{ and } (R, A).$$

In other words,

$$\delta_K(A, \sigma)(R_1, R_2) := [R_1 : A][R_2 : \sigma] + (-1)^{(|R_1|-1)(|R_2|-1)}[R_2 : A][R_1 : \sigma].$$

**Proposition 2.4.6** *Under the Reidemeister move in Fig. 6.V the integer intersection cocycle changes by adding  $\delta_K(A, \sigma)$ .*

Maps  $v_1, v_2 : \tilde{K} \rightarrow \mathbb{Z}$  are called **super-symmetrically cohomologous** if

$$v_1 - v_2 = m_1 \delta_K(A_1, \sigma_1) + \dots + m_k \delta_K(A_k, \sigma_k)$$

for some vertices  $A_1, \dots, A_k$ , edges  $\sigma_1, \dots, \sigma_k$  and integer numbers  $m_1, \dots, m_k$  (not necessarily distinct).

The integral analogues of Lemma 2.4.4 and Proposition 2.4.5 are correct, cf. Lemma 1.5.12.

**Proposition 2.4.7** (cf. Propositions 1.5.13, 2.4.11) *For any 2-hypergraph  $K$  twice the integral intersection cocycle of any general position PL map  $K^{(1)} \rightarrow \mathbb{R}^2$  is super-symmetrically cohomologous to the zero map.*

This follows by the integral analogue of Lemma 2.4.4 and the analogue of Assertion 1.5.10.c for 2-hypergraphs.

### 2.4.3 Elementary Formulation of the Özaydin Theorem

Let  $K = (V, F)$  be a 2-hypergraph and  $f : K^{(1)} \rightarrow \mathbb{R}^2$  a general position PL map. Denote by  $E$  the set of edges. Orient the edges and faces of  $K$ , i.e. choose some cyclic orderings on the subsets that are edges and faces. Denote by  $K^L$  the set of ordered  $r$ -tuples  $(R_1, \dots, R_r)$  of pairwise disjoint sets from  $V \sqcup E \sqcup F$  such that either

- (A) two of the sets  $R_1, \dots, R_r$  are edges and the other are faces, or
- (B) one of the sets  $R_1, \dots, R_r$  is a vertex and the other are faces.<sup>13</sup>

Clearly,

- if  $|V| < 3r - 2$ , then  $K^L = \emptyset$ .
- $(\Delta_{3r-3}^2)^L$  is the set of ordered partitions of  $[3r - 2]$  into  $r$  non-empty subsets, every subset having at most 3 elements.

The  **$r$ -fold intersection cocycle** of  $f$  for  $K$  (and for given orientations) is a map  $K^L \rightarrow \mathbb{Z}$  that assigns to  $r$ -tuple  $(R_1, \dots, R_r)$  the number  $f R_1 \dots \dots f R_r$  or  $-f R_1 \dots \dots f R_r$  in cases (A) or (B) above, respectively.<sup>14</sup>

*Super-symmetric  $r$ -fold elementary coboundary and cohomology* are defined analogously to the case  $r = 2$  considered in Sect. 2.4.2. The  $r$ -fold analogue of Lemma 2.4.4 is correct with a similar proof.

**Remark 2.4.8** It would be interesting to know if the  $r$ -fold analogue of Proposition 2.4.5 is correct.

A map  $K^L \rightarrow \mathbb{Z}$  is called (super-symmetrically cohomologically) **trivial** if it is super-symmetrically cohomologous to the zero map.

Proofs of the Topological Tverberg Theorem in the plane 2.3.2 (mentioned after the statement) show that *if  $r$  is a prime power, then for the complete 2-hypergraph  $K = \Delta_{3r-3}^2$  the  $r$ -fold intersection cocycle of every general position PL map  $K^{(1)} \rightarrow \mathbb{R}^2$  is non-trivial.*

**Remark 2.4.9** The number from Problem 2.3.7 is the sum of some values of the three-fold intersection cocycle (with certain coefficients).

<sup>13</sup> This is the  $d(r - 1)$ -skeleton of the *simplicial  $r$ -fold deleted product* of  $K$ . Cf. [Skopenkov 2018a, Sect. 1.4].

<sup>14</sup> This agrees up to sign with the definition of [Mabillard and Wagner 2015, Lemma 41.b] because by [Mabillard and Wagner 2015, (13) in p. 17]  $\varepsilon_{2,2,\dots,2,0}$  is even and  $\varepsilon_{2,2,\dots,2,1,1}$  is odd.

The  $r$ -fold intersection cocycle depends on an arbitrary choice of orientations, but the triviality condition defined below does not.

**Theorem 2.4.10** (Özaydin) *If  $r$  is not a prime power, then for every 2-hypergraph  $K$  the  $r$ -fold intersection cocycle of any general position PL map  $K^{(1)} \rightarrow \mathbb{R}^2$  is trivial.*

This is implied by the following Proposition 2.4.11.b because when  $r$  is not a prime power, the numbers  $r!/p^{\alpha_{r,p}}$ , for all primes  $p < r$ , have no common multiple. Here  $\alpha_{r,p} = \sum_{k=1}^{\infty} \left\lfloor \frac{r}{p^k} \right\rfloor$  is the power of  $p$  in the prime factorisation of  $r!$ .

**Proposition 2.4.11** (cf. Proposition 2.4.7) *Let  $K$  be a 2-hypergraph and  $f : K^{(1)} \rightarrow \mathbb{R}^2$  a general position PL map.*

- (a) *Threefold intersection cocycle of  $f$  multiplied by 3 is trivial.*
- (b) *If  $r$  is not a power of a prime  $p$ , then the  $r$ -fold intersection cocycle of  $f$  multiplied by  $r!/p^{\alpha_{r,p}}$  is trivial.*

Part (a) is a special case of part (b) for  $r = p + 1 = 3$ .

The usual form of the Özaydin Theorem [Skopenkov 2018a, Theorem 3.3] states the existence of certain *equivariant maps*. Theorem 2.4.10 is equivalent to that statement because the  $r$ -fold intersection cocycle equals to the *obstruction cocycle* [Mabillard and Wagner 2015, Lemma 41.b] which is a complete obstruction to the existence of certain equivariant map [Mabillard and Wagner 2015, Theorem 40]. Analogously Proposition 2.4.11 is equivalent to the corresponding intermediate result from the proof of the ‘usual’ Özaydin Theorem. See simplified exposition in survey [Skopenkov 2018a, Sect. 3.2].

It would be interesting to obtain a direct proof of Proposition 2.4.11, cf. the above direct proofs of Propositions 1.5.13 and 2.4.7.

### 3 Conclusion: Higher-Dimensional Generalizations

#### 3.1 Radon, Tverberg and van Kampen–Flores Theorems

**Theorem 3.1.1** (Radon, cf. Theorem 2.1.1) *For every integer  $d > 0$  any  $d + 2$  points in  $\mathbb{R}^d$  can be decomposed into two groups such that the convex hulls of the groups intersect.*

**Theorem 3.1.2** (Linear van Kampen–Flores, cf. Proposition 1.1.1.a) *For every integer  $k > 0$  from any  $2k + 3$  points in  $\mathbb{R}^{2k}$  one can choose two disjoint  $(k + 1)$ -tuples whose convex hulls intersect.*

This implies linear non-realizability in  $\mathbb{R}^{2k}$  of the complete  $(k + 1)$ -homogeneous hypergraph on  $2k + 3$  vertices.

**Theorem 3.1.3** (Tverberg, see proof in [Matoušek 2008]; cf. Theorem 2.1.5) *For every integers  $d, r > 0$  any  $(d + 1)(r - 1) + 1$  points in  $\mathbb{R}^d$  can be decomposed into  $r$  groups such that all the  $r$  convex hulls of the groups have a common point.*

Here the number  $(d + 1)(r - 1) + 1$  could be remembered by remembering simple examples showing that this number is the least possible [Matoušek 2008, Excercise 2

to Sect. 6.4]. Analogous remark holds for Theorems 3.1.1, 3.1.2, 3.1.5, 3.1.6 and (the proved case when  $r$  is a power of a prime) of Conjecture 3.1.7.

**Conjecture 3.1.4** (Linear  $r$ -fold van Kampen–Flores) For every integers  $k, r > 0$  from any  $(r - 1)(kr + 2) + 1$  points in  $\mathbb{R}^{kr}$  one can choose  $r$  pairwise disjoint  $(k(r - 1) + 1)$ -tuples whose  $r$  convex hulls have a common point.

This is true for a prime power  $r$  [Volovikov 1996b] and is an open problem for other  $r$  [Frick 2017, beginning of Sect. 2].

Here the number  $(r - 1)(kr + 2) + 1$  could be remembered by remembering the following simple examples showing that this number is the least possible. Take in  $\mathbb{R}^{kr}$  the vertices of a  $kr$ -dimensional simplex and its center. Either take every of these  $kr + 2$  points with multiplicity  $r - 1$  or for every point take close  $r - 1$  points in general position. We obtain  $(r - 1)(kr + 2)$  points in  $\mathbb{R}^{kr}$  such that for any  $r$  pairwise disjoint  $(k(r - 1) + 1)$ -tuples all the  $r$  convex hulls of the tuples do not have a common point.

Denote by  $\Delta_N$  the  $N$ -dimensional simplex.

**Theorem 3.1.5** (Topological Radon theorem, [Bajmóczy and Bárány 1979], cf. Theorem 2.2.2) *For any continuous map  $\Delta_{d+1} \rightarrow \mathbb{R}^d$  there are two disjoint faces whose images intersect.*

**Theorem 3.1.6** (van Kampen–Flores, cf. Theorem 1.4.1) *For any continuous map  $\Delta_{2k+2} \rightarrow \mathbb{R}^{2k}$  there are two disjoint  $k$ -dimensional faces whose images intersect.*

This implies non-realizability in  $\mathbb{R}^{2k}$  of the complete  $(k + 1)$ -homogeneous hypergraph on  $2k + 3$  vertices.

The Topological Radon and the van Kampen–Flores Theorems 3.1.5 and 3.1.6 generalize Radon and the Linear van Kampen–Flores Theorems 3.1.1 and 3.1.2. These results are nice in themselves, and are also interesting because they are corollaries of the celebrated Borsuk–Ulam Theorem (see e.g. [Skopenkov 2018a, Sect. 2.1]), of which the topological Radon Theorem 3.1.5 is also a simplicial version. The PL (piecewise-linear) versions of the Topological Radon and the Linear van Kampen–Flores Theorems 3.1.5 and 3.1.6 are as interesting and non-trivial as the stated topological versions, see Remark 3.3.1.

The above results have ‘quantitative version’ analogous to Propositions 1.1.1.b and 2.1.2, Lemmas 1.4.3 and 2.2.3, see e.g. [Skopenkov 2018a, Sect. 4]. For direct proofs of some implications between these results see [Skopenkov 2018a, Sect. 4].

**Conjecture 3.1.7** (topological Tverberg conjecture) For every integers  $r, d$  and any continuous map  $f : \Delta_{(d+1)(r-1)} \rightarrow \mathbb{R}^d$  there are pairwise disjoint faces  $\sigma_1, \dots, \sigma_r \subset \Delta_{(d+1)(r-1)}$  such that  $f\sigma_1 \cap \dots \cap f\sigma_r \neq \emptyset$ .

This conjecture generalizes both the Tverberg and the topological Radon Theorems 3.1.3 and 3.1.5. This conjecture is true for a prime power  $r$  [Bárány et al. 1981, Özaydin 2019, Volovikov 1996a], is false for  $r$  not a prime power and  $d \geq 2r + 1$  by Remark 3.3.1.a and Theorem 3.3.3.a below, and is an open problem for  $r$  not a prime power and  $d \leq 2r$  (e.g. for  $d = 2$  and  $r = 6$ ).

**Conjecture 3.1.8** (*r*-fold van Kampen–Flores) For every integers  $r, k > 0$  and any continuous map  $f: \Delta_{(kr+2)(r-1)} \rightarrow \mathbb{R}^{kr}$  there are pairwise disjoint  $k(r-1)$ -dimensional faces  $\sigma_1, \dots, \sigma_r \subset \Delta_{(kr+2)(r-1)}$  such that  $f\sigma_1 \cap \dots \cap f\sigma_r \neq \emptyset$ .

This is true for a prime power  $r$  [Sarkaria 1991], [Volovikov 1996b, Corollary in Sect. 1], is false for  $r$  not a prime power and  $k \geq 2$  by Theorem 3.3.3.a below, and is an open problem for  $r$  not a prime power and  $k = 1$ .

The arguments for results of this subsection form a beautiful and fruitful interplay between combinatorics, algebra, geometry and topology. Recall that more motivation, detailed description of references and proofs can found in the surveys mentioned in the ‘historical notes’ of the Introduction.

### 3.2 Realizability of Higher-Dimensional Hypergraphs

Definition of a  $k$ -hypergraph is recalled in Sect. 2.4.1. The *body*  $|K|$  of a hypergraph  $K = (V, F)$  is the union of faces corresponding to  $F$  of a simplex with the set of vertices  $V$ . Embeddability (linear, PL or topological) of a hypergraph into  $\mathbb{R}^d$  is defined as the existence of an injective (linear, PL or continuous) map of its body into  $\mathbb{R}^d$ . For  $d = k = 2$  this is equivalent to the definition of Sect. 2.4.1.

Every  $k$ -hypergraph linearly (and then PL and topologically) embeds into  $\mathbb{R}^{2k+1}$ . Here the number  $2k + 1$  is the least possible: for any  $k$  there is a  $k$ -hypergraph topologically (and then PL and linearly) non-embeddable into  $\mathbb{R}^{2k}$ . As an example one can take

- the complete  $(k + 1)$ -hypergraph on  $2k + 3$  vertices, or the  $k$ -skeleton of the  $(2k + 2)$ -simplex (by the van Kampen–Flores Theorem 3.1.6; this hypergraph is  $K_5$  for  $k = 1$ ),
- the  $(k + 1)$ -th join power of the three-point set (see a short proof in [Skopenkov 2008, Sect. 5], [Skopenkov 2019, Sect. 5.8.4 ‘Topological non-realizability of hypergraphs’]; this hypergraph is  $K_{3,3}$  for  $k = 1$ ),
- the  $k$ -th power of a non-planar graph (conjectured by Menger in 1929, proved in [Ummel 1978, Skopenkov 2003], see exposition in [Skopenkov 2014]).

**Theorem 3.2.1** (a) (cf. Proposition 1.1.3) For every fixed  $d, k$  there is an algorithm for recognizing the linear embeddability of  $k$ -hypergraphs in  $\mathbb{R}^d$ .

- (b) (cf. Proposition 1.2.2 and Theorem 2.4.1) For every fixed  $d, k$  such that  $d \geq \frac{3k+3}{2}$  there is a polynomial algorithm for recognizing the PL embeddability of  $k$ -hypergraphs in  $\mathbb{R}^d$ .
- (c) [Matoušek et al. 2018] There is a polynomial algorithm for recognizing the PL embeddability of 2-hypergraphs in  $\mathbb{R}^3$ .

In [Čadek et al. 2019, text after Theorem 1.4], [Skopenkov and Tancer 2017, Sect. 1] it is explained that part (b) follows from [Čadek et al. 2019, Theorem 1.1] and the Weber ‘configuration spaces’ embeddability criterion (stated there or in the survey [Skopenkov 2008, Sect. 8]).

The assumption of part (b) is fulfilled when  $d = 2k \geq 6$ . The idea of proof for  $d = 2k \geq 6$  generalizes proof Proposition 1.2.2.b, see Sect. 1.5 and [Skopenkov 2019,

Sect. 5.6 ‘An algorithm for recognition realizability of hypergraphs’]. Proof of (b) for the general case is more complicated.

**Conjecture 3.2.2** For every fixed  $d, k$  such that  $3 \leq d \leq \frac{3k}{2} + 1$  the algorithmic problem of recognizing linear embeddability of  $k$ -hypergraphs into  $\mathbb{R}^d$  is NP hard.<sup>15</sup>

**Theorem 3.2.3** (a) For every fixed  $d, k$  such that  $3 \leq d \leq \frac{3k}{2} + 1$  the algorithmic problem of recognizing PL embeddability of  $k$ -hypergraphs into  $\mathbb{R}^d$  is NP-hard [de Mesmay et al. 2019, Matoušek et al. 2011].

(b) For every fixed  $d, k$  such that either  $4 \leq k \in \{d - 1, d\}$  or  $8 \leq d \leq \frac{3k+1}{2}$  there is no algorithm recognizing PL embeddability of  $k$ -hypergraphs into  $\mathbb{R}^d$ .

See a simpler exposition of part (a) for  $d \geq 4$  in [Skopenkov and Tancer 2017] (where also a generalization was proved). Part (b) for  $5 \leq d \in \{k, k + 1\}$  is deduced in [Matoušek et al. 2011, Theorem 1.1] from the Novikov theorem on unrecognizability of the sphere, and for  $8 \leq d \leq \frac{3k+1}{2}$  is announced in [Filakovský et al. 2019]. I hope that  $\beta$ -invariant of [Skopenkov 2007] could be used to extend part (b) to  $d = \frac{3k}{2} + 1$ .

For a ‘3- and 2-dimensional explanation’ of ideas of proof see Proposition 1.1.2, [Skopenkov 2018c, Proposition 1.6.1] and [Skopenkov 2019, Sect. 5.6 ‘An algorithm for recognition realizability of hypergraphs’].

The following table summarizes the above results on the algorithmic problem of recognizing PL embeddability of  $k$ -hypergraphs into  $\mathbb{R}^d$  (+ = always embeddable, P = polynomial-time solvable, D = algorithmically decidable, NPh = NP-hard, UD = algorithmically undecidable).

$k \setminus d$	2	3	4	5	6	7	8	9	10	11	12	13	14
1	P	+	+	+	+	+	+	+	+	+	+	+	+
2	P	D,NPh	NPh	+	+	+	+	+	+	+	+	+	+
3		D,NPh	NPh	NPh	P	+	+	+	+	+	+	+	+
4			NPh	UD	NPh	NPh	P	+	+	+	+	+	+
5				UD	UD	NPh	UD	P	P	+	+	+	+
6					UD	UD	UD	UD	NPh	P	P	+	+
7						UD	UD	UD	UD	UD	P	P	P

<sup>15</sup> Here NP-hardness means that using a device which solves this problem EMBED(k,d) at 1 step, we can construct an algorithm which is polynomial in  $n$  and which recognizes if a boolean function of  $n$  variables is identical zero, the function given as a disjunction of some conjunctions of variables or their negations (e.g.  $f(x_1, x_2, x_3, x_4) = x_1x_2\bar{x}_3 \vee \bar{x}_2x_3x_4 \vee \bar{x}_1x_2x_4$ ). M. Tancer suggests that it is plausible to approach the conjecture the same way as in [Matoušek et al. 2011, Skopenkov and Tancer 2017]. Namely, one can possibly triangulate the gadgets in advance and glue them together so that the ‘embeddable gadgets’ would be linearly embeddable with respect to the prescribed triangulations. By using the same triangulation on gadgets of same type, one can achieve polynomial size triangulation. Realization of this idea should be non-trivial.

### 3.3 Algorithmic Recognition of Almost Realizability of Hypergraphs

A (continuous, or PL) map  $f: K \rightarrow \mathbb{R}^d$  from a hypergraph  $K$  is an **almost  $r$ -embedding** if  $f\sigma_1 \cap \dots \cap f\sigma_r = \emptyset$  whenever  $\sigma_1, \dots, \sigma_r$  are pairwise disjoint faces of  $K$ .

**Remark 3.3.1** (a) In this language the Topological Tverberg Conjecture 3.1.7 and the  $r$ -fold van Kampen–Flores Conjecture 3.1.8 state that

( $TT_{r,d}$ ) for every integers  $r, d$  there are no almost  $r$ -embeddings  $\Delta_{(d+1)(r-1)} \rightarrow \mathbb{R}^d$ .

( $VKF_{r,k}$ ) for every integers  $r, k$  there are no almost  $r$ -embeddings of the union of  $k(r-1)$ -faces of  $\Delta_{(kr+2)(r-1)}$  in  $\mathbb{R}^{kr}$ .

We have ( $TT_{r,kr+1}$ )  $\Rightarrow$  ( $VKF_{r,k}$ ). This was proved in [Gromov 2010, 2.9.c] and implicitly rediscovered in [Blagojević et al. 2014, Lemma 4.1.iii and 4.2], [Frick 2015, proof of Theorem 4]; see survey [Skopenkov 2018a, Constraint Lemma 1.8 and Historical Remark 1.10].

(b) The notion of an almost 2-embedding implicitly appeared in studies realizability of graphs and hypergraphs (Theorems 1.4.1, 3.1.6 and 3.2.1.b). It was explicitly formulated in the Freedman–Krushkal–Teichner work on the van Kampen obstruction [Freedman et al. 1994].

(c) Any sufficiently small perturbation of an almost  $r$ -embedding is again an almost  $r$ -embedding. So the existence of a *continuous* almost  $r$ -embedding is equivalent to the existence of a *PL* almost  $r$ -embedding, and to the existence of a *general position PL* almost  $r$ -embedding. Cf. [Skopenkov 2020, Approximation Lemma 1.4.6].

See more introduction in [Skopenkov 2018a, Sect. 1.2].

**Problem 3.3.2** Which 2-hypergraphs admit a PL map to  $\mathbb{R}^2$  without triple points? Which 2-hypergraphs are almost 3-embeddable in  $\mathbb{R}^2$ ? Are there algorithms for checking the above properties of 2-hypergraphs? Same questions for  $\mathbb{R}^2$  replaced by  $\mathbb{R}^3$ , or for ‘triple’ and ‘almost 3-embeddable’ replaced by ‘ $r$ -tuple’ and ‘almost  $r$ -embeddable’.<sup>16</sup>

**Theorem 3.3.3** *If  $r$  is not a prime power, then*

- (a) *for any  $k \geq 2$  there is an almost  $r$ -embedding of any  $k(r-1)$ -hypergraph in  $\mathbb{R}^{kr}$  [Mabillard and Wagner 2015, Avvakumov et al. 2019b],*
- (b) *there is an almost  $r$ -embedding of any  $s$ -hypergraph in  $\mathbb{R}^{s + \lceil \frac{s+3}{r} \rceil}$  [Avvakumov et al. 2019a].*

Part (a) follows from Theorems 3.3.5 and 3.3.6 below.

**Theorem 3.3.4** ([Mabillard and Wagner 2016, Avvakumov et al. 2019b, Skopenkov 2017a, Skopenkov 2017b]) *For every fixed  $k, d, r$  such that either  $rd \geq (r+1)k+3$  or  $d = 2r = k+2 \neq 4$  there is a polynomial algorithm for checking PL almost  $r$ -embeddability of  $k$ -hypergraphs in  $\mathbb{R}^d$ .*

<sup>16</sup> Analogous problems for maps from graphs to the line are investigated in studies of *cutwidth*, see [Thilikos et al. 2005, Lin and Yang 2004, Khoroshavkina 2019] and references therein.



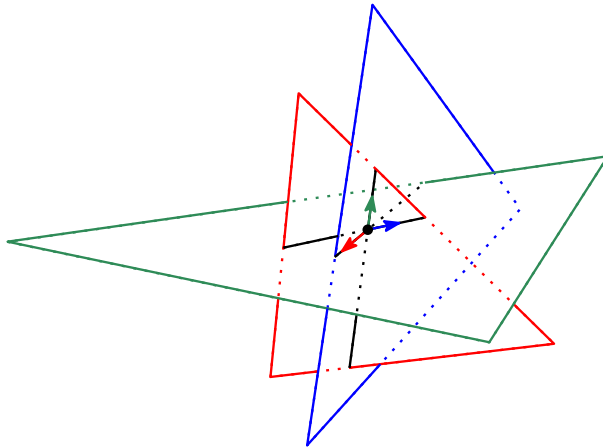


Fig. 10 A 3-fold point and its 3-intersection sign

For  $(r - 1)d = rk$  Theorem 3.3.4 was deduced in [Mabillard and Wagner 2015, Avvakumov et al. 2019b] from Theorem 3.3.5 below.

For a version of Theorem 3.2.3.b with ‘embeddability’ replaced by ‘almost 2-embeddability’ see [Skopenkov and Tancer 2017].

We shall state the Özaydin Theorem [Özaydin 2019] (generalizing the Özaydin Theorem in the plane 2.4.10) in the simplified form of Theorem 3.3.6 below. This is different from the standard form [Skopenkov 2018a, Theorem 3.3] but is equivalent to the standard form by a proposition of Mabillard–Wagner [Skopenkov 2018a, Proposition 3.4]. For the statement we need the following definitions.

Let  $K$  be a  $k(r - 1)$ -hypergraph for some  $k \geq 1, r \geq 2$ , and  $f : K \rightarrow \mathbb{R}^{kr}$  a PL map in general position.

Then preimages  $y_1, \dots, y_r \in K$  of any  $r$ -fold point  $y \in \mathbb{R}^{kr}$  (i.e. of a point having  $r$  preimages) lie in the interiors of  $k(r - 1)$ -dimensional simplices of  $K$ . Choose arbitrarily an orientation for each of the  $k(r - 1)$ -simplices. By general position,  $f$  is affine on a neighborhood  $U_j$  of  $y_j$  for each  $j = 1, \dots, r$ . Take a positive basis of  $k$  vectors in the oriented normal space to oriented  $fU_j$ . The  $r$ -intersection sign of  $y$  is the sign  $\pm 1$  of the basis in  $\mathbb{R}^{kr}$  formed by  $r$  such  $k$ -bases. See Figs. 3 and 10.

This is classical for  $r = 2$ , see Sect. 1.3, and is analogous for  $r \geq 3$ , cf. Sect. 2.3.3, [Mabillard and Wagner 2015, Sect. 2.2].

We call the map  $f$  a  $\mathbb{Z}$ -almost  $r$ -embedding if  $f\sigma_1 \cdot \dots \cdot f\sigma_r = 0$  whenever  $\sigma_1, \dots, \sigma_r$  are pairwise disjoint simplices of  $K$ . Here the algebraic  $r$ -intersection number  $f\sigma_1 \cdot \dots \cdot f\sigma_r \in \mathbb{Z}$  is defined as the sum of the  $r$ -intersection signs of all  $r$ -fold points  $y \in f\sigma_1 \cap \dots \cap f\sigma_r$ . The sign of  $f\sigma_1 \cdot \dots \cdot f\sigma_r$  depends on an arbitrary choice of orientations for each  $\sigma_i$  and on the order of  $\sigma_1, \dots, \sigma_r$ , but the condition  $f\sigma_1 \cdot \dots \cdot f\sigma_r = 0$  does not. See Fig. 4 for  $r = 2$ .

Clearly, an almost  $r$ -embedding is a  $\mathbb{Z}$ -almost  $r$ -embedding.

**Theorem 3.3.5** ([Mabillard and Wagner 2015, Avvakumov et al. 2019b]; cf. Theorems 1.5.3 and 2.4.2) *If  $k \geq 2, k + r \geq 5$  and there is a  $\mathbb{Z}$ -almost  $r$ -embedding of a  $k(r - 1)$ -hypergraph  $K$  in  $\mathbb{R}^{kr}$ , then there is an almost  $r$ -embedding of  $K$  in  $\mathbb{R}^{kr}$ .*

**Theorem 3.3.6** (cf. [Avvakumov et al. 2019a, Theorem 4], [Avvakumov and Karasev 2019, Theorem 5.1]) *If  $r$  is not a prime power and  $k \geq 2$ , then there is a  $\mathbb{Z}$ -almost  $r$ -embedding of any  $k(r - 1)$ -hypergraph in  $\mathbb{R}^{kr}$ .*

## References

- Arnold, V.I.: Topological invariants of plane curves and caustics, University Lecture Series, vol. 5. American Mathematical Society, Providence (1995)
- Avvakumov, S., Karasev, R.: Envy-free division using mapping degree (2019). [arXiv:1907.11183](https://arxiv.org/abs/1907.11183)
- Avvakumov, S., Karasev, R., Skopenkov, A.: Stronger counterexamples to the topological Tverberg conjecture. (2019a). [arxiv:1908.08731](https://arxiv.org/abs/1908.08731) (submitted)
- Avvakumov S, Mabilard, I., Skopenkov, A., Wagner, U.: Eliminating higher-multiplicity intersections, III. Codimension 2, Israel J. Math. (to appear). (2019b). [arxiv:1511.03501](https://arxiv.org/abs/1511.03501)
- Bajmóczy, E.G., Bárány, I.: On a common generalization of Borsuk's and Radon's theorem. Acta Math. Acad. Sci. Hungar. **34**(3), 347–350 (1979)
- Bárány, I., Shlosman, S.B., Szűcs, A.: On a topological generalization of a theorem of Tverberg. J. Lond. Math. Soc. (II. Ser.) **23**, 158–164 (1981)
- Bárány, I., Blagojević, P.V.M., Ziegler, G.M.: Tverberg's theorem at 50: extensions and counterexamples. Not. AMS **63**(7), 732–739 (2016)
- Blagojević, P.V.M., Ziegler, G.M.: Beyond the Borsuk–Ulam theorem: the topological Tverberg story. In: Loeb, M., Nešetřil, J., Thomas, R. (eds.) A Journey Through Discrete Mathematics, pp. 273–341, (2016). [arXiv:1605.07321v2](https://arxiv.org/abs/1605.07321v2)
- Blagojević, P.V.M., Frick, F., Ziegler, G.M.: Tverberg plus constraints. Bull. Lond. Math. Soc. **46**(5), 953–967 (2014). [arXiv: 1401.0690](https://arxiv.org/abs/1401.0690)
- Blagojević, P.V.M., Matschke, B., Ziegler, G.M.: Optimal bounds for the colored Tverberg problem. J. Eur. Math. Soc. **17**(4), 739–754 (2015). [arXiv:0910.4987](https://arxiv.org/abs/0910.4987)
- Boyer, J.M., Myrvold, W.J.: On the cutting edge: simplified  $O(n)$  planarity by edge addition. J. Graph Alg. Appl. **8**(3), 241–273 (2004)
- Čadek, M., Krčál, M., Vokřínek, L.: Algorithmic solvability of the lifting-extension problem (2019). [arXiv:1307.6444](https://arxiv.org/abs/1307.6444)
- Chernov, A., Daynyak, A., Glibichuk, A., Ilyinskiy, M., Kupavskiy, A., Raigorodskiy, A., Skopenkov, A.: Elements of Discrete Mathematics As a Sequence of Problems (in Russian), MCCME, Moscow (2016). <http://www.mccme.ru/circles/oim/discrbook.pdf>. Accessed 7 Nov 2019
- de Mesmay, A., Rieck, Y., Sedgwick, E., Tancer, M.: Embeddability in  $\mathbb{R}^3$  is NP-hard (2019). [arXiv:1708.07734](https://arxiv.org/abs/1708.07734)
- Enne, A., Ryabichev, A., Skopenkov, A., Zaitsev, T.: Invariants of graph drawings in the plane (2019). <http://www.turgor.ru/lktg/2017/6/index.htm>. Accessed 7 Nov 2019
- Filakovskiy, M., Wagner, U., Zhechev, S.: Embeddability of simplicial complexes is undecidable. Oberwolfach reports (2019) (to appear)
- Fokkink, R.: A forgotten mathematician. Eur. Math. Soc. Newsl. **52**, 9–14 (2004)
- Freedman, M.H., Krushkal, V.S., Teichner, P.: Van Kampen's embedding obstruction is incomplete for 2-complexes in  $\mathbb{R}^4$ . Math. Res. Lett. **1**, 167–176 (1994)
- Frick, F.: Counterexamples to the topological Tverberg conjecture, Oberwolfach reports (2015). [arXiv:1502.00947](https://arxiv.org/abs/1502.00947)
- Frick, F.: On affine Tverberg-type results without continuous generalization (2017). [arXiv:1702.05466](https://arxiv.org/abs/1702.05466)
- Gromov, M.: Singularities, expanders and topology of maps. Part 2: from combinatorics to topology via algebraic isoperimetry. Geom. Funct. Anal. **20**(2), 416–526 (2010)
- Gross, J.L., Rosen, R.H.: A linear time planarity algorithm for 2-complexes. J. ACM **26**(4), 611–617 (1979)
- Hopcroft, J., Tarjan, R.E.: Efficient planarity testing. J. Assoc. Comput. Mach. **21**(4), 549–568 (1974)
- Khoroshavkina, N.: A simple characterization of graphs of cutwidth (2019). [arXiv:1811.06716](https://arxiv.org/abs/1811.06716)
- Lin, Y., Yang, A.: On 3-cutwidth critical graphs. Discret. Math. **275**, 339–346 (2004)
- Lovasz, L., Schrijver, A.: A Borsuk theorem for antipodal links and a spectral characterization of linklessly embeddable graphs. Proc. AMS **126**(5), 1275–1285 (1998)
- Mabilard, I., Wagner, U.: Eliminating Higher-Multiplicity Intersections, I. A Whitney trick for Tverberg-Type problems (2015). [arXiv:1508.02349](https://arxiv.org/abs/1508.02349)

- Mabillard, I., Wagner, U.: Eliminating Higher-Multiplicity Intersections, II. The deleted product criterion in the  $r$ -metastable range (2016). [arXiv:1601.00876](#)
- Matoušek, J.: Using the Borsuk-Ulam theorem: lectures on topological methods in combinatorics and geometry. Springer, Berlin (2008)
- Matoušek, J., Tancer, M., Wagner, U.: Hardness of embedding simplicial complexes in  $\mathbb{R}^d$ . *J. Eur. Math. Soc.* **13**(2), 259–295 (2011). [arXiv:0807.0336](#)
- Matoušek, J., Tancer, M., Wagner, U.: A geometric proof of the colored Tverberg theorem. *Discret. Comp. Geom.* **47**(2), 245–265 (2012). [arXiv:1008.5275](#)
- Matoušek, J., Sedgwick, E., Tancer, M., Wagner, U.: Embeddability in the 3-sphere is decidable. *J. ACM* **65**(1), 1–49 (2018). [arXiv:1402.0815](#)
- Özaydin, M.: Equivariant maps for the symmetric group, unpublished (2019). <http://minds.wisconsin.edu/handle/1793/63829>. Accessed 7 Nov 2019
- Sarkaria, K.S.: A generalized Van Kampen-Flores theorem. *Proc. Am. Math. Soc.* **111**, 559–565 (1991)
- Schöneborn, T.: On the topological Tverberg theorem (2004). [arXiv:math/0405393](#)
- Schöneborn, T., Ziegler, G.: The topological Tverberg Theorem and winding numbers. *J. Combin. Theory Ser. A* **112**(1), 82–104 (2005). [arXiv:math/0409081](#)
- Shlosman, S.: Topological Tverberg theorem: the proofs and the counterexamples. *Russ. Math. Surv.* **73**(2), 175–182 (2018). [arXiv:1804.03120](#)
- Skopenkov, M.: Embedding products of graphs into Euclidean spaces. *Fund. Math.* **179**, 191–198 (2003). [arXiv:0808.1199](#)
- Skopenkov, A.: A new invariant and parametric connected sum of embeddings. *Fund. Math.* **197**, 253–269 (2007). [arXiv:math/0509621](#)
- Skopenkov, A.: Embedding and knotting of manifolds in Euclidean spaces. *Lond. Math. Soc. Lect. Notes* **347**, 248–342 (2008). [arXiv:math/0604045](#)
- Skopenkov, A.: Realizability of hypergraphs and Ramsey link theory. (2014). [arxiv:1402.0658](#)
- Skopenkov, A.: Eliminating higher-multiplicity intersections in the metastable dimension range (2017a). [arxiv:1704.00143](#)
- Skopenkov, A.: On the Metastable Mabillard–Wagner Conjecture (2017b). [arxiv:1702.04259](#)
- Skopenkov, A.: A user’s guide to the topological Tverberg Conjecture, *Russian Math. Surveys*, 73:2 (2018a), 323–353. Earlier version: [arXiv:1605.05141v4](#). §4 available as A. Skopenkov, On van Kampen-Flores, Conway-Gordon-Sachs and Radon theorems. [arXiv:1704.00300](#)
- Skopenkov, A.: Stability of intersections of graphs in the plane and the van Kampen obstruction. *Topol. Appl.* **240**, 259–269 (2018b). [arXiv:1609.03727](#)
- Skopenkov, A.: Invariants of graph drawings in the plane. Full author’s version (2018c). [arXiv:1805.10237](#)
- Skopenkov, A.: Algebraic topology from algorithmic viewpoint, draft of a book, mostly in Russian (2019). <http://www.mccme.ru/circles/oim/algorg.pdf>. Accessed 7 Nov 2019
- Skopenkov, A.: Algebraic topology from geometric viewpoint (in Russian), MCCME, Moscow, 2nd edition (2020). <http://www.mccme.ru/circles/oim/obstruct.pdf>. Accessed 7 Nov 2019
- Skopenkov, A., Tancer, M.: Hardness of Almost Embedding Simplicial Complexes in  $\mathbb{R}^d$ . *Discret. Comp. Geom.* (to appear) (2017). [arXiv:1703.06305](#)
- Tamassia, R. (Ed.): *Handbook of Graph Drawing and Visualization*. Chapman and Hall/CRC (2019)
- Thilikos, D.M., Serna, M., Bodlaender, H.L.: Cutwidth I: a linear time fixed parameter algorithm. *J. Algorithms* **56**(1), 1–24 (2005)
- Ummel, B.: The product of nonplanar complexes does not imbed in 4-space. *Trans. Am. Math. Soc.* **242**, 319–328 (1978)
- van Kampen, E.R.: Remark on the address of S. S. Cairns, in *Lectures in Topology*, pp. 311–313. University of Michigan Press, Ann Arbor (1941)
- Volovikov, A.Yu.: On the van Kampen-Flores theorem. *Math. Notes* **59**(5), 477–481 (1996a)
- Volovikov, A.Yu.: On a topological generalization of the Tverberg theorem. *Math. Notes* **59**(3), 324–326 (1996b)
- Vučić, A., Živaljević, R.T.: Note on a conjecture of Sierksma. *Discret. Comput. Geom.* **9**, 339–349 (1993)
- Ziegler, G.M.: 3N colored points in a plane. *Not. AMS* **58**(4), 550–557 (2011)