Arnold Mathematical Journal (2020) 6:313–331
https://doi.org/10.1007/s40598-020-00133-z

RESEARCH CONTRIBUTION

Renormalization in the Golden-Mean Semi-Siegel Hénon Family: Non-Quasisymmetry

Jonguk Yang[1]

Received: 19 August 2019 / Revised: 15 January 2020 / Accepted: 17 January 2020 /
Published online: 3 February 2020

Abstract
For quadratic polynomials of one complex variable, the boundary of the golden-mean Siegel disk must be a quasicircle. We show that the analogous statement is not true for quadratic Hénon maps of two complex variables.

Keywords Renormalization · Dynamics in several complex variables · Siegel disk

1 Introduction

Let $\theta \in (\mathbb{R}\backslash\mathbb{Q})/\mathbb{Z}$ be an irrational rotation number. Then θ can be represented by an infinite continued fraction:

$$\theta = [a_0, a_1, \ldots] = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}}.$$

The *nth partial convergent of* θ is the rational number

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n].$$

The denominator q_n is called the nth *closest return moment*. The sequence $\{q_n\}_{n=0}^{\infty}$ satisfies the following inductive relation:

$$q_0 = 1, \quad q_1 = a_0 \quad \text{and} \quad q_{n+1} = a_n q_n + q_{n-1} \quad \text{for} \quad n \geq 1.$$

We say that θ is *Diophantine of order* d if there exists $C > 0$ such that

✉ Jonguk Yang
jonguk.yang@stonybrook.edu

1 Institute of Mathematical Sciences, Stony Brook University, Stony Brook, USA

$$q_{n+1} < C q_n^{d-1} \quad \text{for all} \quad n \in \mathbb{N}$$

If θ is Diophantine of order 2, then θ is said to be of *bounded type*. It is easy to show that θ is of bounded type if and only if a_n's are uniformly bounded (see e.g. Milnor 2006). The simplest example of a bounded type rotation number is the inverse-golden mean:

$$\theta_* = \frac{\sqrt{5}-1}{2} = [1, 1, \ldots].$$

Observe that for θ_*, the sequence of closest return moments $\{q_n\}_{n=0}^{\infty}$ is the *Fibonacci sequence*.

Consider the standard one-parameter family of quadratic polynomials

$$f_c(z) := z^2 + c \quad \text{for } c \in \mathbb{C}.$$

This is referred to as the *quadratic family*. A quadratic polynomial f_c is determined uniquely by the multiplier $\mu \in \mathbb{C}$ at a fixed point $x_c \in \mathbb{C}$ for f_c. In fact, we have

$$c = \frac{\mu}{2} - \frac{\mu^2}{4}. \tag{1.1}$$

Let c_0 be the unique parameter such that for f_{c_0}, the fixed point x_{c_0} is irrationally indifferent with rotation number θ (i.e. $\mu = e^{2\pi i \theta}$). If θ is equal to the inverse golden-mean θ_*, then we denote f_{c_0} as simply f_*.

The quadratic polynomial f_{c_0} is said to be *Siegel* if it is locally linearizable at x_0. More precisely, f_{c_0} is Siegel if there exist neighborhoods U of 0 and V of x_0, and a conformal change of coordinates

$$\psi : (U, 0) \to (V, x_0)$$

such that

$$\psi^{-1} \circ f_{c_0} \circ \psi(z) = \mu z.$$

A classic theorem of Siegel states, in particular, that f_{c_0} is Siegel whenever θ is Diophantine. Moreover, it is known that the linearizing map ψ has a maximal analytic continuation to a map $\psi : (\mathbb{D}, 0) \to (\Delta, x_0)$ so that its image $\Delta := \psi(\mathbb{D})$ is maximal (see e.g. Milnor 2006). We call Δ and $\partial\Delta$ the *Siegel disk* and the *Siegel boundary* of f_{c_0} respectively. See Fig. 1.

It is natural to ask whether $\partial\Delta$ is a Jordan curve, and if so, whether it is quasisymmetric, or even smooth. The following theorem settles these questions in the case when θ is of bounded type (see Douady 1987; Herman 1987):

Theorem 1.1 (Douady, Ghys, Herman, Shishikura) *Suppose that θ is of bounded type. Then f_{c_0} has its critical point 0 on its Siegel boundary $\partial\Delta$, and the restriction $f_{c_0}|_{\partial\Delta}$:*

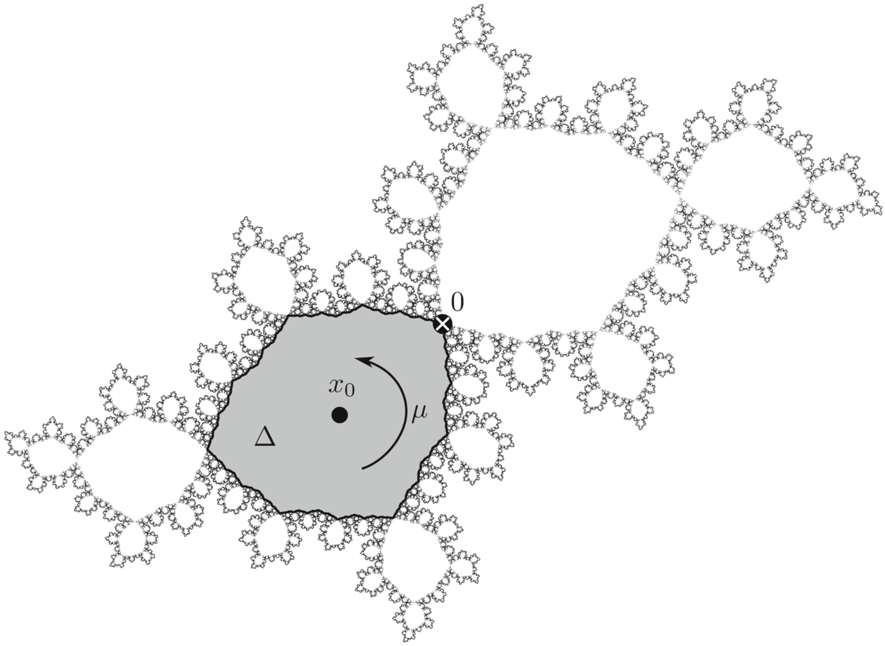

Fig. 1 The Siegel disk Δ of the golden-mean Siegel quadratic polynomial f_*. The critical point 0 is on $\partial\Delta$, and the restriction $f_*|_{\partial\Delta} : \partial\Delta \to \partial\Delta$ is quasisymmetrically conjugate to the rigid rotation of the unit circle $\partial\mathbb{D}$ by the angle $\theta_* = (\sqrt{5}-1)/2$

$\partial\Delta \to \partial\Delta$ *is quasisymmetrically conjugate to the rigid rotation of the unit circle* $\partial\mathbb{D}$ *by* θ.

From Theorem 1.1, it immediately follows that $\partial\Delta$ cannot be smooth if θ is of bounded type, since any curve containing the critical point cannot be both invariant and smooth. It is important to note, however, that there are examples of quadratic Siegel polynomials for which the Siegel boundary does not contain the critical point and is in fact smooth (see Buff et al. 2001; Artur et al. 2004).

The main goal of this paper is to carry the study of Siegel boundaries to a higher dimensional setting. To this end, consider the following two-dimensional extension of the quadratic family:

$$H_{c,b}(x, y) := (x^2 + c - by, x) \quad \text{for } c \in \mathbb{C} \text{ and } b \in \mathbb{C}\backslash\{0\}.$$

This is referred to as the *(complex quadratic) Hénon family*.

It is easy to see that $H_{c,b}$ has constant Jacobian:

$$\operatorname{Jac} H_{c,b} \equiv b.$$

Moreover, for $b = 0$, the map $H_{c,b}$ degenerates to the following embedding of f_c:

$$\iota(f_c)(x, y) := (f_c(x), x).$$

Hence, the parameter b measures how far $H_{c,b}$ is from being a one-dimensional map. In this paper, we will always assume that $H_{c,b}$ is a dissipative map (i.e. $|b| < 1$).

A Hénon map $H_{c,b}$ is determined uniquely by the multipliers $\mu \in \mathbb{C}\backslash\{0\}$ and $\nu \in \mathbb{D}\backslash\{0\}$ at a fixed point $\mathbf{x}_{c,b} \in \mathbb{C}^2$. In fact, we have

$$b = \mu\nu,$$

and

$$c = (1+\mu\nu)\left(\frac{\mu}{2}+\frac{\nu}{2}\right) - \left(\frac{\mu}{2}+\frac{\nu}{2}\right)^2.$$

Compare with (1.1). For any Jacobian $b \in \mathbb{D}\backslash\{0\}$, there exists a unique parameter $c_b \in \mathbb{C}$ such that one of the multipliers μ of the fixed point $\mathbf{x}_b := \mathbf{x}_{c_b,b}$ for $H_b := H_{c_b,b}$ is given by

$$\mu = e^{2\pi i\theta}.$$

Note that in this case, we have $|\nu| = |b|$.

The Hénon map H_b is said to be *semi-Siegel* if it is locally linearizable at $\mathbf{x}_b$. More precisely, H_b is semi-Siegel if there exist neighborhoods $\mathbf{U}$ of $(0, 0)$ and $\mathbf{V}$ of $\mathbf{x}_b$, and a conformal change of coordinates $\Psi_b : (\mathbf{U}, (0, 0)) \to (\mathbf{V}, \mathbf{x}_b)$ such that

$$\Psi_b^{-1} \circ H_b \circ \Psi_b(x, y) = (\mu x, \nu y).$$

Similar to the one-dimensional case, H_b is semi-Siegel whenever θ is Diophantine. Furthermore, the linearizing map Ψ_b can be biholomorphically extended to a map $\Psi_b : (\mathbb{D} \times \mathbb{C}, (0, 0)) \to (\mathcal{C}_b, \mathbf{x}_b)$ so that its image $\mathcal{C}_b := \Psi_b(\mathbb{D} \times \mathbb{C})$ is maximal (see Morosawa et al. 2000). We call $\mathcal{C}_b$ the *Siegel cylinder* of H_b. In the interior of $\mathcal{C}_b$, the dynamics of H_b is conjugate to rotation by θ in one direction, and contraction by ν in the other direction. Clearly, the orbit of every point in $\mathcal{C}_b$ converges to the analytic disk $\mathcal{D}_b := \Psi_b(\mathbb{D} \times \{0\})$ at height 0. We call $\mathcal{D}_b$ the *Siegel disk* of H_b. See Fig. 2.

The study of semi-Siegel Hénon maps had been a wide open subject until a recent work of Gaidashev, Radu, and Yampolsky (see Gaidashev et al. 2016), who proved:

Theorem 1.2 (Gaidashev–Radu–Yampolsky) *Let* $\theta = \theta_* = (\sqrt{5} - 1)/2$. *Then there exists* $\bar{\epsilon} > 0$ *such that for* $b \in \mathbb{D}_{\bar{\epsilon}}\backslash\{0\}$, *the boundary of the Siegel disk* $\mathcal{D}_b$ *for* H_b *is a homeomorphic image of the circle. In fact, the linearizing map*

$$\Psi_b : \mathbb{D} \times \{0\} \to \mathcal{D}_b$$

extends continuously and injectively (but not smoothly) to the boundary.

Recall that H_b is an automorphism of $\mathbb{C}^2$ with constant Jacobian $b \neq 0$. Hence, H_b does not have any definite singularities that would obstruct the smoothness of $\partial\mathcal{D}_b$ like in the one-dimensional case. Nonetheless, in the author's joint paper with Yampolsky (see Yampolsky and Yang 2016), we proved that the Siegel boundary for a Hénon map with golden-mean rotation number is not smooth:

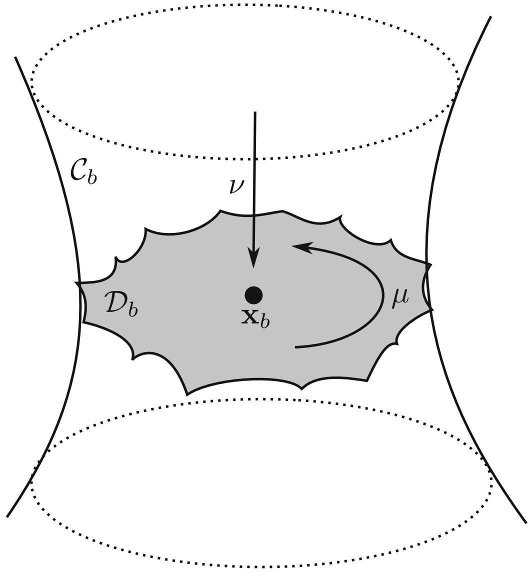

Fig. 2 The Siegel cylinder $\mathcal{C}_b$ and the Siegel disk $\mathcal{D}_b$ of H_b

Theorem 1.3 (Yampolsky-Y.) *Let $\theta = \theta_* = (\sqrt{5}-1)/2$. Then there exists $\bar{\epsilon} > 0$ such that for $b \in \mathbb{D}_{\bar{\epsilon}} \setminus \{0\}$, the boundary of the Siegel disk $\mathcal{D}_b$ for H_b is not C^1-smooth.*

Note that the properties of the Siegel boundary for Hénon maps given in Theorems 1.2 and 1.3 are also true for quadratic polynomials.

Definition 1.4 Let γ be a continuous arc or a simple closed curve. For $x, y \in \gamma$, let $[x, y]_\gamma$ denote the smallest subarc of γ with endpoints at x and y. We say that γ is K-*quasisymmetric* for some $K > 0$ if

$$\operatorname{diam}([x, y]_\gamma) < K \operatorname{dist}(x, y) \quad \text{for all} \quad x, y \in \gamma.$$

If γ is not K-quasisymmetric for all $K > 0$, then we say that γ has *unbounded geometry*.

Our main result states that the similarity between the one and two-dimensional case does not extend to quasisymmetry.

Main Theorem (Non-Quasisymmetry) *Let $\theta = \theta_* = (\sqrt{5}-1)/2$. Then there exists $\bar{\epsilon} > 0$ such that the set of parameter values b for which the boundary of the Siegel disk $\mathcal{D}_b$ for H_b has unbounded geometry contains a dense G_δ subset in the disc $\mathbb{D}_{\bar{\epsilon}} \setminus \{0\}$.*

The proof of the Main Theorem follows the strategy used by de Carvalho et al. (2006) to obtain the analogous result for the limit Cantor sets of Feigenbaum Hénon maps.

2 Preliminaries

In this section, we provide a brief summary of the renormalization theory of semi-Siegel Hénon maps. See Yang (2018) for complete details.

Let $b \in \mathbb{D}_{\bar{\epsilon}} \setminus \{0\}$ for some $\bar{\epsilon} > 0$ sufficiently small. Consider the Hénon map

$$H_b(x, y) = \begin{bmatrix} x^2 + c_b - by \\ x \end{bmatrix}$$

that has a Siegel disc $\mathcal{D}_b \subset \mathbb{C}^2$ with rotation number $\theta_* = (\sqrt{5} - 1)/2$.

2.1 Definition of Renormalization

Let $\hat{\Omega}_0$ and $\hat{\Gamma}_0$ be suitably chosen topological bidisks in $\mathbb{C}^2$ such that $\hat{\Omega}_0 \cap \hat{\Gamma}_0 \ni (0, 0)$ and $\hat{\Omega}_0 \cup \hat{\Gamma}_0 \supset \partial\mathcal{D}_b$. The *pair representation* of H_b is given by

$$\hat{\Sigma}_0 = (\hat{A}_0, \hat{B}_0) := \left(H_b|_{\hat{\Omega}_0}, H_b|_{\hat{\Gamma}_0} \right).$$

Let

$$\Phi_0(x, y) := (\lambda_0 x, \lambda_0 y), \tag{2.1}$$

where $\lambda_0 := c_b$. Observe that

$$\Phi_0^{-1} \circ H_b \circ \Phi_0(0, 0) = (1, 0).$$

The *normalized pair representation* of H_b is defined as

$$\Sigma_0 = (A_0, B_0) := \left(\Phi_0^{-1} \circ \hat{A}_0 \circ \Phi_0, \Phi_0^{-1} \circ \hat{B}_0 \circ \Phi_0 \right).$$

We may assume that for some topological discs $Z, W, V \subset \mathbb{C}$ containing 0, the domains of A_0 and B_0 are given by

$$\Omega = Z \times V := \Phi_0^{-1}(\hat{\Omega}_0) \quad \text{and} \quad \Gamma = W \times V := \Phi_0^{-1}(\hat{\Gamma}_0) \quad \text{respectively.}$$

The nth *renormalization* of H_b:

$$\mathbf{R}^n(H_b) := \Sigma_n = (A_n, B_n),$$

where

$$A_n(x, y) = \begin{bmatrix} a_n(x, y) \\ h_n(x, y) \end{bmatrix} \quad \text{and} \quad B_n(x, y) = \begin{bmatrix} b_n(x, y) \\ x \end{bmatrix},$$

is the pair of rescaled iterates of Σ_0 defined inductively as follows. Denote

$$(a_n)_y(x) := a_n(x, y),$$

and let

$$H_{n+1}(x, y) := \begin{bmatrix} (a_n)_y^{-1}(x) \\ y \end{bmatrix}.$$

Consider the non-linear change of coordinates defined as

$$\Phi_{n+1} := H_{n+1} \circ \Lambda_{n+1},$$

where

$$\Lambda_{n+1}(x, y) = (\lambda_{n+1}x + c_{n+1}, \lambda_{n+1}y) \tag{2.2}$$

is an affine rescaling map to be specified later. The pair $\Sigma_{n+1} = (A_{n+1}, B_{n+1})$ is defined as

$$A_{n+1} = \Phi_{n+1}^{-1} \circ B_n \circ A_n^2 \circ \Phi_{n+1} \quad \text{and} \quad B_{n+1} = \Phi_{n+1}^{-1} \circ B_n \circ A_n \circ \Phi_{n+1}.$$

Lemma 2.1 *For $n \geq 0$, let $\Sigma_n = (A_n, B_n)$ be the nth renormalization of H_b. Then A_n and B_n are bounded analytic maps that are well-defined on $\Omega = Z \times V$ and $\Gamma = W \times V$ respectively. Moreover, the dependence of Σ_n on y decays* super-exponentially *fast. That is, we have*

$$\sup_{(x,y)\in\Omega} \|\partial_y A_n(x, y)\| < C\bar{\epsilon}^{2^n} \quad \textit{and} \quad \sup_{(x,y)\in\Gamma} \|\partial_y B_n(x, y)\| < C\bar{\epsilon}^{2^n}.$$

for some uniform constant $C > 0$.

The one-dimensional projections of A_n, B_n and Σ_n are given by

$$\eta_n(x) := a_n(x, 0), \quad \xi_n(x) := b_n(x, 0) \quad \text{and} \quad \zeta_n := (\eta_n, \xi_n) \quad \text{respectively.}$$

By Lemma 2.1, we see that η_n and ξ_n are bounded analytic functions defined on Z and W respectively. Moreover, the dynamics of Σ_n converges with that of ζ_n super-exponentially fast. It is shown in Yang (2018) that η_n and ξ_n each have a unique simple critical point which are $C\bar{\epsilon}^{2^n}$-close to each other. We choose the normalizing constants λ_n and c_n in (2.2) so that

$$\xi_n(0) = 1 \quad \text{and} \quad \xi_n'(0) = 0.$$

2.2 Renormalization Convergence

Let $\zeta_* = (\eta_*, \xi_*)$ be the fixed point of the one-dimensional renormalization operator $\mathcal{R}$ given in [GaYam]. In particular, we have

$$\lambda_*^{-1}\eta_* \circ \xi_* \circ \eta_*(\lambda_* x) = \eta_*(x) \quad \text{and} \quad \lambda_*^{-1}\eta_* \circ \xi_*(\lambda_* x) = \xi_*(x), \tag{2.3}$$

where

$$\lambda_* := \eta_* \circ \xi_*(0) \in \mathbb{D}$$

is the universal scaling factor.

Convergence under renormalization for semi-Siegel Hénon maps was first obtained in [GaYam]. For the renormalization operator **R** defined above, the proof of convergence is given in Yang (2018).

Theorem 2.2 *As $n \to \infty$, we have the following convergences (each of which occurs at a geometric rate):*

(i) $\zeta_n = (\eta_n, \xi_n) \to \zeta_* = (\eta_*, \xi_*)$;
(ii) $\lambda_n \to \lambda_*$; *and*
(iii) $\Phi_n \to \Phi_*$, *where*

$$\Phi_*(x, y) = \begin{bmatrix} \phi_*(x) \\ \lambda_* y \end{bmatrix} := \begin{bmatrix} \eta_*^{-1}(\lambda_* x) \\ \lambda_* y \end{bmatrix}.$$

2.3 Renormalization Limit Set

Define the *nth microscope map of depth k* by

$$\Phi_n^{n+k} := \Phi_{n+1} \circ \Phi_{n+2} \circ \cdots \circ \Phi_{n+k}.$$

Let

$$\Omega_n^{n+k} := \Phi_n^{n+k}(\Omega) \quad \text{and} \quad \Gamma_n^{n+k} := \Phi_n^{n+k}(\Gamma).$$

Observe that $\{\Omega_n^{n+k} \cup \Gamma_n^{n+k}\}_{k=0}^{\infty}$ is a nested sequence of open sets. See Fig. 3

Proposition 2.3 *Let $\lambda_* \in \mathbb{D}$ be the universal scaling factor. Then for all $0 \leq n, k$, we have*

$$diam\left(\Omega_n^{n+k} \cup \Gamma_n^{n+k}\right) = O\left(|\lambda_*|^k\right).$$

Consequently, there exists a point $(\kappa_n, 0) \in \Omega$ such that

$$\bigcap_{k=0}^{\infty} \Omega_n^{n+k} \cup \Gamma_n^{n+k} = (\kappa_n, 0).$$

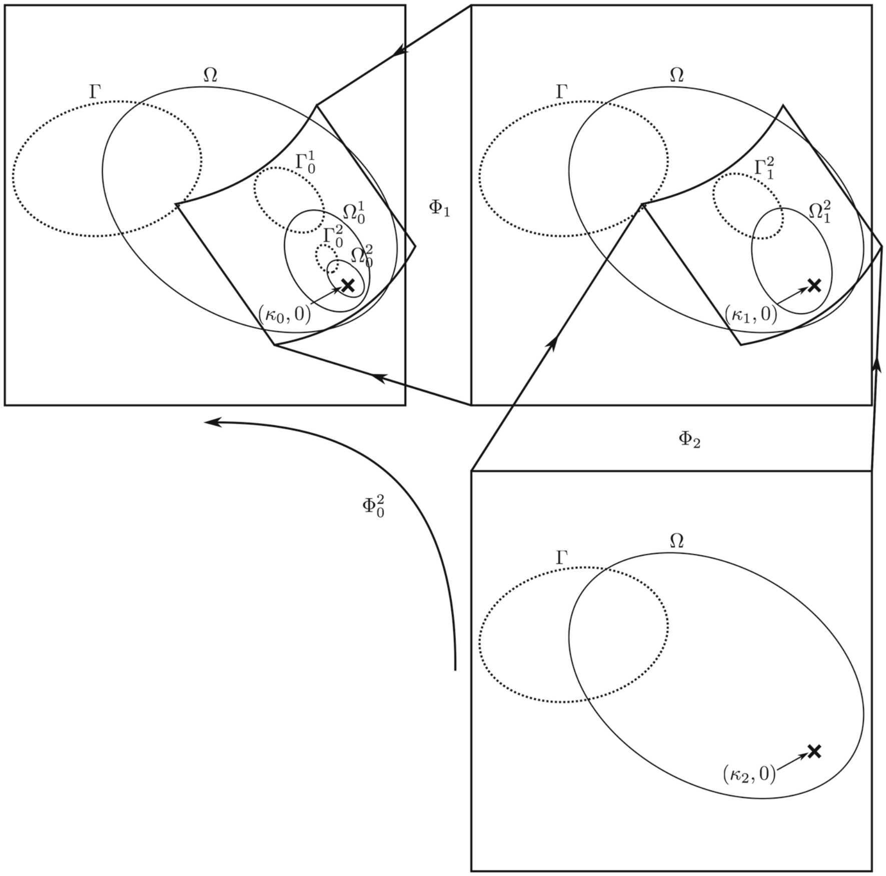

Fig. 3 The renormalization microscope map Φ_0^2 obtained by composing the non-linear changes of coordinates Φ_1 and Φ_2. We have $\Omega_0^1 = \Phi_1(\Omega)$, $\Gamma_0^1 = \Phi_1(\Gamma)$, $\Omega_0^2 = \Phi_0^2(\Omega)$, $\Gamma_0^2 = \Phi_0^2(\Gamma)$, and $(\kappa_0, 0) = \Phi_1((\kappa_1, 0)) = \Phi_0^2((\kappa_2, 0))$

Moreover, κ_n converges to 1 *geometrically fast as $n \to \infty$.*

Definition 2.4 The point $(\kappa_n, 0) \in \Omega$ given in Proposition 2.3 is called the *nth cap*.

Remark 2.5 The cap is a dynamically defined point with the same combinatorial address as the critical value $\xi_*(0) = 1$. In de Carvalho et al. (2006), the analog of the critical value is referred to as the *tip*.

Denote

$$\hat{\Omega}_n := \Phi_0 \circ \Phi_0^n(\Omega) \quad \text{and} \quad \hat{\Gamma}_n := \Phi_0 \circ \Phi_0^n(\Gamma),$$

where Φ_0 is given in (2.1).

Lemma 2.6 *Let* $\{q_n\}_{n=0}^{\infty}$ *be the Fibonacci sequence. Define*

$$\hat{A}_n := H_b^{q_{2n+1}}|_{\hat{\Omega}_n} \quad \text{and} \quad \hat{B}_n := H_b^{q_{2n}}|_{\hat{\Gamma}_n}.$$

Then the nth renormalization $\Sigma_n = (A_n, B_n)$ *is given by*

$$A_n = \left(\Phi_0^n\right)^{-1} \circ \Phi_0^{-1} \circ \hat{A}_n \circ \Phi_0 \circ \Phi_0^n, \quad \text{and} \quad B_n = \left(\Phi_0^n\right)^{-1} \circ \Phi_0^{-1} \circ \hat{B}_n \circ \Phi_0 \circ \Phi_0^n.$$

In Gaidashev et al. (2016), showed that the renormalization limit set for a semi-Siegel Hénon map coincides with its Siegel boundary:

Theorem 2.7 *For* $n \in \mathbb{N}$, *let*

$$X_n := \bigcup_{i=0}^{q_{2n+1}-1} H_b^i(\hat{\Omega}_n) \quad \text{and} \quad Y_n := \bigcup_{i=0}^{q_{2n}-1} H_b^i(\hat{\Gamma}_n).$$

Then the Siegel boundary for H_b *is given by*

$$\partial\mathcal{D}_b := \bigcap_{n=1}^{\infty} X_n \cup Y_n.$$

3 Universality

The proof of the Main Theorem involves giving precise estimates for various geometric quantities that arise when analyzing the dynamics of the semi-Siegel Hénon map H_b near its Siegel boundary $\partial\mathcal{D}_b$. In order to do this, we first need an explicit description for the nth renormalization $\Sigma_n = (A_n, B_n)$ of H_b. In Yang (2018), the author showed that Σ_n has a *universal* first-order approximation in terms of its Jacobian. In this section, we strengthen this result to better suit our application.

As before, we write

$$A_n(x, y) = \begin{bmatrix} a_n(x, y) \\ h_n(x, y) \end{bmatrix} \quad \text{and} \quad B_n(x, y) = \begin{bmatrix} b_n(x, y) \\ x \end{bmatrix}.$$

Recall that A_n and B_n represent the q_{2n+1}th and q_{2n}th iterate of H_b respectively. Accordingly, we expect the Jacobian of A_n and B_n to be on the order of

$$b^{q_{2n+1}} = \text{Jac}\, H_b^{q_{2n+1}} \quad \text{and} \quad b^{q_{2n}} = \text{Jac}\, H_b^{q_{2n}} \quad \text{respectively.}$$

The following theorem is a simplification of Theorem 7.3 in Yang (2018) in the case when the map being renormalized has constant Jacobian. In the general, non-constant Jacobian case, the dependence of b_n on y has a factor of e^{r_n} for some uniformly bounded sequence $\{r_n\}_{n=0}^{\infty} \subset \mathbb{C}$. This is to account for small fluctuations caused by variation in the Jacobian along the renormalization limit set.

Theorem 3.1 *For $n \geq 0$, we have*

$$B_n(x, y) = \begin{bmatrix} \xi_n(x) - b^{q_{2n}}\, \beta(x)\, y\, (1 + O(\rho^n)) \\ x \end{bmatrix},$$

where $0 < \rho < 1$ is a uniform constant; and $\beta(x)$ is a universal function that is uniformly bounded away from 0 *and* ∞*, and has a uniformly bounded derivative and distortion.*

Observe that

$$\begin{aligned} \text{Jac } B_n(x, y) &= \begin{vmatrix} \xi_n'(x) - b^{q_{2n}}\, \beta'(x)\, y & -b^{q_{2n}}\, \beta(x) \\ 1 & 0 \end{vmatrix} + (\text{higher order terms}) \\ &= -b^{q_{2n}}\, \beta(x) + (\text{higher order terms}) \end{aligned} \tag{3.1}$$

as we expected.

Next, we give a first-order approximation of A_n. In Yang (2018), this is only done for the first coordinate a_n (see Corollary 7.4).

Theorem 3.2 *For $n \geq 0$, we have*

$$A_n(x, y) = \begin{bmatrix} \eta_n(x) - b^{q_{2n}}\, \alpha(x)\, y\, (1 + O(\rho^n)) \\ \lambda_n^{-1} \check{\eta}_{n-1}(\lambda_n x) - b^{q_{2n}}\, \check{\alpha}(x)\, y\, (1 + O(\rho^n)) \end{bmatrix},$$

where $0 < \rho < 1$ is a uniform constant; and

$$\check{\eta}_k(x) = \eta_k(x) + O(|b|^{q_{2k}}), \quad \alpha(x) := \frac{\eta_*'(x)}{\xi_*'(x)} \beta(x) \quad \text{and} \quad \check{\alpha}(x) := \frac{\eta_*'(\lambda_* x)}{\eta_*'(x)} \alpha(x).$$

Proof Recall that

$$A_{n+1} := \Lambda_{n+1}^{-1} \circ H_{n+1}^{-1} \circ B_n \circ A_n^2 \circ H_{n+1} \circ \Lambda_{n+1}. \tag{3.2}$$

Denote

$$(\tilde{x}, \tilde{y}) := \Lambda_{n+1}(x, y) = (\lambda_{n+1} x + c_{n+1}, \lambda_{n+1} y).$$

It is not difficult to show, using Theorem 3.1, that $c_{n+1} = O(|b|^{q_{2n}})$. Moreover, we have

$$A_n \circ H_{n+1}(x, y) = \begin{bmatrix} x \\ \tilde{h}_n(x, y) \end{bmatrix},$$

where

$$\tilde{h}_n(x, y) := h_n\left((a_n)_y^{-1}(x), y\right).$$

Hence,

$$h_{n+1}(x, y) = \lambda_{n+1}^{-1} a_n(\tilde{x}, \tilde{h}_n(\tilde{x}, \tilde{y})) = \lambda_{n+1}^{-1} \eta_n(\lambda_{n+1} x) + O(|b|^{q_{2n}}). \tag{3.3}$$

Since A_n and B_n commute, we may rearrange the terms in (3.2) to obtain

$$A_{n+1} = \Lambda_{n+1}^{-1} \circ H_{n+1}^{-1} \circ A_n \circ H_{n+1} \circ \Lambda_{n+1} \circ B_{n+1}.$$

Denote

$$\tilde{\xi}_{n+1}(x) := \lambda_{n+1} \xi_{n+1}(x) + c_{n+1}$$

and

$$\begin{aligned} \tilde{b}_{n+1}(x, y) &:= \lambda_{n+1} b_{n+1}(x, y) + c_{n+1} \\ &= \tilde{\xi}_{n+1}(x) - \lambda_{n+1} b^{q_{2(n+1)}} \beta(x) y (1 + O(\rho^{n+1})). \end{aligned}$$

Then

$$H_{n+1} \circ \Lambda_{n+1} \circ B_{n+1}(x, y) = \begin{bmatrix} (a_n)_{\tilde{x}}^{-1}(\tilde{b}_{n+1}(x, y)) \\ \tilde{x} \end{bmatrix}.$$

Neglecting higher order terms, $(a_n)_{\tilde{x}}^{-1}(\tilde{b}_{n+1}(x, y))$ has the same y-dependence as

$$\eta_n^{-1}(\tilde{b}_{n+1}(x, y)) \approx \eta_n^{-1}(\tilde{\xi}_{n+1}(x)) - \left(\eta_n^{-1}\right)' (\tilde{\xi}_{n+1}(x)) \lambda_{n+1} b^{q_{2(n+1)}} \beta(x) y$$

Let

$$\check{x} := \lambda_n \eta_n^{-1}(\tilde{\xi}_{n+1}(x)).$$

From (3.3), and again neglecting higher order terms, we see that the second coordinate $h_n\big((a_n)_{\tilde{x}}^{-1}(\tilde{b}_{n+1}(x, y)), \tilde{x}\big)$ has the same y-dependence as

$$\lambda_n^{-1} \eta_{n-1}\big(\lambda_n \eta_n^{-1}(\tilde{b}_{n+1}(x, y))\big) \approx \lambda_n^{-1} \eta_{n-1}(\check{x}) - \eta_{n-1}'(\check{x}) \left(\eta_n^{-1}\right)' (\tilde{\xi}_{n+1}(x)) \lambda_{n+1} b^{q_{2(n+1)}} \beta(x) y.$$

Lastly, we compute

$$\begin{aligned} \lim_{n \to \infty} \eta_{n-1}'(\check{x}) \left(\eta_n^{-1}\right)' (\tilde{\xi}_{n+1}(x)) &= \eta_*'\big(\lambda_* \eta_*^{-1}(\lambda_* \xi_*(x))\big) \left(\eta_*^{-1}\right)' (\lambda_* \xi_*(x)) \\ &= \eta_*'\big(\eta_* \circ \xi_*(\lambda_*^2 x)\big) \left(\eta_*^{-1}\right)' (\eta_* \circ \xi_*(\lambda_* x)) \\ &= \frac{\eta_*'\big(\eta_* \circ \xi_* \left(\lambda_*^2 x\right)\big)}{\eta_*'(\xi_*(\lambda_* x))} \end{aligned}$$

$$= \frac{\eta_*'(\lambda_* x)}{\xi_*'(\lambda_* x)} \cdot \frac{\xi_*'(\lambda_* x)}{\xi_*'(x)}$$
$$= \frac{\eta_*'(\lambda_* x)}{\xi_*'(x)},$$

where in the second equality, we used (2.3) and in the third equality, we used the derivative of (2.3). The result follows. □

Observe:

$$\begin{aligned} \operatorname{Jac} A_n(x,y) &= \begin{vmatrix} \eta_n'(x) - b^{q_{2n}}\,\alpha'(x)\,y & -b^{q_{2n}}\,\alpha(x) \\ \check{\eta}_{n-1}'(\lambda_n x) - b^{q_{2n}}\,\check{\alpha}'(x)\,y & -b^{q_{2n}}\,\check{\alpha}(x) \end{vmatrix} + \text{(higher order terms)} \\ &= \big(-\eta_n'(x)\,\check{\alpha}(x) + \check{\eta}_{n-1}'(\lambda_n x)\,\alpha(x)\big) b^{q_{2n}} + \text{(higher order terms)} \\ &= \big(-\eta_*'(x)\,\check{\alpha}(x) + \eta_*'(\lambda_* x)\,\alpha(x)\big) b^{q_{2n}} + \text{(higher order terms)} \\ &= \text{(higher order terms)}. \end{aligned}$$

Hence, we see that the first-order approximation of A_n is not precise enough to "see" the Jacobian of A_n, which we expect to be of higher order ($b^{q_{2n+1}}$ rather than $b^{q_{2n}}$). Fortunately, the approximation can be made much more precise by considering $A_n \circ \Phi_{n+1}$ instead of A_n.

Theorem 3.3 *For $n \geq 0$, we have*

$$A_n \circ \Phi_{n+1}(x,y) = \begin{bmatrix} \lambda_{n+1} x + c_{n+1} \\ \check{\xi}_n^{-1}(\lambda_{n+1} x) - b^{q_{2n+1}}\,\chi(x)\,\lambda_{n+1}\,y(1 + O(\rho^n)) \end{bmatrix},$$

where $0 < \rho < 1$ is a uniform constant; and

$$\check{\xi}_n(x) = \xi_n(x) + O(|b|^{q_{2(n-1)}}) \quad \textit{and} \quad \chi(x) := \frac{\xi_*'(\lambda_* x)}{\xi_*'(x)}\,\frac{\beta(x)}{\beta(\lambda_* x)} = \frac{\eta_*'(\lambda_* x)}{\eta_*'(x)}\,\frac{\alpha(x)}{\alpha(\lambda_* x)}.$$

Proof Write

$$A_n \circ H_{n+1}(x,y) = \begin{bmatrix} x \\ \tilde{h}_n(x,y) \end{bmatrix},$$

where

$$\tilde{h}_n(x,y) := h_n\left((a_n)_y^{-1}(x), y\right).$$

By Theorem 3.2, we have

$$\tilde{h}_n(x,y) = \lambda_n^{-1}\eta_{n-1}\left(\lambda_n \eta_n^{-1}(x)\right) + O(|b|^{q_{2(n-1)}}). \tag{3.4}$$

By definition, $\eta_n(x) = a_n(x, 0)$, where a_n is the first coordinate of A_n. Since A_{n-1} and B_{n-1} commute, we have

$$A_n := \Phi_n^{-1} \circ B_{n-1} \circ A_{n-1}^2 \circ \Phi_n = B_n \circ \Phi_n^{-1} \circ A_{n-1} \circ \Phi_n.$$

Thus,

$$\eta_n = \xi_n \left(\lambda_n^{-1} \eta_{n-1}(\lambda_n x) \right) + O(|b|^{q_{2(n-1)}}). \tag{3.5}$$

Taking the inverse of (3.5) and plugging into (3.4), we obtain

$$\lambda_n^{-1} \eta_{n-1} \left(\lambda_n \eta_n^{-1}(x) \right) = \xi_n^{-1}(x) + O(|b|^{q_{2(n-1)}}).$$

To derive the first-order approximation of the y-dependence of $A_n \circ \Phi_{n+1}$, consider

$$B_{n+1} = \Phi_{n+1}^{-1} \circ B_n \circ A_n \circ \Phi_{n+1}.$$

Taking the Jacobian, we obtain

$$\mathrm{Jac}_{\mathbf{x}_0}(B_{n+1}) = \mathrm{Jac}_{\mathbf{x}_2} \left(\Phi_{n+1}^{-1} \right) \cdot \mathrm{Jac}_{\mathbf{x}_1}(B_n) \cdot \mathrm{Jac}_{\mathbf{x}_0}(A_n \circ \Phi_{n+1}), \tag{3.6}$$

where

$$\mathbf{x}_0 := (x, y), \quad \mathbf{x}_1 = (x_1, y_1) := A_n \circ \Phi_{n+1}(\mathbf{x}_0) \quad \text{and} \quad \mathbf{x}_2 = (x_2, y_2) := B_n(\mathbf{x}_1).$$

Neglecting higher order terms, we have

$$x_1 \approx \lambda_{n+1} x \quad \text{and} \quad x_2 \approx \xi_n(\lambda_{n+1} x).$$

By (3.1), it follows that

$$\mathrm{Jac}_{\mathbf{x}_0}(B_{n+1}) \approx b^{q_{2n+2}} \beta(x) \quad \text{and} \quad \mathrm{Jac}_{\mathbf{x}_1}(B_n) \approx b^{q_{2n}} \beta(\lambda_{n+1} x). \tag{3.7}$$

Observe:

$$\begin{aligned} \mathrm{Jac}_{\mathbf{x}_2} \Phi_{n+1}^{-1} &= \lambda_{n+1}^{-2} \begin{vmatrix} \partial_x a_n(\mathbf{x}_2) & \partial_y a_n(\mathbf{x}_2) \\ 0 & 1 \end{vmatrix} \\ &\approx \lambda_{n+1}^{-2} \eta_n'(\xi_n(\lambda_{n+1} x)). \end{aligned} \tag{3.8}$$

Plugging in (3.7) and (3.8) into (3.6), we obtain

$$\mathrm{Jac}_{\mathbf{x}_0}(A_n \circ \Phi_{n+1}) \approx b^{q_{2n+1}} \lambda_{n+1}^2 \frac{\beta(x)}{\beta(\lambda_{n+1} x)} \eta_n'(\xi_n(\lambda_{n+1} x)). \tag{3.9}$$

Lastly, write:

$$A_n \circ \Phi_{n+1}(x, y) = \begin{bmatrix} \lambda_{n+1} x + c_{n+1} \\ \tilde{h}_n(x, 0) + E_n(x, y) \end{bmatrix},$$

where $E_n(x, y)$ is undetermined. Since

$$\mathrm{Jac}_{(x,y)}(A_n \circ \Phi_{n+1}) = \lambda_{n+1} \partial_y E_n(x, y),$$

plugging in (3.9) and integrating both sides, we obtain the desired formula. □

Recall that the nth cap $(\kappa_n, 0) \in \Omega$ is a dynamically defined point in the renormalization limit set for Σ_n (see Definition 2.4). It is not difficult to see that we have

$$\Phi_n((\kappa_n, 0)) = (\kappa_{n-1}, 0) \quad \text{and} \quad \Phi_n^{n+k}((\kappa_{n+k}, 0)) = (\kappa_n, 0).$$

Denote

$$D_n := D_{(\kappa_n,0)} \Phi_n \quad \text{and} \quad D_n^{n+k} := D_{(\kappa_{n+k},0)} \Phi_n^{n+k}.$$

Observe

$$D_n^{n+k} = D_{n+1} \cdot D_{n+2} \cdot \dots \cdot D_{n+k}.$$

The following estimates on the derivative the microscope maps at the cap is a corollary of Theorem 3.2. The statement is more precise than the one given in Yang (2018), but it follows from the same proof.

Theorem 3.4 *Write*

$$D_n = \begin{bmatrix} 1 & t_n b^{q_{2(n-1)}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_n & 0 \\ 0 & \lambda_n \end{bmatrix}.$$

Then there exist a uniform positive constant $\rho < 1$ such that the following estimates hold for all $n \geq 1$:

(i) $u_n = \lambda_*^2(1 + O(\rho^n))$,
(ii) $\lambda_n = \lambda_*(1 + O(\rho^n))$, *and*
(iii) $t_n = \lambda_* \alpha(1)(1 + O(\rho^n))$.

Consequently, for $0 \leq n, k$, we have

$$D_n^{n+k} = \begin{bmatrix} 1 & t_n^{n+k} b^{q_{2n}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_n^{n+k} & 0 \\ 0 & \lambda_n^{n+k} \end{bmatrix},$$

where

(i) $u_n^{n+k} := u_{n+1} \cdot u_{n+1} \cdot \dots \cdot u_{n+k} = \lambda_*^{2k}(1 + O(\rho^n))$,
(ii) $\lambda_n^{n+k} := \lambda_{n+1} \cdot \lambda_{n+2} \cdot \dots \cdot \lambda_{n+k} = \lambda_*^k(1 + O(\rho^n))$, *and*
(iii) $t_n^{n+k} = t_{n+1}(1 + O(|b|^{q_{2n+1}}))$.

4 Proof of Non-quasisymmetry

We are now ready to prove the Main Theorem stated in Sect. 1.

Proof of the Main Theorem Let $n, k \in \mathbb{N}$ be sufficiently large. To prove the desired result, we analyze the geometry of the Siegel boundary $\partial\mathcal{D}_b$ in three different scales: that of Σ_{n+k}, Σ_n and Σ_0.

We start in the scale of Σ_{n+k}. Consider the dynamically defined points

$$\mathbf{q}_{n+k} := (\kappa_{n+k}, 0) \quad \text{and} \quad \mathbf{p}_{n+k} := A_{n+k}(\mathbf{q}_{n+k}).$$

Next, in the scale of $\Sigma_n = (A_n, B_n)$, consider

$$\mathbf{q}_n^{n+k} := A_n \circ \Phi_n^{n+k}(\mathbf{q}_{n+k}) = A_n((\kappa_n, 0)) \quad \text{and} \quad \mathbf{p}_n^{n+k} := A_n \circ \Phi_n^{n+k}(\mathbf{p}_{n+k}).$$

Lastly, in the scale of Σ_0, consider

$$\mathbf{q}_0^{n+k} := \Phi_0^n\left(\mathbf{q}_n^{n+k}\right) \quad \text{and} \quad \mathbf{p}_0^{n+k} := \Phi_0^n\left(\mathbf{p}_n^{n+k}\right).$$

See Fig. 4.

It is easy to see that

$$\mathbf{q}_0^{n+k} = H_b^i((\kappa_0, 0)) \in \partial\mathcal{D}_b \quad \text{and} \quad \mathbf{p}_0^{n+k} = H_b^j((\kappa_0, 0)) \in \partial\mathcal{D}_b.$$

for some $i, j \in \mathbb{N}$. Denote the x- and y-coordinate of $\mathbf{q}_{n+k} - \mathbf{p}_{n+k}$ by $\Delta_{n+k}x$ and $\Delta_{n+k}y$ respectively. The x- and y-coordinates of $\mathbf{q}_n^{n+k} - \mathbf{q}_n^{n+k}$ and $\mathbf{q}_0^{n+k} - \mathbf{p}_0^{n+k}$ are denoted similarly, but with Δ_n^{n+k} and Δ_0^{n+k} respectively instead of Δ_{n+k}.

First, note that $\Delta_{n+k}\, x$ and $\Delta_{n+k}\, y$ converge to some uniform limits $\Delta_*\, x$ and $\Delta_*\, y$ respectively as $k \to \infty$. By Theorems 3.3 and 3.4, we have:

$$\Delta_n^{n+k}\, x = \lambda_*^{2k-1}\Delta_*\, x(1 + O(\rho^n)) \tag{4.1}$$

and

$$\Delta_n^{n+k}\, y = \lambda_*^{k-1}\left(\lambda_*^k C_1 - b^{q_{2n+1}} C_2\right)(1 + O(\rho^n)), \tag{4.2}$$

where

$$C_1 := \left(\xi_*^{-1}\right)'(\lambda_*)\Delta_*\, x \quad \text{and} \quad C_2 := \lambda_*\chi(1)\Delta_*\, y$$

are uniform constants.

The values of b that solve

$$b^{q_{2n+1}} = \frac{C_1}{C_2}\lambda_*^k$$

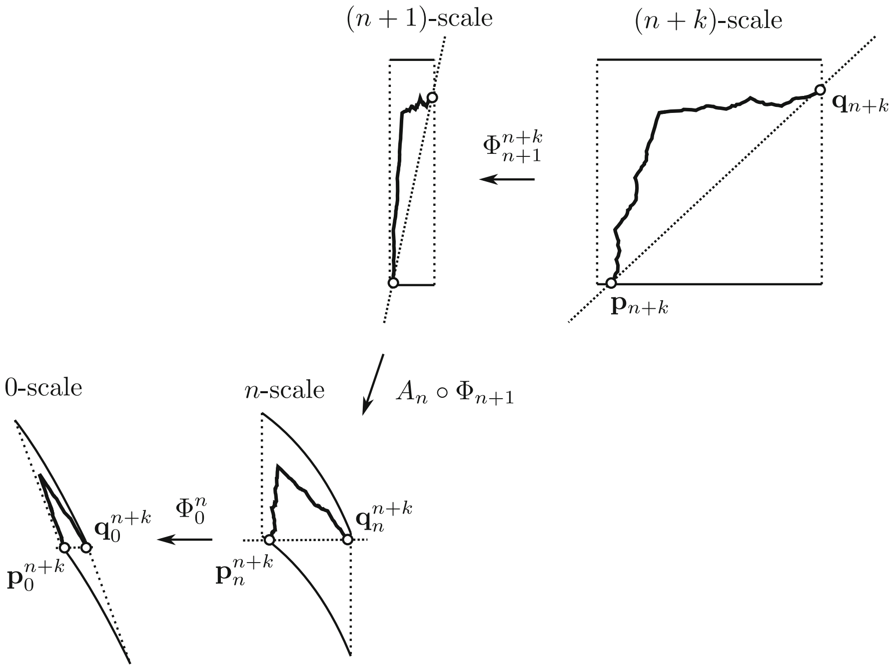

Fig. 4 Exploiting universality to create a cusp in the Siegel boundary $\partial\mathcal{D}_b$

as k and n run through $\mathbb{N}$ is dense in $\mathbb{D}_{\bar{\epsilon}}$. Let $\mathbb{B}_m \subset \mathbb{D}_{\bar{\epsilon}}$ be the set of parameter values such that for some $n \geq m$ and $k \in \mathbb{N}$, we have

$$|b|^{q_{2n+1}} \asymp |\lambda_*|^k$$

and

$$\Delta_n^{n+k} y = O(|\lambda_*|^{n+2k}). \tag{4.3}$$

Then $\mathbb{B}_m$ is a dense open set of $\mathbb{D}_{\bar{\epsilon}}$. Hence, the intersection

$$\mathbb{B}_\infty := \bigcap_{m=N}^{\infty} \mathbb{B}_m$$

is a G_δ-subset of $\mathbb{D}_{\bar{\epsilon}}$, consisting of parameters b for which (4.3) holds for infinitely many $n \in \mathbb{N}$.

Assume that $b \in \mathbb{B}_\infty$ and that (4.3) holds for n and k. By Theorem 3.4, we have:

$$\Delta_0^{n+k} x \asymp \lambda_*^{2n} \Delta_n^{n+k} x \asymp \lambda_*^{2n+2k-1}$$

and

$$\Delta_0^{n+k}\, y \asymp \lambda_*^n \Delta_n^{n+k}\, y = O(|\lambda_*|^{2n+2k})$$

Hence:

$$\text{dist}\left(\mathbf{q}_0^{n+k}, \mathbf{p}_0^{n+k}\right) = O(|\lambda_*|^{2n+2k}). \tag{4.4}$$

We now estimate the diameter of the subarc $[\mathbf{q}_0^{n+k}, \mathbf{p}_0^{n+k}]_{\partial\mathcal{D}_b}$. Consider the points

$$\check{\mathbf{q}}_{n+k} := \Phi_{n+k+1} \circ A_{n+k+1}((\kappa_{n+k+1}, 0)), \quad \check{\mathbf{q}}_n^{n+k} := A_n \circ \Phi_n^{n+k}(\check{\mathbf{q}}_{n+k}) \quad \text{and}$$
$$\check{\mathbf{q}}_0^{n+k} := \Phi_0^n\left(\check{\mathbf{q}}_n^{n+k}\right).$$

It is easy to see that $\check{\mathbf{q}}_0^{n+k} \in [\mathbf{q}_0^{n+k}, \mathbf{p}_0^{n+k}]_{\partial\mathcal{D}_b}$. Denote the x- and y-coordinate of $\check{\mathbf{q}}_{n+k} - \mathbf{q}_{n+k}$ by $\check{\Delta}_{n+k}\, x$ and $\check{\Delta}_{n+k}\, y$ respectively. The x- and y-coordinates of $\check{\mathbf{q}}_n^{n+k} - \mathbf{q}_n^{n+k}$ and $\check{\mathbf{q}}_0^{n+k} - \mathbf{q}_0^{n+k}$ are denoted similarly, but with $\check{\Delta}_n^{n+k}$ and $\check{\Delta}_0^{n+k}$ respectively instead of Δ_{n+k}.

Note that $\check{\Delta}_{n+k}\, x$ and $\check{\Delta}_{n+k}\, y$ converge to $\lambda_*^2 \Delta_*\, x$ and $\lambda_* \Delta_*\, y$ respectively as $k \to \infty$. By similar considerations as for (4.1) and (4.2), we obtain

$$\check{\Delta}_n^{n+k}\, x = \lambda_*^{2k+1} \Delta_*\, x(1 + O(\rho^n))$$

and

$$\begin{aligned}\check{\Delta}_n^{n+k}\, y &= \lambda_*^k \left(\lambda_*^{k+1} C_1 - b^{q_{2n+1}} C_2\right)(1 + O(\rho^n)) \\ &= \lambda_*^{2k} C_1 \left(\lambda_* - \frac{b^{q_{2n+1}} C_2}{\lambda_*^k C_1}\right)(1 + O(\rho^n)).\end{aligned}$$

Since $\lambda_* < 1$, it follows from (4.3) that $\check{\Delta}_n^{n+k}\, y \asymp \lambda_*^{2k}$. Hence,

$$\check{\Delta}_0^{n+k}\, y \asymp \lambda_*^n \check{\Delta}_0^{n+k} \asymp \lambda_*^{n+2k}.$$

Therefore,

$$\frac{\text{diam}\left(\left[\mathbf{q}_0^{n+k}, \mathbf{p}_0^{n+k}\right]_{\partial\mathcal{D}_b}\right)}{\text{dist}\left(\mathbf{q}_0^{n+k}, \mathbf{p}_0^{n+k}\right)} \geq \frac{\text{dist}\left(\check{\mathbf{q}}_0^{n+k}, \mathbf{q}_0^{n+k}\right)}{\text{dist}\left(\mathbf{q}_0^{n+k}, \mathbf{p}_0^{n+k}\right)} \asymp \lambda_*^{-n} \to \infty \quad \text{as} \quad n \to \infty.$$

The result follows. □

References

Artur, A., Buff, X., Chéritat, A.: Siegel disks with smooth boundaries. Acta Math. **193**(1), 1–30 (2004)
Buff, X., Chéritat, A.: Disks, quadratic siegel, with smooth boundaries. Univ, Paul Sabatier, Toulouse, III, Num, p. 242. (2001) **(preprint)**
de Carvalho, A., Lyubich, M., Martens, M.: Renormalization in the Hénon Family, I: Universality but non-rigidity. J. Stat. Phys. **121**(5/6), 611–669 (2006)
Douady, A.: Disques de Siegel et anneaux de Herman, Astérisque, no. 152–153 (1987)
Gaidashev, D., Yampolsky, M.: Renormalization of almost commuting pairs (2016). arXiv:1604.00719
Gaidashev, D., Radu, R., Yampolsky, M.: Renormalization and Siegel disks for complex Henon maps (2016). ArXiv:1604.07069
Herman, M.: Conjugaison quasi symmetrique des homédomorphismes analytiques du cercle a des rotations. Preliminary manuscript (1987)
Milnor, J.: Dynamics in One Complex Variable: Introductory Lectures, 3rd edn. Princeton University Press, Princeton (2006)
Morosawa, S., Nishimura, Y., Taniguchi, M., Ueda, T.: Holomorphic Dynamics, Cambridge Studies in Advanced Mathematics, vol. 66. Cambridge University Press, Cambridge (2000)
Yang, J.: Renormalization in the golden-mean semi-siegel Hénon Family: Universality and non-rigidity. Ergod. Theory Dyn. Syst. **83**, 1–45 (2018)
Yampolsky, M., Yang, J.: The boundaries of golden-mean Siegel disks in the complex quadratic Hénon family are not smooth (2016). arXiv:1609.02600