Arnold Mathematical Journal (2020) 6:231–269
https://doi.org/10.1007/s40598-020-00149-5

RESEARCH CONTRIBUTION

# Double Extensions of Restricted Lie (Super)Algebras

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Received: 2 December 2019 / Revised: 8 June 2020 / Accepted: 1 July 2020 / Published online: 27 July 2020



## Abstract

A *double extension* ($\mathscr{D}$-extension) of a Lie (super)algebra $\mathfrak{a}$ with a non-degenerate invariant symmetric bilinear form $\mathscr{B}$, briefly, a *NIS*-(super)algebra, is an enlargement of $\mathfrak{a}$ by means of a central extension and a derivation; the affine Kac–Moody algebras are the best known examples of double extensions of loops algebras. Let $\mathfrak{a}$ be a restricted Lie (super)algebra with a NIS $\mathscr{B}$. Suppose $\mathfrak{a}$ has a restricted derivation $\mathscr{D}$ such that $\mathscr{B}$ is $\mathscr{D}$-invariant. We show that the double extension of $\mathfrak{a}$ constructed by means of $\mathscr{B}$ and $\mathscr{D}$ is restricted. We show that, the other way round, any restricted NIS-(super)algebra with non-trivial center can be obtained as a $\mathscr{D}$-extension of another restricted NIS-(super)algebra subject to an extra condition on the central element. We give new examples of $\mathscr{D}$-extensions of restricted Lie (super)algebras, and pre-Lie superalgebras indigenous to characteristic 3.


**Keywords** Restricted Lie (super)algebra · $p|2p$-Structure · Double extension · Vectorial Lie (super)algebra

**Mathematics Subject Classification** Primary 17B50; Secondary 17B20

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# 1 Introduction

## 1.1 NIS-Superalgebras

Let $\mathscr{B}$ be a bilinear form on $\mathfrak{a}$. Consider the *upsetting* of bilinear forms $u : \mathrm{Bil}(V, W) \longmapsto \mathrm{Bil}(W, V)$, see [39, Ch.1], given by the formula

$$u(B)(w, v) = (-1)^{p(v)p(w)} B(v, w) \quad \text{for any } v \in V \text{ and } w \in W. \tag{1}$$

If $V = W$, we say that $B$ is *symmetric* (resp. *anti-symmetric*) if $u(B) = B$ (resp. $u(B) = -B$).

Following [18], we call the Lie superalgebra $\mathfrak{a}$ a *NIS-superalgebra* (sometimes used to be called *quadratic* in the literature) if it has a non-degenerate, invariant, and symmetric bilinear form $\mathscr{B}$. We denote such a superalgebra by $(\mathfrak{a}, \mathscr{B})$.

For a list of a wide class of simple modular NIS-superalgebras, see [18].

A NIS-superalgebra $(\mathfrak{a}, \mathscr{B})$ is said to be *reducible* if it can be decomposed into a direct sum of mutually orthogonal ideals, namely $\mathfrak{a} = \oplus I_i$; otherwise, it is called *irreducible*.

## 1.2 Double Extensions in General

Let $\mathfrak{a}$ be a NIS-superalgebra (in particular, a Lie algebra) defined over a field $\mathbb{K}$ of positive characteristic $p$. The notion of a double extension of the Lie superalgebra $\mathfrak{a}$, called $\mathscr{D}$*-extension* in [8], was introduced by Medina and Revoy [40] in the case of

Lie algebras over $\mathbb{R}$. This notion has been *superized* and studied in a series of papers [1–8]; for a succinct summary, see [20].

The double extension of $\mathfrak{a}$, denoted by $\mathfrak{g}$, simultaneously involves three ingredients:

- a derivation $\mathscr{D}$,
- a $\mathscr{D}$-invariant NIS $\mathscr{B}_{\mathfrak{a}}$ on $\mathfrak{a}$,
- the central extension $\mathfrak{a}_x$ of $\mathfrak{a}$ is given by the cocycle $(a, b) \mapsto \mathscr{B}_{\mathfrak{a}}(\mathscr{D}(a), b)x$ for any $a, b \in \mathfrak{a}$ with the center spanned by the vector $x$.

It turns out that for $\mathfrak{g}$ to be non-trivial, i.e., not isomorphic to the direct sum of $\mathfrak{a}$ and two-dimensional ideal $\mathbb{K}x \oplus \mathbb{K}\mathscr{D}$, the central extension must be non-trivial and the derivation $\mathscr{D}$ outer; moreover, if $\mathscr{D}$ is odd, the following conditions must be met:

$$\mathscr{D}^2 = \mathrm{ad}_b \text{ and } \mathscr{D}(b) = 0 \text{ for some } b \in \mathfrak{a}_{\bar{0}}. \tag{2}$$

*Modular* Lie (super)algebras, i.e., Lie (super)algebras defined over the field of characteristic $p > 0$, can be double extended. The inductive description à la Medina and Revoy becomes, however, the most challenging part, because Lie's theorem and the Levi decomposition do not hold if $p > 0$. The formulations of the inductive description differ if $p > 0$ from their counterparts for $p = 0$, see Theorems 3.6, 3.8, 3.10.

Favre and Santharoubane [26] introduced an important ingredient in the study of NIS-algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$—double extensions of the same Lie algebra $\mathfrak{a}$: they suggested to consider them up to an isomorphism $\pi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ (Favre and Santharoubane called $\pi$ an isometry) such that

$$\mathscr{B}_{\tilde{\mathfrak{g}}}(\pi(f), \pi(g)) = \mathscr{B}_{\mathfrak{g}}(f, g) \text{ for any } f, g \in \mathfrak{g}.$$

The equivalence of double extensions up to isomorphisms turns out to be a very important and useful notion, as demonstrated in [8–10,20] by several new examples. In Sect. 3.4.2, we generalize this notion to the case of Lie superalgebras for any $p > 2$, completing the results of [8], where $p = 2$ was considered.

Observe that for Lie algebras over fields of characteristic $p > 0$, and for Lie superalgebras over fields of any characteristic, the Killing form might be degenerate, cf. [11,18,28,46].

Simple NIS-superalgebras having outer derivations are abundant in positive characteristic (see [6,18,22,24]), and therefore might be double extended if they have non-trivial central extensions; for their list, see [12].

In [33], an interesting relation of double extensions with pseudo-Riemannian manifolds is established; the two approaches (all the above and [33]) have an empty intersection.

### 1.3 Double Extensions of Restricted Lie (Super)Algebras

As far as we know, the notion of a *restricted* Lie algebra was introduced by Jacobson [32]. Roughly speaking, one requires the existence of an endomorphism on the modular Lie algebra that resembles the $p$th power mapping $x \mapsto x^p$ in associative algebras. Lie algebras associated with algebraic groups over fields of positive characteristic are restricted, and this class resembles the characteristic 0 case, see [45,46]. Superization of the notion of restrictedness has been studied by several authors, see [15,17,25,41] and especially [19] where new phenomena were observed. Here we consider only the classical restrictedness, see Sect. 2.1.

Let $(\mathfrak{a}, \mathscr{B}_{\mathfrak{a}})$ be a restricted NIS-(super)algebra with a derivation $\mathscr{D}$ such that $\mathscr{B}_{\mathfrak{a}}$ is $\mathscr{D}$-invariant, namely

$$\begin{aligned}&\mathscr{B}_{\mathfrak{a}}(\mathscr{D}(a), b) + (-1)^{p(\mathscr{D})p(a)}\mathscr{B}_{\mathfrak{a}}(a, \mathscr{D}(b)) = 0 \text{ for all } a, b \in \mathfrak{a} \text{ and } p \neq 2;\\ &\text{for } p = 2, \text{ we additionally require}\\ &\mathscr{B}_{\mathfrak{a}}(a, \mathscr{D}(a)) = 0 \text{ for all } a \in \mathfrak{a}, \text{ whenever } p(\mathscr{D}) + p(\mathscr{B}_{\mathfrak{a}}) = \bar{0}.\end{aligned} \tag{3}$$

In particular, if $\mathscr{D} = \mathrm{ad}_c$, then

$$\mathscr{B}_{\mathfrak{a}}([c, a], b) + (-1)^{p(c)p(a)}\mathscr{B}_{\mathfrak{a}}(a, [c, b]) = 0 \text{ for any } a, b, c \in \mathfrak{a}.$$

In this paper, we answer under what condition

(i) the $p$-structure on $\mathfrak{a}$ with NIS $\mathscr{B}_{\mathfrak{a}}$ can be extended to a $p$-structure on the double extension $\mathfrak{g}$ of $(\mathfrak{a}, \mathscr{B}_{\mathfrak{a}})$.
(ii) a restricted NIS-superalgebra $\mathfrak{g}$ is a double extension of a restricted NIS-(super)algebra $\mathfrak{a}$.

Conditions for (i): for all characteristics, provided the derivation $\mathscr{D}$ is restricted (see condition (6) in Sect. 2.1) and satisfies the following conditions:

(a) In the case of Lie algebras, $\mathscr{D}$ must satisfy condition (8), called *p-property*, which says that the derivations $\mathscr{D}^p$ and $\gamma\mathscr{D}$, where $\gamma \in \mathbb{K}$, have to be cohomologous, see Sect. 2.1.2.
(b) In the case of Lie superalgebras and $p(\mathscr{D}) = \bar{1}$, condition (2) is required; if $p(\mathscr{B}) = \bar{0}$, then condition (19) is required, see Theorems 3.7 and 3.9.
(c) In the case of Lie superalgebras and $p(\mathscr{D}) = \bar{0}$, the derivation $\mathscr{D}$ must satisfy condition (10), a superversion of condition (8), i.e., have $p$-property, see Theorems 3.7 and 3.9.

The NIS on $\mathfrak{g}$ is given by Eq. (12).

Observe that there is a large class of Lie algebras whose restricted derivations have *p-property* (8). For instance, for *nilpotent* restricted Lie algebras, it was proved in [27] that there are outer derivations for which $\gamma = 0$ and $a_0 = 0$, see Eq. (9); for examples of *simple* restricted Lie algebras with $p$-property, see Sect. 4.

Conditions for (ii): any restricted NIS-superalgebra $\mathfrak{g}$ can be obtained as a double extension of a restricted NIS-superalgebra $\mathfrak{a}$ provided the center of $\mathfrak{g}$ is not trivial, and the orthogonal complement of the central element is a $p$-ideal, see Sect. 2.1.

We show that the condition on a restricted derivation $\mathscr{D}$ of the restricted Lie (super)algebra $\mathfrak{a}$ to have *p-property* (8) or (10) is necessary to get a double extension of $\mathfrak{a}$.

We introduce the notion of *equivalent double extensions* that takes the $p$-structures into account, see Theorems 3.13, 3.16, 3.17.

If $\mathfrak{h}$ is a restricted Lie algebra, and $p = 2$, then the Manin triple $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^*$ is also restricted, and the $p$-structure is induced from $\mathfrak{h}$, see Sect. 4.3.1. (Is the same true for $p > 2$?)

### 1.4 Example, See Sect. 4, Summary

These examples, based on the list of **known** simple Lie algebras and superalgebras with NIS, see [18], and the descriptions of their derivations and central extensions, see [12], are classified up to an isomorphism. The cases admitting double extensions are collected in Table (4).

Lie algebras. Ibraev, see [31], proved that only for $\mathfrak{g} = \mathfrak{psl}(3)$, among finite-dimensional Lie algebras $\mathfrak{g}$ with Cartan matrix and their simple subquotients, the dimension of the Lie algebra of outer derivations is $> 1$. More precisely, $\mathfrak{out}(\mathfrak{psl}(3)) \simeq \mathfrak{psl}(3)$.

Among known simple restricted $\mathbb{Z}$-graded vectorial Lie algebras, see [14], only $\mathfrak{svect}^{(1)}(3)$ can be double extended, see Sect. 4.4. Since the vectorial Lie algebras whose shearing vector $\underline{N}$ has at least one coordinate $> 1$ can not be restricted, we assume that $\underline{N} = (1, \ldots, 1)$. For open cases, see Sect. 4.4.3.

The Manin triple $\mathfrak{g} = \mathfrak{hei}(2) \oplus \mathfrak{hei}(2)^*$, although not simple, provides us with an exceptional (see (9)) example in which the $p$-property (8) is satisfied with $\gamma = 1$ and $a_0 \neq 0$, see Eq. (39).

Lie superalgebras. Among Lie superalgebras with Cartan matrix and their simple subquotients, only $\mathfrak{g}(2,3)^{(1)}/\mathfrak{z}$ and $\mathfrak{g}(3,3)^{(1)}/\mathfrak{z}$ can be double extended to Lie superalgebras, see [12]. Amazingly, the double extensions of $\mathfrak{osp}(1|2)$ and $\mathfrak{brj}(2,3)$ for $p = 3$ are **pre-Lie** superalgebras, not Lie superalgebras, see Appendix.

| The Lie (super)algebra | Its double extensions | $p$ | $p(\mathscr{B})$ |
|---|---|---|---|
| $\mathfrak{psl}(3)$ | $\mathfrak{gl}(3),\ \widetilde{\mathfrak{gl}}(3),\ \widehat{\mathfrak{gl}}(3)$ | 3 | $\bar{0}$ |
| $\mathfrak{svect}^{(1)}(3)$ | $\widehat{\mathfrak{svect}}^{(1)}(3)$ | 3 | $\bar{0}$ |
| $\mathfrak{hei}(2) \oplus \mathfrak{hei}(2)^*$ | $\widehat{\mathfrak{hei}(2) \oplus \mathfrak{hei}(2)^*}$ | 2 | $\bar{0}$ |
| $\mathfrak{g}(2,3)^{(1)}/\mathfrak{z}$ | $\mathfrak{g}(2,3),\ \widetilde{\mathfrak{g}}(2,3),\ \widehat{\mathfrak{g}}(2,3)$ | 3 | $\bar{0}$ |
| $\mathfrak{g}(3,3)^{(1)}/\mathfrak{z}$ | $\mathfrak{g}(3,3),\ \widetilde{\mathfrak{g}}(3,3)$ | 3 | $\bar{0}$ |
| $\mathfrak{psq}(n)$ for $n > 2$ | $\mathfrak{q}(n)$ | $> 2$ | $\bar{1}$ |

(4)

None of the simple vectorial Lie superalgebras considered in [18] can be non-trivially double extended. Certain double extensions of $\mathfrak{psq}(n)$ for $n > 2$ and $p = 2$ are described in [35].

## 2 Main Definitions

Hereafter, $\mathbb{K}$ is an arbitrary field of characteristic $\text{char}(\mathbb{K}) = p$.

### 2.1 Restricted Lie Algebras

Let $\mathfrak{a}$ be a finite-dimensional modular Lie algebra over $\mathbb{K}$. For a comprehensive study of modular Lie algebras, see [45,46].

Following [32,46], a mapping $[p] : \mathfrak{a} \to \mathfrak{a}, \quad a \mapsto a^{[p]}$ is called a *p-structure* of $\mathfrak{a}$ and $\mathfrak{a}$ is said to be *restricted if*

$$\begin{aligned}
&\mathrm{ad}_{a^{[p]}} = (\mathrm{ad}_a)^p \text{ for all } a \in \mathfrak{a};\\
&(\alpha a)^{[p]} = \alpha^p a^{[p]} \text{ for all } a \in \mathfrak{a} \text{ and } \alpha \in \mathbb{K};\\
&(a+b)^{[p]} = a^{[p]} + b^{[p]} + \sum_{1\le i\le p-1} s_i(a,b), \text{ where the } s_i(a,b) \text{ can be obtained from}\\
&(\mathrm{ad}_{\lambda a+b})^{p-1}(a) = \sum_{1\le i\le p-1} i s_i(a,b)\lambda^{i-1}.
\end{aligned} \tag{5}$$

The following theorem, due to Jacobson, is very useful to us.

**Theorem 2.1** ([32]) *Let $(e_j)_{j\in J}$ be a basis of $\mathfrak{a}$ such that there are $f_j \in \mathfrak{a}$ satisfying $(\mathrm{ad}_{e_j})^p = \mathrm{ad}_{f_j}$. Then there exists exactly one p-mapping $[p] : \mathfrak{a} \to \mathfrak{a}$ such that*

$$e_j^{[p]} = f_j \quad \text{for all } j \in J.$$

Let $(\mathfrak{a}, [p]_{\mathfrak{a}})$ and $(\tilde{\mathfrak{a}}, [p]_{\tilde{\mathfrak{a}}})$ be two restricted Lie algebras. A linear map $\pi : \mathfrak{a} \to \tilde{\mathfrak{a}}$ is called a *p-homomorphism* if $\pi$ is a homomorphism of Lie algebras and

$$\pi(x^{[p]_{\mathfrak{a}}}) = (\pi(x))^{[p]_{\tilde{\mathfrak{a}}}} \quad \text{for all } x \in \mathfrak{a}.$$

An ideal $I$ of $\mathfrak{a}$ is called a *p-ideal* if $x^{[p]} \in I$ for all $x \in I$.

For an arbitrary subset $S \subset \mathfrak{a}$, we denote

$$S^{[p]^i} := \{x^{[p]^i} \mid x \in S\},$$

where the expression $[p]^i$ stands for the composition $[p] \circ \cdots \circ [p]$ applied $i$ times.

For an arbitrary ideal $I$, we denote

$$I_{(p)} := \sum_{i\ge 0} \mathrm{Span}(I^{[p]^i}).$$

One can show that $I_{(p)}$ is a $p$-ideal of $(\mathfrak{a}, [p])$ (see, e.g., [46, Prop. 1.3]). By definition, $I_{(p)}$ is the smallest $p$-ideal containing the ideal $I$. In particular, $\mathfrak{z}(\mathfrak{a})_{(p)} = \mathfrak{z}(\mathfrak{a})$, where $\mathfrak{z}(\mathfrak{a})$ is the center of $\mathfrak{a}$, a consequence of the first condition of (5).

If $I$ is a $p$-ideal, then the quotient Lie algebra $\mathfrak{a}/I$ has a $p$-structure defined by

$$(a+I)^{[p]} := a^{[p]} + I \quad \text{for any } a \in \mathfrak{a},$$

and the natural map $\pi : \mathfrak{a} \to \mathfrak{a}/I$ is a $p$-homomorphism.

For every restricted Lie algebra $\mathfrak{a}$, one can construct its *p-enveloping algebra*

$$u(\mathfrak{a}) := U(\mathfrak{a})/I,$$

where $I$ is the ideal generated by the central elements $a^{[p]} - a^p \in U(\mathfrak{a})$.

An $\mathfrak{a}$-module $M$ is called *restricted* if

$$\underbrace{a \cdot \ldots a}_{p \text{ times}} \cdot m = a^{[p]} \cdot m \quad \text{for all } a \in \mathfrak{a} \text{ and any } m \in M.$$

A derivation $\mathscr{D} \in \mathfrak{der}(\mathfrak{a})$ is called *restricted* if

$$\mathscr{D}(a^{[p]}) = (\mathrm{ad}_a)^{p-1}(\mathscr{D}(a)) \quad \text{for all } a \in \mathfrak{a}. \tag{6}$$

Denote the space of restricted derivations by $\mathfrak{der}^p(\mathfrak{a})$. Every inner derivation $\mathrm{ad}_a$, where $a \in \mathfrak{a}$, is a restricted derivation. Set $\mathfrak{out}^p(\mathfrak{a}) := \mathfrak{der}^p(\mathfrak{a})/\mathrm{ad}_{\mathfrak{a}}$.

#### 2.1.1 Restricted Lie Algebra Cohomology

We denote by $\mathrm{H}^n(\mathfrak{a}; M)$ the usual Chevalley–Eilenberg cohomology of the Lie algebra $\mathfrak{a}$ with coefficient in the $\mathfrak{a}$-module $M$. Following Hochschild [30], the *restricted* cohomology of a restricted Lie algebra $\mathfrak{a}$ with coefficients in a restricted module $M$ is given by

$$\mathrm{H}^n_{\mathrm{res}}(\mathfrak{a}; M) := \mathrm{Ext}^n_{u(\mathfrak{a})}(\mathbb{K}, M), \quad \text{where } n \geq 0.$$

Hochschild [30] showed that there is an exact sequence

$$\begin{aligned} 0 &\to \mathrm{H}^1_{\mathrm{res}}(\mathfrak{a}; M) \to \mathrm{H}^1(\mathfrak{a}; M) \to S(\mathfrak{a}; M^{\mathfrak{a}}) \to \mathrm{H}^2_{\mathrm{res}}(\mathfrak{a}; M) \\ &\to \mathrm{H}^2(\mathfrak{a}; M) \to S(\mathfrak{a}; \mathrm{H}^1(\mathfrak{a}; M)), \end{aligned} \tag{7}$$

where $S(X, Y)$ is the space of $p$-semi-linear[1] maps $X \to Y$, and $M^{\mathfrak{a}} := \{m \in M \mid \mathfrak{a} \cdot m = 0\}$ is the space of $\mathfrak{a}$-invariants.

An explicit description of the space of cochains $C^k(\mathfrak{a}; M)$ for $k \leq 3$ was carried out in [23]. This description was used to classify extensions of restricted modules and infinitesimal deformations of restricted Lie algebras.

The canonical homomorphism

$$\mathrm{H}_{\mathrm{res}}(\mathfrak{a}; M) \to \mathrm{H}(\mathfrak{a}; M)$$

isomorphically maps $\mathrm{H}^1_{\mathrm{res}}(\mathfrak{a}; M)$ onto the subspace of $\mathrm{H}^1(\mathfrak{a}; M)$ whose elements are represented by the 1-cocycles which satisfy the relation

$$x^{p-1} \cdot f(x) = f(x^{[p]_{\mathfrak{a}}}),$$

[1] A map $f : X \to Y$ is called *p-semi-linear* if $f(x + \lambda y) = f(x) + \lambda^p f(y)$ for all $x, y \in X$ and $\lambda \in \mathbb{K}$.

see [30, Theorem 2.1, page 563]. In particular, $\mathrm{H}^1_{\mathrm{res}}(\mathfrak{a};\mathfrak{a}) \simeq \mathrm{H}^1(\mathfrak{a};\mathfrak{a})$ if $\mathfrak{z}(\mathfrak{a}) = 0$.

### 2.1.2 Restricted Outer Derivations

Clearly, see [23,30],

$$\mathfrak{out}^p(\mathfrak{a}) \simeq \mathrm{H}^1_{\mathrm{res}}(\mathfrak{a};\mathfrak{a}).$$

In all the examples we provide in Sect. 4, we do have $\mathfrak{out}^p(\mathfrak{a}) \neq 0$. Recall that $\mathfrak{out}(\mathfrak{g}) \simeq \mathrm{H}^1(\mathfrak{a};\mathfrak{a})$ while $\mathfrak{out}^p(\mathfrak{g}) \simeq \mathrm{H}^1_{\mathrm{res}}(\mathfrak{a};\mathfrak{a})$. For the simple Lie algebras, we have $\mathrm{H}^1(\mathfrak{a};\mathfrak{a}) = \mathrm{H}^1_{\mathrm{res}}(\mathfrak{a};\mathfrak{a})$. For Lie algebras with center, such as the Manin triple in Sect. 4.3.1, we have to compute $\mathrm{H}^1_{\mathrm{res}}(\mathfrak{a};\mathfrak{a})$ to capture restricted outer derivations.

We say that $\mathscr{D} \in \mathfrak{der}^p(\mathfrak{a})$ has *p-property* if there exist $\gamma \in \mathbb{K}$ and $a_0 \in \mathfrak{a}$ such that

$$\begin{aligned} \mathscr{D}^p &= \gamma\mathscr{D} + \mathrm{ad}_{a_0}, \text{ (or, equivalently, } \mathscr{D}^p \simeq \gamma\mathscr{D} \text{ in } \mathrm{H}^1_{\mathrm{res}}(\mathfrak{a};\mathfrak{a})) \\ \mathscr{D}(a_0) &= 0. \end{aligned} \tag{8}$$

The restricted outer derivations we provide as main examples in Sect. 4 do have $p$-property, same as nilpotent Lie algebras, see [27]:

$$\begin{gathered} \text{Any restricted derivation of any torus identically vanishes ([FSW, Prop. 3.1]);} \\ \text{Every restricted derivation of } \mathfrak{hei}(2) \text{ for } p=2 \text{ is inner}^2 \text{ ([FSW, Prop. 3.2]);} \\ \text{Apart from a torus, and } \mathfrak{hei}(2) \text{ for } p=2, \text{ every outer restricted derivation } \mathscr{D} \\ \text{of any } \textit{nilpotent} \text{ restricted Lie algebra satisfies } \mathscr{D}^2 = 0 \text{ ([FSW, Theorem 3.3]).} \end{gathered} \tag{9}$$

## 2.2 Restricted Lie Superalgebras

Let $\mathfrak{a}$ be a Lie superalgebra defined over a field of characteristic $p > 2$. We denote the parity of a given non-zero homogenous element $a \in \mathfrak{a}$ by $p(a)$; no confusion with the characteristic of the ground field is possible.

Following [41], we say that $\mathfrak{a}$ has a *p*|2*p-structure* if $\mathfrak{a}_{\bar{0}}$ is restricted and

$$\mathrm{ad}_{a^{[p]}}(b) = (\mathrm{ad}_a)^p(b) \text{ for all } a \in \mathfrak{a}_{\bar{0}} \text{ and } b \in \mathfrak{a}.$$

Recall that the bracket of two odd elements of the Lie superalgebra is polarization of squaring $a \mapsto a^2$. We set

$$[2p] : \mathfrak{a}_{\bar{1}} \to \mathfrak{a}_{\bar{0}}, \quad a \mapsto (a^2)^{[p]} \text{ for any } a \in \mathfrak{a}_{\bar{1}}.$$

The pair $(\mathfrak{a}, [p|2p])$ is referred to as a *restricted* Lie superalgebra.

The following theorem is a straightforward superization of Jacobson's theorem 2.1.

[2] Recall that the Heisenberg Lie algebra $\mathfrak{hei}(2n)$ is spanned by elements $c_i, a_i$ for $i = 1, \dots, n$, and a central element $z$ with the only non-zero relations $[c_i, a_i] = z$.

**Theorem 2.2** *Let $(e_j)_{j\in J}$ be a basis of $\mathfrak{a}_{\bar 0}$, let the elements $f_j \in \mathfrak{a}_{\bar 0}$ be such that $(\mathrm{ad}_{e_j})^p = \mathrm{ad}_{f_j}$. Then there exists exactly one $p|2p$-mapping $[p|2p] : \mathfrak{a} \to \mathfrak{a}$ such that*

$$e_j^{[p]} = f_j \quad \textit{for all } j \in J.$$

### 2.2.1 Restricted Derivations

A derivation $\mathscr{D} \in \mathfrak{der}(\mathfrak{a})$ is called *restricted* if

$$\mathscr{D}(a^{[p]}) = (\mathrm{ad}_a)^{p-1}(\mathscr{D}(a)) \quad \text{for all } a \in \mathfrak{a}_{\bar 0}.$$

Consequently, we have

$$\mathscr{D}(a^{[2p]}) = (\mathrm{ad}_{a^2})^{p-1}(\mathscr{D}(a^2)) \quad \text{for all } a \in \mathfrak{a}_{\bar 1}.$$

As in the non-super setting, the space of restricted derivation is denoted by $\mathfrak{der}^p(\mathfrak{a})$.

A super version of condition (8): $\mathscr{D} \in \mathfrak{der}^p_{\bar 0}(\mathfrak{a})$ has *p-property* if there exist $\gamma \in \mathbb{K}$ and $a_0 \in \mathfrak{a}_{\bar 0}$ such that

$$\mathscr{D}^p = \gamma\mathscr{D} + \mathrm{ad}_{a_0}, \quad \mathscr{D}(a_0) = 0. \tag{10}$$

## 3 The Main Results

Let $\mathscr{K}$ and $\mathscr{K}^*$ be one-dimensional vector spaces spanned by the vectors $x$ and $x^*$, respectively.

### 3.1 The Case of Lie Algebras

The following theorem was proved in [40] for $\mathbb{K} = \mathbb{R}$. Passing to a field of characteristic $p \neq 2$, the proof is absolutely the same. The case where $p = 2$ has been completely described in [8].

**Theorem 3.1** [8,40] *Let $\mathscr{B}_\mathfrak{a}$ be $\mathscr{D}$-invariant, where $\mathscr{D} \in \mathfrak{der}(\mathfrak{a})$. Then, there exists a NIS-algebra structure on $\mathfrak{g} := \mathscr{K} \oplus \mathfrak{a} \oplus \mathscr{K}^*$, where the bracket is defined by*

$$\begin{aligned} &[\mathscr{K}, \mathfrak{g}]_\mathfrak{g} := 0, \qquad [a, b]_\mathfrak{g} := [a, b]_\mathfrak{a} + \mathscr{B}_\mathfrak{a}(\mathscr{D}(a), b)x \textit{ for any } a, b \in \mathfrak{a}, \\ &[x^*, a]_\mathfrak{g} := \mathscr{D}(a) \textit{ for any } a \in \mathfrak{a}. \end{aligned} \tag{11}$$

*The NIS form $\mathscr{B}_\mathfrak{g}$ on $\mathfrak{g}$ is defined as follows:*

$$\begin{aligned} &\mathscr{B}_\mathfrak{g}|_{\mathfrak{a}\times\mathfrak{a}} := \mathscr{B}_\mathfrak{a}, \quad \mathscr{B}_\mathfrak{g}(x, x^*) := 1, \quad \mathscr{B}_\mathfrak{g}(\mathfrak{a}, x) = \mathscr{B}_\mathfrak{g}(\mathfrak{a}, x^*) = \mathscr{B}_\mathfrak{g}(x, x) := 0, \\ &\mathscr{B}_\mathfrak{g}(x^*, x^*) := \begin{cases} \textit{arbitrary}, & \textit{if } p = 2 \\ 0, & \textit{if } p > 2 \end{cases} \end{aligned} \tag{12}$$

We call the Lie algebra $(\mathfrak{g}, \mathscr{B}_\mathfrak{g})$ constructed in Theorem 3.1 a *$\mathscr{D}$-extension* of $(\mathfrak{a}, \mathscr{B}_\mathfrak{a})$ by means of $\mathscr{D}$.

***Remark 3.2*** If the derivation $\mathscr{D}$ is inner, the double extension is isomorphic to $\mathfrak{a} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is a two-dimensional center, see Theorem 3.11.

The converse of Theorem 3.1 is given by the following.

**Theorem 3.3** ([40]) *If $\mathfrak{z}(\mathfrak{g}) \neq 0$, then $(\mathfrak{g}, \mathscr{B}_\mathfrak{g})$ can be obtained from a NIS-algebra by means of a $\mathscr{D}$-extension.*

Denote by $\sigma_i^\mathfrak{a}(a, b)$ the coefficient obtained from the expansion

$$\mathscr{B}_\mathfrak{a}(\mathscr{D}(\lambda a + b), (\mathrm{ad}^\mathfrak{a}_{\lambda a+b})^{p-2}(a)) = \sum_{1 \leq i \leq p-1} i\sigma_i^\mathfrak{a}(a, b)\lambda^{i-1}.$$

For instance,

- If $p = 2$, then $\sigma_1^\mathfrak{a}(a, b) = \mathscr{B}_\mathfrak{a}(\mathscr{D}(b), a)$.
- If $p = 3$, then $\sigma_1^\mathfrak{a}(a, b) = \mathscr{B}_\mathfrak{a}(\mathscr{D}(b), [b, a])$ and $\sigma_2^\mathfrak{a}(a, b) = 2\mathscr{B}_\mathfrak{a}(\mathscr{D}(a), [b, a])$.

**Lemma 3.4** *Under assumptions of Theorem 3.1, we have*

$$s_i^\mathfrak{g}(a, b) = s_i^\mathfrak{a}(a, b) + \sigma_i^\mathfrak{a}(a, b)x \quad \textit{for all } a, b \in \mathfrak{a}.$$

***Proof*** Indeed,

$$(\mathrm{ad}^\mathfrak{g}_{\lambda a+b})^{p-1}(a) = (\mathrm{ad}^\mathfrak{a}_{\lambda a+b})^{p-1}(a) + \mathscr{B}_\mathfrak{a}(\mathscr{D}(\lambda a + b), (\mathrm{ad}^\mathfrak{a}_{\lambda a+b})^{p-2}(a))x. \qquad \square$$

#### 3.1.1 The *p*-Structures on Double Extensions: The Map $\mathscr{P}$

Note that for $p = 2$, the map $\mathscr{P}$ is quadratic, and therefore it was denoted by $\mathfrak{q}$ in [8,20].

**Theorem 3.5** ($p$-structure on DE). *Let $\mathscr{B}_\mathfrak{a}$ be $\mathscr{D}$-invariant, where $\mathscr{D} \in \mathfrak{Der}^p(\mathfrak{a})$ and let $\mathscr{D}$ have the $p$-property. For arbitrary $m, l \in \mathbb{K}$, the $p$-structure on $\mathfrak{a}$ can be extended to its $\mathscr{D}$-extension $\mathfrak{g} = \mathscr{K} \oplus \mathfrak{a} \oplus \mathscr{K}^*$ as follows (for any $a \in \mathfrak{a}$):*

$$\begin{aligned} a^{[p]_\mathfrak{g}} &:= a^{[p]_\mathfrak{a}} + \mathscr{P}(a)x, \\ (x^*)^{[p]_\mathfrak{g}} &:= a_0 + lx + \gamma x^*, \\ x^{[p]_\mathfrak{g}} &:= mx + b_0, \end{aligned}$$

*where $a_0$ and $\gamma$ are as in (8) (the $p$-property), $b_0 \in \mathfrak{z}(\mathfrak{a})$ such that $\mathscr{D}(b_0) = 0$, and $\mathscr{P}$ is a map satisfying (for any $a, b \in \mathfrak{a}$ and any $\lambda \in \mathbb{K}$):*

$$\begin{aligned} \mathscr{P}(\lambda a) &= \lambda^p \mathscr{P}(a), \\ \mathscr{P}(a + b) - \mathscr{P}(a) - \mathscr{P}(b) &= \sum_{1 \leq i \leq p-1} \sigma_i^\mathfrak{a}(a, b). \end{aligned} \tag{13}$$

***Proof*** Using Jacobson's Theorem 2.1, it suffices to show that

$$(\mathrm{ad}_a^{\mathfrak{g}})^p = \mathrm{ad}_{a^{[p]_{\mathfrak{a}}}+\mathscr{P}(a)x}, \quad (\mathrm{ad}_{x^*}^{\mathfrak{g}})^p = \mathrm{ad}_{a_0+lx+\gamma x^*}, \quad (\mathrm{ad}_x^{\mathfrak{g}})^p = \mathrm{ad}_{mx+b_0}. \qquad \square$$

The converse of Theorem 3.5 is the following.

**Theorem 3.6** (Converse of Theorem 3.5) *Let* $(\mathfrak{g}, \mathscr{B}_{\mathfrak{g}})$ *be an irreducible and restricted NIS-algebra of dim* $> 1$ *such that* $\mathfrak{z}(\mathfrak{g}) \neq 0$*. Let* $0 \neq x \in \mathfrak{z}(\mathfrak{g})$ *be such that* $\mathscr{K}^\perp$ *is a p-ideal. Then,* $(\mathfrak{g}, \mathscr{B}_{\mathfrak{g}})$ *is a* $\mathscr{D}$*-extension of a restricted NIS-algebra* $(\mathfrak{a}, \mathscr{B}_{\mathfrak{a}})$*, where* $\mathscr{D}$ *is a restricted derivation with p-property.*

***Proof*** The subspace $\mathscr{K}$ is an ideal in $(\mathfrak{g}, \mathscr{B}_{\mathfrak{g}})$ because $x$ is central in $\mathfrak{g}$. Moreover, $\mathscr{K}^\perp$ is also an ideal in $(\mathfrak{g}, \mathscr{B}_{\mathfrak{g}})$, see [8,40]. Since $\mathfrak{g}$ is irreducible, it follows that $\mathscr{K} \subset \mathscr{K}^\perp$ and $\dim(\mathscr{K}^\perp) = \dim(\mathfrak{g}) - 1$. Therefore, there exists a non-zero $x^* \in \mathfrak{g}$ such that

$$\mathfrak{g} = \mathscr{K}^\perp \oplus \mathscr{K}^*.$$

We normalize $x^*$ so that $\mathscr{B}_{\mathfrak{g}}(x, x^*) = 1$. Besides, $\mathscr{B}_{\mathfrak{g}}(x, x) = 0$ since $\mathscr{K} \cap \mathscr{K}^\perp = \mathscr{K}$.

Set $\mathfrak{a} := (\mathscr{K} + \mathscr{K}^*)^\perp$. We then obtain a decomposition $\mathfrak{g} = \mathscr{K} \oplus \mathfrak{a} \oplus \mathscr{K}^*$.

There exists a NIS-algebra structure on the vector space $\mathfrak{a}$ for which $\mathfrak{g}$ is its double extension by Theorem 3.3. We denote a NIS on $\mathfrak{a}$ by $\mathscr{B}_{\mathfrak{a}}$.

It remains to show that there is a $p$-structure on $\mathfrak{a}$. Since $\mathfrak{a} \subset \mathscr{K}^\perp$, then

$$a^{[p]_{\mathfrak{g}}} \in \mathscr{K}_p^\perp = \mathscr{K}^\perp = \mathscr{K} \oplus \mathfrak{a} \quad \text{for any } a \in \mathfrak{a}.$$

It follows that

$$a^{[p]_{\mathfrak{g}}} = \mathscr{P}(a)x + s(a), \quad \text{where } s(a) \in \mathfrak{a}.$$

Define a $p$-structure on $\mathfrak{a}$ by setting

$$s : \mathfrak{a} \to \mathfrak{a}, \quad a \mapsto s(a).$$

The fact that $(\lambda a)^{[p]_{\mathfrak{g}}} = \lambda^p(a^{[p]_{\mathfrak{g}}})$ implies $s(\lambda a) = \lambda^p s(a)$ and $\mathscr{P}(\lambda a) = \lambda^p \mathscr{P}(a)$. Besides,

$$\begin{aligned} 0 &= [a^{[p]_{\mathfrak{g}}}, b]_{\mathfrak{g}} - (\mathrm{ad}_a^{\mathfrak{g}})^p(b) \\ &= [s(a), b]_{\mathfrak{a}} + \mathscr{B}_{\mathfrak{a}}(\mathscr{D}(s(a)), b)x - (\mathrm{ad}_a^{\mathfrak{a}})^p(b) - \mathscr{B}_{\mathfrak{a}}(\mathscr{D}(a), (\mathrm{ad}_a^{\mathfrak{a}})^{p-1}(b))x. \end{aligned}$$

Therefore,

$$\mathscr{D}(s(a)) = (\mathrm{ad}_a^{\mathfrak{a}})^{p-1} \circ (\mathscr{D}(a)), \quad [s(a), b]_{\mathfrak{a}} = (\mathrm{ad}_a^{\mathfrak{a}})^p(b).$$

Now, since

$$\sum_{1\le i\le p-1} i s_i^{\mathfrak{g}}(a,b)\lambda^{i-1} = (\mathrm{ad}^{\mathfrak{g}}_{\lambda a+b})^{p-1}(a),$$
$$= (\mathrm{ad}^{\mathfrak{a}}_{\lambda a+b})^{p-1}(a) + \mathscr{B}_{\mathfrak{a}}(\mathscr{D}(\lambda a+b), (\mathrm{ad}^{\mathfrak{a}}_{\lambda a+b})^{p-2}(a))x,$$

it follows that

$$s_i^{\mathfrak{g}}(a,b) = s_i^{\mathfrak{a}}(a,b) + \sigma_i^{\mathfrak{a}}(a,b)x.$$

Moreover,

$$\begin{aligned}0 &= (a+b)^{[p]_{\mathfrak{g}}} - a^{[p]_{\mathfrak{g}}} - b^{[p]_{\mathfrak{g}}} - \sum_{1\le i\le p-1} s_i^{\mathfrak{g}}(a,b)\\ &= (\mathscr{P}(a+b) - \mathscr{P}(a) - \mathscr{P}(b) - \sum_{1\le i\le p-1}\sigma_i^{\mathfrak{a}}(a,b))x - \sum_{1\le i\le p-1} s_i^{\mathfrak{a}}(a,b)\\ &\quad + s(a+b) - s(a) - s(b).\end{aligned}$$

Consequently,

$$s(a+b) = s(a) + s(b) + \sum_{1\le i\le p-1} s_i^{\mathfrak{a}}(a,b),$$
$$\mathscr{P}(a+b) - \mathscr{P}(a) - \mathscr{P}(b) = \sum_{1\le i\le p-1}\sigma_i^{\mathfrak{a}}(a,b).$$

It follows that $s$ defines a $p$-mapping on $\mathfrak{a}$, that $\mathscr{D}$ is a restricted derivation of $\mathfrak{a}$ (relative to the $p$-mapping $s$), and $\mathscr{P}$ is a mapping on $\mathfrak{a}$ satisfying Eq. (13).

Suppose that

$$(x^*)^{[p]_{\mathfrak{g}}} = a_0 + \beta x + \gamma x^*, \text{ where } a_0 \in \mathfrak{a} \text{ and } \beta, \gamma \in \mathbb{K}. \tag{14}$$

For all $a \in \mathfrak{a}$, we have

$$0 = [(x^*)^{[p]}, a]_{\mathfrak{g}} - (\mathrm{ad}^{\mathfrak{g}}_{x^*})^p(a) = [a_0, a] + \mathscr{B}_{\mathfrak{a}}(\mathscr{D}(a_0), a)x + \gamma\mathscr{D}(a) - \mathscr{D}^p(a).$$

Since $\mathscr{B}_{\mathfrak{g}}$ is non-degenerate, it follows that $\mathscr{D}^p = \gamma\mathscr{D} + \mathrm{ad}_{a_0}$ and $\mathscr{D}(a_0) = 0$.

Suppose now that

$$x^{[p]_{\mathfrak{g}}} = b_0 + mx + \delta x^* \text{ for some } m, \delta \in \mathbb{K} \text{ and } b_0 \in \mathfrak{a}. \tag{15}$$

For any $b \in \mathfrak{a}$, we have

$$0 = [x^{[p]_{\mathfrak{g}}}, b]_{\mathfrak{g}} - (\mathrm{ad}^{\mathfrak{g}}_x)^p(b) = [b_0, b]_{\mathfrak{a}} + \mathscr{B}_{\mathfrak{a}}(\mathscr{D}(b_0), b)x + \delta\mathscr{D}(b).$$

Since $\mathscr{B}_{\mathfrak{a}}$ is non-degenerate, it follows that $\delta\mathscr{D}(b)+[b_0,b]_{\mathfrak{a}}=0$ and $\mathscr{D}(b_0)=0$.

The case where $\mathscr{D}\not\simeq 0$ in $\mathrm{H}^1_{\mathrm{res}}(\mathfrak{a};\mathfrak{a})$ : It follows that $\delta=0$ and $b_0\in\mathfrak{z}(\mathfrak{a})$. Therefore, $\mathfrak{g}$ can be obtained from the restricted Lie algebra $\mathfrak{a}$ by means of the restricted derivation $\mathscr{D}$ as in Theorem 3.5

The case where $\mathscr{D}\simeq 0$ in $\mathrm{H}^1_{\mathrm{res}}(\mathfrak{a};\mathfrak{a})$ : Without loss of generality, we can assume that $\mathscr{D}=0$, cf. Theorem 3.11. In this case, $\mathfrak{g}\simeq\mathfrak{a}\oplus\mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$ spanned by $x$ and $x^*$, and the $p$-structure is given by Eqs. (14) and (15), where $a_0,b_0\in\mathfrak{z}(\mathfrak{a})$, while $\gamma,\delta,m$ and $\beta$ are arbitrary. □

### 3.2 The Case of Lie Superalgebras: $p(\mathscr{B}_{\mathfrak{a}})=\bar{0}$

Let $(\mathfrak{a},\mathscr{B}_{\mathfrak{a}})$ be a NIS-superalgebra, $p\neq 2$. A double extension by means of an even derivation is called a $\mathscr{D}_{\bar{0}}$-*extension*, while a double extension by means of an odd derivation is called a $\mathscr{D}_{\bar{1}}$-*extension*.

For every $a,b\in\mathfrak{a}_{\bar{0}}$, denote by $\sigma_i^{\mathfrak{a}}(a,b)$ the coefficient obtained from the expansion

$$\mathscr{B}_{\mathfrak{a}}(\mathscr{D}(\lambda a+b),(\mathrm{ad}^{\mathfrak{a}}_{\lambda a+b})^{p-2}(a))=\sum_{1\leq i\leq p-1} i\sigma_i^{\mathfrak{a}}(a,b)\lambda^{i-1}.$$

Let $\mathscr{B}_{\mathfrak{a}}$ be $\mathscr{D}$-invariant, where $\mathscr{D}\in\mathfrak{der}^p(\mathfrak{a})$. If $p(\mathscr{D})=\bar{1}$, assume that $\mathscr{D}$ satisfies the following conditions for some $b_0\in\mathfrak{a}_{\bar{0}}$:

$$\mathscr{D}(b_0)=0,\quad \mathscr{D}^2=\mathrm{ad}_{b_0},\quad \mathscr{B}_{\mathfrak{a}}(b_0,b_0)=0. \tag{16}$$

For $p=3$, let us suppose further that[3]

$$\mathscr{B}_{\mathfrak{a}}(\mathscr{D}(a),[a,a]_{\mathfrak{a}})=0\quad\text{for all } a\in\mathfrak{a}_{\bar{1}}. \tag{17}$$

Let $p(x)=p(x^*)=p(\mathscr{D})$. Let $b_0$ be as in Eq. (16). Following [1,5], there exists a NIS-superalgebra structure on $\mathfrak{g}:=\mathscr{K}\oplus\mathfrak{a}\oplus\mathscr{K}^*$, where the bracket is defined by

$$\begin{aligned}
[\mathscr{K},\mathfrak{g}]_{\mathfrak{g}}&:=0,\\
[a,b]_{\mathfrak{g}}&:=[a,b]_{\mathfrak{a}}+(-1)^{p(\mathscr{D})}\mathscr{B}_{\mathfrak{a}}(\mathscr{D}(a),b)x \text{ for any } a,b\in\mathfrak{a},\\
[x^*,a]_{\mathfrak{g}}&:=\mathscr{D}(a)-\begin{cases}2\mathscr{B}_{\mathfrak{a}}(a,b_0)x & \text{if } p(\mathscr{D})=\bar{1}\\ 0 & \text{if } p(\mathscr{D})=\bar{0}\end{cases}\text{ for any } a\in\mathfrak{a}\\
[x^*,x^*]_{\mathfrak{g}}&:=\begin{cases}2b_0 & \text{if } p(\mathscr{D})=\bar{1}\\ 0 & \text{if } p(\mathscr{D})=\bar{0}\end{cases}
\end{aligned} \tag{18}$$

The NIS form $\mathscr{B}_{\mathfrak{g}}$ on $\mathfrak{g}$ is defined as in (12).

**Theorem 3.7** ($p$-structure on DE for $p(\mathscr{B}_{\mathfrak{a}})=\bar{0}$). *Let $\mathscr{B}_{\mathfrak{a}}$ be $\mathscr{D}$-invariant, where $\mathscr{D}\in\mathfrak{der}^p(\mathfrak{a})$ and let $p(\mathscr{B}_{\mathfrak{a}})=\bar{0}$.*

[3] This condition is automatically satisfied for $p>3$, since $\mathscr{D}$ is a derivation.

(i) *If* $p(\mathscr{D})=\bar{1}$, $\mathscr{D}$ *satisfies the conditions* (16) *and* (17) *and*

$$2\mathscr{B}_{\mathfrak{a}}(a^{[p]_{\mathfrak{a}}},b_0)-\mathscr{B}_{\mathfrak{a}}(\mathscr{D}(a),(\mathrm{ad}_a)^{p-2}\circ\mathscr{D}(a))=0 \quad \text{for all } a\in\mathfrak{a}_{\bar{0}}, \tag{19}$$

*then the* $p|2p$*-structure on* $\mathfrak{a}$ *can be extended to its double extension* $\mathfrak{g}$ *as follows:*

$$a^{[p]_{\mathfrak{g}}}:=a^{[p]_{\mathfrak{a}}} \quad \text{for any } a\in\mathfrak{a}_{\bar{0}}.$$

(ii) *If* $p(\mathscr{D})=\bar{0}$ *and* $\mathscr{D}$ *has p-property, then for arbitrary* $m,l\in\mathbb{K}$, *the* $p|2p$*-structure on* $\mathfrak{a}$ *can be extended to its double extension* $\mathfrak{g}$ *as follows (for any* $a\in\mathfrak{a}_{\bar{0}}$*):*

$$\begin{aligned}a^{[p]_{\mathfrak{g}}}&:=a^{[p]_{\mathfrak{a}}}+\mathscr{P}(a)x,\\(x^*)^{[p]_{\mathfrak{g}}}&:=a_0+lx+\gamma x^*,\\x^{[p]_{\mathfrak{g}}}&:=mx+c_0,\end{aligned}$$

*where* $\gamma,a_0$ *are defined as in* (10) *(the p-property),* $c_0\in\mathfrak{z}_{\bar{0}}(\mathfrak{a})$ *such that* $\mathscr{D}(c_0)=0$, *and* $\mathscr{P}$ *is a map satisfying (for any* $a,b\in\mathfrak{a}_{\bar{0}}$ *and for any* $\lambda\in\mathbb{K}$*):*

$$\begin{aligned}\mathscr{P}(\lambda a)&=\lambda^p\mathscr{P}(a),\\\mathscr{P}(a+b)-\mathscr{P}(a)-\mathscr{P}(b)&=\sum_{1\le i\le p-1}\sigma_i^{\mathfrak{a}}(a,b).\end{aligned} \tag{20}$$

***Proof*** Similar to that of Theorem 3.5, minding the Sign Rule. □

**Theorem 3.8** (Converse of Theorem 3.7). *Let* $(\mathfrak{g},\mathscr{B}_{\mathfrak{g}})$ *be an irreducible and restricted NIS-superalgebra of dim* $>1$.

(i) *Suppose there exists* $0\neq x\in\mathfrak{z}_{\bar{0}}(\mathfrak{g})$ *such that* $\mathscr{K}^{\perp}$ *is a p-ideal. Then* $(\mathfrak{g},\mathscr{B}_{\mathfrak{g}})$ *is a* $\mathscr{D}_{\bar{0}}$*-extension of a restricted NIS-superalgebra* $(\mathfrak{a},\mathscr{B}_{\mathfrak{a}})$*, where* $\mathscr{D}$ *is a restricted derivation with p-property.*
(ii) *Let* $\mathfrak{z}_{\bar{1}}(\mathfrak{g})\neq 0$. *Then* $(\mathfrak{g},\mathscr{B}_{\mathfrak{g}})$ *is a* $\mathscr{D}_{\bar{1}}$*-extension of a restricted NIS-superalgebra* $(\mathfrak{a},\mathscr{B}_{\mathfrak{a}})$*, where* $\mathscr{D}_{\bar{1}}$ *is a restricted derivation that satisfies conditions* (16) *and* (17).

***Proof*** Similar to that of Theorem 3.6, minding the Sign Rule. □

### 3.3 The Case of Lie Superalgebras: $p(\mathscr{B}_{\mathfrak{a}})=\bar{1}$

Let $(\mathfrak{a},\mathscr{B}_{\mathfrak{a}})$ be a NIS-Lie superalgebra, $p\neq 2$. Let $\mathscr{B}_{\mathfrak{a}}$ be $\mathscr{D}$-invariant for some $\mathscr{D}\in\mathfrak{der}^p(\mathfrak{a})$ that satisfies the following conditions:

$$\text{If } p(\mathscr{D})=\bar{1}, \text{ we assume that } \mathscr{D}^2=\mathrm{ad}_{b_0} \text{ and } \mathscr{D}(b_0)=0. \tag{21}$$

$$\text{If } p(\mathscr{D})=\bar{0} \text{ and } p=3, \text{ we assume that } \mathscr{B}_{\mathfrak{a}}(\mathscr{D}(a),[a,a]_{\mathfrak{a}})=0 \quad \text{for any } a\in\mathfrak{a}_{\bar{1}}. \tag{22}$$

Let $\mathfrak{g} := \mathscr{K} \oplus \mathfrak{a} \oplus \mathscr{K}^*$, where $p(x) = p(\mathscr{D}) + \bar{1}$ and $p(x^*) = p(\mathscr{D})$. As shown in [2], there exists a NIS-superalgebra structure on $\mathfrak{g}$. Let $\lambda_0 \in \mathbb{K}$ and $b_0$ be as in Eq. (21). Set

$$\begin{aligned}[][\mathscr{K}, \mathfrak{g}]_\mathfrak{g} &:= 0,\\ [a, b]_\mathfrak{g} &:= [a, b]_\mathfrak{a} + \mathscr{B}_\mathfrak{a}(\mathscr{D}(a), b)x \text{ for any } a, b \in \mathfrak{a}\\ [x^*, x^*]_\mathfrak{g} &:= \begin{cases} 2b_0 + \lambda_0 x, & \text{if } p(\mathscr{D}) = \bar{1} \\ 0, & \text{if } p(\mathscr{D}) = \bar{0} \end{cases},\\ [x^*, a]_\mathfrak{g} &:= \mathscr{D}(a) - \begin{cases} (-1)^{p(a)} 2\mathscr{B}_\mathfrak{a}(a, b_0)x, & \text{if } p(\mathscr{D}) = \bar{1} \\ 0, & \text{if } p(\mathscr{D}) = \bar{0} \end{cases} \text{ for any } a \in \mathfrak{a}. \end{aligned} \tag{23}$$

The NIS form $\mathscr{B}_\mathfrak{g}$ on $\mathfrak{g}$ is defined as in (12).

**Theorem 3.9** ($p$-structure on DE for $p(\mathscr{B}_\mathfrak{a}) = \bar{1}$). *Let $\mathscr{B}_\mathfrak{a}$ be $\mathscr{D}$-invariant, where $\mathscr{D} \in \mathfrak{der}^p(\mathfrak{a})$, and let $p(\mathscr{B}_\mathfrak{a}) = \bar{1}$.*

(i) *If $p(\mathscr{D}) = \bar{1}$ and condition (21) is satisfied, then the $p|2p$-structure on $\mathfrak{a}$ can be extended to $\mathfrak{g}$ as follows* (for any $a \in \mathfrak{a}_{\bar{0}}$, $m \in \mathbb{K}$ and $c_0 \in \mathfrak{z}_{\bar{0}}(\mathfrak{a})$ such that $\mathscr{D}(c_0) = 0$)

$$a^{[p]_\mathfrak{g}} := a^{[p]_\mathfrak{a}} + \mathscr{P}(a)x, \quad x^{[p]_\mathfrak{g}} := mx + c_0.$$

*where $\mathscr{P}$ is a mapping on $\mathfrak{a}_{\bar{0}}$ satisfying* (*for any $a, b \in \mathfrak{a}_{\bar{0}}$ and for any $\lambda \in \mathbb{K}$*)*:*

$$\begin{aligned} \mathscr{P}(\lambda a) &= \lambda^p \mathscr{P}(a),\\ \mathscr{P}(a + b) - \mathscr{P}(a) - \mathscr{P}(b) &= \sum_{1 \le i \le p-1} \sigma_i^\mathfrak{a}(a, b). \end{aligned} \tag{24}$$

(ii) *If $p(\mathscr{D}) = \bar{0}$, has $p$-property, and satisfies condition (22), then the $p|2p$-structure on $\mathfrak{a}$ can be extended to $\mathfrak{g}$ as follows* (for any $a \in \mathfrak{a}_{\bar{0}}$)

$$a^{[p]_\mathfrak{g}} := a^{[p]_\mathfrak{a}}, \quad (x^*)^{[p]_\mathfrak{g}} := \gamma x^* + a_0,$$

*where $a_0$ and $\gamma$ are as in the $p$-property.*

***Proof*** Similar to that of Theorem 3.7. □

**Theorem 3.10** (Converse of Theorem 3.9) *Let $(\mathfrak{g}, \mathscr{B}_\mathfrak{g})$ be an irreducible restricted NIS-superalgebra of dim $> 1$ such that $p(\mathscr{B}_\mathfrak{a}) = \bar{1}$.*

(i) *Suppose that $\mathfrak{z}_{\bar{0}}(\mathfrak{g}) \neq 0$. Then $(\mathfrak{g}, \mathscr{B}_\mathfrak{g})$ is a $\mathscr{D}_{\bar{1}}$-extension of a restricted NIS-superalgebra $(\mathfrak{a}, \mathscr{B}_\mathfrak{a})$, where $\mathscr{D}_{\bar{1}}$ is an odd restricted derivation.*

(ii) *Suppose there exists $0 \neq x \in \mathfrak{z}_{\bar{1}}(\mathfrak{g})$ such that $\mathscr{K}^\perp$ is a $p$-ideal. Then $(\mathfrak{g}, \mathscr{B}_\mathfrak{g})$ is a $\mathscr{D}_{\bar{0}}$-extension of a restricted NIS-superalgebra $(\mathfrak{a}, \mathscr{B}_\mathfrak{a})$ such that $\mathscr{B}_\mathfrak{a}$ is $\mathscr{D}$-invariant, where $\mathscr{D}_{\bar{0}}$ is a restricted derivation with $p$-property and (only for $p = 3$) satisfying condition (22).*

***Proof*** Similar to that of Theorem 3.6, minding the Sign Rule. □

### 3.4 Isomorphisms of NIS-Superalgebras

For a NIS-superalgebra $\mathfrak{a}$ with NIS $\mathscr{B}_{\mathfrak{a}}$, denote by $\mathfrak{g}$ (resp. $\tilde{\mathfrak{g}}$) the double extension of $\mathfrak{a}$ by means of a derivation $\mathscr{D}$ (resp. $\tilde{\mathscr{D}}$). An isomorphism $\pi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ should satisfy (see [8,26]):

$$\begin{aligned} \pi([f,g]_{\mathfrak{g}}) &= [\pi(f),\pi(g)]_{\tilde{\mathfrak{g}}} \text{ for any } f,g \in \mathfrak{g},\\ \mathscr{B}_{\tilde{\mathfrak{g}}}(\pi(f),\pi(g)) &= \mathscr{B}_{\mathfrak{g}}(f,g) \quad\quad \text{for any } f,g \in \mathfrak{g}.\end{aligned}$$

We will further assume that $\pi(\mathscr{K} \oplus \mathfrak{a}) = \tilde{\mathscr{K}} \oplus \tilde{\mathfrak{a}}$, and call $\pi$ an *adapted isomorphism*. We will see how the derivations $\mathscr{D}$ and $\tilde{\mathscr{D}}$ are related with each other when $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are isomorphic.

#### 3.4.1 The Case of Lie Algebras

The following theorem was proved in [26] in the case where $\mathbb{K} = \mathbb{R}$. The passage to any $p \neq 2$, the proof is absolutely the same. The case where $p = 2$ was studied in [8].

**Theorem 3.11** ([8,26]) *Let $\mathscr{B}_{\mathfrak{a}}$ be $\mathscr{D}$- and $\tilde{\mathscr{D}}$-invariant, where $\mathscr{D}, \tilde{\mathscr{D}} \in \mathfrak{der}(\mathfrak{a})$. There exists an adapted isomorphism $\pi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ if and only if there exists an automorphism $\pi_0 : \mathfrak{a} \to \mathfrak{a}$, a scalar $\lambda \in \mathbb{K}^{\times}$ and $\varkappa \in \mathfrak{a}$ (all unique) such that*

$$\pi_0^{-1} \circ \tilde{\mathscr{D}} \circ \pi_0 = \lambda \mathscr{D} + \mathrm{ad}_{\varkappa}; \tag{25}$$

$$\textit{(Only when } p = 2\textit{)} \quad \mathscr{B}_{\mathfrak{g}}(x^*, x^*) = \lambda^{-2}\left(\mathscr{B}_{\mathfrak{a}}(\varkappa,\varkappa) + \mathscr{B}_{\tilde{\mathfrak{g}}}(\tilde{x}^*, \tilde{x}^*)\right); \tag{26}$$

*and*

$$\begin{aligned} \pi &= \pi_0 + \mathscr{B}_{\mathfrak{a}}(\varkappa, \cdot)\tilde{x} \quad \textit{on } \mathfrak{a};\\ \pi(x) &= \lambda \tilde{x};\\ \pi(x^*) &= \lambda^{-1}(\tilde{x}^* - \pi_0(\varkappa) - \rho\tilde{x}), \quad \textit{where}\\ \rho &= \begin{cases} \textit{arbitrary} & \textit{if } p = 2,\\ \frac{1}{2}\mathscr{B}_{\mathfrak{a}}(\varkappa,\varkappa) & \textit{if } p \neq 2. \end{cases}\end{aligned} \tag{27}$$

***Remark 3.12*** If $\pi_0 = \mathrm{Id}_{\mathfrak{a}}$, then condition (25) means that $\mathscr{D} \simeq \tilde{\mathscr{D}}$ in $\mathrm{H}^1(\mathfrak{a};\mathfrak{a})$. Moreover, if the derivations are restricted, then condition (25) means $\mathscr{D} \simeq \tilde{\mathscr{D}}$ in $\mathrm{H}^1_{\mathrm{res}}(\mathfrak{a};\mathfrak{a})$.

Suppose now that $\mathfrak{a}$ is restricted with a $p$-structure $[p]_{\mathfrak{a}}$. In Theorem 3.5, we proved that it is possible to extend the $p$-structure to any double extension. Let us denote by $[p]_{\mathfrak{g}}$ (resp. $[p]_{\tilde{\mathfrak{g}}}$) the $p$-structure on $\mathfrak{g}$ (resp. $\tilde{\mathfrak{g}}$) written in terms of $m, l, a_0, b_0, \gamma$ and $\mathscr{P}$ (resp. $\tilde{m}, \tilde{l}, \tilde{a}_0, \tilde{b}_0, \tilde{\gamma}$ and $\tilde{\mathscr{P}}$). The following theorem characterizes the equivalence class of $p$-structures on double extensions; but we prove it only for $p = 2, 3$.

An analog of Theorem 3.13 for $p > 3$ is still out of reach.

**Theorem 3.13** (Isomorphism of DEs and $p$-homomorphism) *For* $p = 2, 3$*, the adapted isomorphism* $\pi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ *given in Theorem 3.11 defines a* $p$*-homomorphism if and only if*

$$\pi_0(a^{[p]_\mathfrak{a}}) = (\pi_0(a))^{[p]_\mathfrak{a}} + \mathscr{B}_\mathfrak{a}(\varkappa, a)^p \tilde{b}_0,$$

*and*

$$\begin{aligned}
\tilde{m} &= \lambda^{-p}(\lambda m + \mathscr{B}_\mathfrak{a}(\varkappa, b_0)),\\
\tilde{b}_0 &= \lambda^{-p}\pi_0(b_0),\\
\tilde{\mathscr{P}} \circ \pi_0 &= \lambda \mathscr{P} + \mathscr{B}_\mathfrak{a}(\varkappa, (\cdot)^{[p]_\mathfrak{a}}) - \mathscr{B}_\mathfrak{a}(\varkappa, \cdot)^p \tilde{m}\\
\tilde{\gamma} &= \lambda^{p-1}\gamma,\\
\tilde{l} &= \lambda^p(\mathscr{B}_\mathfrak{a}(\varkappa, a_0) + \lambda l - \lambda^{-1}\gamma\rho) + \rho^p \tilde{m} + \mathscr{P}(\pi_0(\varkappa))\\
&\quad - \mathscr{B}_\mathfrak{a}(\tilde{\mathscr{D}}^{p-1}(\pi_0(\varkappa)), \pi_0(\varkappa)),\\
\tilde{a}_0 &= \lambda^p(\pi_0(a_0) - \lambda^{-1}\gamma\pi_0(\varkappa)) + (\pi_0(\varkappa))^{[p]_\mathfrak{a}} + \rho^p \tilde{b}_0\\
&\quad + \tilde{\mathscr{D}}^{p-1}(\pi_0(\varkappa)) - \begin{cases} \lambda\pi_0([\mathscr{D}(\varkappa), \varkappa]_\mathfrak{a}), & \textit{if } p = 3,\\ 0, & \textit{if } p = 2.\end{cases}
\end{aligned}$$

*Moreover, if* $\mathfrak{z}(\mathfrak{a}) = 0$*, then the automorphism* $\pi_0$ *of* $\mathfrak{a}$ *is also a* $p$*-homomorphism.*

***Proof*** Let us study the conditions for which $\pi$ is a $p$-homomorphism. We have

$$\begin{aligned}
\pi(x^{[p]_\mathfrak{g}}) - (\pi(x))^{[p]_{\tilde{\mathfrak{g}}}} &= \pi(mx + b_0) - (\lambda\tilde{x})^{[p]_{\tilde{\mathfrak{g}}}}\\
&= m\lambda\tilde{x} + \pi_0(b_0) + \mathscr{B}_\mathfrak{a}(\varkappa, b_0)\tilde{x} - \lambda^p(\tilde{m}\tilde{x} + \tilde{b}_0).
\end{aligned}$$

Therefore, $\tilde{m} = \lambda^{-p}(\lambda m + \mathscr{B}_\mathfrak{a}(\varkappa, b_0))$, and $\tilde{b}_0 = \lambda^{-p}\pi_0(b_0)$.

Let $a \in \mathfrak{a}$. We have

$$\pi(a^{[p]_\mathfrak{g}}) = \pi(a^{[p]_\mathfrak{a}} + \mathscr{P}(a)x) = \pi_0(a^{[p]_\mathfrak{a}}) + \mathscr{B}_\mathfrak{a}(\varkappa, a^{[p]_\mathfrak{a}})\tilde{x} + \mathscr{P}(a)\lambda\tilde{x}.$$

On the other hand,

$$\begin{aligned}
(\pi(a))^{[p]_{\tilde{\mathfrak{g}}}} &= (\pi_0(a) + \mathscr{B}_\mathfrak{a}(\varkappa, a)\tilde{x})^{[p]_{\tilde{\mathfrak{g}}}}\\
&= (\pi_0(a))^{[p]_\mathfrak{a}} + \tilde{\mathscr{P}}(\pi_0(a))\tilde{x} + \mathscr{B}_\mathfrak{a}(\varkappa, a)^p(\tilde{m}\tilde{x} + \tilde{b}_0).
\end{aligned}$$

It follows that (for every $a \in \mathfrak{a}$)

$$\begin{aligned}
&\mathscr{B}_\mathfrak{a}(\varkappa, a)^p \tilde{m} + \tilde{\mathscr{P}}(\pi_0(a)) - \mathscr{B}_\mathfrak{a}(\varkappa, a^{[p]_\mathfrak{a}}) - \mathscr{P}(a)\lambda = 0,\\
&\pi_0(a^{[p]_\mathfrak{a}}) - (\pi_0(a))^{[p]_\mathfrak{a}} - \mathscr{B}_\mathfrak{a}(\varkappa, a)^p \tilde{b}_0 = 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}\pi((x^*)^{[p]_{\mathfrak{g}}}) &= \pi(a_0 + lx + \gamma x^*)\\ &= \pi_0(a_0) + \mathscr{B}_{\mathfrak{a}}(\varkappa, a_0)\tilde{x} + l\lambda\tilde{x} + \gamma\lambda^{-1}(\tilde{x}^* - \pi_0(\varkappa) - \rho\tilde{x})\\ &= (\mathscr{B}_{\mathfrak{a}}(\varkappa, a_0) + \lambda l - \lambda^{-1}\gamma\rho)\tilde{x} - \lambda^{-1}\gamma\pi_0(\varkappa) + \pi_0(a_0) + \lambda^{-1}\gamma\tilde{x}^*.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\pi(x^*))^{[p]_{\tilde{\mathfrak{g}}}} &= \lambda^{-p}(\tilde{x}^* - \pi_0(\varkappa) - \rho\tilde{x})^{[p]_{\tilde{\mathfrak{g}}}}\\ &= \lambda^{-p}(\tilde{a}_0 + \tilde{\gamma}\tilde{x}^* - \pi_0(\varkappa)^{[p]_{\mathfrak{a}}} - \rho^p\tilde{b}_0 - \tilde{\mathscr{D}}^{p-1}(\pi_0(\varkappa))\\ &\quad + (\tilde{l} - \rho^p\tilde{m} - \tilde{\mathscr{P}}(\pi_0(\varkappa)))\tilde{x}) + \lambda^{-p}\mathscr{B}_{\mathfrak{a}}(\tilde{\mathscr{D}}^{p-1}(\pi_0(\varkappa)), \pi_0(\varkappa))\tilde{x}\\ &\quad + \begin{cases}\lambda^{-p}[\tilde{\mathscr{D}}(\pi_0(\varkappa)), \pi_0(\varkappa)]_{\mathfrak{a}} & \text{if } p = 3,\\ 0 & \text{if } p = 2.\end{cases}\end{aligned}$$

Therefore,

$$\begin{aligned}\lambda^{-1}\gamma &= \lambda^{-p}\tilde{\gamma},\\ \mathscr{B}_{\mathfrak{a}}(\varkappa, a_0) + \lambda l - \lambda^{-1}\gamma\rho &= \lambda^{-p}(\tilde{l} - \rho^p\tilde{m} - \mathscr{P}(\pi_0(\varkappa)) + \mathscr{B}_{\mathfrak{a}}(\tilde{\mathscr{D}}^{p-1}(\pi_0(\varkappa)), \pi_0(\varkappa))),\\ \pi_0(a_0) - \lambda^{-1}\gamma\pi_0(\varkappa) &= \lambda^{-p}(\tilde{a}_0 - \pi_0(\varkappa)^{[p]_{\mathfrak{a}}} - \rho^p\tilde{b}_0 - \tilde{\mathscr{D}}^{p-1}(\pi_0(\varkappa)))\\ &\quad + \begin{cases}\lambda^{-p}[\tilde{\mathscr{D}}(\pi_0(\varkappa)), \pi_0(\varkappa)]_{\mathfrak{a}} & \text{if } p = 3,\\ 0 & \text{if } p = 2.\end{cases}\end{aligned}$$

If $\mathfrak{z}(\mathfrak{a}) = 0$, we have $b_0 = \tilde{b}_0 = 0$ and the automorphism $\pi_0$ is a $p$-homomorphism. □

### 3.4.2 The Case of Lie Superalgebras

The main goal of this subsection is to superize Theorem 3.11. The case $p = 2$ has already been studied in [8], so we assume that $p \neq 2$. For $p = 3$, the factor 3 in all expressions should be understood as zero.

Let $\mathrm{pr} : \tilde{\mathscr{K}} \oplus \mathfrak{a} \to \mathfrak{a}$ be the projection, and $\pi_0 := \mathrm{pr} \circ \pi$. The map $\pi_0$ is obviously linear. Let $a \in \mathfrak{a}$. Since $\pi(a) - \pi_0(a) \in \mathrm{Ker}(\mathrm{pr})$, it follows that $\pi(a) - \pi_0(a) \in \tilde{\mathscr{K}}$. Since $\mathscr{B}_{\mathfrak{a}}$ is non-degenerate, there exists a unique $t_\pi \in \mathfrak{a}_{p(\tilde{\mathscr{D}})}$ (depending only in $\pi$ and $p(\tilde{\mathscr{D}})$) such that

$$\pi(a) - \pi_0(a) = \mathscr{B}_{\mathfrak{a}}(t_\pi, a)\tilde{x} \quad \text{for any } a \in \mathfrak{a}.$$

Besides, $\pi(x) = \lambda\tilde{x}$ for some $\lambda$ in $\mathbb{K}$. Indeed, let us write $\pi(x) = \lambda\tilde{x} + a$, where $a \in \mathfrak{a}$. We have

$$0 = \mathscr{B}_{\mathfrak{g}}(x, b) = \mathscr{B}_{\tilde{\mathfrak{g}}}(\pi(x), \pi(b)) = \mathscr{B}_{\tilde{\mathfrak{g}}}(\lambda\tilde{x} + a, \pi_0(b) + \mathscr{B}_{\mathfrak{a}}(t_\pi, b)\tilde{x}) = \mathscr{B}_{\mathfrak{a}}(a, \pi_0(b)).$$

Since $\pi_0$ is surjective and $\mathscr{B}_\mathfrak{a}$ is nondegenerate, it follows that $a = 0$. Let us show that $\pi_0$ preserves $\mathscr{B}_\mathfrak{a}$. Indeed, for any $a, b \in \mathfrak{a}$, we have

$$\begin{aligned}\mathscr{B}_\mathfrak{a}(a, b) &= \mathscr{B}_{\tilde{\mathfrak{g}}}(\pi(a), \pi(b)) \\ &= \mathscr{B}_{\tilde{\mathfrak{g}}}(\pi_0(a) + \mathscr{B}_\mathfrak{a}(t_\pi, a)\tilde{x}, \pi_0(b) + \mathscr{B}_\mathfrak{a}(t_\pi, b)\tilde{x}) \\ &= \mathscr{B}_{\tilde{\mathfrak{g}}}(\pi_0(a), \pi_0(b)) = \mathscr{B}_\mathfrak{a}(\pi_0(a), \pi_0(b)).\end{aligned}$$

Let us show that $\pi_0$ is an automorphism of $(\mathfrak{a}, \mathscr{B}_\mathfrak{a})$. Let $a, b \in \mathfrak{a}$. We get

$$\begin{aligned}\pi([a, b]_\mathfrak{g}) &= \pi([a, b]_\mathfrak{a}) + (-1)^{p(\mathscr{D})(p(\mathscr{B}_\mathfrak{a})+1)}\lambda\mathscr{B}_\mathfrak{a}(\mathscr{D}(a), b)\tilde{x} \\ &= \pi_0([a, b]_\mathfrak{a}) + \mathscr{B}_\mathfrak{a}(t_\pi, [a, b]_\mathfrak{a})\tilde{x} + (-1)^{p(\mathscr{D})(p(\mathscr{B}_\mathfrak{a})+1)}\lambda\mathscr{B}_\mathfrak{a}(\mathscr{D}(a), b)\tilde{x}\end{aligned}$$

and

$$\begin{aligned}[\pi(a), \pi(b)]_{\tilde{\mathfrak{g}}} &= [\pi_0(a) + \mathscr{B}_\mathfrak{a}(t_\pi, a)\tilde{x}, \pi_0(b) + \mathscr{B}_\mathfrak{a}(t_\pi, b)\tilde{x}]_{\tilde{\mathfrak{g}}} \\ &= [\pi_0(a), \pi_0(b)]_{\tilde{\mathfrak{g}}} \\ &= [\pi_0(a), \pi_0(b)]_\mathfrak{a} + (-1)^{p(\tilde{\mathscr{D}})(p(\mathscr{B}_\mathfrak{a})+1)}\mathscr{B}_\mathfrak{a}(\tilde{\mathscr{D}}(\pi_0(a)), \pi_0(b))\tilde{x}.\end{aligned}$$

It follows that

$$\mathscr{B}_\mathfrak{a}(\pi_0^{-1}\tilde{\mathscr{D}}(\pi_0(a)), b) = (-1)^{p(\mathscr{D})(p(\mathscr{B}_\mathfrak{a})+1)}\mathscr{B}_\mathfrak{a}([t_\pi, a]_\mathfrak{a}, b) + \lambda\mathscr{B}_\mathfrak{a}(\mathscr{D}(a), b).$$

Therefore,

$$\pi_0^{-1} \circ \tilde{\mathscr{D}} \circ \pi_0 = (-1)^{p(\mathscr{D})(p(\mathscr{B}_\mathfrak{a})+1)}\mathrm{ad}_{t_\pi} + \lambda\mathscr{D}. \tag{28}$$

Let us write

$$\pi(x^*) = \mu\tilde{x}^* + a + \nu\tilde{x}, \quad \text{where } a \in \mathfrak{a} \text{ and } \nu = 0 \text{ if } \mathscr{B}_\mathfrak{a} \text{ is odd.}$$

Because $\pi$ is an isomorphism, we have

$$1 = \mathscr{B}_{\tilde{\mathfrak{g}}}(\pi(x), \pi(x^*)) = \mathscr{B}_{\tilde{\mathfrak{g}}}(\lambda\tilde{x}, \mu\tilde{x}^* + a + \nu\tilde{x}) = \lambda\mu.$$

Therefore, $\mu = \lambda^{-1}$. Besides,

$$\begin{aligned}0 = \mathscr{B}_{\tilde{\mathfrak{g}}}(\pi(x^*), \pi(b)) &= \mathscr{B}_{\tilde{\mathfrak{g}}}(\tilde{x}^* + a + \nu\tilde{x}, \pi_0(b) + \mathscr{B}_\mathfrak{a}(t_\pi, b)\tilde{x}) \\ &= \mu(-1)^{p(\tilde{x})p(\tilde{x}^*)}\mathscr{B}_\mathfrak{a}(t_\pi, b) + \mathscr{B}_\mathfrak{a}(a, \pi_0(b)).\end{aligned}$$

Hence, $\mathscr{B}_\mathfrak{a}(\mu(-1)^{p(\tilde{x})p(\tilde{x}^*)}t_\pi + \pi_0^{-1}a, b) = 0$, and $\mu(-1)^{p(\tilde{x})p(\tilde{x}^*)}t_\pi + \pi_0^{-1}(a) = 0$ implying

$$a = -(-1)^{p(\tilde{x})p(\tilde{x}^*)}\mu\pi_0(t_\pi). \tag{29}$$

Finally, we get

$$\pi(x^*) = \lambda^{-1}(\tilde{x}^* - (-1)^{p(\tilde{x})p(\tilde{x}^*)}\pi_0(t_\pi)) + \nu\tilde{x}.$$

Besides,

$$\begin{aligned} 0 &= \mathscr{B}_{\tilde{\mathfrak{g}}}(\pi(x^*), \pi(x^*)) = \mathscr{B}_{\tilde{\mathfrak{g}}}(\mu\tilde{x}^* + a + \nu\tilde{x}, \mu\tilde{x}^* + a + \nu\tilde{x}) \\ &= \mu\nu(1 + (-1)^{p(\tilde{x})p(\tilde{x}^*)}) + \mathscr{B}_{\mathfrak{a}}(a, a). \end{aligned}$$

We distinguish two cases:

(i) If $\mathscr{B}_{\mathfrak{a}}$ is odd, then $\nu = 0$, and therefore $\mathscr{B}_{\mathfrak{a}}(a, a) = 0$.
(ii) If $\mathscr{B}_{\mathfrak{a}}$ is even, then $\mathscr{B}_{\mathfrak{a}}(a, a) = 0$ and $\nu$ is arbitrary, except for $p(x) = p(x^*) = \bar{0}$. In the latter case (recall that $p \neq 2$),

$$\nu = -\frac{1}{2}\lambda^{-1}\mathscr{B}_{\mathfrak{a}}(t_\pi, t_\pi).$$

Let $p(x^*) = \bar{1}$ (and therefore $p(\mathscr{D}) = p(\tilde{\mathscr{D}}) = \bar{1}$). We have

$$\begin{aligned} [\pi(x^*), \pi(x^*)]_{\tilde{\mathfrak{g}}} &= [\mu\tilde{x}^* + a + \nu x,\ \mu\tilde{x}^* + a + \nu\tilde{x}]_{\tilde{\mathfrak{g}}} \\ &= 2\mu(\tilde{\mathscr{D}}(a) - (-1)^{p(a)p(\mathscr{B}_{\mathfrak{a}})}2\mathscr{B}_{\mathfrak{a}}(a, \tilde{b}_0)\tilde{x}) + [a, a]_{\mathfrak{a}} \\ &\quad - (-1)^{p(\mathscr{B}_{\mathfrak{a}})}\mathscr{B}_{\mathfrak{a}}(\tilde{\mathscr{D}}(a), a)\tilde{x} + \mu^2[\tilde{x}^*, \tilde{x}^*]_{\tilde{\mathfrak{g}}}. \end{aligned}$$

On the other hand,

$$\pi([x^*, x^*]_{\mathfrak{g}}) = \begin{cases} 2\pi_0(b_0) + 2\mathscr{B}_{\mathfrak{a}}(t_\pi, b_0)\tilde{x} + \lambda\lambda_0\tilde{x} & \text{if } p(\mathscr{B}_{\mathfrak{a}}) = \bar{1}, \\ 2\pi_0(b_0) & \text{if } p(\mathscr{B}_{\mathfrak{a}}) = \bar{0}. \end{cases}$$

It follows that

$$2\tilde{b}_0 = \mu^{-2}(2\pi_0(b_0) - [a, a]_{\mathfrak{a}} - 2\mu\tilde{\mathscr{D}}(a)). \tag{30}$$

Therefore,

$$\tilde{b}_0 = \lambda^2\pi_0(b_0) + (-1)^{p(x)}\lambda\pi_0(\mathscr{D}(t_\pi)) + \frac{1}{2}\pi_0([t_\pi, t_\pi]_{\mathfrak{a}}). \tag{31}$$

In addition, in the case where $p(\mathscr{B}_{\mathfrak{a}}) = \bar{1}$, we have

$$\lambda_0\lambda + 2\mathscr{B}_{\mathfrak{a}}(t_\pi, b_0) = \mu^2\tilde{\lambda}_0 + 4\mu\mathscr{B}_{\mathfrak{a}}(a, \tilde{b}_0) + \mathscr{B}_{\mathfrak{a}}(\tilde{\mathscr{D}}(a), a).$$

After computation, we get

$$\tilde{\lambda}_0 = \lambda_0\lambda^3 + \mathscr{B}_{\mathfrak{a}}(t_\pi, [t_\pi, t_\pi]_{\mathfrak{a}}) + 3\lambda\mathscr{B}_{\mathfrak{a}}(t_\pi, \mathscr{D}(t_\pi)) + 6\lambda^2\mathscr{B}_{\mathfrak{a}}(t_\pi, b_0).$$

We arrive at the following two theorems.

**Theorem 3.14** (Isomorphism of DEs for even derivations) *Let $\mathscr{B}_{\mathfrak{a}}$ be $\mathscr{D}$- and $\tilde{\mathscr{D}}$-invariant, where $\mathscr{D}, \tilde{\mathscr{D}} \in \mathfrak{der}_{\bar{0}}(\mathfrak{a})$ (and satisfy condition (22) if $p = 3$ and $p(\mathscr{B}_{\mathfrak{a}}) = \bar{1}$).*

*Then there exists an adapted isomorphism $\pi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ if and only if there exists an automorphism $\pi_0 : \mathfrak{a} \to \mathfrak{a}$, a $\lambda \in \mathbb{K}^\times$, and $\varkappa \in \mathfrak{a}$ (all unique) such that*

$$\begin{aligned} \pi_0^{-1} \circ \tilde{\mathscr{D}} \circ \pi_0 &= \lambda \mathscr{D} + \mathrm{ad}_\varkappa; \\ \mathscr{B}_{\mathfrak{a}}(\varkappa, \varkappa) &= 0 \quad \text{if } p(\mathscr{B}_{\mathfrak{a}}) = \bar{1}, \end{aligned}$$

*and*

$$\begin{aligned} \pi &= \pi_0 + \mathscr{B}_{\mathfrak{a}}(\varkappa, \cdot)\tilde{x} \quad \text{on } \mathfrak{a}; \\ \pi(x) &= \lambda \tilde{x}; \\ \pi(x^*) &= \lambda^{-1}(\tilde{x}^* - \pi_0(\varkappa) - \frac{1}{4}(1 + (-1)^{p(\mathscr{B}_{\mathfrak{a}})})\mathscr{B}_{\mathfrak{a}}(\varkappa, \varkappa)x). \end{aligned}$$

**Theorem 3.15** (Isomorphism of DEs for odd derivations) *Let $\mathscr{B}_{\mathfrak{a}}$ be $\mathscr{D}$- and $\tilde{\mathscr{D}}$-invariant, where $\mathscr{D}, \tilde{\mathscr{D}} \in \mathfrak{der}_{\bar{1}}(\mathfrak{a})$ (and satisfy condition (17) if $p = 3$ and $p(\mathscr{B}_{\mathfrak{a}}) = \bar{1}$). Then there exists an adapted isomorphism $\pi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ if and only if there exists an automorphism $\pi_0 : \mathfrak{a} \to \mathfrak{a}$, a $\lambda \in \mathbb{K}^\times$, and $\varkappa \in \mathfrak{a}$ (all unique) such that*

$$\begin{aligned} \pi_0^{-1} \circ \tilde{\mathscr{D}} \circ \pi_0 &= \lambda \mathscr{D} - (-1)^{p(\mathscr{B}_{\mathfrak{a}})} \mathrm{ad}_\varkappa; \\ \mathscr{B}_{\mathfrak{a}}(\varkappa, \varkappa) &= 0; \end{aligned}$$

*and*

$$\begin{aligned} \pi &= \pi_0 + \mathscr{B}_{\mathfrak{a}}(\varkappa, \cdot)\tilde{x} \quad \text{on } \mathfrak{a}; \\ \pi(x) &= \lambda \tilde{x}; \\ \pi(x^*) &= \lambda^{-1}(\tilde{x}^* - (-1)^{p(x)}\pi_0(\varkappa)) + (1 + (-1)^{p(\mathscr{B}_{\mathfrak{a}})})\nu x, \text{ where } \nu \text{ is arbitrary.} \end{aligned}$$

*Moreover,*

$$\tilde{b}_0 = \lambda^2 \pi_0(b_0) + (-1)^{p(x)} \lambda(\pi_0(\mathscr{D}(\varkappa))) + \frac{1}{2}\pi_0([\varkappa, \varkappa]_{\mathfrak{a}}).$$

*If $p(\mathscr{B}_{\mathfrak{a}}) = \bar{1}$, we additionally have (see Eq. (23))*

$$\tilde{\lambda}_0 = \lambda_0 \lambda^3 + \mathscr{B}_{\mathfrak{a}}(\varkappa, [\varkappa, \varkappa]) + 3\lambda \mathscr{B}_{\mathfrak{a}}(\varkappa, \mathscr{D}(\varkappa)) + 6\lambda^2 \mathscr{B}_{\mathfrak{a}}(\varkappa, b_0).$$

***Proof of Theorem 3.14 and Theorem 3.15*** To check that $\pi$ preserves the Lie bracket, it remains to check that $\pi([x^*, a]_{\mathfrak{g}}) = [\pi(x^*), \pi(a)]_{\tilde{\mathfrak{g}}}$. We distinguish two cases:

(i) The case where $p(x^*) = \bar{1}$. We have

$$\begin{aligned} \pi([x^*, a]_{\mathfrak{g}}) &= \pi(\mathscr{D}(a) - (-1)^{p(\mathscr{B}_{\mathfrak{a}})p(a)} 2\mathscr{B}_{\mathfrak{a}}(a, b_0)x) \\ &= \pi_0(\mathscr{D}(a)) + \mathscr{B}_{\mathfrak{a}}(\varkappa, \mathscr{D}(a))\tilde{x} - (-1)^{p(\mathscr{B}_{\mathfrak{a}})p(a)} 2\lambda \mathscr{B}_{\mathfrak{a}}(a, b_0)\tilde{x}. \end{aligned}$$

On the other hand,

$$
\begin{aligned}
[\pi(x^*), \pi(a)]_{\tilde{\mathfrak{g}}} &= \pi_0(\mathscr{D}(a)) - (-1)^{p(a)p(\mathscr{B}_{\mathfrak{a}})} 2\lambda \mathscr{B}_{\mathfrak{a}}(a, b_0)\tilde{x} \\
&\quad + (-2(-1)^{p(x)+p(a)p(\mathscr{B}_{\mathfrak{a}})} - 1)\mathscr{B}_{\mathfrak{a}}(a, \mathscr{D}(\varkappa))\tilde{x} \\
&\quad + \lambda^{-1}((-1)^{p(\mathscr{B}_{\mathfrak{a}})} - (-1)^{p(a)p(\mathscr{B}_{\mathfrak{a}})})\mathscr{B}_{\mathfrak{a}}(a, [\varkappa, \varkappa]_{\mathfrak{a}})\tilde{x}.
\end{aligned}
$$

The result follows since $p(\mathscr{B}_{\mathfrak{a}}) = p(x) + p(x^*) = p(x) + 1$, and (for all $a \in \mathfrak{a}$)

$$
((-1)^{p(\mathscr{B}_{\mathfrak{a}})} - (-1)^{p(a)p(\mathscr{B}_{\mathfrak{a}})})\mathscr{B}_{\mathfrak{a}}(a, [\varkappa, \varkappa]_{\mathfrak{a}}) = 0.
$$

(ii) The case where $p(x^*) = \bar{0}$. Similar computation.

There is no need to check the remaining brackets because they are certainly satisfied as shown by the previous computations prior formulations of Theorems 3.14 and 3.15.

Let us show that $\pi$ preserves $\mathscr{B}_{\mathfrak{g}}$. For every element $a \in \mathfrak{a}$, we have $\mathscr{B}_{\mathfrak{g}}(x^*, a) = 0$. On the other hand,

$$
\begin{aligned}
\mathscr{B}_{\tilde{\mathfrak{g}}}(\pi(a), \pi(x^*)) &= \mathscr{B}_{\tilde{\mathfrak{g}}}(\pi_0(a) + \mathscr{B}_{\mathfrak{a}}(\varkappa, a)\tilde{x}, \pi(x^*)) \\
&= \lambda^{-1}\mathscr{B}_{\mathfrak{a}}(\varkappa, a) - (-1)^{p(x)p(x^*)}\mathscr{B}_{\mathfrak{a}}(\pi_0(a), \lambda^{-1}\pi_0(\varkappa)) \\
&= \lambda^{-1}((-1)^{p(x)p(a)} - (-1)^{p(x)p(x^*)})\mathscr{B}_{\mathfrak{a}}(\varkappa, a) = 0.
\end{aligned}
$$

Let $\mathscr{D}(b_0) = 0$. Let us show that $\tilde{\mathscr{D}}(\tilde{b}_0) = 0$. Indeed (recall that $p(\mathscr{D}) = p(\tilde{\mathscr{D}}) = \bar{1}$),

$$
\begin{aligned}
\pi_0^{-1}\tilde{\mathscr{D}}(\tilde{b}_0) &= \pi_0^{-1}\left(\tilde{\mathscr{D}}(\lambda^2\pi_0(b_0) + (-1)^{p(x)p(x^*)}\lambda(\pi_0(\mathscr{D}(\varkappa))) + \frac{1}{2}\pi_0([\varkappa, \varkappa]_{\mathfrak{a}})\right) \\
&= (\lambda\mathscr{D} - (-1)^{p(\mathscr{B}_{\mathfrak{a}})}\mathrm{ad}_{\varkappa})\left(\lambda^2 b_0 + (-1)^{p(x)p(x^*)}\lambda\mathscr{D}(\varkappa) + \frac{1}{2}[\varkappa, \varkappa]_{\mathfrak{a}}\right) = 0.
\end{aligned}
$$

Let us show that $\tilde{\mathscr{D}}^2 = \mathrm{ad}_{\tilde{b}_0}$. Indeed,

$$
\begin{aligned}
\tilde{\mathscr{D}}^2 &= \pi_0(\lambda\mathscr{D} - (-1)^{p(\mathscr{B}_{\mathfrak{a}})}\mathrm{ad}_{\varkappa})^2\pi_0^{-1} \\
&= \pi_0(\lambda^2\mathscr{D}^2 - (-1)^{p(\mathscr{B}_{\mathfrak{a}})}(\lambda\mathscr{D}\mathrm{ad}_{\varkappa} + \lambda\mathrm{ad}_{\varkappa}\mathscr{D}) + \mathrm{ad}_{\varkappa}^2)\pi_0^{-1} \\
&= \pi_0\left(\lambda^2\mathrm{ad}_{b_0} - \lambda(-1)^{p(\mathscr{B}_{\mathfrak{a}})}\mathrm{ad}_{\mathscr{D}(\varkappa)} + \frac{1}{2}\mathrm{ad}_{[\varkappa,\varkappa]_{\mathfrak{a}}}\right)\pi_0^{-1} = \mathrm{ad}_{\tilde{b}_0}.
\end{aligned}
$$

Besides (since $\tilde{b}_0 = \lambda^2\pi_0(b_0) + (-1)^{p(x)}\lambda(\pi_0(\mathscr{D}(\varkappa))) + \frac{1}{2}\pi_0([\varkappa, \varkappa]_{\mathfrak{a}})$ and $\pi_0$ is an isomorphism),

$$
\begin{aligned}
\mathscr{B}_{\tilde{\mathfrak{g}}}(\tilde{b}_0, \tilde{b}_0) &= \lambda^4\mathscr{B}_{\mathfrak{a}}(b_0, b_0) + \lambda^2\mathscr{B}_{\mathfrak{a}}(b_0, [\varkappa, \varkappa]_{\mathfrak{a}}) + \lambda^2\mathscr{B}_{\mathfrak{a}}(\mathscr{D}(\varkappa), \mathscr{D}(\varkappa)) \\
&\quad + (-1)^{p(x)}\lambda\mathscr{B}_{\mathfrak{a}}(\mathscr{D}(\varkappa), [\varkappa, \varkappa]_{\mathfrak{a}}) = 0.
\end{aligned}
$$

□

Suppose now that $\mathfrak{a}$ is restricted. In Theorems 3.7 and 3.9, we proved that it is possible to extend its $p|2p$-structure to any of its double extension. Let us denote by $[p|2p]_{\mathfrak{g}}$ (resp. $[p|2p]_{\tilde{\mathfrak{g}}}$) the $p|2p$-structure on $\mathfrak{g}$ (resp. $\tilde{\mathfrak{g}}$) written in terms of $m, l, a_0, b_0, \gamma$ and $\mathscr{P}$ (resp. $\tilde{m}, \tilde{l}, \tilde{a}_0, \tilde{b}_0, \tilde{\gamma}$ and $\tilde{\mathscr{P}}$). The following theorems characterize the equivalence class of $p|2p$-structures on double extensions. Theorem 3.16 is proved only for $p = 3$.

**Theorem 3.16** (Isomorphism of DEs and $p$-homomorphism for even derivations) *The adapted isomorphism $\pi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ given in Theorem 3.14 defines a $p|2p$-homomorphism, if and only if the equations below are satisfied:*

(i) *If $p(\mathscr{B}_{\mathfrak{a}}) = \bar{0}$ (here $p = 3$), we have*

$$\pi_0(a^{[p]_{\mathfrak{a}}}) = (\pi_0(a))^{[p]_{\mathfrak{a}}} + \mathscr{B}_{\mathfrak{a}}(\varkappa, a)^p \tilde{c}_0 \quad \textit{for all } a \in \mathfrak{a}_{\bar{0}},$$

*and*

$$\begin{aligned}
\tilde{m} &= \lambda^{-p}(\lambda m + \mathscr{B}_{\mathfrak{a}}(\varkappa, c_0)),\\
\tilde{c}_0 &= \lambda^{-p}\pi_0(c_0),\\
\tilde{\mathscr{P}} \circ \pi_0 &= \lambda \mathscr{P} + \mathscr{B}_{\mathfrak{a}}(\varkappa, (\cdot)^{[p]_{\mathfrak{a}}}) - \mathscr{B}_{\mathfrak{a}}(\varkappa, \cdot)^p \tilde{m} \textit{ on } \mathfrak{a}_{\bar{0}}\\
\tilde{\gamma} &= \lambda^{p-1}\gamma,\\
\tilde{l} &= \lambda^p(\mathscr{B}_{\mathfrak{a}}(\varkappa, a_0) + \lambda l - \lambda^{-1}\gamma\rho) + \rho^p \tilde{m} + \mathscr{P}(\pi_0(\varkappa))\\
&\quad - \mathscr{B}_{\mathfrak{a}}(\tilde{\mathscr{D}}^{p-1}(\pi_0(\varkappa)), \pi_0(\varkappa))),\\
\tilde{a}_0 &= \lambda^p(\pi_0(a_0) - \lambda^{-1}\gamma\pi_0(\varkappa)) + (\pi_0(\varkappa))^{[p]_{\mathfrak{a}}}\\
&\quad + \rho^p \tilde{c}_0 + \tilde{\mathscr{D}}^{p-1}(\pi_0(\varkappa)) - \lambda\pi_0([\mathscr{D}(\varkappa), \varkappa]_{\mathfrak{a}}).
\end{aligned}$$

*Moreover, if $\mathfrak{z}(\mathfrak{a}) = 0$, then the automorphism $\pi_0$ of $\mathfrak{a}$ is a also a $p|2p$-homomorphism.*

(ii) *If $p(\mathscr{B}_{\mathfrak{a}}) = \bar{1}$ (here $p = 3$), we have*

$$\pi_0(a^{[p]_{\mathfrak{a}}}) = (\pi_0(a))^{[p]_{\mathfrak{a}}} \quad \textit{for all } a \in \mathfrak{a}_{\bar{0}},$$

*and*

$$\begin{aligned}
\tilde{\gamma} &= \lambda^{p-1}\gamma,\\
\tilde{a}_0 &= \pi_0(\varkappa^{[p]_{\mathfrak{a}}}) + \lambda^2\pi_0(\mathscr{D}^2(\varkappa)) - \gamma\lambda^{p-1}\pi_0(\varkappa) + \lambda^p\pi_0(a_0) + 2\lambda\pi_0([\varkappa, \mathscr{D}(\varkappa)]_{\mathfrak{a}}).
\end{aligned}$$

**Theorem 3.17** (Isomorphism of DEs and $p$-homomorphism for odd derivations) *The adapted isomorphism $\pi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ given in Theorem 3.15 defines a $p|2p$-homomorphism, if and only if the following conditions are satisfied:*

(i) *If $p(\mathscr{B}_{\mathfrak{a}}) = \bar{1}$, then*

$$\pi_0(a^{[p]_{\mathfrak{a}}}) = (\pi_0(a))^{[p]_{\mathfrak{a}}} + \mathscr{B}_{\mathfrak{a}}(\varkappa, a)^p \tilde{c}_0 \quad \textit{for all } a \in \mathfrak{a}_{\bar{0}},$$

*and*

$$\begin{aligned}\tilde{m} &= \lambda^{-p}(\lambda m + \mathscr{B}_{\mathfrak{a}}(\varkappa, c_0)),\\ \tilde{c}_0 &= \lambda^{-p}\pi_0(c_0),\\ \tilde{\mathscr{P}} \circ \pi_0 &= \lambda \mathscr{P} + \mathscr{B}_{\mathfrak{a}}(\varkappa, (\cdot)^{[p]_{\mathfrak{a}}}) - \mathscr{B}_{\mathfrak{a}}(\varkappa, \cdot)^p \tilde{m} \quad \text{on } \mathfrak{a}_{\bar{0}}.\end{aligned}$$

*Moreover, if* $\mathfrak{z}(\mathfrak{a}) = 0$*, then the automorphism* $\pi_0$ *of* $\mathfrak{a}$ *is also a* $p|2p$*-homomorphism.*

(ii) *If* $p(\mathscr{B}_{\mathfrak{a}}) = \bar{0}$*, then* $\pi_0$ *defines a* $p|2p$*-homomorphism on* $\mathfrak{a}$.

***Proof*** The proof of Theorems 3.16 and 3.17 is similar to that of Theorem 3.13, minding the Sign Rule. □

## 4 Examples

The calculations performed with the aid of *SuperLie* code [29] are called **Claims**. The 1-cochain $\hat{x} \in C^1(\mathfrak{g})$ denotes the dual of $x \in \mathfrak{g}$.

### 4.1 $\mathfrak{psl}(3)$ for $p = 3$

Let us fix a basis of $\mathfrak{psl}(3)$ generated by the root vectors $x_1, x_2, x_3 = [x_1, x_2]$ (positive) and $y_1, y_2, y_3 = [y_1, y_2]$ (negative). In the ordered basis $e_1 = [x_1, y_1]$, $e_2 = x_1$, $e_3 = x_2$, $e_4 = x_3$, $e_5 = y_1$, $e_6 = y_2$, $e_7 = y_3$ of $\mathfrak{psl}(3)$, NIS has the Gram matrix

$$\mathscr{B}_{\mathfrak{psl}(3)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & I_{2,1} \\ 0 & I_{2,1} & 0 \end{pmatrix},$$

where $I_{r,s} = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$ with $r$-many 1s and $s$ many $(-1)$s, see [18]. The 3-structure on $\mathfrak{psl}(3)$ is given by the formulas:

$$e_1^{[3]} = e_1, \quad e_i^{[3]} = 0 \quad \text{for all } i > 1.$$

**Claim 4.1** ([31]) *The space* $\mathrm{H}^1_{\mathrm{res}}(\mathfrak{psl}(3); \mathfrak{psl}(3))$ *is spanned by the 7 cocycles of which we only need the following ones, their degrees in the subscript.* (In fact, we have $\mathfrak{out}(\mathfrak{psl}(3)) \simeq \mathfrak{psl}(3)$.)

$$\begin{aligned}\mathscr{D}^1_{-3} &= y_1 \otimes \widehat{x_3} + y_3 \otimes \widehat{x_1},\\ \mathscr{D}^2_0 &= 2\, x_1 \otimes \widehat{x_2} + y_2 \otimes \widehat{y_1},\\ \mathscr{D}^3_0 &= x_1 \otimes \widehat{x_1} + x_3 \otimes \widehat{x_3} + 2\, y_1 \otimes \widehat{y_1} + 2\, y_3 \otimes \widehat{y_3}.\end{aligned} \tag{32}$$

An easy computation shows that $\mathscr{B}_{\mathfrak{psl}(3)}$ is $\mathscr{D}$-invariant for any $\mathscr{D}$ in (32).

The isomorphism

$$y_1 \leftrightarrow x_1,\; y_2 \leftrightarrow x_2,\; y_1 \leftrightarrow y_2,\; y_3 \leftrightarrow -y_3$$

sends $\mathscr{D}^1_{-3}$ to a new one $\mathscr{D}^1_3$. The root system of $\mathfrak{psl}(3)$ is symmetric, so the isomorphism

$$x_1 \leftrightarrow x_2,\; x_3 \leftrightarrow -x_3,\; y_1 \leftrightarrow y_2,\; y_3 \leftrightarrow -y_3$$

sends the cocycles described in (32) and $\mathscr{D}^1_{-3}$ into the new ones:

$$\mathscr{D}^2_0 \leftrightarrow \mathscr{D}^1_0,\; \mathscr{D}^1_{-3} \leftrightarrow \mathscr{D}^2_{-3},\; \mathscr{D}^1_3 \leftrightarrow \mathscr{D}^2_3.$$

Therefore, we can confine ourselves to $\mathscr{D}^1_{-3}$, $\mathscr{D}^2_0$ and $\mathscr{D}^3_0$ only. Let us summarize and introduce notation:

| Derivation | $\mathscr{P}(a)$ | $\gamma$ | $a_0$ | Double extension |
|---|---|---|---|---|
| $\mathscr{D}^2_0$ | $\lambda_3^2\lambda_7 + 2\lambda_1\lambda_5\lambda_3 + \lambda_4\lambda_5^2$ | 0 | 0 | $\widetilde{\mathfrak{gl}}(3)$ |
| $\mathscr{D}^1_{-3}$ | $\lambda_4^2\lambda_6 + 2\lambda_1\lambda_2\lambda_4 + 2\lambda_2^2\lambda_3$ | 0 | 0 | $\widehat{\mathfrak{gl}}(3)$ |
| $\mathscr{D}^3_0$ | $\lambda_1\lambda_2\lambda_5 + \lambda_4\lambda_6\lambda_5 + 2\lambda_2\lambda_3\lambda_7 + \lambda_1\lambda_4\lambda_7$ | 1 | 0 | $\mathfrak{gl}(3)$ |

(33)

**Claim 4.2** $\dim \mathrm{H}^2(\mathfrak{gl}(3)) = 0$, $\dim \mathrm{H}^2(\widetilde{\mathfrak{gl}}(3)) = 3$, *and* $\dim \mathrm{H}^2(\widehat{\mathfrak{gl}}(3)) = 4$; *hence,* $\mathfrak{gl}(3)$, $\widetilde{\mathfrak{gl}}(3)$, *and* $\widehat{\mathfrak{gl}}(3)$ *are pairwise non-isomorphic.*

### 4.2 $\mathfrak{brj}(2;3)$ for $p = 3$

Let us realize $\mathfrak{brj}(2;3)$ (for more details, see [15,16]) by the Cartan matrix and the positive root vectors (odd ones are underlined)

$$\begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix} \quad \begin{array}{l} \underline{x_1}, \underline{x_2},\; x_3 = [x_1, x_2],\; x_4 = [x_2, x_2],\; \underline{x_5} = [x_2, x_3], \\ x_6 = [x_3, x_4],\quad \underline{x_7} = [x_4, x_5],\quad x_8 = [x_5, x_5]. \end{array}$$

As shown in [18], the Lie superalgebra $\mathfrak{brj}(2;3)$ admits a NIS given in the ordered basis

$$h_1 := [x_1, y_1],\quad h_2 := [x_2, y_2],\quad x_1, \ldots, x_8,\quad y_1, \ldots, y_8 \tag{34}$$

by the Gram matrix

$$\mathscr{B}_{\mathfrak{brj}(2;3)} = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & B \\ 0 & C & 0 \end{pmatrix},\quad \text{where } A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix},$$

$$B = \operatorname{diag}\{1, 2, 2, 2, 2, 2, 1, 1\} \text{ and } C = \operatorname{diag}\{2, 1, 2, 2, 1, 2, 2, 1\}.$$

There exists a 3|6 structure on $\mathfrak{brj}(2;3)$ that we express in the basis (34) as follows:

$h_1^{[3]} = h_1,\ h_2^{[3]} = h_2,\ v^{[3]} = w^{[6]} = 0$ for any root vectors $v$ (even) and $w$ (odd) in (34).

**Claim 4.3** *The space* $\mathrm{H}^1(\mathfrak{brj}(2;3); \mathfrak{brj}(2;3))$ *is spanned by the odd cocycles*[4]*:*

$$\begin{aligned}
&\deg = -3 : \\
&\ \mathscr{D}_{-3} = x_1 \otimes \widehat{x_6} + x_3 \otimes \widehat{x_7} + 2y_2 \otimes \widehat{x_4} + y_4 \otimes \widehat{x_2} + 2y_6 \otimes \widehat{y_1} + y_7 \otimes \widehat{y_3}, \\
&\deg = 3 : \\
&\ \mathscr{D}_3 = x_2 \otimes \widehat{y_4} + x_4 \otimes \widehat{y_2} + x_6 \otimes \widehat{x_1} + 2x_7 \otimes \widehat{x_3} + y_1 \otimes \widehat{y_6} + y_3 \otimes \widehat{y_7}.
\end{aligned} \tag{35}$$

A direct computation shows that $\mathscr{B}_{\mathfrak{brj}(2;3)}$ is $\mathscr{D}$-invariant, where $\mathscr{D}$ is any of the derivations given in (35). However, the condition (17) is violated for these derivations. Indeed,

$$\begin{aligned}
\mathscr{B}_{\mathfrak{a}}(\mathscr{D}_{-3}(x_2), [x_2, x_2]) &= \mathscr{B}_{\mathfrak{a}}(2y_4, x_4) = 1 \neq 0, \\
\mathscr{B}_{\mathfrak{a}}(\mathscr{D}_3(y_2), [y_2, y_2]) &= \mathscr{B}_{\mathfrak{a}}(2x_4, y_4) = 1 \neq 0.
\end{aligned}$$

It follows that the double extension of $\mathfrak{brj}(2;3)$ is a *pre-Lie superalgebra*, see Appendix.

**Claim 4.4** *For* $\mathfrak{a} = \mathfrak{sl}(1|2)$, $\mathfrak{osp}(2|3)$, $\mathfrak{osp}(1|4)$ *for* $p > 2$, *the Brown algebra* $\mathfrak{br}(2, \varepsilon)$ *for* $p = 3$ *and* $\mathfrak{brj}(2;5)$ *for* $p = 5$, *we have* $\mathrm{H}^1(\mathfrak{a}; \mathfrak{a}) = 0$.

Therefore, these Lie superalgebras do not have non-trivial double extensions.

## 4.3 Manin Triples Related to $\mathfrak{hei}(2)$ for $p = 2$

### 4.3.1 2-Structures on Manin Triples

Let $(\mathfrak{h}, [2]_{\mathfrak{h}})$ be a finite-dimensional restricted Lie algebra (not necessarily "NIS"), and let $\mathfrak{h}^*$ have the structure of an abelian Lie algebra. Set $\mathfrak{a} := \mathfrak{h} \oplus \mathfrak{h}^*$, and define the bracket of two elements in $\mathfrak{a}$ as follows:

$$\begin{aligned}
&[h + \pi, h' + \pi']_{\mathfrak{a}} := [h, h']_{\mathfrak{h}} + \pi \circ \mathrm{ad}_{h'} + \pi' \circ \mathrm{ad}_h \\
&\quad \text{for any } h + \pi,\ h' + \pi' \in \mathfrak{a}.
\end{aligned} \tag{36}$$

It is easy to show that the bracket $[\cdot, \cdot]_{\mathfrak{a}}$ defined by Eq. (36) satisfies the Jacobi identity.

Define the 2-structure on $\mathfrak{a}$ as follows (for any $h \in \mathfrak{h}$ and $\pi \in \mathfrak{h}^*$; hence, for any $h + \pi \in \mathfrak{a}$):

$$(h + \pi)^{[2]_{\mathfrak{a}}} := h^{[2]_{\mathfrak{h}}} + \pi \circ \mathrm{ad}_h. \tag{37}$$

[4] Recall that $c(a_k) = (-1)^{p(a_k)} \delta_j^k a_i$ for the 1-cochain $c = a_i \otimes \widehat{a_j}$,

Define a bilinear form on $\mathfrak{a}$ as follows:

$$\mathscr{B}_{\mathfrak{a}}(h+\pi, h'+\pi') := \pi(h') + \pi'(h) \quad \text{for any } h+\pi,\ h'+\pi' \in \mathfrak{a}. \tag{38}$$

It is easy to show that the bilinear form $\mathscr{B}_{\mathfrak{a}}$ is NIS.

### 4.3.2 Heisenberg Algebra $\mathfrak{hei}(2)$

Consider the Heisenberg algebra $\mathfrak{hei}(2)$ spanned by $p, q$ and $z$, with the only nonzero bracket: $[p, q] = z$. Let us consider a 2-structure given by

$$p^{[2]} = q^{[2]} = 0, \quad z^{[2]} = z.$$

We consider the NIS-algebra $\mathfrak{a} := \mathfrak{hei}(2) \oplus \mathfrak{hei}(2)^*$ constructed as in Sect. 4.3.1. A direct computation using Eq. (37) shows that (for any $s, w, u, v \in \mathbb{K}$)

$$(rz + sp + wq + up^* + vq^* + tz^*)^{[2]_{\mathfrak{a}}} = (r^2 + sw)z + stq^* + wtp^*.$$

A direct computation using Eqs. (37) and (36) shows that the only nonzero brackets are

$$[p, q]_{\mathfrak{a}} = z, \quad [p, z^*]_{\mathfrak{a}} = q^*, \quad [q, z^*]_{\mathfrak{a}} = p^*.$$

**Claim 4.5** *The space* $\mathrm{H}^1_{\text{res}}(\mathfrak{a}; \mathfrak{a})$ *is spanned by the (classes of the) following cocycles:*

$$\begin{array}{lll}
\mathscr{D}_1 = q^* \otimes \widehat{p}, & \mathscr{D}_2 = q^* \otimes \widehat{q}, & \mathscr{D}_3 = q^* \otimes \widehat{z^*}, \\
\mathscr{D}_4 = p^* \otimes \widehat{p}, & \mathscr{D}_5 = p^* \otimes \widehat{z^*}, & \mathscr{D}_6 = z \otimes \widehat{z^*}, \\
\mathscr{D}_7 = p \otimes \widehat{p} + q^* \otimes \widehat{q^*} + z \otimes \widehat{z}, & \mathscr{D}_8 = q \otimes \widehat{q} + p^* \otimes \widehat{p^*} + z \otimes \widehat{z}, & \\
\mathscr{D}_9 = q^* \otimes \widehat{q^*} + p^* \otimes \widehat{p^*} + z^* \otimes \widehat{z^*}. & &
\end{array}$$

Let us fix an ordered basis as follows: $p, q, z, p^*, q^*, z^*$. In this basis, the Gram matrix of the bilinear form $\mathscr{B}_{\mathfrak{a}}$ in (38) is

$$\text{antidiag}(I_3, I_3), \text{ where } I_n \text{ denotes the } n \times n \text{ unit matrix.}$$

**Proposition 4.6** *Up to an isomorphism,* $\mathfrak{a}$ *admits a unique non-trivial double extension given by the derivation* $\mathscr{D}_7 + \mathscr{D}_9$.

***Proof*** Any derivation $\mathscr{D}$ has the following representation as a supermatrix in the standard format:

$$\mathscr{D} = \left(\begin{array}{c|c} A & B \\ \hline C & F \end{array}\right).$$

It follows that $\mathscr{B}_{\mathfrak{a}}$ is $\mathscr{D}$-invariant if and only if $F = A^t$, $B^t = B$ and $C^t = C$. Let us consider the most general derivation $\mathscr{D} = \sum_{1 \le i \le 9} \mu_i \mathscr{D}_i$, where $\mathscr{D}_i$ are the cocycles given in Claim 4.5. In the same basis $p, q, z, p^*, q^*, z^*$, we have

$$\mathscr{D} = \left( \begin{array}{c|c} \mu_7 E^{1,1} + (\mu_7+\mu_8)E^{3,3} + \mu_8 E^{2,2} & \mu_6 E^{3,3} \\ \hline \mu_4 E^{1,1} + \mu_1 E^{2,1} + \mu_2 E^{2,2} & \mu_8 E^{1,1} + \mu_7 E^{2,2} + \mu_3 E^{2,3} + \mu_5 E^{1,3} + \mu_9 I \end{array} \right),$$

where $E^{i,j}$ is the $(i, j)$th $3 \times 3$ matrix unit.

It follows that $\mathscr{B}_{\mathfrak{a}}$ is $\mathscr{D}$-invariant if and only if $\mu_1 = \mu_3 = \mu_5 = 0$ and $\mu_9 = \mu_7 + \mu_8$.

The most general derivation is of the form

$$\mu_2 \mathscr{D}_2 + \mu_4 \mathscr{D}_4 + \mu_6 \mathscr{D}_6 + \mu_7 \mathscr{D}_7 + \mu_8 \mathscr{D}_8 + (\mu_7 + \mu_8)\mathscr{D}_9.$$

Since $p = 2$, we also need to check that $\mathscr{B}_{\mathfrak{a}}(a, \mathscr{D}(a)) = 0$ for all $a \in \mathfrak{a}$. This condition implies that $\mu_2 = \mu_4 = \mu_6 = 0$.

Now, we define two quadratic forms on $\mathfrak{a}$ as follows:

$$\mathscr{P}_1(rz + sp + wq + tz^* + up^* + vq^*) = rt + su,$$
$$\mathscr{P}_2(rz + sp + wq + tz^* + up^* + vq^*) = rt + wv.$$

We have two $\mathscr{D}$-extensions given by the following data (where $\alpha, \beta \in \mathbb{K}$):

$$\begin{aligned} &(\mathscr{D} = \mathscr{D}_7 + \mathscr{D}_9, \;\; a_0 = \beta q^*, \;\; b_0 = \alpha q^*, \;\; \mathscr{P}_1, \;\; m, \;\; l, \;\; \gamma = 1), \\ &(\mathscr{D} = \mathscr{D}_8 + \mathscr{D}_9, \;\; a_0 = \beta p^*, \;\; b_0 = \alpha p^*, \;\; \mathscr{P}_2, \;\; \tilde{m}, \;\; \tilde{l}, \;\; \tilde{\gamma} = 1). \end{aligned} \tag{39}$$

Let us show that these two $\mathscr{D}$-extensions are isomorphic if $m, l, \tilde{m}$ and $\tilde{l}$ are suitably chosen. Indeed, the isomorphism is given by (for notation, see Theorem 3.11)

$$\begin{aligned} \pi_0(z) &= z, \;\; \pi_0(z^*) = z^*, \quad \pi_0(p) = q, \\ \pi_0(q) &= p \;\; \pi_0(q^*) = p^*, \; \pi_0(p^*) = q^*, \\ \varkappa &= 0, \qquad \lambda = 1, \qquad \rho = 0. \end{aligned}$$

On the other hand, let us show that the $\mathscr{D}$-extension by means of $\mathscr{D}_7 + \mathscr{D}_9$ is not a trivial one, namely it is not isomorphic to the one by means of $\mathrm{ad}_T$ for some $T \in \mathfrak{a}$. Suppose there is such an isomorphism, say $\pi$. Let us write

$$\pi_0(z) = mz, \quad \pi_0(z^*) = m_1 z + m_2 p + m_3 q + m_4 p^* + m_5 q^* + m^{-1} z^*.$$

Now, because $q^* = [z^*, p]$, it follows that for some $c_1, c_2, c_3 \in \mathbb{K}$, we have

$$\begin{aligned} \pi_0(q^*) &= [m_1 z + m_2 p + m_3 q + m_4 p^* + m_5 q^* + m^{-1} z^*, \pi_0(p)] \\ &= c_1 p^* + c_2 q^* + c_3 z. \end{aligned}$$

Similarly, since $p^* = [z^*, q]$, it follows that for some $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3 \in \mathbb{K}$, we have

$$\pi_0(p^*) = [m^{-1} z^*, \pi_0(q)] = \tilde{c}_1 p^* + \tilde{c}_2 q^* + \tilde{c}_3 z.$$

Let $T = W_1 p + W_2 p^* + W_3 q + W_4 q^* + W_5 z + W_6 z^*$, where $W_i \in \mathbb{K}$. We have

$$\begin{aligned}
&((\mathscr{D}_7 + \mathscr{D}_9) \circ \pi_0 - \pi_0 \circ \mathrm{ad}_T)(z^*) \\
&\quad = (\mathscr{D}_7 + \mathscr{D}_9)(m_1 z + m_2 p + m_3 q + m_4 p^* + m_5 q^* + m^{-1} z^*) - \pi_0[T, z^*] \\
&\quad = m^{-1} z^* + m_1 z + m_2 p + m_4 p^* - W_1 \pi_0(q^*) - W_3 \pi_0(p^*) \\
&\quad = m^{-1} z^* + m_1 z + m_2 p + m_4 p^* - W_1(c_1 p^* + c_2 q^* + c_3 z) \\
&\qquad - W_3(\tilde{c}_1 p^* + \tilde{c}_2 q^* + \tilde{c}_3 z).
\end{aligned}$$

But this is never zero; hence, a contradiction. □

### 4.4 Vectorial Lie (Super)Algebras

Over any field $\mathbb{K}$ of characteristic $p > 0$, consider not polynomial coefficients but divided powers in $n$ indeterminates, whose powers are bounded by the shearing vector $\underline{N} = (N_1, \ldots, N_n)$. We get a commutative algebra (here $p^\infty := \infty$)

$$\mathscr{O}(n; \underline{N}) := \mathbb{K}\left[u; \underline{N}\right] := \mathrm{Span}_{\mathbb{K}}(u^{(r)} \mid 0 \le r_i < p^{N_i}),$$

where $u^{(r)} = \prod_{1 \le i \le n} u_i^{(r_i)}$. The addition in $\mathscr{O}(n; \underline{N})$ is natural; the multiplication is defined by

$$u_i^{(r_i)} \cdot u_i^{(s_i)} = \binom{r_i + s_i}{r_i} u_i^{(r_i + s_i)}.$$

Set $\mathbf{1} := (1, \ldots, 1)$ and set $\tau(\underline{N}) := (p^{N_1} - 1, \ldots, p^{N_n} - 1)$ often abbreviated to $\tau$.

The *distinguished* partial derivatives $\partial_i$, each of them serving as several partial derivatives at once, for each of the generators $u_i, u_i^{(p)}, u_i^{(p^2)}, \ldots$ (or, in terms of $y_{i,j} := u_i^{(p^j - 1)}$) are defined by the formula

$$\partial_i(u_j^{(k)}) := \delta_{ij} u_j^{(k-1)} \text{ for all } k, \text{ i.e., } \partial_i = \sum_{j \ge 1} (-1)^{j-1} y_{i,1}^{p-1} \cdots y_{i,j-1}^{p-1} \partial_{y_{i,j}}.$$

#### 4.4.1 $\mathfrak{vect}(n; \underline{N})$

The *general vectorial Lie algebra*, known as the *Jacobson–Witt algebra*:

$$\mathfrak{vect}(n; \underline{N}) := \left\{ \sum_i f_i \partial_i \mid f_i \in \mathscr{O}(n; \underline{N}) \right\},$$

where the bracket is given by the Lie bracket of vector fields. The Lie algebra $\mathfrak{vect}(n; \underline{N})$ has a NIS ([46, Theorems 6.3 and 6.4 in Ch. 4]) if and only if either

$n = 1$ and $p = 3$, in which case NIS is

$$\mathscr{B}(u^{(a)}\partial, u^{(b)}\partial) := \int u^{(a)}u^{(b)}\mathrm{d}u, \tag{40}$$

or $n = p = 2$, in which case NIS is

$$\mathscr{B}(u^{(a)}\partial_i, u^{(b)}\partial_j) := (i + j)\int u^{(a)}u^{(b)}\mathrm{d}u_1 \wedge \mathrm{d}u_2,$$

where $\int f(u)\mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_n :=$ coefficient of $u^{(\tau(\underline{N}))}$ in the Taylor series expansion of $f(u)$.

***Remark 4.7*** It was shown in [18] that there is no NIS on the simple Lie algebras denoted there by $\mathfrak{svect}_{exp_i}(n; \underline{N})$ and $\mathfrak{svect}^{(1)}_{1+\bar{u}}(n; \underline{N})$—deforms of the divergence-free subalgebra of $\mathfrak{vect}(n; \underline{N})$. For examples of NISes on simple $\mathbb{Z}$-graded vectorial Lie algebras in characteristic $p > 0$, see [22,24]; these examples are reproduced in [18].

Since we are interested only in restricted Lie (super)algebras which can only exist for $\underline{N} = \mathbf{1}$, we mostly skip $\underline{N}$.

**Claim 4.8** (i) [46, Theorem 8.5] *For* $p = 3$, $\mathrm{H}^1(\mathfrak{vect}(1); \mathfrak{vect}(1)) = 0$. *Hence, no non-trivial double extensions of* $\mathfrak{vect}(1)$.
(ii) *For* $p = 2$, $\mathrm{H}^1(\mathfrak{vect}(2); \mathfrak{vect}(2)) = 0$. *Hence, no non-trivial double extensions of* $\mathfrak{vect}(2)$.

#### 4.4.2 $\mathfrak{svect}(3; \underline{N})$ for $p = 2$

Set

$$\mathfrak{svect}(n; \underline{N}) := \left\{\mathscr{D} \in \mathfrak{vect}(n; \underline{N}) \mid \mathrm{div}(\mathscr{D}) = 0\right\}.$$

This Lie algebra is not simple; however, its first derived subalgebra $\mathfrak{svect}^{(1)}(n; \underline{N})$ is simple for $n \geq 3$, see [45,46]. Consider the maps

$$D_{i,j} : \mathscr{O}(n; \underline{N}) \to \mathfrak{vect}(n; \underline{N}), \quad f \mapsto D_{i,j}(f) = \partial_j(f)\partial_i - \partial_i(f)\partial_j.$$

Clearly,

$$\mathfrak{svect}^{(1)}(n; \underline{N}) = \mathrm{Span}\{D_{i,j}(f) \mid f \in \mathscr{O}(n, \underline{N}), 1 \leq i < j \leq n\}.$$

The Lie algebra $\mathfrak{svect}^{(1)}(n; \underline{N})$ has a NIS if and only if $n = 3$; explicitly, we set

$$\mathscr{B}(\partial_i, D_{j,k}(u^{(\tau(\underline{N}))})) = \mathrm{sign}(i, j, k), \tag{41}$$

and extend the form $\mathscr{B}$ to other pairs of elements by invariance and linearity.

We have the following exceptional isomorphisms:

$$\mathfrak{svect}^{(1)}(3;\mathbf{1}) \simeq \mathfrak{psl}(4) \simeq \mathfrak{h}^{(1)}_{\omega}(4;\mathbf{1}) \quad \text{for } p=2 \text{ (as shown in [21])}.$$

The case of the Hamiltonian Lie superalgebra $\mathfrak{h}_\omega$ preserving the symplectic form $\omega$ with constant coefficients is completely investigated in [20] for all possible types of $\omega$, see Sect. 4.4.3.

**Claim 4.9** $\dim \mathrm{H}^1_{\mathrm{res}}(\mathfrak{svect}^{(1)}(3); \mathfrak{svect}^{(1)}(3)) = 2$ *is spanned by the cocycles* $\mathscr{D}_3$ *of degree 3 and* $\mathscr{D}_0$ *of degree 0* (for details, see the arXiv version of this paper).

A direct computation shows that the bilinear form given in Eq. (41) is **not** $\mathscr{D}_3$-invariant. Therefore, the Lie algebra $\mathfrak{svect}^{(1)}(3)$ cannot be double extended by means of $\mathscr{D}_3$. However, a direct computation shows that this bilinear form is $\mathscr{D}_0$-invariant and, moreover, $(\mathscr{D}_0)^3 = 0$ (cube of the operator). Therefore, $a_0 = \gamma = 0$, see Sect. 2.1.2.

Now we define the cubic form $\mathscr{P}(\sum_{1\leq i\leq 52} \lambda_i e_i)$ to be equal to

$$\begin{aligned}&2\lambda_{32}\lambda_2^2 + \lambda_9\lambda_{12}\lambda_2 + \lambda_{10}\lambda_{17}\lambda_2 + 2\lambda_4\lambda_{20}\lambda_2 + 2\lambda_8\lambda_{21}\lambda_2\\ &\quad + \lambda_3\lambda_{37}\lambda_2 + \lambda_7\lambda_8^2 + \lambda_4^2\lambda_9 + \lambda_4\lambda_8\lambda_{10} + 2\lambda_4\lambda_8\lambda_{11} + 2\lambda_1\lambda_8\lambda_{12}\\ &\quad + \lambda_3\lambda_{11}\lambda_{12} + \lambda_3\lambda_8\lambda_{15} + \lambda_1\lambda_4\lambda_{17} + 2\lambda_3\lambda_7\lambda_{17} + 2\lambda_3\lambda_4\lambda_{22} + 2\lambda_3^2\lambda_{28},\end{aligned} \tag{42}$$

where it suffices to describe the $e_i$ that appear in the expression of $\mathscr{P}(a)$:

$$\begin{array}{lll}
e_2 = \partial_2, & e_3 = \partial_3, & e_4 = 2D_{1,2}(u^{(\tau-2\epsilon_2-2\epsilon_3)}),\\
e_7 = D_{2,3}(u^{(\tau-2\epsilon_1-2\epsilon_2)}), & e_8 = D_{1,3}(u^{(\tau-2\epsilon_2-2\epsilon_3)}), & e_9 = D_{2,3}(u^{(\tau-2\epsilon_1-2\epsilon_3)}),\\
e_{10} = D_{1,2}(u^{(\tau-\epsilon_1-\epsilon_2-2\epsilon_3)}), & e_{11} = D_{1,3}(u^{(\tau-2\epsilon_2-\epsilon_3-\epsilon_1)}), & e_{12} = D_{2,3}(u^{(\tau-\epsilon_3-2\epsilon_2)}),\\
e_{15} = 2D_{1,2}(u^{(\tau-\epsilon_3-2\epsilon_2)}), & e_{17} = D_{2,3}(u^{(\tau-2\epsilon_3-\epsilon_2)}), & e_{20} = D_{1,3}(u^{(\tau-2\epsilon_3-\epsilon_2)}),\\
e_{21} = D_{1,2}(u^{(\tau-2\epsilon_3-\epsilon_2)}), & e_{22} = D_{1,3}(u^{(\tau-\epsilon_3-2\epsilon_2)}) & e_{28} = D_{2,3}(u^{(\tau-2\epsilon_2)}),\\
e_{32} = D_{2,3}(u^{(\tau-2\epsilon_3)}), & e_{37} = D_{2,3}(u^{(\tau-\epsilon_3-\epsilon_2)}). &
\end{array}$$

Let us summarize and introduce notation:

| Derivation | $\mathscr{D}$-invariance | $\mathscr{P}(a)$ | $\gamma$ | $a_0$ | Double extension |
|---|---|---|---|---|---|
| $\mathscr{D}_3$ | No | – | – | – | – |
| $\mathscr{D}_0$ | Yes | see Eq. (42) | 0 | 0 | $\widehat{\mathfrak{svect}}^{(1)}(3)$ |

(43)

### 4.4.3 The Hamiltonian Lie (Super)Algebras $\mathfrak{h}^{(1)}_{\omega}(a; 1|b)$ for $a+b>3$ and $p=2$

There are four types of symplectic forms $\omega$ with constant coefficients, as Lebedev showed, see [36]. For the double extensions of $\mathfrak{h}^{(1)}_{\omega}(a;\mathbf{1}|b)$ for $a+b>3$ and $p=2$, see [20].

For $p>2$ and in the non-super case, Skryabin classified the normal shapes of symplectic forms $\omega$ with non-constant coefficients, see [43,44]; some of these shapes exist for $p = 2$. If $p = 2$, no classification is known; for new examples, see [34]. Some of these $\mathfrak{h}_\omega(2n; \underline{N})$ have a NIS, see [18]. To consider these cases, as well as deformations of other simple vectorial Lie algebras and superalgebras is an open problem.

#### 4.4.4 The Exceptional Lie Superalgebra $\mathfrak{vas}$ for $p = 3$, See [18], Has no Double Extensions: Sketch of the Proof

The Lie superalgebra $\mathfrak{g} := \mathfrak{vas}$ has a grading operator $D$; hence, the central extension should be graded. If, moreover, we wish this central extension be inherited from $\mathfrak{sle}^{(1)}(4)$, see [42], the central element must be odd of degree $-2$. This means that the bracket should define projectively $\mathfrak{g}_0$-invariant 2-form $\omega$ on $\mathfrak{g}_{-1}$.

If $\omega$ were degenerate, its kernel would have been $\mathfrak{g}_0$-invariant subspace of $\mathfrak{g}_{-1}$, but the latter is irreducible.

Therefore, $\omega$ is a non-degenerate odd projectively $\mathfrak{g}_0$-invariant bilinear form on $\mathfrak{g}_{-1}$. Hence, $\mathfrak{g}_0 \subset \mathfrak{pe}(4) \ltimes \mathbb{K}D$. Precisely as 0th component of $\mathfrak{le}(4)$, see [38]; but $\mathfrak{as} = \mathfrak{vas}_0$ (for its definition, see [42]) can not be embedded into $\mathfrak{pe}(4) \ltimes \mathbb{K}D$.

### 4.5 $\mathfrak{osp}(1|2)$, $\mathfrak{g}^{(1)}(2,3)/\mathfrak{z}$ and $\mathfrak{g}^{(1)}(3,3)/\mathfrak{z}$ for $p = 3$

#### 4.5.1 $\mathfrak{osp}(1|2)$

Let us fix a basis of $\mathfrak{osp}(1|2)$ generated by the root vectors $x_1$, $x_2 = [x_1, x_1]$ (positive), $y_1$, $y_2 = [y_1, y_1]$ (negative), and $h_1 = [x_1, y_1]$. The Lie superalgebra $\mathfrak{osp}(1|2)$ admits a NIS given in the ordered basis $e_1 = [x_1, y_1]$, $e_2 = x_1$, $e_3 = x_2$, $e_4 = y_1$, $e_5 = y_2$ by the Gram matrix (here $E^{i,j}$ is the $(ij)$th matrix unit)

$$\mathscr{B}_{\mathfrak{osp}(1|2)} = 2E^{1,1} + 2E^{2,4} + 2E^{3,5} + E^{4,2} + 2E^{5,3}.$$

There is a $3|6$-structure on $\mathfrak{osp}(1|2)$ given by

$$e_1^{[3]} = e_1, \quad e_2^{[6]} = e_4^{[6]} = e_3^{[3]} = e_5^{[3]} = 0.$$

**Claim 4.10** ([12]) (i) *For $p = 3$, the space $\mathrm{H}^1(\mathfrak{osp}(1|2); \mathfrak{osp}(1|2))$ is spanned by the odd cocycles:*

$$\begin{aligned} \deg = -3 &: \mathscr{D}_{-3} = 2y_1 \otimes \widehat{x_2} + y_2 \otimes \widehat{x_1}, \\ \deg = 3 &: \mathscr{D}_3 = x_1 \otimes \widehat{y_2} + x_2 \otimes \widehat{y_1}. \end{aligned} \tag{44}$$

(ii) *For $p > 3$,* $\mathrm{H}^1(\mathfrak{osp}(1|2); \mathfrak{osp}(1|2)) = 0$.

A long, but easy, computation shows that $\mathscr{B}_{\mathfrak{osp}(1|2)}$ is both $\mathscr{D}_{-3}$-invariant and $\mathscr{D}_3$-invariant. Moreover, it is easy to show that $(\mathscr{D}_{-3})^2 = (\mathscr{D}_3)^2 = 0$ (squares of the operators). Therefore, conditions (16) are satisfied because $\mathfrak{osp}(1|2)$ is simple. However, condition (17) is violated. Indeed, for the derivation $\mathscr{D}_{-3}$, we have

$$\mathscr{B}_{\mathfrak{osp}(1|2)}(\mathscr{D}_{-3}(x_1), [x_1, x_1]) = \mathscr{B}_{\mathfrak{osp}(1|2)}(2y_2, x_2) = 1 \neq 0.$$

Therefore, the double extension of $\mathfrak{osp}(1|2)$ is a pre-Lie superalgebra, see Appendix. The isomorphism $\pi$ defined by

$$\pi(x_1) = 2y_1,\ \pi(y_1) = x_1,$$

shows that the derivations $\mathscr{D}_{-3}$ and $\mathscr{D}_3$ produce the same pre-Lie superalgebra.

### 4.5.2 $\mathfrak{g}(2, 3)^{(1)}/\mathfrak{z}$, See [15]

Consider the Lie superalgebra $\mathfrak{g}(2, 3)$ with the Cartan matrix

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 0 \end{pmatrix} \text{ with positive roots } \begin{array}{l} x_1, \quad x_2, \quad x_3, \\ x_4 = [x_1, x_2], \quad x_5 = [x_1, x_3], \quad x_6 = [x_2, x_3], \\ x_7 = [x_3, [x_1, x_2]], \\ x_8 = [[x_1, x_2], [x_1, x_3]], \quad x_9 = [[x_1, x_2], [x_2, x_3]], \\ x_{10} = [[x_1, x_2], [x_3, [x_1, x_2]]], \\ x_{11} = [[x_3, [x_1, x_2]], [x_3, [x_1, x_2]]]. \end{array}$$

Denote by $y$'s the corresponding negative roots, and set $h_i = [x_i, y_i]$ for $i = 1, 2, 3$.

The simple core $\mathfrak{a} := \mathfrak{g}(2, 3)^{(1)}/\mathfrak{z}$ admits a NIS given in the ordered basis

$$h_2 := [x_2, y_2],\ h_3 := [x_3, y_3],\ x_1, \dots x_{11},\ y_1, \dots, y_{11}, \tag{45}$$

by the Gram matrix

$$\mathscr{B}_{\mathfrak{a}} = \begin{pmatrix} S & 0 & 0 \\ 0 & 0 & T \\ 0 & U & 0 \end{pmatrix}, \quad \text{where } S = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and}$$
$$T = \text{diag}\{2, 2, 2, 1, 1, 1, 1, 1, 1, 2, 1\},\ U = \text{diag}\{2, 2, 1, 1, 2, 2, 2, 2, 2, 1, 2\}.$$

There is a 3|6-structure on $\mathfrak{g}(2, 3)^{(1)}/\mathfrak{z}$ that we express in the basis (45) as follows:

$$h_2^{[3]} = h_2,\ h_3^{[3]} = h_3,\ v^{[3]} = w^{[6]} = 0 \text{ for any } v \text{ (even) and } w \text{ (odd) root vectors in (45).}$$

**Claim 4.11** ([12]) *The space* $\mathrm{H}^1(\mathfrak{g}(2,3)^{(1)}/\mathfrak{z};\mathfrak{g}(2,3)^{(1)}/\mathfrak{z})$ *is spanned by the 7 even cocycles of which we indicate the ones we need:*

$$\begin{array}{ll} \deg = -3: & \mathscr{D}^1_{-3} = 2\,x_3 \otimes \widehat{x}_8 + x_6 \otimes \widehat{x}_{10} + y_1 \otimes \widehat{x}_4 + y_4 \otimes \widehat{x}_1 + 2\,y_8 \otimes \widehat{y}_3 + y_{10} \otimes \widehat{y}_6,\\ \deg = 0: & \mathscr{D}^1_0 = x_2 \otimes \widehat{x}_1 + 2\,x_6 \otimes \widehat{x}_5 + 2\,x_9 \otimes dx_8 + 2\,y_1 \otimes \widehat{y}_2 + y_5 \otimes \widehat{y}_6 + y_8 \otimes \widehat{y}_9,\\ & \mathscr{D}^2_0 = x_1 \otimes \widehat{x}_1 + 2\,x_2 \otimes \widehat{x}_2 + 2\,x_5 \otimes dx_5 + x_6 \otimes \widehat{x}_6 + 2\,x_8 \otimes \widehat{x}_8 + x_9 \otimes \widehat{x}_9 +\\ & 2\,y_1 \otimes \widehat{y}_1 + y_2 \otimes \widehat{y}_2 + y_5 \otimes \widehat{y}_5 + 2\,y_6 \otimes \widehat{y}_6 + y_8 \otimes \widehat{y}_8 + 2\,y_9 \otimes \widehat{y}_9. \end{array} \tag{46}$$

We easily see that $\mathfrak{out}(\mathfrak{g}(2,3)^{(1)}/\mathfrak{z}) \simeq \mathfrak{psl}(3)$.

A direct computation shows that the bilinear form $\mathscr{B}_{\mathfrak{a}}$ is $\mathscr{D}$-invariant for the derivations $\mathscr{D}$ given in (46). The isomorphism

$$x_1 \leftrightarrow y_1, \quad x_2 \leftrightarrow y_2, \quad x_3 \leftrightarrow y_3, \quad y_3 \leftrightarrow 2x_3$$

permutes the derivations $\mathscr{D}^1_{-3} \leftrightarrow \mathscr{D}^1_3$ introducing $\mathscr{D}^1_3$. The isomorphism

$$x_1 \leftrightarrow x_2, \quad y_1 \leftrightarrow y_2, \quad x_3 \leftrightarrow x_3, \quad y_3 \leftrightarrow y_3, \tag{47}$$

permutes the derivations $\mathscr{D}^1_{-3} \leftrightarrow \mathscr{D}^2_{-3}$, $\mathscr{D}^1_0 \leftrightarrow \mathscr{D}^3_0$, and $\mathscr{D}^1_3 \leftrightarrow \mathscr{D}^2_3$ turning the ones defined in (46) and $\mathscr{D}^1_3$ into new ones. The derivations $\mathscr{D}^1_{-3}$, $\mathscr{D}^1_0$ and $\mathscr{D}^2_0$ have the $p$-property.

Let us describe now the cubic forms. We fix an ordered basis of $\mathfrak{a}_{\bar{0}}$ as follows:

$$\begin{aligned} &e_1 = h_2, e_2 = h_3, e_3 = x_1, e_4 = x_2, e_5 = x_4, e_6 = x_{11},\\ &e_7 = y_1, e_8 = y_2, e_9 = y_4, e_{10} = y_{11}. \end{aligned}$$

For every $a = \sum_{1\le i\le 10} \lambda_i e_i$, we define

$$\mathscr{P}_1(a) := \lambda_1\lambda_3\lambda_5 + \lambda_2\lambda_3\lambda_5 + \lambda_3^2\lambda_4 + 2\lambda_5^2\lambda_8, \tag{48}$$

$$\mathscr{P}_2(a) := 2\lambda_1\lambda_3\lambda_8 + 2\lambda_2\lambda_3\lambda_8 + 2\lambda_3^2\lambda_9 + 2\lambda_5\lambda_8^2, \tag{49}$$

$$\mathscr{P}_3(a) := 2\lambda_1\lambda_3\lambda_7 + \lambda_1\lambda_4\lambda_8 + 2\lambda_2\lambda_3\lambda_7 + \lambda_2\lambda_4\lambda_8 + 2\lambda_3\lambda_4\lambda_9 + \lambda_5\lambda_7\lambda_8. \tag{50}$$

For each of these cubic forms, a direct computation shows that the condition (20) is satisfied. Let us summarize and introduce notation:

| Derivation | $\mathscr{P}(a)$ | $\gamma$ | $a_0$ | Double extension |
|---|---|---|---|---|
| $\mathscr{D}^1_{-3}$ | see Eq. (48) | 0 | 0 | $\widetilde{\mathfrak{g}}(2,3)$ |
| $\mathscr{D}^1_0$ | see Eq. (49) | 0 | 0 | $\widehat{\mathfrak{g}}(2,3)$ |
| $\mathscr{D}^2_0$ | see Eq. (50) | 1 | 0 | $\mathfrak{g}(2,3)$ |

(51)

**Claim 4.12** $\dim \mathrm{H}^3(\mathfrak{g}(2,3)) = 15$, $\dim \mathrm{H}^3(\widetilde{\mathfrak{g}}(2,3)) = 20$ *and* $\dim \mathrm{H}^3(\widehat{\mathfrak{g}}(2,3)) = 18$, *hence* $\mathfrak{g}(2,3)$, $\widetilde{\mathfrak{g}}(2,3)$ *and* $\widehat{\mathfrak{g}}(2,3)$ *are pairwise not isomorphic.*

### 4.5.3 $\mathfrak{g}(3,3)^{(1)}/\mathfrak{z}$, See [15]

Consider the Lie superalgebra $\mathfrak{g}(3,3)$ with the Cartan matrix and positive roots

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{array}{l} x_1,\ x_2,\ x_3,\ x_4, \\ x_5 = [x_1, x_2],\ x_6 = [x_2, x_3],\ x_7 = [x_3, x_4], \\ x_8 = [x_3, [x_1, x_2]],\ x_9 = [x_3, [x_2, x_3]], \\ x_{10} = [x_4, [x_2, x_3]], x_{11} = [x_3, [x_3, [x_1, x_2]]], \\ x_{12} = [[x_1, x_2], [x_3, x_4]], x_{13} = [[x_2, x_3], [x_3, x_4]], \\ x_{14} = [[x_2, x_3], [x_3, [x_1, x_2]]], \\ x_{15} = [[x_3, x_4], [x_3, [x_1, x_2]]], \\ x_{16} = [[x_3, [x_1, x_2]], [x_4, [x_2, x_3]]], \\ x_{17} = [[x_4, [x_2, x_3]], [x_3, [x_3, [x_1, x_2]]]]. \end{array}$$

Denote by $y$'s the corresponding negative roots, and set $h_i = [x_i, y_i]$ for $i = 1, 2, 3$.

The simple core $\mathfrak{a} := \mathfrak{g}(3,3)^{(1)}/\mathfrak{z}$ admits a NIS given in the ordered basis

$$h_2 := [x_2, y_2],\ h_3 := [x_3, y_3],\ h_4 := [x_4, y_4],\ x_1, \ldots x_{17},\ y_1, \ldots, y_{17} \tag{52}$$

by the Gram matrix

$$\mathscr{B}_{\mathfrak{a}} = \begin{pmatrix} S & 0 & 0 \\ 0 & 0 & T \\ 0 & U & 0 \end{pmatrix}, \quad \text{where}$$

$$\begin{aligned} T &= \operatorname{diag}\{2, 2, 1, 2, 1, 2, 1, 1, 2, 2, 1, 1, 1, 2, 2, 1, 2, \}, \\ U &= \operatorname{diag}\{2, 2, 1, 1, 1, 2, 2, 1, 2, 1, 1, 2, 2, 2, 1, 2, 1\}, \\ S &= E^{1,1} + 2E^{2,1} + 2E^{2,1} + 2E^{2,2} + E^{3,1} + E^{1,3}. \end{aligned}$$

There is a $3|6$-structure on $\mathfrak{g}(2,3)^{(1)}/\mathfrak{z}$ that we express in the basis (52) as follows:

$$\begin{aligned} &h_2^{[3]} = h_2,\ h_3^{[3]} = h_3,\ h_4^{[3]} = h_4, \\ &v^{[3]} = w^{[6]} = 0 \text{ if } v \text{ is even and } w \text{ is odd roots in (52).} \end{aligned}$$

**Claim 4.13** ([12]) *The space* $\mathrm{H}^1(\mathfrak{g}(3,3)^{(1)}/\mathfrak{z}; \mathfrak{g}(3,3)^{(1)}/\mathfrak{z})$ *is spanned by the following cocycles:*

$$\begin{aligned} \mathscr{D}_{-8} &= y_4 \otimes \widehat{x}_{17} + y_7 \otimes \widehat{x}_{16} + 2\, y_{10} \otimes \widehat{x}_{15} + y_{12} \otimes \widehat{x}_{13} + y_{13} \otimes \widehat{x}_{12} \\ &\quad +2\, y_{15} \otimes \widehat{x}_{10} + y_{16} \otimes \widehat{x}_7 + y_{17} \otimes \widehat{x}_4, \\ \mathscr{D}_0 &= 2\, x_4 \otimes \widehat{x}_4 + 2\, x_7 \otimes \widehat{x}_7 + 2\, x_{10} \otimes \widehat{x}_{10} + 2\, x_{12} \otimes \widehat{x}_{12} + 2\, x_{13} \otimes \widehat{x}_{13} \\ &\quad +2\, x_{15} \otimes \widehat{x}_{15} + 2\, x_{16} \otimes \widehat{x}_{16} + 2\, x_{17} \otimes \widehat{x}_{17} + y_4 \otimes \widehat{y}_4 \\ &\quad +y_7 \otimes \widehat{y}_7 + y_{10} \otimes \widehat{y}_{10} + y_{12} \otimes \widehat{y}_{12} + y_{13} \otimes \widehat{y}_{13} \\ &\quad +y_{15} \otimes \widehat{y}_{15} + y_{16} \otimes \widehat{y}_{16} + y_{17} \otimes \widehat{y}_{17}, \\ \mathscr{D}_8 &= x_4 \otimes \widehat{y}_{17} + x_7 \otimes \widehat{y}_{16} + 2\, x_{10} \otimes \widehat{y}_{15} + x_{12} \otimes \widehat{y}_{13} + x_{13} \otimes \widehat{y}_{12} \\ &\quad +2\, x_{15} \otimes \widehat{y}_{10} + x_{16} \otimes \widehat{y}_7 + x_{17} \otimes \widehat{y}_4. \end{aligned}$$

We will omit the details here. By symmetry, the cases $\mathscr{D}_{-8}$ and $\mathscr{D}_8$ give isomorphic double extensions. There are two non-isomorphic double extensions of $\mathfrak{g}(3,3)^{(1)}/\mathfrak{z}$ given as in the table below:

| Derivation | The cubic form $\mathscr{P}$ | $\gamma$ | $a_0$ | Double extension (DE) | dim $\mathrm{H}^2$(DE) |
|---|---|---|---|---|---|
| $\mathscr{D}_{-8}$ | 0 | 0 | 0 | $\widetilde{\mathfrak{g}}(3,3)$ | 1 |
| $\mathscr{D}_0$ | 0 | 0 | 0 | $\mathfrak{g}(3,3)$ | 0 |

(53)

### 4.6 $\mathfrak{psq}(n)$ for $n > 2$ and $p \neq 2$

The case $p = 2$ is completely different: there is a $1|1$-dimensional space of NISes on $\mathfrak{psq}(n)$, see [35]; to classify double extensions of $\mathfrak{psq}(n)$ is an open problem; to solve it, one should extend the results of [12]. For completeness, note that for $p = 0$, the double extension of $\mathfrak{psq}(n)$ is $\mathfrak{q}(n)$.

**Conjecture 4.14** (Verified for $n = 3, 4$ and $5$ and $p = 3, 5$) *The space* $\mathrm{H}^1(\mathfrak{psq}(n); \mathfrak{psq}(n))$ *is spanned by one odd cocycle. Therefore, the only double extension of* $\mathfrak{psq}(n)$ *is* $\mathfrak{q}(n)$.

**Acknowledgements** We heartily thank A. Lebedev for clarifying to us that the double extension of $\mathfrak{osp}(1|2)$ in characteristic 3 is not a Lie superalgebra, but a pre-Lie superalgebra, and the referee for meticulous job and huge help. We also thank D. Leites for discussions and clarifications. S. Bouarroudj was supported by the grant AD 065 NYUAD.

## 5 Appendix: Pre-Lie Superalgebras in Characteristic 3

A Lie superalgebra $\mathfrak{a}$ over a field of characteristic $p$, where $p \neq 2, 3$, is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $\mathfrak{a} = \mathfrak{a}_{\bar{0}} \oplus \mathfrak{a}_{\bar{1}}$, endowed with a bilinear operation $[-, -]$ that satisfies anti-commutativity and Jacobi identities amended by the Sign Rule.

The **Lie superalgebra** $\mathfrak{a}$ **in characteristic 3** must satisfy one more condition ([38])

$$[a, [a, a]] = 0 \quad \text{for all}\, a \in \mathfrak{a}_{\bar{1}}. \tag{54}$$

Obviously, condition (54) is a consequence of the Jacobi identity if $p \neq 3$. Accordingly, the 2-cocycle, say $\omega$, describing central extensions of $\mathfrak{a}$ should satisfy the following conditions:

$$\omega(a, b) = -(-1)^{p(a)p(b)}\omega(b, a) \quad \text{for all}\, a, b \in \mathfrak{a}, \tag{55}$$

$$\omega(a, [b, c]) = \omega([a, b], c) + (-1)^{p(a)p(b)}\omega(b, [a, c]) \quad \text{for all}\, a, b, c \in \mathfrak{a}, \tag{56}$$

$$(\text{Only for } p = 3) \quad \omega(a, [a, a]) = 0 \quad \text{for all}\, a \in \mathfrak{a}_{\bar{1}}. \tag{57}$$

It was shown in [12] that

$$\mathrm{H}^2(\mathfrak{osp}(1|2)) = 0 \quad \text{and} \quad \mathrm{H}^2(\mathfrak{brj}(2;3)) = 0.$$

Since $\mathfrak{osp}(1|2)$ and $\mathfrak{brj}(2;3)$ do not admit non-trivial central extensions, they have no double extensions.

On the other hand, in Sects. 4.2 and 4.5.1, we observe the following interesting fact: both $\mathfrak{osp}(1|2)$ and $\mathfrak{brj}(2;3)$ are NIS superalgebras and admit an odd outer derivation $\mathscr{D}$, given explicitly (recall that $p = 3$) in (35) and (44). For any of these $\mathscr{D}$, the map

$$\omega(a, b) := \mathscr{B}(\mathscr{D}(a), b)$$

does satisfy the 2-cocycle conditions (55) and (56), but not the condition (57) since for any of these $\mathscr{D}$, we have $\mathscr{B}(\mathscr{D}(a), [a, a]) \neq 0$ for some $a \in \mathfrak{a}_{\bar{1}}$. Moreover, there is no $\alpha \in \mathfrak{a}^*$ such that $\omega(a, b) = \alpha([a, b])$; otherwise, this would have implied that $\mathscr{D}$ was an inner derivation.

It follows that the bracket, see Eq. (18), defined on the double extensions of $\mathfrak{osp}(1|2)$ and $\mathfrak{brj}(2;3)$ satisfies the Jacobi identity but not (54). Leites suggested to call these algebras for which condition (54) is violated, but the Jacobi identity holds, *pre-Lie superalgebras*.

The double extensions of $\mathfrak{osp}(1|2)$ and $\mathfrak{brj}(2;3)$, both for $p = 3$, are **pre-Lie** superalgebras. Currently, these are the **only** examples known in the literature of central extensions of simple Lie superalgebras which are pre-Lie superalgebras but not Lie superalgebras.

A similar situation occurs in characteristic 2, where a central extension of $\mathfrak{h}_I(n; \underline{N})$ is not a Lie algebra, but a **Leibniz** algebra, see [13].

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