



Cutpoints of Invariant Subcontinua of Polynomial Julia Sets

Alexander Blokh¹ · Lex Oversteegen¹ · Vladlen Timorin^{2,3} 

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Abstract

We prove fixed point results for branched covering maps f of the plane. For complex polynomials P with Julia set J_P these imply that periodic cutpoints of some invariant subcontinua of J_P are also cutpoints of J_P . We deduce that, under certain assumptions on invariant subcontinua Q of J_P , every Riemann ray to Q landing at a periodic repelling/parabolic point $x \in Q$ is isotopic to a Riemann ray to J_P relative to Q .

Keywords Complex dynamics · Julia set · External rays

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1 Introduction

The plane fixed point problem is a central problem in continuum theory. While solving it in full generality is still elusive, advances have been recently made in certain cases.

Dedicated to Misha Lyubich's 60-th birthday.

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✉ Vladlen Timorin
vtimorin@hse.ru

Alexander Blokh
ablokh@math.uab.edu

Lex Oversteegen
overstee@uab.edu

¹ Department of Mathematics University of Alabama at Birmingham, Birmingham, AL 35294-1241, USA

² Faculty of Mathematics, HSE University, Russian Federation, 6 Usacheva St., 119048 Moscow, Russia

³ Independent University of Moscow, Bolshoy Vlasievskiy Pereulok 11, 119002 Moscow, Russia

In particular, by [2], for *positively oriented branched covering maps of the plane* the existence of a fixed point *with specific properties* can be established even inside non-invariant continua (provided certain conditions hold). In this paper, we use tools from [2], prove new fixed point results in the same spirit, and rely upon them to tackle some topological problems of polynomial dynamics. In the Introduction we assume knowledge of complex dynamics and give preliminary versions of our main results; later we make more detailed statements.

Standing Notation Throughout the Paper

Write \mathbb{R}_+ for the set $\{x \in \mathbb{R}, x > 0\}$ of positive reals, \mathbb{C} for the plane of complex numbers, $\widehat{\mathbb{C}}$ for the Riemann sphere, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for the open unit disk centered at the origin, $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ for the corresponding closed disk (more generally, \overline{A} will refer to the closure of $A \subset \mathbb{C}$), and $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ for the unit circle. Fix a polynomial P of degree $\deg(P) > 1$ with connected Julia set $J_P = J$ and filled Julia set $K_P = K$ (if it does not cause ambiguity, we do not refer to P in our notation). We always assume that the term $z^{\deg(P)}$ has coefficient 1 in $P(z)$, where $\deg(P)$ is the degree of P (this can be achieved by a coordinate change of the form $z \mapsto \lambda z$). The boundary of a set E is denoted by $\text{Bd}(E)$. Given a compact set $Q \subset \mathbb{C}$, denote by $U_\infty(Q)$ the unbounded complementary domain of Q and by $\text{Th}(Q)$ the set $\mathbb{C} \setminus U_\infty(Q)$ called the *topological hull* of Q . Given a continuum Q , consider the topological disk $\widehat{\mathbb{C}} \setminus \text{Th}(Q)$ and a conformal isomorphism between this disk and $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. We will always assume that the conformal isomorphism takes ∞ to ∞ and has a real positive derivative at ∞ . We refer to such an isomorphism as the *Riemann map* for $\widehat{\mathbb{C}} \setminus \text{Th}(Q)$. The images under the Riemann map of straight radial rays connecting \mathbb{S}^1 with ∞ are called *Q -rays*.

Assume that K is connected. A periodic repelling/parabolic point x of P is said to be *regular*; a point eventually mapped to a regular periodic point is called a *regular (pre)periodic* point.

The main theorem of this paper relies on a fixed point result which partially strengthens a theorem of [11]. Consider the union Σ of all invariant P -rays and the set Fix of their landing points. According to [11], every complementary component A of Σ contains a unique invariant rotational object (either a rotational fixed point of P or an invariant parabolic domain). We prove Theorem 2.15, in which the component A is replaced by a (not even necessarily invariant) continuum in $A \cup \text{Fix}$ that is locally invariant near $\text{Bd}(A)$ (this makes Theorem 2.15 slightly stronger than the quoted result of [11] as far as the *existence* of an invariant rotational object is concerned).

Let $Q \subset J$ be a continuum. A point $x \in Q$ is a *cutpoint of Q of order n* if $Q \setminus \{x\}$ has exactly n components. Suppose that either Q is full, or $Q = \text{Bd}(\text{Th}(Q))$.

Main Theorem *Let $x \in K$ be a regular periodic point and all K -rays to x form m wedges W_i , where $1 \leq i \leq m$. Moreover, suppose that $x \in Q$ is a cutpoint of order n of an invariant continuum $Q \subset J$. Then $n \leq m$, each wedge W_i intersects Q over a connected (possibly empty) set, and every Q -ray to x is isotopic rel. Q to a K -ray that lands at x .*

Observe that a point may be a cutpoint of a subcontinuum while not being a cutpoint of a big continuum. E.g., if a continuum X is formed by the graph of $\sin(\frac{1}{x})$ on the segment $(0, \pi]$ and segment I connecting points $(0, -1)$ and $(0, 1)$, then any non-endpoint x of I is a cutpoint of I of order 2, but *not* a cutpoint of X . The main theorem implies that such behavior is impossible for invariant continua of polynomial Julia sets.

2 Rotational Fixed Points in Non-invariant Continua

2.1 Weakly Repelling Fixed Points

Let $f : X \rightarrow X$ be a continuous map of a locally compact metric space (X, d) to itself. Call an f -fixed point x *weakly repelling in the sense of the metric d* if, for an open neighborhood U of x , the restriction $f|_U$ is a homeomorphism, and $d(f(y), x) > d(y, x)$ for any $y \in U, y \neq x$. Now, let X be a locally compact topological space, $x \in X$ be a point, and $f : X \rightarrow X$ be a continuous map. The point x is *weakly repelling* if it is weakly repelling in the sense of some metric on X that induces the given topology. The orbit of any point $y \in U, y \neq x$ escapes any compact subset of U . Indeed, otherwise the sequence of distances $d(x, f^n(y))$ is increasing and bounded for some $y \neq x$. This implies that for any limit point z of the orbit of y we have, by continuity, $d(x, z) = d(x, f(z))$, a contradiction. Let us call this property the *escaping property at x* (in U). Thus, there are no fixed points of f in U except for x .

If $\psi : \mathbb{R}_+ \rightarrow \mathbb{C}$ is an embedding, call $T = \psi(\mathbb{R}_+)$ a (*topological*) ray. If also $\lim_{t \rightarrow \infty} \psi(t) = \infty$, say that T is a ray from infinity. If Q is a full continuum or the boundary of a full continuum, Q -rays are rays from infinity. If $\lim_{t \rightarrow 0} \psi(t) = y \notin T$, say that T lands at y . A ray R is said to be (f -)invariant if $f(R) \supset R$; if R is a ray from infinity, call it (f -)invariant if $f(R) = R$. Consider a positively oriented branched covering map $f : \mathbb{C} \rightarrow \mathbb{C}$ and an f -invariant ray T landing at a fixed point y . If y is weakly repelling, then, for some r and any $s \in (0, r)$, the point $\psi(s)$ is mapped to $\psi(s')$ with $s' > s$. (In this case, we say that points of T move away from y .) Otherwise the absence of fixed points of f close to x implies that for all small s the point $\psi(s)$ is mapped to $\psi(s')$ with $s' < s$ which contradicts the escaping property at y .

Definition 2.1 Suppose that $f : \mathbb{C} \rightarrow \mathbb{C}$ is a positively oriented branched covering map, $X \subset \mathbb{C}$ is a full continuum, and $x \in X$ is an f -fixed point such that, for some neighborhood U of x , we have that $f|_U$ is a homeomorphism and $f(U \cap X) \subset X$. Then x is called *non-rotational for X* if there exists an invariant topological ray $T \subset \mathbb{C} \setminus X$ that lands at x such that points of T sufficiently close to x move away from x . If such ray T does not exist, then a fixed point x is called *rotational for X* .

Examples of rotational fixed points are points contained in the interior of X , or points in the boundary of X but not accessible, or accessible points x at which no invariant topological ray lands such that its points close to x move away from x along the ray (*accessible* always means accessible from $U_\infty(X)$).

We do not talk of (non-)rotational fixed points x in the absence of the continuum X as then the definitions are too inclusive: if d is a metric on \mathbb{C} defining the usual topology, if x is a fixed point at which $f : \mathbb{C} \rightarrow \mathbb{C}$ is a homeomorphism, and if, for some $\lambda > 1$, we have $d(x, f(y)) > \lambda d(x, y)$, then we can always construct an invariant topological ray landing at x . Indeed, choose a small open disk D_0 around x such that a pullback D_1 of D_0 is compactly contained in D_0 . Choose a topological arc I_1 in $\overline{D_0} \setminus D_1$ connecting a point $y_1 \in \text{Bd}(D_1)$ with $y_0 = f(y_1) \in \text{Bd}(D_0)$ but otherwise lying in $D_0 \setminus \overline{D_1}$. Consider a pullback I_2 of I_1 connecting y_1 with y_2 , a pullback of I_2 connecting y_2 with y_3 , etc. The countable union of these iterated pullbacks of I_1 is an invariant topological ray landing at x .

The Riemann map $\varphi : \widehat{\mathbb{C}} \setminus K_P \rightarrow \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ conjugates $P|_{\mathbb{C} \setminus K_P}$ with the restriction of $z \mapsto z^{\deg(P)}$ to $\mathbb{C} \setminus \overline{\mathbb{D}}$, cf. [19, Theorem 9.5].

Lemma 2.2 *A fixed point $x \in K_P$ is non-rotational for K_P if and only if there exists an invariant K_P -ray landing at x (hence x is regular). Thus, a fixed point $x \in K$ is rotational if it is either non-regular (i.e., attracting, Cremer or Siegel) or regular with non-zero combinatorial rotation number at x .*

Proof If there is an invariant K_P -ray landing at x , then x is non-rotational for K_P . If now x is non-rotational for K_P , then $x \in J = \text{Bd}(K_P)$ is accessible from $\mathbb{C} \setminus K_P$, and an invariant topological ray $R \subset \mathbb{C} \setminus K_P$ lands at x . Let $\psi : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \rightarrow \widehat{\mathbb{C}} \setminus K_P$ be the Riemann map. The topological ray $\psi^{-1}(R)$ lands at a point $e^{2\pi i \theta} \in \mathbb{S}^1$. Then $d\theta = \theta$ modulo 1, and the K_P -ray of argument θ lands at x . □

2.2 Plane Continua and Fixed Points

By a *continuum* we mean a compact and connected metric space and by a *full continuum* X in the plane \mathbb{C} we mean a continuum $X \subset \mathbb{C}$ such that $\mathbb{C} \setminus X$ is connected. A famous theorem by Brouwer [8] states that every continuous map from a disk to itself has a fixed point (this property is called the *fixed point property*). This result motivated the following long-standing problem in topology.

Plane Fixed Point Problem [26]. Does a continuous map of a full plane continuum to itself always have a fixed point?

Even though this problem is not solved yet, certain progress has been made. By [2], full plane continua do have the fixed point property for a restricted class of maps: all *positively oriented branched covering maps of the plane* (see Definition 2.3.3 in [2]). Evidently, every map $f : [a, b] \rightarrow \mathbb{R}$ so that $f(a) > a$ and $f(b) < b$ has a fixed point (even though the interval $[a, b] \subset \mathbb{R}$ is not required to map into itself). It was shown in [2] that similar results hold for full plane continua under a positively oriented branched covering map of the plane. We will extend these results in this section. We will also provide, in certain cases, more information about the local behavior of the map near the fixed point.

2.3 Background from Complex Dynamics and Continuum Theory

If K is a connected filled Julia set of a polynomial P , the Riemann map for $\mathbb{C} \setminus K$ has also a dynamical meaning as it conjugates $P|_{\mathbb{C} \setminus K}$ and $z \mapsto z^d$ restricted to $\mathbb{C} \setminus \overline{\mathbb{D}}$. Call a K -ray *rational* if has rational argument.

Theorem 2.3 [9] *Each rational K -ray lands; its landing point is a periodic or preperiodic regular point. Conversely, each regular periodic point is the landing point of a non-empty finite collection of rational K -rays.*

In [11], the authors consider the partition of the plane by all invariant K -rays united with their landing points (by definition and Lemma 2.2, these landing points are exactly all non-rotational P -fixed points). Denote the union of all invariant K -rays and their landing points by Σ . Define a *rotational object* as either a rotational fixed point or an invariant parabolic domain.

Theorem 2.4 [11] *Every component A of $\mathbb{C} \setminus \Sigma$ contains exactly one rotational object.*

In the next section, we prove a stronger result.

Theorem 2.5 is well-known (see, e.g., Corollary 5.9 of [21]).

Theorem 2.5 (Boundary Bumping Theorem) *Let X be a continuum, V a proper nonempty open subset of X , and F any component of V . Then the closure of F must intersect the boundary of V .*

Let $Q \subset J$ be an invariant continuum and $x \in Q$ be a fixed point. The next lemma follows from Theorem 2.5.

Lemma 2.6 *Let F be a component of $Q \setminus \{x\}$. Then $\overline{F} = F \cup \{x\}$ is a continuum.*

Proof By Theorem 2.5, the closure of F contains x . On the other hand, $\overline{F} \setminus F$ cannot contain any other points. \square

Lemma 2.7 follows from the fact that branched covering maps of the plane are open maps. It is a topological version of the Maximum Modulus Principle; we leave its proof to the reader.

Lemma 2.7 *If $f : \mathbb{C} \rightarrow \mathbb{C}$ is a branched covering map and E is a continuum then $f(\text{Th}(E)) = \text{Th}(f(E))$. In particular, if E is f -invariant then so is $\text{Th}(E)$.*

In the end of this section, we state the following important result.

Theorem 2.8 (Theorem 6.6 [18]) *Suppose x is a point in a continuum Q such that $Q \setminus \{x\}$ has at least $n > 1$ connected components. Then there are n external Q -rays that land at x and separate these components of $Q \setminus \{x\}$. Thus, at least n external Q -rays land at x .*

2.4 An Overview of Some Fixed Point Theorems

We need a number of definitions from [2], some of which are fairly standard.

Let $p : \mathbb{R} \rightarrow \mathbb{S}^1$ be the covering map $p(x) = e^{2\pi i x}$. Let $g : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a continuous map. By the *degree* $\text{deg}(g)$ of g we mean the number $G(1) - G(0)$, where $G : \mathbb{R} \rightarrow \mathbb{R}$ is a lift of the map g to the universal covering space \mathbb{R} of \mathbb{S}^1 (i.e., $p \circ G = g \circ p$). It is well-known that $\text{deg}(g)$ is independent of the choice of the lift.

Let $g : \mathbb{S}^1 \rightarrow \mathbb{C}$ be a continuous map and $f : g(\mathbb{S}^1) \rightarrow \mathbb{C}$ be a fixed point free continuous map. Define the map $v : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ by

$$v(t) = \frac{f(g(t)) - g(t)}{|f(g(t)) - g(t)|}.$$

Define the index $\text{Ind}(f, g)$ of f with respect to g , by $\text{Ind}(f, g) = \text{deg}(v)$; the index $\text{Ind}(f, g)$ measures the net number of revolutions of the vector $f(g(t)) - g(t)$ as t travels through the unit circle one revolution in the positive direction. If S is a Jordan curve and $f : S \rightarrow \mathbb{C}$ is a fixed point free map, we can define $\text{Ind}(f, S)$ by using any positively oriented homeomorphic parameterization g of S by \mathbb{S}^1 as in this case the index does not depend on g . If x is a point, S is the boundary of a Jordan disk D around x , and $F : D \rightarrow \mathbb{C}$ is a positively oriented homeomorphism such that $F(x) = x$, and $F(S)$ is disjoint from \overline{D} , then it is easy to see that $\text{Ind}(F, S) = 1$. If x above is a unique fixed point in D , then for all small Jordan disks $D' \subset D$ with fixed point free boundaries S' we have $\text{Ind}(F, S') = 1$ or 0 depending on whether $x \in D'$ or $x \in D \setminus \overline{D}'$. To see this note that in the first case we can isotope S' to S so that x is always contained in the bounded complementary domain and, in the second case, we can isotope S' to a point while always avoiding x . We talk about the *local index* $\text{Ind}(F, x)$ of F at x so that $\text{Ind}(F, x) = 1$ in the case just described.

Let us now combine Definition 7.4.5 and Lemma 7.4.9 from [2] and define a type of fixed points to which our results will be applied.

Definition 2.9 Suppose that f is a positively oriented map of the plane to itself, $X \subset \mathbb{C}$ is a full continuum and $p \in \text{Bd}(X)$ is a fixed point of f such that:

1. there exists a neighborhood U of p such that $f|_U$ is one-to-one and $f(U \cap X) \subset X$,
2. there exists a ray $R \subset \mathbb{C} \setminus X$ from infinity to p such that $\overline{R} = R \cup p$, and points of R move away from p along R ;
3. there exists a nested sequence of closed disks $D_j \subset U$ with boundaries S_j such that $p \in D_j \setminus S_j$ and $\bigcap D_j = \{p\}$ and $f(S_j \setminus X) \cap D_j = \emptyset$.

Then we say that f *repels outside X at p in the narrow sense*.

We also need to define a class of non-invariant continua to which the results of [2] apply.

Definition 2.10 Suppose that $f : \mathbb{C} \rightarrow \mathbb{C}$ is a positively oriented branched covering map and $X \subset \mathbb{C}$ is a full continuum. Assume that there exist $n \geq 0$ disjoint full continua Z_i such that the following properties hold:

1. $f(X) \setminus X \subset \cup_i Z_i$;
2. for all i , the intersection $Z_i \cap X = K_i$ is a full continuum;
3. for all i , either K_i is a non-rotational weakly repelling fixed point or $f(K_i) \cap Z_i = \emptyset$.

Then we say that f *strongly scrambles the boundary* (of X) and the continua K_i are called *exit continua* (of X).

If $n = 0$ in Definition 2.10, then X is invariant (i.e., $f(X) \subset X$).

Remark 2.11 Since Z_i and $Z_i \cap X = K_i \neq \emptyset$ are full continua and the sets Z_i are pairwise disjoint, then $X \cup (\bigcup Z_i)$ is a full continuum. Loosely, strongly scrambling the boundary means that $f(X)$ can only “grow” off X *within* the sets Z_i and *through* the sets $K_i \subset X$ while either K_i is a non-rotational fixed point, or the image of the set K_i is disjoint from Z_i .

The next theorem helps find certain fixed points in some continua.

Theorem 2.12 (Theorem 7.4.7 [2]) *Suppose that f is a positively oriented branched covering map of the plane with only isolated fixed points, $X \subset \mathbb{C}$ is a full continuum, and the following holds.*

1. *Each fixed point $p \in X$ belongs to the boundary of X , the index $\text{Ind}(f, p)$ at p equals 1, and f repels outside X at p in the narrow sense.*
2. *The map f strongly scrambles the boundary of X . Moreover, in the notation from Definition 2.10, for each i , either $f(K_i) \cap Z_i = \emptyset$, or there exists a neighborhood U_i of K_i with $f(U_i \cap X) \subset X$.*

Then X is a point.

Recall that we consider only metrics in \mathbb{C} that generate the same topology as the Euclidian metric.

Lemma 2.13 *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a positively oriented branched covering; let x be a weakly repelling fixed point of f in the sense of a metric d . For a small open d -disk D_1 of radius ε centered at x with boundary S_1 take iterated pullbacks of D_1 under the map f choosing the branch of the inverse function that fixes x . Denote the n -th pullback of D_1 by D_n , and set $\text{Bd}(D_n) = S_n$; then, for every n , the set D_n is a Jordan disk, and $\overline{D_{n+1}} \subset D_n$. Moreover, $\{x\} = \bigcap D_i$.*

In the above case we say that the sequence $\{D_j\}$ is a *repelling sequence* for x .

Proof We may assume that ε is small enough so that $d(f(y), x) > d(y, x)$ for all $y \in \overline{D_1}$ and f is one-to-one on $\overline{D_1}$. Since x is weakly repelling and x is fixed, $f(\text{Bd}(D_1)) \cap D_1 = \emptyset$ and it is easy to see that $\overline{D_1} \subset f(D_1)$. Since f at x is a local homeomorphism, $\overline{D_{n+1}} \subset D_n$. We claim that $\{x\} = \bigcap D_i$. Indeed let $\{x\} \subsetneq \bigcap D_i$ (observe that $\bigcap D_i = \bigcap \overline{D_i}$). Clearly, $f(\bigcap D_i) = \bigcap D_i$. On the other hand, let $\delta = \max\{d(y, x) : y \in \bigcap D_i\}$. Then $\bigcap D_i$ is contained in the closed disk E of radius δ centered at x , and some points of $\bigcap D_i$ belong to $\text{Bd}(E)$. Since x is weakly repelling, $\text{Bd}(E)$ is mapped outside E which makes $f(\bigcap D_i) = \bigcap D_i$ impossible, a contradiction. \square

2.5 New Fixed Point Results

We return to the case of a complex polynomial P with connected K_P . The following construction allows to replace parabolic points with attracting ones in the topological category.

Proposition 2.14 *There exists a topological branched covering \widehat{P} of the sphere such that the following holds.*

- *The maps \widehat{P} and P coincide outside of periodic parabolic domains of P .*
- *For a periodic parabolic domain U of P of period n , all periodic points of \widehat{P} of period n in $\text{Bd}(U)$ are weakly repelling.*
- *The map \widehat{P} has a unique (super)attracting periodic point of period n in any periodic parabolic domain U of P of period n , and U is an immediate attracting basin for this point with respect to \widehat{P} .*

Observe that the action of P and of \widehat{P} on K -rays and on J is the same.

All parabolic points of P are changed to repelling points of \widehat{P} so that Theorem 2.12 can be applied.

Proof We will prove the proposition for the case when there is only one periodic parabolic domain U (in which case it must be invariant); in the general case the arguments are similar. Recall that by [24] the boundary of U is a Jordan curve. Let $\psi : \mathbb{D} \rightarrow \overline{U}$ be a homeomorphism. Consider the map $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ given by $h = \psi^{-1} \circ P \circ \psi$.

We claim that all fixed points of h are weakly repelling. Indeed, let u be a fixed point of h ; then $x = \psi(u)$ is a fixed point of P . Let P_x^{-1} be a local inverse of P near x , and W a repelling petal at x . Then $P_x^{-1}(W) \subset \overline{W}$, and $P_x^{-n}(W)$ converges to x (here P_x^{-n} is the n th iterate of P_x^{-1}). Set $A = \psi^{-1}(\overline{W} \cap \text{Bd}(U))$. Then A contains a neighborhood of u in \mathbb{S}^1 , we have $h_u^{-1}(A) \subset A$, and $h_u^{-n}(A)$ converges to u , where h_u^{-1} is a local inverse of h near u , and h_u^{-n} is the n th iterate of it. It follows that u is weakly repelling in \mathbb{S}^1 .

Consider the map $H : \mathbb{D} \rightarrow \mathbb{D}$ given by $w \mapsto |w|^2 h(w/|w|)$ for $w \neq 0$; set $H(0) = 0$ by continuity. It is easy to see that all fixed points of h in \mathbb{S}^1 are weakly repelling for H . Set $\widehat{P} = \psi \circ H \circ \psi^{-1}$ on U and $\widehat{P} = P$ everywhere else. □

Theorem 2.15 strengthens Theorem 2.4; recall that by Theorem 2.4 there is a unique rotational object of P in any component of $\mathbb{C} \setminus \Sigma$.

Theorem 2.15 *Let A be a component of $\mathbb{C} \setminus \Sigma$, let $F \subset A \cup \text{Fix}$ be a non-degenerate continuum, and let $P(F) \setminus F$ be disjoint from A . Then the rotational object of P in A is contained in $\text{Th}(F)$.*

Proof Apply Theorem 2.12 to the map $f = \widehat{P}$ and the continuum $X = \text{Th}(F)$. Suppose that X does not contain a fixed rotational point of f . Then the assumptions of Theorem 2.12 are satisfied. The conclusion is that X is a point, a contradiction. Therefore, there is a fixed rotational point of f in X . This point is either a fixed rotational point of P , or the superattracting fixed point created in a parabolic domain of P . In both cases, we obtain the desired. □

3 Locally Fully Invariant Continua

A continuum $Y \subset J_P$ is said to be *locally fully invariant* (LFI) if Y is a component of $P^{-1}(Y)$. In this case, as is easy to see, $P(Y) = Y$, i.e., the set Y is forward invariant.

Conceptually, LFI continua are related to the *locally maximal* invariant sets (cf [1] or even [25]). Given an LFI set Y let $v_Y(z)$ be the number of all P -preimages of z in Y counted with multiplicities. The following lemma is a particular case of [4, Lemma 4.1].

Lemma 3.1 (Lemma 4.1 [4]) *If Y is locally fully invariant, then v_Y is constant on Y .*

This constant is called the *degree* of Y .

Lemma 3.2 *Let $T \subset J_P$ be a continuum such that $P(T) = T$. Then there is a minimal by inclusion LFI continuum $Y \supset T$. Every periodic cutpoint of Y is also a cutpoint of T .*

The LFI continuum Y is called the *LFI hull* of T .

Proof Define a family of continua T_α indexed by ordinals α by transfinite induction. Set $T_0 = T$ and $T_{\alpha+1}$ to be the component of $P^{-1}(T_\alpha)$ containing T . Finally, if α is a limit ordinal, let T_α be the closure of the union of all T_β with $\beta < \alpha$. Then $T_\beta \subset T_\alpha$ whenever $\alpha < \beta$. It is clear that the transfinite sequence T_α stabilizes at an ordinal of cardinality at most that of the continuum. That is, there is an ordinal α_0 such that $T_\alpha = T_{\alpha_0}$ for all $\alpha > \alpha_0$. Then T_{α_0} is the desired LFI continuum.

Now consider a periodic cutpoint z of Y . Replacing P with a suitable iterate, we may assume that $P(z) = z$. Let W be the wedge formed by two adjacent external rays of P landing at z such that $T \cap W \neq \emptyset$. If z is not a cutpoint of T , then $T \subset W$. Then $T_\alpha \subset \overline{W}$ for all α , a contradiction with z being a cutpoint of Y . □

Let us study properties of LFI continua.

Lemma 3.3 *Let Y be an LFI continuum for P . If $P : Y \rightarrow Y$ is one-to-one, then there are no critical points of P in $\text{Th}(Y)$.*

Proof Suppose that $P : Y \rightarrow Y$ is one-to-one but there is a critical point c of P in $\text{Th}(Y)$. First, assume that a critical point c of P is in a bounded complementary component U of Y . Then U is a Fatou component on which P is not 1-to-1, hence, by the classification of Fatou components, P cannot be 1-1 on $\text{Bd}(U) \subset Y$, a contradiction. Finally, suppose that $c \in Y$. Choose $y_i \in Y$ so that $\lim y_i = P(c)$. Since Y is an LFI continuum, for i large enough, multiple preimages of y_i near c must belong to Y , a contradiction. □

Recall the notion of an external map due to Douady and Hubbard [10] in the polynomial-like setup and to Haïssinsky [12] in a more general setup of *polynomial silhouettes*.

Definition 3.4 (External map) Let Y be a *full* LFI continuum for P , and let $\phi : \mathbb{D} \rightarrow \mathbb{C}^* \setminus Y$ be a Riemann map. Since Y is LFI, there exists a neighborhood $U \supset Y$ such that no point of $U \setminus Y$ can map into Y . Hence we can choose $\varepsilon > 0$ so that the map $F = \phi^{-1} \circ P \circ \phi$ is defined and holomorphic on the annulus $A_\varepsilon^- = \{z : 1 - \varepsilon < |z| < 1\}$. By the Schwarz reflection principle, extend F to a holomorphic map of the annulus $A_\varepsilon = \{z : 1 - \varepsilon < |z| < (1 - \varepsilon)^{-1}\}$ to \mathbb{C} preserving \mathbb{S}^1 and taking A_ε to another annulus around \mathbb{S}^1 . The real analytic map $F : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is the *external map* of Y with

respect to P . Since F takes A_ε^- to a subset of the disk $|z| < 1$, it has no critical points on \mathbb{S}^1 . Abusing the terminology, we will sometimes call the analytic continuation of F to A_ε also the external map.

In Lemma 3.5, we let Y denote an LFI continuum for P , and F be the external map of $\text{Th}(Y)$.

Lemma 3.5 *The external map $F : A_\varepsilon \rightarrow \mathbb{C}$ has neither attracting periodic points in \mathbb{S}^1 nor parabolic periodic points in \mathbb{S}^1 whose basins intersect \mathbb{S}^1 .*

Proof Assume the contrary: $z \in \mathbb{S}^1$ is an attracting fixed point of F^m or a parabolic point whose basin intersects \mathbb{S}^1 . Then there is a disk U such that $z \in \overline{U} \cap \mathbb{S}^1$ and $F^m(U) \subset U$. It follows that $P^m(V) \subset V$, where $V = \phi(U \cap \mathbb{D})$ and ϕ is as in Definition 3.4. In particular, a circle arc containing z is contained in \overline{U} . By the Denjoy-Wolff theorem (cf. Theorem 5.4 of [19]), all P^m orbits in V converge to a P^m -fixed point $y \in Y$. Then y is parabolic or attracting, and $V \subset W$ where W is a parabolic or attracting basin of y . In both cases, the parabolic or attracting basins of y are completely inside or completely outside $\text{Th}(Y)$ (because $Y \subset J_P$). Since points of V do not belong to $\text{Th}(Y)$ by construction, then $W \cap \text{Th}(Y) = \emptyset$. Thus, y is parabolic. The fact that $Y \subset J_P$ now implies that points of Y close to y are repelled from y under P^m . A contradiction with the fact that $\overline{V} \cap Y$ is mapped to \overline{V} under all iterates of P^m . □

We can now deduce that F is topologically expanding.

Corollary 3.6 *Let Y be an LFI continuum for P of degree $k > 1$. Then the external map $F : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ of $\text{Th}(Y)$ is topologically conjugate to the k -th tupling map $z \mapsto z^k$.*

Proof By Lemma 3.5, it follows from [3,16] (see also [17]) that for every arc $I \subset \mathbb{S}^1$ there exists a number $n > 0$ such that $F^n(I) \cap I \neq \emptyset$. This, in turn, implies (again together with Lemma 3.5) that $F|_{\mathbb{S}^1}$ is topologically conjugate to the map z^k with appropriate k (see, e.g., [20]). □

Let Y be an LFI continuum for P . Following Douady and Hubbard [10], say that P is *polynomial-like* (PL) near $\text{Th}(Y)$ if P maps some neighborhood of $\text{Th}(Y)$ onto a strictly bigger neighborhood as a branched covering. In this case, points of $\mathbb{C} \setminus \text{Th}(Y)$ near $\text{Th}(Y)$ are repelled from $\text{Th}(Y)$.

Definition 3.7 Say an LFI set Y contains an *outward* parabolic periodic point (or cycle) if Y contains a parabolic periodic point x such that the corresponding parabolic Fatou domains (whose points are attracted to the orbit of x) are otherwise disjoint from $\text{Th}(Y)$.

Theorem 3.8 is, basically, a folklore result.

Theorem 3.8 (Theorem B [4]) *If Y is an LFI continuum that contains no outward parabolic cycles, then P is PL near $\text{Th}(Y)$.*

Consider a sequence of sets E_n . Say that E_n *accumulate on* Y if, for every open neighborhood U of Y , we have $E_n \subset U$ for large n (how large may depend on U).

Proposition 3.9 *Consider an LFI set Y for P of degree > 1 . Let $x \in Y$ be a fixed point. Let $E \subset J_P$ be a continuum such that sets $E_n = P^n(E)$ share x with Y and are otherwise disjoint from Y . If x is not a cutpoint of Y , then E_n cannot accumulate in Y .*

Proof Assume the contrary: E_n accumulate on Y . Let ϕ be as in Definition 3.4. The sets $\phi^{-1}(E_n \setminus Y)$ accumulate in \mathbb{S}^1 . Define $T \subset \mathbb{S}^1$ as the set of points $z \in \mathbb{S}^1$ such that $\phi^{-1}(E_n \setminus Y)$ come arbitrarily close to z . Clearly, T is closed, connected and F -invariant, where $F : A_\varepsilon \rightarrow \mathbb{C}$ is the external map for $\text{Th}(Y)$.

If T is a singleton, then T attracts some nearby continua. Hence T is an attracting or parabolic fixed point of F . By Lemma 3.5 it must be a parabolic fixed point of F , and nearby continua must be contained in the corresponding parabolic domain. However then all points close to these continua must also converge to T under F while on the P -plane there are points close to $E_n \subset J_P$ that escape to infinity, a contradiction. It follows by Corollary 3.6 that $T = \mathbb{S}^1$. Consider a pair of different regular (pre)periodic points a, b in Y . If there are no outward parabolic points in Y , then the existence of a and b follows from the Douady–Hubbard straightening theorem, since P is PL near Y . On the other hand, if there are outward parabolic points in Y , then choose a and b among the pullbacks of these points. There are external K_P -rays R_a, R_b landing at a and b , respectively.

We claim that all $E_n \setminus Y$ lie in one complementary component W of $Y \cup R_a \cup R_b$. This follows from the fact that x is not a cutpoint of Y . Thus $\phi^{-1}(E_n \setminus Y) \subset \phi^{-1}(W)$. However, $\phi^{-1}(W)$ is bounded by two topological rays to \mathbb{S}^1 landing in \mathbb{S}^1 , and $\phi^{-1}(W) \cap \mathbb{S}^1$ is an arc that is a proper subset of \mathbb{S}^1 . On the other hand, this arc must include $T = \mathbb{S}^1$, a contradiction. □

4 Invariant Continua and Their Cutpoints

This section concludes the proof of the Main Theorem.

More standing notation. Assume that $Q \subset J$ is an invariant continuum, $x \in Q$ is a regular fixed non-rotational point of P . Denote by W the wedge between invariant K -rays R', R'' landing at x so that there are no K -rays in W landing at x . Assume that the movement within W from R'' to R' is in the clockwise direction. Finally, assume that $Q \cap W \neq \emptyset$.

We want to prove that $Q \cap W$ consists of just one component. The idea of the proof is based upon the dynamics of such components. To study it we need a concept of a *thread*. Namely, a connected set $E \subset W$ such that $\overline{E} = E \cup \{x\}$ is called a *thread*. A *thread of Q* is a thread that is a subset of Q . A thread of Q is contained in a component of $Q \setminus \{x\}$ contained in W ; by Lemma 2.6, each such component of $Q \setminus \{x\}$ is a thread of Q .

If E_1 and E_2 are disjoint threads of Q , then by Theorem 2.8 there is a topological ray T that separates E_1 from E_2 inside W ; write $E_1 < E_2$ if T separates E_2 from R' in W . Say that $E_1 < E_2$ *locally* if $E'_1 < E'_2$ for some threads $E'_1 \subset E_1$ and $E'_2 \subset E_2$.

If E is a thread of Q containing no preimages of x , then its P -image $P(E)$ is a thread too (e.g., this is the case if E is sufficiently small). The set $P(E)$ may also contain smaller threads. Let us study their location with respect to E . Recall that

threads are by definition subsets of W even though W is omitted from the notation and terminology. Also, recall that a *topological ray to Q* is a topological ray disjoint from Q but accumulating in Q (as the parameter on the ray tends to 0).

Lemma 4.1 *Let E' and E'' be threads of Q such that $E' < E''$ locally but $E' \cap E'' \neq \emptyset$. There are no threads F of Q disjoint from $E' \cup E''$ and such that $E' < F < E''$.*

Proof Let V be the complementary component of $\overline{E'} \cup \overline{E''}$ containing F . If V is unbounded, then there exists a $E' \cup E'' \cup F$ -ray T from a point of F to infinity. Evidently, $F \cup T$ separate E' from E'' in W , a contradiction with $E' \cap E'' \neq \emptyset$. Thus V is bounded. Since no point $x \in J$ belongs to V , we have a contradiction with $F \subset V$. □

The following lemma allows to find a “monotone” infinite sequence of disjoint threads.

Lemma 4.2 *Let $E < F$ be disjoint threads of Q such that $x \notin P(E)$ and $F \subset P(E)$. Then there exists a thread $T \subset E$ such that all its images $T < P(T) < \dots$ are pairwise disjoint threads that never contain a preimage of x .*

Proof Construct a null sequence of threads $E \supset E^1 \supset \dots$. For each i , choose n_i such that $E^i, P(E^i), \dots, P^{n_i-1}(E^i)$ are all pairwise disjoint threads not containing immediate preimages of x while $P^{n_i}(E^i)$ contains an immediate preimage of x (if for some j no image of E^j contains an immediate preimage of x , set $T = E^j$). By continuity and construction, $n_i \nearrow \infty$ as $i \rightarrow \infty$. Let $y \in P^{-1}(x) \cap W$. Suppose that $y \in P^{n_i}(E^i)$. Then by Lemma 4.1 we may assume that for $j > i$ no image of E^j contains y . Hence if j is sufficiently large, immediate preimages of x do not belong to images of E^j . Set $T = E^j$. □

Lemma 4.3 describes the dynamics of threads.

Lemma 4.3 *Suppose that E is a thread of Q . Then its image $P(E)$ cannot contain a thread disjoint from E .*

Proof If there is a thread E_1 contained in $P(E)$ and disjoint from E , and E_0 is a pullback of E_1 that is a thread in E , then $E_0 \cap E_1 = \emptyset$. Assume that $E_0 < E_1$ and consider possible location of sets $E_n = P^n(E_0)$; call them *E-sets* for brevity. By Lemma 4.2, we may assume that all E_n are disjoint threads of Q not containing preimages of x .

Let T be the *topological upper limit* of the sequence of sets E_n , i.e., the set of all limit points of sequences of points $z_i \in E_i, i = 1, 2, \dots$. Clearly, T is closed, $P(T) = T$, and $x \in T$. Note that T coincides with the union of all Hausdorff limits of sequences of sets E_{i_n} with $i_n \nearrow \infty$; also, all *E-sets* are disjoint from T by Lemma 4.1. Let Y be the LFI hull of T . Then Y is disjoint from the *E-sets* because if, say, $E_n \cap Y \neq \emptyset$, then, again by Lemma 4.1, the sets E_m cannot be contained in J for $m > n$.

By Proposition 3.9, the map $P : Y \rightarrow Y$ is one-to-one. Hence $P : T \rightarrow T$ is one-to-one. By Lemma 3.3, there are no critical points of P in $\text{Th}(Y)$. In particular, $\text{Th}(Y)$ cannot include any attracting or parabolic domains. By Theorem 2.15, there is

a P -fixed rotational point $z \in \text{Th}(Y)$. If z is a cutpoint of T , then all E_n with large n are in the same wedge bounded by external P -rays landing at z . A contradiction. Hence z is Siegel or Cremer. Let O be a neighborhood of $\text{Th}(Y)$ such that \overline{O} contains no critical points. Then the set of all points whose forward orbits stay forever in \overline{O} contains a component $\tilde{Y} \supset Y$. By [22,23], the set \tilde{Y} is a so-called *hedghehog* of z . It cannot contain fixed points other than z , a contradiction with $x \in Y$. \square

This implies the following corollary.

Corollary 4.4 *If F is a component of $Q \cap W$ then all small threads of F map back to F . Thus, if F contains no immediate preimages of x then $P(F) \subset F$. In particular, there are at most finitely many components of $Q \cap W$.*

Proof The first claim is immediate. It implies the second one. Since there are at most finitely many immediate preimages of x in W , the last claim follows. \square

We can now prove Lemma 4.5 which allows us to use Theorem 2.12.

Lemma 4.5 *If F is a component of $Q \cap W$, then all points of F sufficiently close to x map to F .*

Proof Suppose that $y \in F$ maps to a component $F' \neq F$ of $Q \cap W$. By Corollary 4.4, there are preimages x' of x that belong to F . Let $X' = P^{-1}(x) \cap F$; clearly, X' is finite. Choose a component F_y of $F \setminus X'$ that contains y . Then $P(F_y) \subset F'$. Hence $P(\overline{F_y}) \subset F' \cup x$. Now, by way of contradiction suppose that there is a sequence $y_i \in F$ converging to x such that $P(y_i) \notin F$. Passing to a subsequence and using that, by Corollary 4.4, there are only finitely many components of $Q \cap W$, we may assume that for some component F' of $Q \cap W$ and for some point $x' \in X'$ the sets $\overline{F_{y_i}}$ converge to a continuum $B \subset \overline{F}$ containing x and x' such that $P(B) \subset \overline{F'} = F' \cup \{x\}$. However this contradicts Lemma 4.3. Thus, all points of F sufficiently close to x map to F . \square

We are ready to prove Theorem 4.6.

Theorem 4.6 *The set $Q \cap W$ is connected.*

Proof Let F be a component of $Q \cap W$. Denote the union of all invariant K -rays and their landing points by Σ . Let A be the component of $\mathbb{C} \setminus \Sigma$ that contains points of F close to x . By Theorem 2.15, the set $\text{Th}(F)$ contains the rotational object of P in A . Since this applies to any component of $Q \cap W$, we conclude that there is just one component, i.e., that $F = Q \cap W$. \square

Evidently, Theorem 4.6 implies Main Theorem.

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