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RESEARCH CONTRIBUTION

An Extension of the $\mathfrak{sl}_2$ Weight System to Graphs with $n \leq 8$ Vertices

Evgeny Krasilnikov[1]

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Abstract
Chord diagrams and 4-term relations were introduced by Vassiliev in the late 1980. Various constructions of weight systems are known, and each of such constructions gives rise to a knot invariant. In particular, weight systems may be constructed from Lie algebras as well as from the so-called 4-invariants of graphs. A Chmutov–Lando theorem states that the value of the weight system constructed from the Lie algebra $\mathfrak{sl}_2$ on a chord diagram depends on the intersection graph of the diagram, rather than the diagram itself. This inspired a question due to Lando about whether it is possible to extend the weight system $\mathfrak{sl}_2$ to a graph invariant satisfying the four term relations for graphs. We show that for all graphs with up to 8 vertices such an extension exists and is unique, thus answering in affirmative to Lando's question for small graphs.

Keywords Chord diagram · Intersection graph · 4-term relations · Vassiliev invariants · Weight system

1 Introduction

Chord diagrams and 4-term relations appear naturally in studying knot invariants of finite type—the notion introduced by Vassiliev [12] in the late 1980s. Vassiliev's theory of finite-type knot invariants [12] describes these invariants in terms of weight systems, which are functions on chord diagrams satisfying so-called 4-term relations.

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✉ Evgeny Krasilnikov
evgeny12@mail.ru

1 International Laboratory of Cluster Geometry National Research University Higher School of Economics, Moscow, Russia

Kontsevich [8] proved that over a field of characteristic 0 this correspondence is one-to-one, that is, to each weight system a finite-type knot invariant is associated.

A weight system can be associated with any finite dimensional complex Lie algebra endowed with a nondegenerate invariant bilinear form [1,8]. For the simplest nontrivial case, that of the Lie algebra $\mathfrak{sl}_2$, the corresponding weight system comes from the colored Jones polynomial, a well-known knot invariant. This weight system takes values in the algebra of polynomials in one variable, the Casimir element of $\mathfrak{sl}_2$. Even in this simplest case, the weight system is not understood well, and computation of its value on a sufficiently big chord diagram is a hard task.

To each chord diagram, one associates a simple graph, its intersection graph. In [4], it is proved that the value of the $\mathfrak{sl}_2$ weight system on a chord diagram depends on the intersection graph of the diagram rather than the diagram itself (that is, if two chord diagrams have isomorphic intersection graphs, then the value of the $\mathfrak{sl}_2$ weight system on them is the same). Due to computational complexity, explicit formulas for the values of this weight system on graphs are known only for small graphs or for simple infinite families of graphs, like trees or cycles. Even in the case of complete graphs, a conjecture of Lando about the explicit form of the answer is proved only for the linear term of the polynomial [2].

On the other hand, in [10], a 4-term relation for simple graphs was introduced in such a way that the mapping taking a chord diagram to its intersection graph takes any 4-term relation for chord diagrams to a one for graphs. This construction naturally leads to the following question posed explicitly by Lando:

whether there exists a graph invariant satisfying 4*-term relations that coincides with the* $\mathfrak{sl}_2$ *weight system on intersection graphs*?

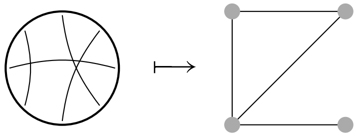
The goal of the present paper is to give an affirmative answer to this question for all graphs with up to 8 vertices. Moreover, we prove that for these graphs, such an extension is unique. Our main tool in computation of the $\mathfrak{sl}_2$ weight system is the Chmutov–Varchenko recurrence relation [5]. We start with computing these values on all intersection graphs with up to 8 vertices. Then we find, for each graph that is not an intersection graph, its expression as a linear combination of intersection graphs modulo 4-term relations. This gives us a presumable extension. Finally, we check that this extension indeed satisfies all 4-term relations for graphs. As a byproduct, we get a table of values of the $\mathfrak{sl}_2$-weight system on all graphs with up to 8 vertices.

Our results can be considered as a part of a bigger project whose goal is to find the answer to Lando's question. Other achievements in this direction include the proof in [9] of extendability to arbitrary graphs of the leading coefficient in the projection of the $\mathfrak{sl}_2$ weight system to primitives, and the computation [6,7] of the values of this weight system on an infinite family of graphs that are not intersection graphs under the assumption that such an extension exists.

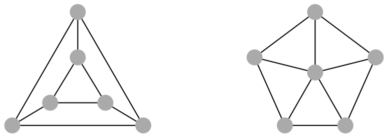
The paper is organized as follows. In Sect. 2, we recall the notions of chord diagrams, 4-term relations, weight systems. Our exposition follows [11], see also [3]. In Sect. 3, we describe the construction of weight systems from Lie algebras, the $\mathfrak{sl}_2$-weight system and the Chmutov–Varchenko relations for it. Sections 4 and 5 are devoted to a description of the computer algorithm we developed for extending the weight system to graphs that are not intersection graphs and the results of implementing this algorithm.

2 Chord Diagrams and Intersection Graphs

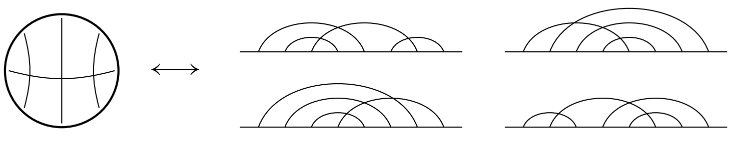
Chord diagrams are combinatorial objects, namely, oriented circles together with a collection of chords whose initial and terminal points are pairwise distinct, considered up to an orientation-preserving diffeomorphism of the circles. The *intersection graph* $\Gamma(A)$ of a chord diagram A is a graph whose vertices correspond to the chords of the diagram A, and there is an edge connecting two vertices, provided that the corresponding chords intersect. It is clear that to any chord diagram, there corresponds its intersection graph. An example of such correspondence is depicted below. In the pictures, we always assume that the circle is oriented counterclockwise.



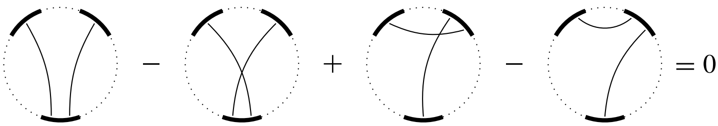
In contrast, not each graph is the intersection graph of a chord diagram. All graphs with 0 through 5 vertices are intersection graphs, whereas there exist two graphs with 6 vertices that are not intersection graphs, namely,



An *arc diagram* is a representation of chord diagram, in which the vertices of the chord diagram are placed along an oriented line with edges drawn as semicircles in one of the two halfplanes bounded by the line. Each of the arc diagrams corresponds to a single chord diagram. In contrast, a chord diagram with n chords admits up to $2n$ representations as an arc diagram. For example, to the following chord diagram four arc diagrams are associated.



Denote by $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \ldots$ the graded vector space of chord diagrams; the component of grading n is the vector space $\mathcal{C}_n$ generated by chord diagrams with n chords. *The four-term relation* on the vector space of chord diagrams is the equality:



Two chords in each of the diagrams depicted above are located as they appear in the figure. The other chords are located arbitrarily, but in the same way in each of the four diagrams shown above; their initial and terminal points connect the dashed arcs.

Denote by $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus ...$ the quotient of the vector space $\mathcal{C}$ by the 4-term relation. The multiplication m on the space $\mathcal{A}$ is defined by concatenating two arc diagrams corresponding to the factors and extending this operation by linearity. The multiplication is well-defined modulo the 4-term relation,

$$m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}.$$

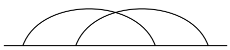
The multiplication respects the grading.

3 Weight System $\mathfrak{sl}_2$

Let H be a Lie algebra endowed with a nondegenerate invariant bilinear form $(\cdot, \cdot)$. Invariance means that $(x, [y, z]) = ([x, y], z)$ for all $x, y, z \in H$. Let $U(H)$ be an universal enveloping algebra of the algebra H.

Pick an orthonormal basis $\{e_1, \dots e_n\}$ with respect to the scalar product $(\cdot, \cdot)$. Consider the mapping $w : \mathcal{A} \to U(H)$ of the algebra of chord diagrams modulo 4-term relations to the universal enveloping algebra of H. For a given chord diagram d and an arc representation a of d we construct an element of $U(H)$ as follows. For a given mapping of the set of arcs of the diagram a to the set $\{1, \dots, m\}$, at the ends of each arc, we place an element $e_i \in H$ if this arc goes to i. The summation over all such mappings gives us the image of the chord diagram d in $U(H)$. We extend this mapping to the whole $\mathcal{A}$ by linearity.

For example, for $m = 3$:



$$\longmapsto \quad e_1^4 + e_1e_2e_1e_2 + e_1e_3e_1e_3 + e_2e_1e_2e_1 + e_2^4 + \\ + e_2e_3e_2e_3 + e_3e_1e_3e_1 + e_3e_2e_3e_2 + e_3^4$$

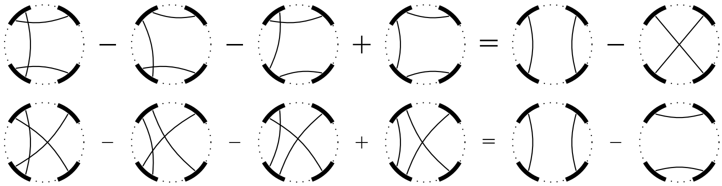
Theorem 1 [1,8] *Let H be a Lie algebra together with the nondegenerate invariant scalar product $(\cdot, \cdot)$. Then the mapping $w : \mathcal{A} \to U(H)$ possesses the following properties:*

(1) *w does not depend on the choice of the orthonormal basis $e_1, \dots, e_m$;*
(2) *w does not depend on choosing an arc representation of chord diagram d;*
(3) *the image of w lies in the center of $U(H)$;*
(4) *w satisfies the 4-term relation for chord diagrams.*

Note that if a chord diagram is the product of two nonempty diagrams, then the value of the map w on it is the product of its values on the factors, so that w is an algebra homomorphism.

In the simplest nontrivial case, namely, in the case of the Lie algebra $\mathfrak{sl}_2$ and the Killing form, the center of the universal enveloping algebra $U(\mathfrak{sl}_2)$ is generated by a single element, the quadratic Casimir $c = e_1^2 + e_2^2 + e_3^2$. In this particular case, the function is determined by the Chmutov–Varchenko recurrence relations, which can be considered as an alternative definition of the weight system $\mathfrak{sl}_2$.

Let v denote the weight system $\mathfrak{sl}_2$. It associates to a chord diagram with n chords a polynomial of degree n in the variable c. The value of v on a chord diagram with one chord equals c. If a chord diagram contains a chord that intersects precisely one other chord, in which case we call the former chord a *leaf*, then the value of v on the initial chord diagram is equal to that on the chord diagram obtained from the initial one by deleting the leaf times $(c-1)$. If a chord diagram contains no leaves, then it contains a triple of chords of either of the leftmost diagrams below, and the Chmutov–Varchenko recurrence relations for the values of v on it hold:



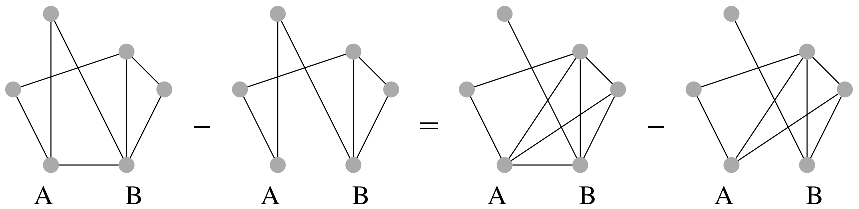
By means of these relations, the value of v on any chord diagram can be computed recursively. However, complexity of such a computation is exponential: at each step, the diagram is replaced with 5 simpler diagrams.

Theorem 2 [5]

(1) *The function v defined by the recurrence relations above is well-defined.*
(2) *The function v coincides with the weight system* $\mathfrak{sl}_2$ *constructed from the Lie algebra* $\mathfrak{sl}_2$.

4 Extending the Weight System $\mathfrak{sl}_2$ to Arbitrary Graphs

In [10] the *four-term relation for graphs* is introduced.



Choose an arbitrary edge, say AB, of the graph—it is the first graph on the left-hand side in the relation above. The second graph on the left-hand side is the same graph with the AB deleted. We now describe how the graphs on the right-hand side are obtained. Consider the set of edges (excluding AB) sharing a common vertex B. Denote them by $BC_1, \ldots, BC_n$. Now if the vertices A and C_i are connected by an edge in the initial graph, we delete this edge; otherwise—if the vertices A and C_i lack an edge connecting them—we add this edge. In this way, the first term on the right-hand side is obtained. The second term differs from the first one in that it lacks the edge AB.

For four chord diagrams forming a 4-term relation, the corresponding intersection graphs form a 4-term relation for graphs. In contrast to, say, contraction-deletion relations, 4-term relations do not simplify graphs (in any known sense), whence cannot be used for recursive computation of a graph invariant satisfying them.

The following assertion allows one to define the value of the weight system $\mathfrak{sl}_2$ on intersection graphs.

Theorem 3 [4] *The value of the weight system* $\mathfrak{sl}_2$ *on a chord diagram is determined by the intersection graph of the diagram.*

In other words, two different chord diagrams with isomorphic intersection graphs have the same value of the weight system $\mathfrak{sl}_2$.

Here is the main result of the present paper.

Theorem 4 *There is a polynomial graph invariant of graphs with up to* 8 *vertices satisfying the 4-term relations whose values on intersection graphs coincide with that of the weight system* $\mathfrak{sl}_2$*; such a graph invariant is unique.*

5 Algorithms and Computational Results

Theorem 4 follows from a computer computation. The numbers of chord diagrams, graphs and intersection graphs for small orders are given in the table below, see, e.g., [3].

Grading	# Chord diagrams	# Graphs	# Intersection graphs
1	1	1	1
2	2	2	2
3	5	4	4
4	18	11	11
5	105	34	34
6	902	156	154
7	9749	1044	978
8	127,072	12,346	9497
9	1,915,951	274,668	127,954

To prove Theorem 4 we compute the values of the weight system $\mathfrak{sl}_2$ on graphs with up to 8 vertices and check that the computed values satisfy all the 4-term relations.

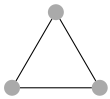
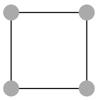
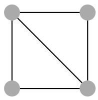
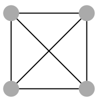
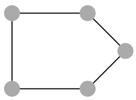
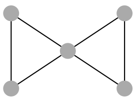
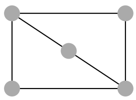
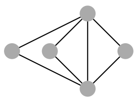
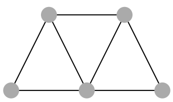
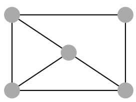
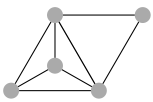
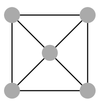
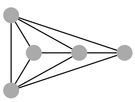
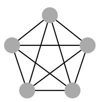
The first step is the recursive computation of the values of the weight system $\mathfrak{sl}_2$ on chord diagrams. We order the diagrams of a given degree in such a way that the number of chord intersections (that is, the number of edges in the intersection graph) is nondecreasing with respect to the chosen order. The value on a chord diagram that is a product of two smaller diagrams is computed as the product of the values on the factors. The value on a chord diagram d with a leaf is $(c-1)$ times the value on the result of deleting the leaf in d. We repeat the removal of leaves until no leaves remain. Then, we use one of the two Chmutov–Varchenko relations and replace the initial diagram with 5 simpler diagrams, the values on which we already know.

For each chord diagram, we construct the corresponding intersection graph. According to Theorem 3, the values of the weight system $\mathfrak{sl}_2$ on chord diagrams with isomorphic intersection graphs are equal, and we obtain a table of the values of the weight system $\mathfrak{sl}_2$ on intersection graphs.

Denote by

- G_n the set of graphs with n vertices;
- $G_n^{(0)}$ the set of intersection graphs with n vertices;
- $G_n^{(1)}$ the set of graphs in G_n that are not intersection graphs that admit a 4-term relation expressing them as a linear combination of intersection graphs;
- $G_n^{(k)}$, $k > 1$ the set of graphs in G_n that admit a 4-term relation expressing them as a linear combination of graphs in $G_8^{(0)} \sqcup G_8^{(1)} \sqcup \cdots \sqcup G_8^{(k-1)}$, but not belong to the latter union.

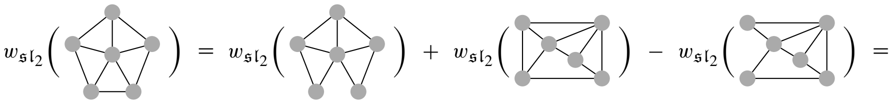
For $n = 1, 2, 3, 4, 5$, all the graphs in G_n are intersection graphs, $G_n = G_n^{(0)}$, and no additional computations are required. Here are a few examples—the values of the weight system $\mathfrak{sl}_2$ for all connected graphs without leaves up to 5 vertices.

Graph	$w_{\mathfrak{sl}_2}$	Graph	$w_{\mathfrak{sl}_2}$
	c		$c^3 - 3c^2 + 2c$
	$c^4 - 4c^3 + 8c^2 - 4c$		$c^4 - 5c^3 + 10c^2 - 5c$
	$c^4 - 6c^3 + 13c^2 - 7c$		$c^5 - 5c^4 + 10c^3 - 13c^2 + 6c$
	$c^5 - 6c^4 + 13c^3 - 12c^2 + 4c$		$c^5 - 6c^4 + 21c^3 - 30c^2 + 13c$
	$c^5 - 6c^4 + 16c^3 - 21c^2 + 9c$		$c^5 - 7c^4 + 24c^3 - 33c^2 + 14c$
	$c^5 - 7c^4 + 22c^3 - 29c^2 + 12c$		$c^5 - 7c^4 + 25c^3 - 38c^2 + 17c$
	$c^5 - 8c^4 + 29c^3 - 43c^2 + 19c$		$c^5 - 8c^4 + 34c^3 - 56c^2 + 26c$
	$c^5 - 9c^4 + 39c^3 - 66c^2 + 31c$		$c^5 - 10c^4 + 45c^3 - 79c^2 + 38c$

Still, one have to check that the graph invariant thus defined satisfies all the 4-term relations.

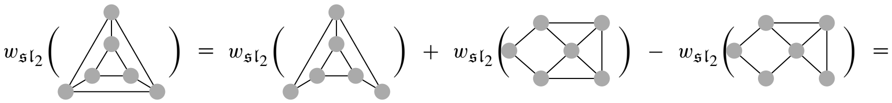
For $n = 6$, the set $G_n^{(1)}$ consists of two graphs that are not intersection graphs. The values of the weight system $\mathfrak{sl}_2$ on these graphs were obtained by Netrusova, see e.g. [9].

For the two graphs below, any 4-term relation expresses them as a linear combination of intersection graphs; all such decompositions lead to the same value of the weight system $\mathfrak{sl}_2$.



$$c^6 - 9c^5 + 38c^4 - 83c^3 + 83c^2 - 29c \qquad c^6 - 10c^5 + 53c^4 - 155c^3 + 205c^2 - 86c \qquad c^6 - 9c^5 + 41c^4 - 99c^3 + 112c^2 - 43c$$

$$= c^6 - 10c^5 + 50c^4 - 139c^3 + 176c^2 - 72c$$



$$c^6 - 8c^5 + 31c^4 - 73c^3 + 87c^2 - 35c \qquad c^6 - 9c^5 + 39c^4 - 97c^3 + 121c^2 - 50c \qquad c^6 - 8c^5 + 30c^4 - 62c^3 + 62c^2 - 22c$$

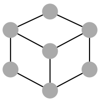
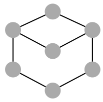
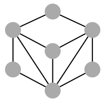
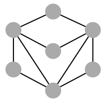
$$= c^6 - 9c^5 + 40c^4 - 108c^3 + 146c^2 - 63c$$

The cases of graphs on 7 and 8 vertices are more complicated. Computational iterations correspond to the following statement.

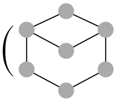
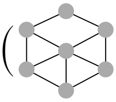
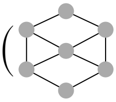
Proposition 1 *The set* G_n, $n = 4, 5, 6, 7, 8$, *consists of the sets with the following cardinalities*

n	# G_n	# $G_n^{(0)}$	# $G_n^{(1)}$	# $G_n^{(2)}$	# $G_n^{(3)}$	# $G_n^{(4)}$	# $G_n^{(5)}$	# $G_n^{(6)}$	# $G_n \setminus \{\cup G_n^{(k)}\}$
4	11	11	0	0	0	0	0	0	0
5	34	34	0	0	0	0	0	0	0
6	156	154	2	0	0	0	0	0	0
7	1044	978	51	15	0	0	0	0	0
8	12,346	9497	985	665	606	131	4	0	458

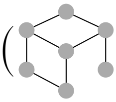
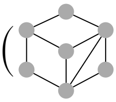
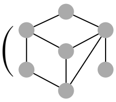
In the case of 7 vertices, we choose such a 4-term relation in order to calculate the values of the weight system $\mathfrak{sl}_2$ for each of the 51 graphs in $G_7^{(1)}$ and then the rest 15 graphs in $G_7^{(2)}$. The final step consists in verification that the obtained extension satisfies all the 4-term relations. For example, the four possible 4-term representations for the following graph in $G_7^{(2)}$ lead to the same result:

$w_{\mathfrak{sl}_2}\Big($  $\Big) = w_{\mathfrak{sl}_2}\Big($  $\Big) + w_{\mathfrak{sl}_2}\Big($  $\Big) - w_{\mathfrak{sl}_2}\Big($  $\Big) =$

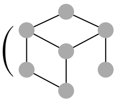
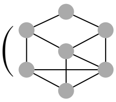
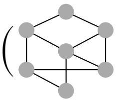
$c^7 - 8c^6 + 30c^5 - 60c^4 + 81c^3 - 76c^2 + 29c$; $c^7 - 11c^6 + 61c^5 - 196c^4 + 357c^3 - 324c^2 + 108c$; $c^7 - 10c^6 + 49c^5 - 138c^4 + 231c^3 - 204c^2 + 68c$

$= w_{\mathfrak{sl}_2}\Big($  $\Big) + w_{\mathfrak{sl}_2}\Big($  $\Big) - w_{\mathfrak{sl}_2}\Big($  $\Big) =$

$c^7 - 8c^6 + 30c^5 - 60c^4 + 81c^3 - 76c^2 + 29c$; $c^7 - 11c^6 + 59c^5 - 184c^4 + 350c^3 - 354c^2 + 130c$; $c^7 - 10c^6 + 47c^5 - 126c^4 + 224c^3 - 234c^2 + 90c$

$= w_{\mathfrak{sl}_2}\Big($  $\Big) + w_{\mathfrak{sl}_2}\Big($  $\Big) - w_{\mathfrak{sl}_2}\Big($  $\Big) =$

$c^7 - 8c^6 + 32c^5 - 74c^4 + 97c^3 - 65c^2 + 17c$; $c^7 - 10c^6 + 49c^5 - 148c^4 + 270c^3 - 253c^2 + 87c$; $c^7 - 9c^6 + 39c^5 - 104c^4 + 160c^3 - 122c^2 + 35c$

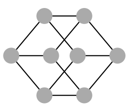
$= w_{\mathfrak{sl}_2}\Big($  $\Big) + w_{\mathfrak{sl}_2}\Big($  $\Big) - w_{\mathfrak{sl}_2}\Big($  $\Big) =$

$c^7 - 8c^6 + 32c^5 - 74c^4 + 97c^3 - 65c^2 + 17c$; $c^7 - 11c^6 + 62c^5 - 212c^4 + 436c^3 - 460c^2 + 172c$; $c^7 - 10c^6 + 52c^5 - 168c^4 + 326c^3 - 329c^2 + 120c$

$$= \quad c^7 - 9c^6 + 42c^5 - 118c^4 + 207c^3 - 196c^2 + 69c$$

In the case of 8 vertices, we also successively construct the sets $G_8^{(0)}, G_8^{(1)}, \dots, G_8^{(5)}$. The union $G_8^{(0)} \sqcup G_8^{(1)} \sqcup G_8^{(2)} \sqcup G_8^{(3)} \sqcup G_8^{(4)} \sqcup G_8^{(5)}$ contains 11888 graphs. To compute the value of $w_{\mathfrak{sl}_2}$ on the rest 458 graphs, we constructed a system of linear equations, one for each pair consisting of such a graph and a 4-term relation for it. It happened that the system, although overdetermined, possesses a unique solution. The final step consists in checking that the computed values satisfy all the 4-term relations for G_8, which completes the proof of Theorem 4.

Analysis of the results of the computations led to a discovery of a new phenomenon. In contrast to the case of smaller number of vertices, it happened that not all the coefficient of the extended polynomial invariant are integers. For example,

$w_{\mathfrak{sl}_2}\Bigg($  $\Bigg) = c^8 - 12c^7 + 78c^6 - 332c^5 + 1025c^4 - 2004c^3 + \frac{4001}{2}c^2 - 717c$

There are 458 graphs on 8 vertices with fractional coefficients; all these coefficients are half-integers and of monomial c^2. This phenomenon shows that even if the answer to Lando's question is positive, one cannot expect that a resulting graph invariant satisfies a simple contraction–deletion-type recursive relation. Still, however, the coefficients of the extended invariant have alternating signs.

In the case $n = 9$, there are 274668 graphs, 127954 of which are intersection graphs. The computation of the weight system $\mathfrak{sl}_2$ on graphs with 9 vertices is out of reach for computer algorithms known to the author.

6 Computer Data

The following files with the computer data are available at link https://drive.google.com/drive/folders/1bINcS2ZWYw6gCVpANsrfRXZgvdEO_GYJ

The file "AllGraphsSl2.txt" contains the list of all graphs with 4 to 8 vertices as well as the values of the weight system $\mathfrak{sl}_2$ on each of them.

The file "FourRelationsList.txt" contains the list of all 4-term relations for these graphs.

The file "Computations.nb" is the Wolfram Mathematica computer program demonstrating sample computations that confirm the results of the previous section.

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