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RESEARCH CONTRIBUTION

Equivalence of Neighborhoods of Embedded Compact Complex Manifolds and Higher Codimension Foliations

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Abstract

We consider an embedded n-dimensional compact complex manifold in $n+d$ dimensional complex manifolds. We are interested in the holomorphic classification of neighborhoods as part of Grauert's formal principle program. We will give conditions ensuring that a neighborhood of C_n in M_{n+d} is biholomorphic to a neighborhood of the zero section of its normal bundle. This extends Arnold's result about neighborhoods of a complex torus in a surface. We also prove the existence of a holomorphic foliation in M_{n+d} having C_n as a compact leaf, extending Ueda's theory to the high codimension case. Both problems appear as a kind of linearization problems involving *small divisors condition* arising from solutions to their cohomological equations.

Keywords Neighborhood of a complex manifold · Normal bundle · Solution of cohomological equations with bounds · Holomorphic extension · Holomorphic linearization · Resonances · Small divisors condition · Holomorphic foliations

Mathematics Subject Classification 32Q57 · 32L30 · 32L10 · 37F50 · 58F36

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Contents

1 Introduction

We are interested in the classification of the germs of neighborhood of an embedded compact complex manifold C in a complex manifold M. Here, two germs (M, C) and $(\tilde{M}, C)$ are holomorphically equivalent if there is a biholomorphic mapping F fixing C pointwise and sending a neighborhood V of C in M into a neighborhood $\tilde{V}$ of C in $\tilde{M}$. These considerations can be useful to extend holomorphic objects such as cohomology classes of holomorphic sections of bundles over C or functions on C to a neighborhood of C in M. Indeed, it might be that such an extension problem is much easy to solve on an equivalent neighborhood. We are also interested in the existence of a non-singular holomorphic foliation of the germ of neighborhood of C in a complex manifold having C as a compact leaf. We refer to it as a "horizontal foliation".

A neighborhood V of an embedded complex manifold C_n in M_{n+d} has local holomorphic charts $(h_j, v_j) = \Phi_j$ mapping V_j onto $\hat{V}_j$ in $\mathbf{C}^{n+d}$ with $n = \dim C$. Here $\cup V_j$ is a neighborhood of C and $U_j := V_j \cap C$ is defined by $v_j = 0$. The above-mentioned classification of the germs of neighborhoods of C is then the classification of transition functions $\Phi_{kj} := \Phi_k \Phi_j^{-1}$ under holomorphic conjugacy $F_k^{-1} \Phi_{kj} F_j$. To such an embedding, one can associate the normal bundle $N_C(M)$ of C in M, which has the transition matrices $g_{kj}(p)$, $p \in U_k \cap U_j$. To this embedding one can associate another natural embedding, namely the embedding of C as the zero section of $N_C(M)$. Under a mild assumption, this last embedding $(N_C(M), C)$ naturally serves as a first order approximation of (M, C). Let $\varphi_j = \Phi_j|_{U_j}$ and let $\varphi_{kj} = \varphi_k\varphi_j^{-1}$ be the transition functions of C. To have a neighborhood of C in M equivalent to a neighborhood of the zero section in $N_C(M)$ is equivalent to seeking F_j such that $\hat{\Phi}_{kj} = F_k^{-1}\Phi_{kj}F_j$ are of the form $\hat{\Phi}_{kj}(h_j, v_j) = (\varphi_{kj}(h_j), t_{kj}(h_j)v_j)$ with $t_{kj}(h_j) = g_{kj}$, the latter being regarded as the transition functions of a neighborhood of the zero section of $N_C(M)$. We call this process a "full linearization" of the neighborhood. The above-mentioned "horizontal foliation" will be obtained as a consequence of a "vertical linearization" of the neighborhood which amounts to seeking F_j such that $\hat{\Phi}_{kj} = (\varphi_{kj}(h_j) + \hat{\phi}^h_{kj}(h_j, v_j), t_{kj}(h_j)v_j)$.

Without even considering holomorphic equivalence problem, it is known that there are formal obstructions to linearizing [16,34] or to linearizing vertically [42] a neighborhood; see Sect. 2. Part of the Grauert formal principle [6,13,18,29] is to seek geometry conditions that ensure a holomorphic linearization when the formal obstructions are absent. In this paper, we will obtain a linearization of a neighborhood of an embedded compact complex manifold C_n at the absence of formal obstructions under *small divisor conditions* in the form of bounds of solutions of cohomology equations involving all symmetric powers of N_C^*, the dual of the normal bundle N_C of C_n in M_{n+d}. Because of the very nonlinear nature of the problem, we need to work with a *family of nested domains* on which we solve and eventually bound the solutions of 1-cohomological equations. Indeed, we are naturally led to dealing with shrinking of the domains as we need to get estimates of derivatives of sections (by Cauchy estimates for instance). To be more precise, assume that a 1-cocycle f with value in the sheaf of sections of holomorphic bundle (involving symmetric power $S^mN_C^*$ for some $m \geq 2$) on C vanishes in the 1st cohomology group over a covering $\mathcal{W}$. Then there is a 0-cochain w over $\mathcal{W}$ such that $\delta w = f$. Nevertheless, we need to prove the existence of a (possibly different) solution u satisfying the linear equation $\delta u = f$ and a "linear" estimate of the form $\|u\|_{\mathcal{W}} \leq K\|f\|_{\mathcal{W}}$ (the norm is either L^2 or the sup-norm). Because of the nonlinear nature of our problem, we need to solve the linear equation iteratively and estimate solutions of the form $\delta u_m = F_m(f_2, \ldots, f_m, u_2, \ldots, u_{m-1})$, $m \geq 2$. Here $F_m(f_2, \ldots, f_m, u_2, \ldots, u_{m-1})$ is a nonlinear function and vanishes in a first cohomology group. Therefore, the bound K, depending on m, will compound, which leads to a problem on non-linear estimates. Here come some of the main issues : we need that, at the limit, the sequence of nested domains, over which the solutions are estimated iteratively, remains to cover the manifold. And we need to control the growth of the bound K with respect to m, that gives rise to the so-called *small divisors condition*. Therefore, the existence of any bound K for linear solutions u *without* shrinking the covering $\mathcal{W}$ is a basic question. The latter was solved affirmatively by Kodaira–Spencer [26, eq. (9), p. 499] for the case of line bundles for a general covering. For higher rank vector bundles, we provide a positive solution in the following result :

Proposition 1.1 *Let C be a compact complex manifold. There exists a family of coverings $\mathcal{U}^r = \{U_j^r\}$ of C with $r_* \leq r < r^*$ and $U_j^r = \varphi_j^{-1}(\Delta_r^n)$ via a holomorphic* *coordinate map φ_j mapping $U_j^{r^*}$ onto the polydisc $\Delta^n_{r^*}$ such that for any holomorphic vector bundle E over C, and each $f \in C^1(\mathcal{U}^{r'}, E)$, the space of 1-cochains on $\mathcal{U}^{r'}$ of holomorphic sections with values in E, satisfying $f = \delta u_0$ for some $u_0 \in C^0(\mathcal{U}^{r'}, E)$, there exist $u \in C^0(\mathcal{U}^{r'}, E)$ and $v \in C^0(\mathcal{U}^{r''}, E)$ such that $\delta u = f$ and $\delta v = f$, and*

$$|u|_{r'} \leq K(E)|f|_{r'}, \tag{1.1}$$

$$|v|_{r''} \leq \frac{D(E)}{(r' - r'')^{\tau}}|f|_{r'}. \tag{1.2}$$

Here r', r'' are any numbers satisfying $r_* < r'' < r' \leq \tilde{r} < r^*$ and $r' - r'' \leq r^* - \tilde{r}$, and τ, $K(E)$, $D(E)$ are independent of r', r''.

Here, we have used the sup-norm (or L^2-norm) of cochains of holomorphic sections of bundles (see Sect. A.2 for specific notations). We do not know if $K(E)$ and $D(E)$ are comparable when they are applied to the symmetric powers of N_C^* except when N_C is unitary. We note that Hörmander [20,35] obtained solutions with bounds for cohomology groups with respect to the $\bar{\partial}$ operator acting on the sheaf of (p, q)-forms with L^2 coefficients on bounded pseudoconvex domains in $\mathbf{C}^n$.

The estimate (1.2) was proved by Donin [9] for a special family of coverings by the L^2 theory. He also raised the question if estimate (1.1) exists, which is the basic question mentioned above. Proposition 1.1 gives us a more flexible kind of results and ultimately an estimate that holds without any shrinking for higher rank vector bundles via the above mentioned nested coverings. We also use the L^2-theory. We first obtain (1.2) by Theorem A.9. Then (1.1) is obtained by Lemma A.2. The constant $K(E)$ is defined for the kind of bundles we need in Definition A.5. This is summarized in Theorem A.12. The main results of this paper are based on the existence of *nested finite coverings* proved in subsection A.5.

Proposition 1.1 will be a useful tool in this paper. We now formulate our main results. We say that $T_CM = TM|_C$ *splits* if $T_CM = TC \oplus N_C$ holomorphically. For instance, T_CE splits for any holomorphic vector bundle E over C, where T_CE is the restriction of TE to its zero section identified with C. Here and in the sequel, we identify C with the zero section of E. We say that N_C is *flat* if the transition matrices of N_C are locally constant. We say that N_C is *unitary* if its transition matrices are unitary. Note that the maximum principle implies that a unitary N_C is flat; see a proof following Definiton 2.2. We have the following "vertical linearization" result:

Theorem 1.2 *Let C_n be a compact submanifold of M_{n+d} with splitting T_CM and unitary N_C. Let $\eta_0 = 1$ and*

$$\eta_m := K(N_C \otimes S^m(N_C^*)) \max_{m_1+\cdots+m_p+s=m} \eta_{m_1} \cdots \eta_{m_p},$$

where the maximum is taken in $1 \leq m_i < m$ for all i and $s \in \mathbb{N}$. Assume that there are positive constants L, L_0 such that

$$\eta_m \leq L_0 L^m, \quad m = 1, 2 \ldots.$$

Assume that $H^0(C, N_C \otimes S^\ell(N_C^*)) = 0$ for all $\ell > 1$. Assume that either $H^1(\mathcal{U}, N_C \otimes S^\ell(N_C^*)) = 0$ for all $\ell > 1$ or a neighborhood of C is formally vertically linearizable by a formal holomorphic mapping that is tangent to the identity (see Definition 2.5). Then the embedding is actually vertically linearizable by a holomorphic mapping that is tangent to the identity.

When C is a compact holomorphic curve embedded in a complex surface M with a unitary normal bundle N_C, the above vertical linearization is one of main results in Ueda [42] where $H^0(C, N_C \otimes S^\ell(N_C^*)) = 0$ for all $\ell > 1$ follows from his small-divisor condition. This has been generalized by Koike in higher codimension case under a strong assumption that N_C is a direct sum of unitary line bundles [27,28]; see also the direct sum condition stated in [28, Lemma 3.4 and Remark 3.5]. The Ueda theory for codimension-one foliations has also been extended by Claudon–Loray–Pereira–Touzet [7] and Loray–Thom–Touzet [31]. We remark that Theorem 1.2 via the flatness of N_C ensures the existence of a "horizontal" foliation :

Corollary 1.3 *Under assumptions of Theorem 1.2, there exists a neighborhood of C_n in M_{n+d} that admits an n-dimensional smooth holomorphic foliation having C_n as a leaf.*

The following results can be understood in the context of the Grauert formal principle for rigidity: If (M, C) is formal equivalent to (N_C, C), then they are holomorphically equivalent under suitable assumptions. We first consider the unitary case.

Theorem 1.4 *Let C_n be a compact submanifold of M_{n+d}. Suppose that N_C is unitary. Let $\eta_0 = 1$ and*

$$\eta_m := \max\left(K(N_C \otimes S^m(N_C^*)), K(TC \otimes S^m(N_C^*))\right) \max_{m_1+\cdots+m_p+s=m} \eta_{m_1} \cdots \eta_{m_p},$$

where the maximum is taken in $1 \le m_i < m$ for all i and $s \in \mathbb{N}$. Assume that there are positive constants L, L_0 such that

$$\eta_m \le L_0 L^m, \quad m = 1, 2 \dots. \tag{1.3}$$

If $T_C M$ splits and $H^1(\mathcal{U}, T_C M \otimes S^\ell(N_C^*)) = 0$ for all $\ell > 1$ or more generally if a neighborhood of C in M is linearizable by a formal holomorphic mapping which is tangent to the identity, then there exists a neighborhood of C in M which is equivalent to a neighborhood of C (i.e the 0th section) in N_C by a holomorphic mapping that is tangent to the identity. In that case, we say that the embedding $C \hookrightarrow M$ is holomorphically linearizable.

More generally, the following result treats more general cases, including the case where N_C is not necessarily flat.

Theorem 1.5 *Let C_n be a compact submanifold of M_{n+d}. Suppose that*

$$\sum_{k \ge 1} \frac{\log D_*(2^{k+1})}{2^k} < +\infty, \tag{1.4}$$

where $D_*(2^{k+1})$ is defined by (5.27). Suppose that either $H^0(C, TC \otimes S^\ell(N_C^*)) = 0$ for all $\ell > 1$, or N_C is flat. Assume further that either $T_C M$ splits and $H^1(\mathcal{U}, T_C M \otimes S^\ell(N_C^*)) = 0$ for all $\ell > 1$ or (M, C) and (N_C, C) are equivalent by a formal holomorphic mapping which is tangent to the identity. Then (M, C) and (N_C, C) are actually holomorphically equivalent.

The previous results can be seen as “full linearization” results. Theorem 1.4 is proved using a majorant method while Theorem 1.5 is based on a Newton scheme. It is not clear how to compare the two ”small divisors conditions” (1.3) and (1.4) although the counterparts in theory of dynamical systems are equivalent [4,5].

Let us mention a few results for the above-mentioned Grauert formal principle. The formal principle holds in the following cases: (a) negative N_C in the sense of Grauert, by results of Grauert [13] and Hironaka-Rossi [18]. In Grauert’s case, C_n has a system of strictly pseudoconvex neighborhoods and consequently C_n is the only compact n-submanifold near C_n. In the same spirit, Savelev proved that all neighborhoods of embeddings of $\mathbb{P}^1$ in complex surfaces with a unitary flat normal bundle are holomorphically equivalent [38]. (b) sufficiently positive N_C and $\dim C > 2$, by a result of Griffiths [16, Thm II (i)] showing that a neighborhood is determined by a finite-order neighborhood. In other words, under this condition the holomorphic classification of neighborhoods is “finitely determined”. (c) $H^1(C, N_C) = 0$ and the case that for each $x \in C$ there is $x' \in C$ such that the fiber of N_C at x is generated by global sections of N_C vanishing at x', by a result of Hirschowitz (see [19] for more general results)[1]. (d) 1-positive N_C, by a result of Commichau-Grauert [8].

We should remark that the above “full linearization” result was obtained by Arnol’d when C is an elliptic curve and M is a surface, where the vanishing of $H^0(X, T_C M \otimes S^\ell M)$ follows from the non-vanishing of “small divisors” [2,3]. Ilyashenko and Pyartli [23] proved an analogous result for special embeddings of the product flat tori under a strong assumption that N_C is a direct sum of flat line bundles. We emphasize that in our Theorem 1.5, for general compact manifolds C_n, we impose the vanishing of $H^0(X, T_C M \otimes S^\ell M)$ for all integers $\ell \geq 2$ whereas there is no restriction on H^0 when C has affine transition functions for coordinate charts and N_C is flat.

As a simple consequence, we have the following

Corollary 1.6 *Under assumptions of Theorem 1.5 on C and M, any holomorphic section of a holomorphic vector bundle E over C extends to a holomorphic section of a holomorphic-vector-bundle extension of E over a neighborhood of C in M.*

Corollary 1.7 *Let C be a compact complex manifold. Let (M, C) be equivalent to $(C \times \mathbf{C}^d, C)$ by a formal holomorphic mapping which is tangent to the identity. Suppose that the small-divisor condition in Theorem 1.5 is satisfied. Then (M, C) is holomorphically equivalent to $(C \times \mathbf{C}^d, C)$.*

We now give an outline of the paper.

[1] Recently, Jun-Muk Hwang proved instances of Hirschowitz’s conjecture on the Formal Principle [22]. The authors thank Takeo Ohsawa for acknowledging this work.

In Sect. 2, we study the formal obstructions to the full linearization and vertical linearization problems. The formal obstructions are known from work of Nirenberg-Spencer [34], Griffiths [16], Morrow–Rossi [32], for the the full linearization problem and by Ueda [42] (see also Neeman [33] and among others) for the vertical linearization problem. The obstructions are described in $H^1(C, E \otimes S^\ell N_C^*)$ for a natural vector bundle E that is either $T_C M$ or N_C. In this paper we emphasize the role of $H^0(C, T_C M \otimes S^\ell N_C^*)$. In local dynamical systems, the elements in the analogous group appear as finite symmetries in the Ecalle–Voronin theory [1] and centralizers for the linearizations [12]. The small divisors in local dynamics emerge in the form of the bounds $K(N_C \otimes S^\ell N_C^*)$ and $D(T_C M \otimes S^\ell N_C^*)$ in Proposition 1.1. In work of Arnol'd [2] and Ueda [42], the vanishing condition of the corresponding zero-th cohomology groups is not explicit; however, it follows from their small-divisor conditions.

In Sect. 3, we prove Theorem 1.2 by using Ueda's majorization method [42]. In our case the majorization relies on an important tool of the (modified) Fischer norm which is invariant under a unitary change of coordinates. The invariance allows us to overcome the main difficulty in our majorization proof to deal with the transition functions of N_C^* when they are unitary, but not necessarily diagonal. The (modified) Fischer norms have also been useful in other convergence proofs [24,30,40]; see also Koike [28] for a recent use of Fischer norms in the diagonal case. In Sect. 4, we also extend the majorant method to the full linearization problem for the special case where N_C is unitary. In Sect. 5, we obtain the full linearization in the general case by introducing a Newton scheme, i.e. a rapid convergence scheme as in Brjuno's work [4,5]; see also [37,41]. However, we must cope with the domains of transition functions which are not so regular. These domains, when carefully chosen, have nevertheless a disc structure. This allows us to obtain a proof using sup-norm estimates.

Finally, the paper contains an appendix which has interests in its own right. It has two results, namely the existence of the two bounds stated in Proposition 1.1 and the existence of nested coverings (see Definition A.1). The existence of bound $K(E)$ was employed by Ueda [42] through the complete system of Kodaira–Spencer [26] when $\dim C = 1$ and $\operatorname{codim}_M C = 1$.

We will prove Proposition 1.1 using some techniques developed by Donin [9]. Our proof also relies on a "quantified" version of Grauert–Remmert finiteness theorem [15]. The existence of bound $D(E' \otimes S^\ell E'')$ was proved by Donin [9] for the so-called "normal" coverings. We have used nested coverings in the proof of Proposition 1.1 as well as the convergence proof in Theorem 1.5. We believe that the methods and tools developed in this article will be useful for other kinds of problems.

2 Full Linearizations, Horizontal Foliations, and Vertical Linearizations

In this section, we describe the problem of equivalence of a neighborhood of a complex compact submanifold C of M with a neighborhood of the zero section in the normal bundle of C in M as a "full" linearization problem of the transition functions of this neighborhood. We also describe the existence of a holomorphic foliation of a neighborhood of C having C as a leaf as a consequence of a *vertical* linearization problem of the transition functions of this neighborhood.

We will first describe the formal coordinate changes in terms of cohomological groups of holomorphic sections of a suitable sequence of holomorphic vector bundles.

2.1 Transition Functions

We recall basic facts on vector bundles, which we refer to [17, Chap. 0, Sect. 5].

We first set up notation. If a vector space E has a basis $e = \{e_1, \dots, e_d\}$, then a vector v in E can be expressed as

$$v = \xi^\mu e_\mu, \quad \xi = (\xi^1, \dots, \xi^d)^t.$$

Here, we use the summation notation: $\xi^\mu e_\mu$ stands for $\sum_{\mu=1}^d \xi^\mu e_\mu$. The ξ^μ's are the coordinates or components of v in the basis e.

We recall that a holomorphic vector bundle $\mathbf{E}$ over a complex manifold X is defined by a projection $\pi : \mathbf{E} \to X$ and holomorphic trivializations $\Psi_j : \pi^{-1}(D_j) \to D_j \times \mathbf{C}^r$ such that each $\Psi_j : \pi^{-1}(D_j) \to D_j \times \mathbf{C}^r$ is a biholomorphism, and $\Psi_j(\mathbf{E}_p) = \{p\} \times \mathbf{C}^r$ for $\mathbf{E}_p := \pi^{-1}(p)$. Furthermore $\{D_j\}$ is an open covering of X and the maps $\Psi_{kj} = \Psi_k \Psi_j^{-1} : D_k \cap D_j \times \mathbf{C}^r \to D_k \cap D_j \times \mathbf{C}^r$ satisfy

$$\Psi_{kj}(p, \xi_j) = (p, g_{kj}(p)\xi_j), \tag{2.1}$$

where g_{kj} are transition matrices which are holomorphic and invertible on $D_k \cap D_j$. Thus for $\xi_k^\mu e_{k,\mu} = \xi_j^\mu e_{j,\mu}$, we have

$$\xi_k^\mu = g_{kj,\nu}^\mu \xi_j^\nu, \quad e_{j,\mu} = g_{kj,\mu}^\nu e_{k,\nu}, \tag{2.2}$$

$$\xi_k = g_{kj} \xi_j, \quad e_k = (g_{kj}^{-1})^t e_j. \tag{2.3}$$

They satisfy the cocycle conditions,

$$g_{kj} g_{jk} = \mathrm{Id}, \quad \text{on } D_k \cap D_j; \quad g_{ki} g_{ij} = g_{kj}, \quad \text{on } D_k \cap D_j \cap D_i, \tag{2.4}$$

where Id is the identity matrix. We also need to consider the dual bundle E^*. Let e_j^* be the basis dual to e_j so that $(e_{j,\mu}^*(e_{j,\nu}))_{1 \le \mu,\nu \le r}$ is the identity matrix. Suppose $\zeta_j^\mu e_{j,\mu}^* = \zeta_k^\mu e_{k,\mu}^* \in E^*$. Corresponding to (2.3), we have

$$e_k^* = g_{kj} e_j^*, \quad \zeta_k = (g_{kj}^{-1})^t \zeta_j. \tag{2.5}$$

Let us also express transition functions for various vector bundles in coordinate charts as above. Let C_n be a compact complex manifold embedded in complex manifold M_{n+d}. We cover a neighborhood of C in M by open sets V_j so that we can choose coordinate charts (z_j, w_j) on V_j for M such that

$$U_j := C \cap V_j = \{w_j = 0\}.$$

Then $\mathcal{U} = \{U_i\}$ is a finite covering of C by open sets on which the coordinate charts $z_i = (z_i^1, \dots, z_i^n)$ are defined. Let

$$z_k = \varphi_{kj}(z_j) = \varphi_k \varphi_j^{-1}(z_j) \tag{2.6}$$

be the transition function of C on $U_{kj} := U_k \cap U_j$. It is a biholomorphic mapping from $\varphi_j(U_{kj})$ onto $\varphi_k(U_{kj})$ in $\mathbf{C}^n$. Then TC has a basis

$$e_{j,\alpha} := \frac{\partial}{\partial z_j^\alpha}, \quad 1 \leq \alpha \leq n$$

over U_j and its transition matrices s_{kj} have the form

$$s_{kj,\beta}^\alpha(z_j) := \frac{\partial z_k^\alpha}{\partial z_j^\beta}\Big|_{U_j \cap U_k}. \tag{2.7}$$

Thus for $\eta_k^\alpha \frac{\partial}{\partial z_k^\alpha} = \eta_j^\alpha \frac{\partial}{\partial z_j^\alpha}$ on $U_j \cap U_k$, we have $\eta_k = s_{kj}(z_j)\eta_j$.

Regarding the normal bundle N_C, its transition matrices $t_{kj,\nu}^\mu(z_j) := \frac{\partial w_k^\mu}{\partial w_j^\nu}|_{U_j \cap U_k}$ on $U_j \cap U_k$ are for the basis

$$f_{j,\mu} := \frac{\partial}{\partial w_j^\mu} \mod TC, \quad 1 \leq \mu \leq d.$$

Thus for $\xi_k^\mu f_{j,\mu} = \xi_j^\mu f_{k,\mu}$, we have $\xi_k = t_{kj}(z_j)\xi_j$. With notation (2.1), the transition matrices of $TM|_C$ are then of the form

$$g_{kj} := \begin{pmatrix} s_{kj} & l_{kj} \\ 0 & t_{kj} \end{pmatrix}(z_j) \quad \text{on } U_j \cap U_k$$

for some $n \times d$ matrices l_{jk}. Note that $\frac{\partial w_j}{\partial z_k}|_C = 0$.

Throughout the paper, $\tau_{kj}(z_j)$ are the transition matrices of N_C^* for the base dw_j. Note that

$$\tau_{kj} = (t_{kj}^{-1})^t.$$

More specifically, if $w_{j,\mu}^* := dw_j^\mu|_{U_j}$ and $\zeta_j^\mu w_{j,\mu}^* = \zeta_k^\mu w_{k,\mu}^*$, then (2.5) becomes

$$\zeta_k^* = (t_{kj}^{-1}(z_j))^t \zeta_j^*, \quad w_k^* = t_{kj}(z_j) w_j^*. \tag{2.8}$$

We remark that the cocycle conditions (2.4) for N_C now take the form

$$t_{kj}(z_j)t_{jk}(z_k) = \text{Id on } U_j \cap U_k, \quad t_{kj}(z_j)t_{j\ell}(z_\ell) = t_{k\ell}(z_\ell) \text{ on } U_j \cap U_k \cap U_\ell. \tag{2.9}$$

We say that TM *splits* on C, if there is a (non-canonical) decomposition

$$TM|_C = TC \oplus \tilde{N}_C, \quad \tilde{N}_C \cong N_C. \tag{2.10}$$

Equivalently, there exists a system of coordinate charts such that on C, the transition matrices of $TM|_C$ are of the form

$$g_{kj} = \begin{pmatrix} s_{kj} & 0 \\ 0 & t_{kj} \end{pmatrix}(z_j) \quad \text{on } U_j \cap U_k.$$

In other words, $\left.\frac{\partial z_j}{\partial w_k}\right|_C = 0$.

Throughout the paper, we assume that TM splits on C and we fix a splitting (2.10). Then the change of bases of the normal bundle N_C has a simple form

$$z_k = \varphi_{kj}(z_j), \quad \frac{\partial}{\partial w_k^\nu} = t_{jk,\nu}^{\mu}(z_k)\frac{\partial}{\partial w_j^\mu}, \quad \text{on } U_j \cap U_k.$$

In summary, for a neighborhood of the embedded manifold C in M with splitting $T_C M$, we can find a covering $\mathcal{V} = \{V_i\}$, with $\Phi_j(V_j) = \tilde{U}_i \times \tilde{W}_i$, by open sets on M and coordinates (z_i, w_i) defined on V_i. We assume that $U_j := C \cap V_i$ is defined by $\{w_i = 0\}$. A neighborhood of C will then be described by transition functions on V_{kj} of the form

$$\Phi_{kj}: \begin{aligned} z_k &= \Phi_{kj}^h(z_j, w_j) := \varphi_{kj}(z_j) + \phi_{kj}^h(z_j, w_j), \\ w_k &= \Phi_{kj}^v(z_j, w_j) := t_{kj}(z_j)w_j + \phi_{kj}^v(z_j, w_j). \end{aligned} \tag{2.11}$$

Here, ϕ_{kj}^h (resp. ϕ_{kj}^v) are holomorphic functions of vanishing order ≥ 2 along $w_j = 0$:

$$\phi_{kj}^h(z_j, w_j) = O(|w_j|^2), \quad \phi_{kj}^v(z_j, w_j) = O(|w_j|^2). \tag{2.12}$$

That ϕ_{kj}^h vanishes to order ≥ 2 follows from the fact that $TM|_C$ splits as $TC \oplus N_C$ (see above and [32, proposition 2.9]). An interested reader can also refer to [32] for a non-splitting example. Define

$$N_{kj}(h_j, v_j) := (\varphi_{kj}(z_j), t_{kj}(h_j)v_j).$$

Our goals are to apply changes of coordinates to simplify ϕ_{kj}^h, ϕ_{kj}^v, or one of them, according to the problem we study.

2.2 The Equivalence of Transition Functions

The germ of neighborhood of an embedded manifold is well-defined. For the formal normalization, we need to introduce (semi) formal charts and formal neighborhoods of an embedded manifold in a (semi) formal manifold.

Definition 2.1 We call $\hat{M}$ an (admissible and splitting) formal neighborhood of C if there are holomorphic coordinate charts φ_j on U_j where $\{U_j\}$ is a covering of C and there are formal power series

$$(z_j, w_j) = \hat{\Phi}_j(p, w) := (\varphi_j(p), t_j(p)w) + \sum_{|Q|\geq 2} \Phi_{j,Q}(p)w^Q,$$

where $\Phi_{j,Q}$ are holomorphic functions in U_j and each t_j is an invertible holomorphic $d \times d$ matrix on U_j. Note that the formal transition functions $\hat{\Phi}_{kj} = \hat{\Phi}_k \hat{\Phi}_j^{-1}$ have the form

$$\hat{\Phi}_{kj}(z_j, w_j) = (\varphi_{kj}(z_j), t_{kj}(z_j)w_j) + \sum_{|Q|>1} \hat{\Phi}_{kj,Q}(z_j)w_j^Q, \quad z_j \in \varphi_j(U_j \cap U_k).$$

(a) When all Φ_j are holomorphic, the formal neighborhood $\hat{M}$ is called the germ of a (holomorphic) neighborhood of C.
(b) $\hat{M}$ is called a linear neighborhood of C if additionally

$$\hat{\Phi}_{kj}(z_j, v_j) = (\varphi_{kj}(z_j), t_{kj}(z_j)v_j) \tag{2.13}$$

and each t_{kj} is an invertible holomorphic matrix in $U_k \cap U_j$. The terminology is meaningful since the $\hat{\Phi}_{kj}$ can be realized as the transition functions of a holomorphic vector bundle over C, namely the normal bundle of C in M.

We are mainly interested in the classification of a neighborhood of C for a given C. Therefore, it is reasonable to assume that the local trivialization of C are fixed. In other words, φ_{kj} are fixed and we will only consider mappings sending a neighborhood of C into another neighborhood of C that fix C pointwise.

Definition 2.2 We shall say that N_C is flat (resp. unitary flat), if we can find constant (resp. with values in group of unitary matrices U_d) transition functions in a possibly refined covering. If $T_C M := (TM)|_C$ is *holomorphically flat*, or *flat*, i.e. in some coordinates both transition functions N_C and TC are constant matrices, then by (2.7)

$$\varphi_{kj}(z_j) = s_{kj}z_j + c_{kj}$$

where s_{kj} are constant matrices and c_{kj} are constant vectors. Then, the transition functions of a neighborhood of the zero section of the normal bundle, $\hat{\Phi}_{kj}$ as defined in (2.13) read

$$A_{kj}(z_j, w_j) := (s_{kj}z_j + c_{kj}, t_{kj}w_j).$$

We will use the following notation: When N_C is flat, we write its transition matrices $t_{kj}(z_j)$ as t_{kj}, indicating that they are independent of z_j.

As mentioned in the introduction, a unitary holomorphic vector bundle is flat. Indeed, let t_{kj} be unitary and holomorphic transition matrices. Let $(f_1, \dots, f_d)$ be a row vector of the matrices. We have

$$|f_1|^2 + \cdots + |f_d|^2 = 1. \tag{2.14}$$

Fix a point $p \in U_k \cap U_j$. Conjugating the matrix by a constant unitary matrix, we may assume that $(f_1, \dots, f_d) = (1, \dots, 0)$ at p. By the maximum principle, (2.14) implies that near p, $f_1 = 1$ and hence $f_j = 0$ for $j > 1$.

Definition 2.3 We shall say that a change of coordinates $\{F_j\}$ preserves the germ of a neighborhood of the zero section of N_C with transition maps $\{N_{kj}\}$ if each F_j is biholomorphic and fixes $v_j = 0$ pointwise and $F_k N_{kj} = N_{kj} F_j$, in which case we say that $\{F_j\}$ preserves $\{N_{kj}\}$ for simplicity.

We further observe the following.

Lemma 2.4 *Let M, $\hat{M}$ be two (admissible) neighborhoods of C, of which coordinate charts are $\{\Phi_j\}$, $\{\hat{\Phi}_j\}$, respectively. Let $\Phi_{kj} = \Phi_k \Phi_j^{-1}$ and $\hat{\Phi}_{kj} = \hat{\Phi}_k \hat{\Phi}_j^{-1}$.*

(a) *There is a biholomorphic mapping $F \colon M \to \hat{M}$, defined near C and fixing C, if and only if there are biholomorphic mappings F_j satisfying*

$$F_k \hat{\Phi}_{kj}(z_j, w_j) = \Phi_{kj} F_j(z_j, w_j), \quad F_j(z_j, 0) = (z_j, 0). \tag{2.15}$$

(b) *If F_j satisfies (2.15), then*

$$\begin{aligned} &F_j(z_j, w_j) = LF_j(z_j, w_j) + O(|w_j|^2), \quad LF_j = (z_j + s_j(z_j)w_j, u_j(z_j)w_j),\\ &s_k(\varphi_{kj}(z_j))t_{kj}(z_j) = D\varphi_{kj}(z_j)s_j(z_j),\\ &u_k(\varphi_{kj}(z_j))t_{kj}(z_j) = t_{kj}(z_j)u_j(z_j). \end{aligned}$$

Assume further that F preserves the splitting. Then $s_j = 0$.

(c) *Let TC and N_C be flat and let F_j be (semi) formal biholomorphism fixing C pointwise. Suppose that $F_k^{-1}\Phi_{kj}F_j = N_{kj} + O(|v|_j^2)$. Then $\{LF_j\}$ preserves $\{N_{kj}\}$, i.e. $LF_k N_{kj}(LF_j)^{-1} = N_{kj}$, where*

$$F_j(h_j, v_j) = LF_j(h_j, v_j) + O(|v_j|^2), \quad LF_j(h_j, v_j) = (h_j + s_j(h_j)v_j, u_j(h_j)v_j).$$

Proof The points (a),(b) can be verified easily. For (c), let us expand $F_k \Phi_{kj}(h_j, v_j) = N_{kj} \circ F_j(h_j, v_j) + O(|v_j|^2)$ and compare the constant and linear terms in v_j. We obtain

$$\begin{aligned} &\varphi_{kj}(h_j) + s_j(\varphi_{kj}(h_j))t_{kj}v_j = \varphi_{kj}(h_j + s_j(h_j)v_j) + O(|v_j|^2),\\ &u_k(\varphi_{kj}(h_j))t_{kj}v_j = t_{kj}u_j(h_j)v_j + O(|v_j|^2). \end{aligned}$$

Here we have used the assumption that t_{kj} are constant. Since φ_{kj} are affine, the two identities still hold if we drop $O(|v_j|^2)$ from them. This shows that $LF_k N_{kj} = N_{kj} LF_j$, again using the fact that t_{kj} are constant and φ_{kj} are affine. □

Finally, we mention that we will choose the atlas of C so that each φ_j is a biholomorphism from U_j onto the unit polydisc Δ_n in $\mathbf{C}^n$ and from a neighborhood $\tilde{U}_j$ of $\overline{U_j}$ onto another larger polydisc. When C is embedded in a complex manifold M, we can extend φ_j to V_j to get a coordinate chart Φ_j on V_j such that Φ_j maps V_j onto $U_j \times \Delta_\delta^d$. This can be achieved since any holomorphic vector bundle over $\tilde{U}_j$ is holomorphically trivial. Thus $N_C|_{U_j}$ splits. Consequently, we can use a flow box of holomorphic normal vector fields to construct the required Φ_j. Therefore, if C is embedded into another complex manifold $\tilde{M}$, we will choose the atlas of a neighborhood of C in $\tilde{M}$ such that the restriction of the chart on U_j agrees with φ_j.

Therefore, we introduce the following.

Definition 2.5 We say that a formal neighborhood $\{\Phi_{kj}\}$ of C is equivalent to a neighborhood $\{\hat{\Phi}_{kj}\}$ of C in M by a formal holomorphic mapping F that is tangent to the identity, if there are formal maps $F_j(z_j) = (z_j, w_j) + \sum_{|Q|>1} F_{j,Q}(z_j) w_j^Q$ such that $F_{j,Q}(z_j)$ are holomorphic functions in U_j and as power series in w_j

$$F_k \hat{\Phi}_{kj}(z_j, w_j) = \Phi_{kj} F_j(z_j, w_j).$$

We take $F = \hat{\Phi}_j^{-1} F_j \Phi_j$, which is well-defined, when $\Phi_{kj} = \Phi_k \Phi_j^{-1}$ and $\hat{\Phi}_{kj} = \hat{\Phi}_k \hat{\Phi}_j^{-1}$.

2.3 The Full Linearization of a Neighborhood

In this case, our goal is to seek new coordinates (h_k, v_k) so that all ϕ_{kj}^h, ϕ_{kj}^v are 0.

Let us consider a change of coordinates in a neighborhood of C by modifying the old coordinate charts (z_k, w_k) via F_k. We write it as

$$F_k : \begin{aligned} z_k &= F_k^h(h_k, v_k) := h_k + f_k^h(h_k, v_k), \\ w_k &= F_k^v(h_k, v_k) := v_k + f_k^v(h_k, v_k). \end{aligned} \tag{2.16}$$

Here, $f_k^h(h_k, v_k)$ and $f_k^v(h_k, v_k)$ are holomorphic functions vanishing to order ≥ 2 at $v_k = 0$. In particular, C is pointwise fixed by the change as $z_k = h_k$ on C (i.e. for $v_k = 0$). We require that the inverse of F_k is defined in a possibly smaller open sets $\hat{V}_k \subset \varphi_k(U_k)$ such that the union of $\Phi_k^{-1}(\hat{V}_k)$ remains a neighborhood of C in M.

We recall that the cocycle condition (2.9) on the transition matrices t_{kj} has the form

$$\begin{aligned} &t_{kj}(z_j) t_{jk}(\varphi_{kj}(z_j)) = \mathrm{Id}, \\ &t_{kj}(\varphi_{j\ell}(z_\ell)) t_{j\ell}(z_\ell) = t_{k\ell}(z_\ell). \end{aligned} \tag{2.17}$$

Let us assume that the (*a priori* formal) change of coordinates (2.16) maps a neighborhood C to a neighborhood of the zero section in the normal bundle. This means that, in these new coordinates, we have

$$N_{kj} := F_k^{-1}\Phi_{kj}F_j : \begin{array}{l} h_k = \varphi_{kj}(h_j), \\ v_k = t_{kj}(z_j)v_j. \end{array}$$

Let us write down the above "conjugacy equations". We first consider the horizontal equation of

$$F_k N_{kj} = \Phi_{kj} F_j.$$

On the left side of the equation, we have

$$z_k = h_k + f_k^h(h_k, v_k) = \varphi_{kj}(h_j) + f_k^h(\varphi_{kj}(h_j), t_{kj}(h_j)v_j).$$

On the other side, we have

$$z_k = \varphi_{kj}(h_j + f_j^h(h_j, v_j)) + \phi_{kj}^h(h_j + f_j^h, v_j + f_j^v).$$

Let us define the *horizontal cohomological operator* to be

$$\mathcal{L}_{kj}^h(f_j^h) := f_k^h(\varphi_{kj}(h_j), t_{kj}(h_j)v_j) - s_{kj}(h_j)f_j^h(h_j, v_j). \tag{2.18}$$

Recall that $s_{kj}(h_j) = D\varphi_{kj}(h_j)$ is the Jacobian matrix of φ_{kj}. Hence, we can write the previous horizontal equation as

$$\begin{aligned}\mathcal{L}_{kj}^h(f_j^h) = {} & \phi_{kj}^h(h_j + f_j^h, v_j + f_j^v) \\ & + \varphi_{kj}(h_j + f_j^h(h_j, v_j)) - \varphi_{kj}(h_j) - D\varphi_{kj}(h_j)f_j^h(h_j, v_j).\end{aligned} \tag{2.19}$$

Let us consider the vertical equation. We have, on one side of the equation,

$$w_k = v_k + f_k^v(h_k, v_k) = t_{kj}(h_j)v_j + f_k^v(\varphi_{kj}(h_j), t_{kj}(h_j)v_j).$$

On the other side, we have

$$w_k = t_{kj}(h_j + f_j^h)(v_j + f_j^v) + \phi_{kj}^v(h_j + f_j^h, v_j + f_j^v).$$

Let us define the *vertical cohomological operator* to be

$$\mathcal{L}_{kj}^v(f_j^v) := f_k^v(\varphi_{kj}(h_j), t_{kj}(h_j)v_j) - t_{kj}(h_j)f_j^v. \tag{2.20}$$

Hence, we can write the previous vertical equation as

$$\begin{aligned}\mathcal{L}_{kj}^v(f_j^v) = {} & \phi_{kj}^v(h_j + f_j^h, v_j + f_j^v) \\ & + \left(t_{kj}(h_j + f_j^h(h_j, v_j)) - t_{kj}(h_j)\right) f_j^v \\ & + \left(t_{kj}(h_j + f_j^h(h_j, v_j)) - t_{kj}(h_j)\right) v_j.\end{aligned} \tag{2.21}$$

2.4 Horizontal Foliations and Vertical Trivializations

Let us assume that there exists a non singular holomorphic foliation having C as a leaf. We seek holomorphic functions $f_j = (f_{j,1}, \ldots, f_{j,d})$ defined in a neighborhood V_j of U_j such that $f_j = 0$ on U_j and $df_{j,1} \wedge \cdots \wedge df_{j,d} \neq 0$. Then, we may use $(h_j, v_j) = (z_j, f_j(z_j, w_j))$ as a coordinate mapping on V_j, which changes variables in vertical components. We then prove that in these new coordinates, the transition functions of a neighborhood of C are of the form $\hat{\Phi}_{kj} = (\hat{\Phi}^h_{kj}, \hat{\Phi}^v_{kj})$ such that Φ^v_{kj} are independent of h_j. We remark that N_C must be flat if a horizontal foliation exists.

Proposition 2.6 *Assume that there is smooth holomorphic horizontal foliation defined in a neighborhood V of C in M. By a refinement of U_j, then there exists a change of variables of the form*

$$z_k = h_k \quad w_k = s(h_j)v_j + O(|v_j|^2)$$

so that in the new variables, we have

$$\begin{aligned} h_k &= \varphi_{kj}(h_j) + \phi^h_{kj}(h_j, v_j), \\ v_k &= \tilde{t}_{kj} v_j + \sum_{|Q|>1} c_{kj,Q} v_j^Q, \end{aligned}$$

where $\tilde{t}_{kj}, c_{kj,Q}$ are constants.

Proof By a refinement, we may assume that the foliation on V_j is given $W_j(h_j, v_j) = cst$ by holomorphic functions $W_j = (W_{j,1}, \ldots, W_{j,d})$ such that $W_j = 0$ on U_j and $dW_{j,1} \wedge \cdots \wedge dW_{j,d} \neq 0$. We have $W_k = \tilde{\Phi}^v_{kj} W_j$, where $\tilde{\Phi}^v_{kj}$ is a biholomorphism of $(\mathbb{C}^d, 0)$ with $\tilde{\Phi}^v_{kj}(0) = 0$. Then $\tilde{W}_j = (z_j, W_j)$ is a biholomorphism defined on V_j and fixing $C \cap V_j$ pointwise, by shrinking V_j if necessary in the vertical direction. Since $\tilde{W}_j$ is invertible, we can define $\tilde{\Phi}^h_{kj} = z_k \tilde{W}_j^{-1}$ Then we have $\tilde{\Phi}^h_{kj} \tilde{W}_j = z_k$. Therefore,

$$\tilde{W}_k \tilde{W}_j^{-1}(h_j, v_j) = (\tilde{\Phi}^h_{kj}(h_j, v_j), \tilde{\Phi}^v_{kj}(v_j)).$$

Set $F_j = \Phi_j \tilde{W}_j^{-1}$. We have $F^h_j(h_j, v_j) = h_j$. We now get

$$F_k^{-1} \Phi_k \Phi_j^{-1} F_j = \tilde{W}_k \tilde{W}_j^{-1} = \tilde{\Phi}_{kj}.$$

□

In this paper, we will approach the horizontal foliation problem via the following vertical linearization when N_C is unitary.

2.5 The Vertical Linearization

Here we seek new coordinates (h_j, v_j) from (z_j, w_j) such that the vertical component of the new Φ_{kj} agrees with the vertical component of N_{kj}. In Lemma 2.17 we will show that if such *formal* coordinates exist, then the vertical linearization can be achieved by changing vertical coordinates only, i.e. a coordinate change of the form

$$w_k = F_k^v(h_k, v_k) := v_k + f_k^v(h_k, v_k), \quad z_k = h_k.$$

For the vertical linearization, we only need to consider the *vertical part* of transition functions so that in the new variables, we have

$$\begin{aligned} h_k &= \hat{\Phi}_{kj}^h(h_j, v_j) := \varphi_{kj}(h_j) + \hat{\phi}_{kj}^h(h_j, v_j) \\ v_k &= t_{kj}(h_j)v_j. \end{aligned}$$

Here, $\hat{\phi}_{kj}^h(h_j, v_j)$ vanishes up to order 2 at $v_j = 0$. The vertical equation reads

$$\begin{aligned} t_{kj}(h_j)(v_j + f_j^v) + \phi_{kj}^v(h_j, v_j + f_j^v) &= w_k \\ &= t_{kj}(h_j)v_j + f_k^v(\hat{\Phi}_{kj}^h(h_j, v_j), t_{kj}(h_j)v_j). \end{aligned}$$

Using the previous notation, we finally obtain the following "conjugacy equations"

$$\mathcal{L}_{kj}^v(f_j^v) = \phi_{kj}^v(h_j, v_j + f_j^v) - \Big(f_k^v(\hat{\Phi}_{kj}^h(h_j, v_j), t_{kj}(h_j)v_j) - f_k^v(\varphi_{kj}(h_j), t_{kj}(h_j)v_j)\Big). \tag{2.22}$$

Having determined the coordinate change, let us find the horizontal component $\hat{\phi}_{kj}^h$ from the horizontal equation

$$\varphi_{kj}(h_j) + \phi_{kj}^h(h_j, v_j + f_j^v) = z_k = \hat{\Phi}_{kj}^h(h_j, v_j) = \varphi_{kj}(h_j) + \hat{\phi}_{kj}^h(h_j, v_j).$$

We get

$$\hat{\phi}_{kj}^h(h_j, v_j) = \phi_{kj}^h(h_j, v_j + f_j^v). \tag{2.23}$$

2.6 An Open Problem on the Horizontal Linearization

In this paper, we will not study this analogous linearization problem which is interest in its own right. Namely, one could seek coordinate changes so that the new transition functions of M near C have the form

$$\widetilde{N}_{kj} := F_k^{-1}\Phi_{kj}F_j : \begin{array}{l} h_k = \varphi_{kj}(h_j), \\ v_k = \tilde{t}_{kj}(z_j, v_j). \end{array}$$

The existence of such a horizontal linearization ensures that a neighborhood of C in M admits a holomorphic foliation with leaves transversal to C. If one follows the approach in this paper for $\tilde{t}_{kj}(z_j, h_j)$ not to be $t_{kj}h_j$ where $t_{kj}(h_j)$ are unitary, constant or non-constant functions in general, it leads to an interesting and new kind of difficulty.

2.7 Coboundary Operators in Symmetric Powers and Coordinates

In this subsection, we establish the connections between coordinate changes and formal obstructions to the full linearization and vertical linearization via cohomological groups. In local dynamics, the resonant terms play an important role in the construction of normal forms at least at the formal level, while non-resonant terms play another important role in coordinate changes. In all problems, obstructions are described via the first cohomological groups, while the coordinate changes are described via solutions to the cohomological equations of first order approximation.

Let E' be a vector bundle of rank τ over C. Let $\mathcal{U} = \{U_i\}$ be a covering of C as above. Let $e_j := \{e_{j,1}, \dots, e_{j,\tau}\}$ be a basis over U_j and let $\xi_j := (\xi_j^1, \dots, \xi_j^\tau)^t$ be coordinates in e_j. Let $s_{kj}(z_j)$ be the transition matrices of E' over $U_k \cap U_j$. Using notation in (2.3), we have

$$\xi_k^\alpha = s_{kj,\beta}^\alpha(z_j)\xi_j^\beta, \quad e_{k;\alpha} = s_{jk,\alpha}^\beta(z_k)e_{j,\beta}, \tag{2.24}$$

$$z_k = \varphi_{kj}(z_j), \quad \xi_k = s_{kj}(z_j)\xi_j, \quad e_k = (s_{kj}^{-1}(z_j))^t e_j, \tag{2.25}$$

where φ_{kj} are the transition functions of C. For N_C^*, by (2.8) we have

$$\zeta_k = (t_{kj}^{-1})^t(z_j)\zeta_j, \quad w_k^* = t_{kj}(z_j)w_j^*, \quad z_k = \varphi_{kj}(z_j).$$

The following fact is well-known. We provide a proof for the reader's convenience. Let us first introduce

$$\tilde{f}_{i_0\cdots i_q}^\lambda(z_{i_q}, \zeta_{i_q}) := \sum_{|Q|=L} f_{i_0\cdots i_q;Q}^\lambda(z_{i_q})\zeta_{i_q}^Q, \tag{2.26}$$

for a cochain $\{f_I\} \in C^q(\{U_j\}, \mathcal{O}(E \otimes S^L(N_C^*)))$ given by

$$f_{i_0\cdots i_q}(p) = \sum_{\lambda=1}^{\tau} \sum_{|Q|=L} f_{i_0\cdots i_q;Q}^\lambda(z_{i_q}(p))e_{i_0,\lambda}(p) \otimes (w_{i_q}^*(p))^Q, \tag{2.27}$$

where each $f_{i_0\dots i_q;Q}^\lambda$ is a holomorphic function on $\varphi_{i_q}(U_{i_0\cdots i_q})$, and $U_{i_0\cdots i_q}$ denotes as usual $U_{i_0} \cap \cdots \cap U_{i_q}$. Here we have chosen a representation of cochains in bases that arise from the linearized equations for the problems described above.

Let $f_{i_0\cdots \hat{i}_\ell \cdots i_{q+1}}$ denote $f_{i_0\cdots i_{\ell-1}i_{\ell+1}\cdots i_{q+1}}$. Then $(\delta f)_{i_0\cdots i_{q+1}} = \sum (-1)^\ell f_{i_0\cdots \hat{i}_\ell \cdots i_{q+1}}$ becomes

$$
\begin{aligned}
(\delta f)_{i_0\cdots i_{q+1}} &= \sum_{\ell=1}^{q}(-1)^{\ell}\sum_{\lambda=1}^{\tau}\sum_{|Q|=L} f^{\lambda}_{i_0\cdots\hat{i}_\ell\cdots i_{q+1};Q}(z_{i_{q+1}}(p))e_{i_0,\lambda}(p)\otimes(w^*_{i_{q+1}}(p))^Q\\
&\quad+\sum_{\lambda=1}^{\tau}\sum_{|Q|=L} f^{\lambda}_{i_1\cdots i_{q+1};Q}(z_{i_{q+1}}(p))e_{i_1,\lambda}(p)\otimes(w^*_{i_{q+1}}(p))^Q\\
&\quad-(-1)^q\sum_{\lambda=1}^{\tau}\sum_{|Q|=L} f^{\lambda}_{i_0\cdots i_q;Q}(z_{i_q}(p))e_{i_0,\lambda}(p)\otimes(w^*_{i_q}(p))^Q\\
&=:\sum_{\lambda=1}^{\tau}\sum_{|Q|=L} g^{\lambda}_{i_0\cdots i_{q+1}}(z_{q+1})e_{i_0,\lambda}(p)\otimes(w^*_{i_{q+1}}(p))^Q.
\end{aligned}
$$

By (2.24), we have $e_{i_1,\lambda}=s^{\mu}_{i_0 i_1,\lambda}e_{i_0,\mu}$. In notation (2.26), we can express

$$
\begin{aligned}
\tilde{g}^{\lambda}_{i_0\cdots i_{q+1}}(z_{i_{q+1}},\zeta_{i_{q+1}}) &= \sum_{\ell=1}^{q}(-1)^{\ell}\tilde{f}^{\lambda}_{i_0\cdots\hat{i}_\ell\cdots i_{q+1}}(z_{i_{q+1}},\zeta_{i_{q+1}})\\
&\quad+s^{\lambda}_{i_0 i_1,\mu}(\varphi_{i_1 i_{q+1}}(z_{q+1}))\tilde{f}^{\mu}_{i_1\cdots i_{q+1}}(z_{i_{q+1}},\zeta_{i_{q+1}})\\
&\quad-(-1)^q f^{\lambda}_{i_0\cdots i_q}(\varphi_{i_q i_{q+1}}(z_{i_{q+1}}),t_{i_q i_{q+1}}(z_{i_{q+1}})\zeta_{i_{q+1}})).
\end{aligned}
$$

The above computation especially gives us the following formulae for 0 and 1-cochains.

Lemma 2.7 *Let $\{U_j\}$ be an open covering of C. Let t_{kj} be the transition matrices for N_C with respect to basis w_j and let s_{kj} be the transitions functions of E with respect to base e_j. Let*

$$
\begin{aligned}
f_{ij}(p) &= \sum_{\lambda=1}^{d}\sum_{|Q|=L} f^{\lambda}_{ij;Q}(z_j(p))e_{i,\lambda}(p)\otimes(w^*_j(p))^Q,\quad \tilde{f}^{\lambda}_{ij}(z_j,\zeta_j):=\sum_{|Q|=L} f^{\lambda}_{ij;Q}(z_j)\zeta_j^Q,\\
u_j(p) &= \sum_{\lambda=1}^{d}\sum_{|Q|=L} u^{\lambda}_{j,Q}(z_j(p))e_{j,\lambda}(p)\otimes(w^*_j(p))^Q,\quad \tilde{u}^{\lambda}_{j}(z_j,\zeta_j):=\sum_{|Q|=L} u^{\lambda}_{j;Q}(z_j)\zeta_j^Q.
\end{aligned}
$$

The following hold:

(a) $f:=\{f_{ij}\}\in Z^1(\mathcal{U},\mathcal{O}(E\otimes S^L(N_C^*)))$ *if and only if*

$$
\tilde{f}^{\lambda}_{ij}(\varphi_{jk}(z_k),t_{jk}(z_k)\zeta_k)-\tilde{f}^{\lambda}_{ik}(z_k,\zeta_k)+s^{\lambda}_{ij,\ell}(z_j)\tilde{f}^{\lambda}_{jk}(z_k,\zeta_k)=0.
$$

(b) $u:=\{u_j\}$ *solves the first order cohomological equation* $\delta u=f$ *if and only if*

$$
s^{\lambda}_{ij,\ell}(z_j)\tilde{u}^{\ell}_{j}(z_j,\zeta_j)-\tilde{u}^{\lambda}_{i}(\varphi_{ij}(z_j),t_{ij}(z_j)\zeta_j)=\tilde{f}^{\lambda}_{ij}(z_j,\zeta_j).
$$

We notice that according to (2.18) and (2.20), we have

$$-\mathcal{L}(f) = -(\mathcal{L}^h(f^h), \mathcal{L}^v(f^v)) = \delta(f) := (\delta^h(f^h), \delta^v(f^v)).$$

2.8 Formal Obstructions in Cohomology Groups

Recall that

$$N_{kj}(h_j, v_j) := (\varphi_{kj}(z_j), t_{kj}(h_j)v_j). \tag{2.28}$$

Let us denote the properties depending on an order $m \geq 1$:
$(L_m(\mathcal{U}))$: the neighborhood of C matches the neighborhood of zero section of the normal bundle up to order m.
$(V_m(\mathcal{U}))$: the vertical components of the transition functions of neighborhoods of C in M and in N_C match up to order m.

That embedding of C has property (L_m) (resp. (V_m)) means that the order along $v_j = 0$ of $(\phi^h_{kj}(h_j, v_j), \phi^v_{kj}(h_j, v_j))$ (resp. $\phi^v_{kj}(h_j, v_j)$) as defined in (2.11) is $\geq m+1$.

Definition 2.8 We shall say that N_C is a flat (resp. unitary flat), if we can find constant (resp. with values in group of unitary matrices U_d) transition functions in a possibly refined covering.

We will use the following notation: When N_C is flat, we write its transition matrices $t_{kj}(z_j)$ as t_{kj}, indicating that they are independent of z_j.

Definition 2.9 We shall say that a change of coordinates $\{F_j\}$ preserves the germ of a neighborhood of the zero section of N_C with transition maps $\{N_{kj}\}$ if $F_k N_{kj} = N_{kj} F_j$, in which case we says that $\{F_j\}$ preserves $\{N_{kj}\}$ for simplicity.

Lemma 2.10 *Let the transition functions Φ_{kj} of a neighborhood of C be given by (2.11)–(2.12).*

(a) *Assume that C satisfies L_m. Then the horizontal and vertical components satisfy*

$$[\phi^h_{kj}]^\ell \in Z^1(\mathcal{U}, TC \otimes S^\ell(N_C^*)), \quad \text{if } m < \ell \leq 2m;$$
$$[\phi^v_{kj}]^\ell \in Z^1(\mathcal{U}, N_C \otimes S^\ell(N_C^*)), \quad \text{if } \ell = m+1.$$

Furthermore, if $[\phi^\bullet_{kj}]^{m+1} = 0$ in $H^1(\mathcal{U}, T_C M \otimes S^{m+1}(N_C^*))$, then there exist $\{F_j = Id + f_j\}$ such that $F_k \Phi_{kj} F_j^{-1} \in L_{m+1}$ and f_j are homogeneous of degree $m+1$.
If N_C is flat, then the vertical component of Φ_{kj} further satisfies

$$[\phi^v_{kj}]^\ell \in Z^1(\mathcal{U}, N_C \otimes S^\ell(N_C^*)), \quad m+1 < \ell \leq 2m.$$

(b) *Let C satisfy V_m. Assume that N_C is flat. Then*

$$[\phi^v_{kj}]^\ell \in Z^1(\mathcal{U}, N_C \otimes S^\ell(N_C^*)), \quad \ell = m+1. \tag{2.29}$$

Proof When $\ell = m+1$, (a) is in Griffiths [16], Morrow–Rossi [32] and (b) is proved in Ueda [42] for flat line bundle N_C^* over a compact curve C.

(a) The general case can be verified using Lemma 2.7 to compare coefficients of w_j^α on both sides of $\Phi_{ij}(z_j, w_j) = \Phi_{ik} \circ \Phi_{kj}(z_j, w_j)$ for $|\alpha| \leq 2m$. Indeed, we have $\Phi_{ik} = N_{ik} + (\phi_{ik}^h, \phi_{ik}^v)$ and $(\phi_{ik}^h, \phi_{ik}^v)(z_k, w_k) = O(|w_k|^{m+1})$ with $m \geq 1$. Thus

$$
\begin{aligned}
N_{ik} \circ \Phi_{kj}(z_j, w_j) &= \left\{N_{ik} \circ N_{kj} + DN_{ik} \circ N_{kj} \cdot (\phi_{kj}^h, \phi_{kj}^v)\right\}(z_j, w_j) + O(|w_j|^{2m+1}) \\
&= N_{ik} \circ N_{kj}(z_j, w_j) + (s_{ik}(\varphi_{kj}(z_j))\phi_{kj}^h, t_{ik}(\varphi_{kj}(z_j))\phi_{kj}^v) \\
&\quad + (0, Dt_{ik}(\varphi_{kj}(z_j))\phi_{kj}^h(z_j)t_{kj}(z_j)w_j) + O(|w_j|^{2m+1}).
\end{aligned}
$$

Here s_{kj} are the transition matrices of TC given by (2.7). Therefore,

$$
\begin{aligned}
\Phi_{ik} \circ \Phi_{kj}(z_j, w_j) &= N_{ik} \circ \Phi_{kj}(z_j, w_j) + (\phi_{ik}^h, \phi_{ik}^v) \circ \Phi_{kj}(z_j, w_j) \\
&= \left\{N_{ik} \circ N_{kj} + (\phi_{ik}^h, \phi_{ik}^v) \circ N_{kj}\right\}(z_j, w_j) \\
&\quad + \Big(s_{ik}(\varphi_{kj}(z_j))\phi_{kj}^h(z_j, w_j), t_{ik}(\varphi_{kj}(z_j))\phi_{kj}^v(z_j, w_j)\Big) \\
&\quad + (0, Dt_{ik}(\varphi_{kj}(z_j))\phi_{kj}^h(z_j)t_{kj}(z_j)w_j) + O(|w_j|^{2m+1}).
\end{aligned}
$$

Comparing both sides of $\Phi_{ij}(z_j, w_j) = \Phi_{ik} \circ \Phi_{kj}(z_j, w_j)$ for the coefficients in w_j of order $\ell = m+1$, we obtain the desired conclusion by Lemma 2.7.

(b) We have $\Phi_{kj}(z_j, w_j) = (\varphi_{kj}(z_j) + \phi_{kj}^h(z_j, w_j), t_{kj}w_j + \phi_{kj}^v(z_j, w_j))$ with $\phi_{kj}^v(z_j, w_j) = O(|w_j|^{m+1})$. Here t_{kj} are constant. We get from the vertical components of $\Phi_{kj} = \Phi_{ki}\Phi_{ij}$ that

$$
\begin{aligned}
\phi_{kj}^v(z_j, w_j) &= t_{ki}\phi_{ij}^v(z_j, w_j) + \phi_{ki}^v(\Phi_{ij}(z_j, w_j)) \\
&= t_{ki}\phi_{ij}^v(z_j, w_j) + \varphi_{ki}(N_{ij}(z_j, w_j)) + O(|w_j|^{m+2}),
\end{aligned}
$$

since $(\Phi_{ij} - N_{ij})(z_j, w_j) = O(|w_j|^2)$. This shows that $\{[\phi_{kj}^v]^\ell\} \in Z^1(\mathcal{U}, N_C \otimes N_C^{*\ell})$ for $\ell = m+1$ by Lemma 2.7 (a). This gives us (2.29). ◻

2.9 Automorphisms of Neighborhood of the Zero Section of Flat Vector Bundles

Let ϕ_{kj} defined on $U_k \cap U_j$ be the transition functions of C. Let Φ_{kj}, defined on $V_k \cap V_j$, be the transition functions of M, and let N_{kj}, defined on $\tilde{V}_k \cap \tilde{V}_j$ be the transition functions of N_C, with $\tilde{V}_k = \pi^{-1}U_k$. We identify (C, U_j) as subsets of $\tilde{V}_j$ via the zero-section. Recall Φ_{kj}, N_{kj}, and ϕ_{kj} are the same on $U_k \cap U_j$. By Cartan-Serre theorem, for any integer m, the space of global sections, $H^0(C, T_C M \otimes S^m N_C^*)$, is finite dimensional.

We say that a vector bundle is *flat* if its transition matrices are locally constant.

Definition 2.11 (1) A formal tangent vector field Y_j on $\tilde{V}_j$ vanishing at U_j is identified with $Y_j = \sum_{\ell \geq 1} Y_j^\ell$ with $Y_j^\ell \in \Gamma(U_j, T_C M \otimes S^\ell N_C^*)$ via

$$\sum_{|Q|=\ell} a_Q^\alpha(h_j) v_j^Q \frac{\partial}{\partial h_j^\alpha} + b_Q^\lambda(h_j) v_j^Q \frac{\partial}{\partial v_j^\lambda} \mapsto \sum_{|Q|=\ell} a_Q^\alpha(z_j)(w_j^*)^Q \frac{\partial}{\partial z_j^\alpha} + b_Q^\lambda(z_j)(w_j^*)^Q \frac{\partial}{\partial w_j^\lambda}.$$

Here (h_j, v_j) is the coordinate map for $v_j^\lambda \frac{\partial}{\partial w_j^\lambda} \in (N_C)_p$ and we identity h_j with $z_j|_{U_j}$ and $\frac{\partial}{\partial v_j}$ with $\frac{\partial}{\partial w_j}|_{U_j}$.

(2) A formal automorphism of $\tilde{V}_j$ at U_j that is tangent to the identity is an automorphism of a formal neighborhood of the 0-section of $\tilde{V}_j$, fixing U_j pointwise.

Lemma 2.12 *Let $\{F_j\}_j$ be a collection of formal automorphisms of $\tilde{V}_j$ fixing U_j pointwise. Let $\{Y_j\}_j$ be a collection of formal tangent vector fields of $\tilde{V}_j$ vanishing at U_j. We have*

(1) *$\{F_j\}_j$ defines an automorphism F of a formal neighborhood of the 0-section in N_C if and only $F_k \circ N_{kj} = N_{kj} \circ F_j$ for all k, j.*
(2) *Suppose that N_C is flat. Then $\{Y_j\}_j$ defines a vector field Y on a formal neighborhood of the 0-section in N_C if and only if $\{Y_j^\ell\} \in H^0(C, T_C M \otimes S^\ell N_C^*)$ for all ℓ.*
(3) *Suppose that N_C is not flat. Then $\{Y_j\}_j$ defines a vector field on a formal neighborhood of the 0-section in N_C if and only if $\{Y_j\} \in H^0_{twisted}(C, T_C M \otimes \oplus_{\ell \geq 2} S^\ell N_C^*)$ with respect to the linear operator $\delta_{nf}(\{(Y_j^h, Y_j^v)\}) = \{(\tilde{Y}_{kj}^h, \tilde{Y}_{kj}^v)\}$ with*

$$\tilde{Y}_{kj}^h = Y_k^h(N_{kj}(h_j, v_j)) - D\phi_{kj}(h_j) Y_j^h(h_j, v_j),$$
$$\tilde{Y}_{kj}^v = Y_k^v(N_{kj}(h_j, v_j)) - t_{kj}(h_j) Y_j^v(h_j, v_j) - Dt_{kj}(h_j) v_j . Y_j^h(h_j, v_j).$$

Proof Let (h_j, v_j) be the coordinates in N_C over U_j. Note that $\{Y_j\}$ defines a global tangent vector filed of N_C if and only if $DN_{kj}(Y_j) = Y_k$. A homogeneous vector field of degree ℓ on $\tilde{V}_j$ is an element $Y_j^\ell \in C^0(U_j, T_C M \otimes S^\ell N_C^*)$ defined by

$$Y_j^\ell(h_j, v_j) = \sum_{m=1}^n Y_{j,m}^{\ell,h}(h_j, v_j) \frac{\partial}{\partial h_{j,m}} + \sum_{r=1}^d Y_{j,r}^{\ell,v}(h_j, v_j) \frac{\partial}{\partial v_{j,r}} =: Y_j^{\ell,h} + Y_j^{\ell,v}.$$

Recall that $N_{kj}(h_j, v_j) = (\phi_{kj}(h_j), t_{kj}(h_j) v_j)$. Thus

$$DN_{kj}\left(Y_j^{\ell,h} + Y_j^{\ell,v}\right) = D\phi_{kj}(h_j) Y_j^{\ell,h}(h_j, v_j) + t_{kj}(h_j) Y_j^{\ell,v}(h_j, v_j) + \sum_{j=1}^n \sum_{r,s=1}^d \frac{\partial t_{kj,rs}(h_j)}{\partial h_{j,m}} Y_{j,m}^{\ell,h}(h_j, v_j) v_{j,s} \frac{\partial}{\partial v_{k,r}},$$

where the last term is in $C^0(U_k \cap U_j, N_C \otimes S^{\ell+1}N_C^*)$. When N_C is flat, we see that $DN_{kj}Y_j = Y_k$ if and only if $DN_{kj}Y_j^\ell = Y_k^\ell$ for each ℓ and that the latter holds if and only if

$$Y_k^{\ell,h}(\phi_{kj}(h_j), t_{kj}v_j) = D\phi_{kj}(h_j)Y_j^{\ell,h}(h_j, v_j), \quad Y_k^{\ell,v}(\phi_{kj}(h_j), t_{kj}v_j) = t_{kj}Y_j^{\ell,v}(h_j, v_j). \tag{2.30}$$

In other words, $\{Y_j^\ell\}_j$ defines a global section of $T_C M \otimes S^\ell N_C^*$. □

Lemma 2.13 *Let F_j be a formal automorphism of $\tilde{V}_j$ in N_C, which is tangent to identity. Then, F_j is the time-1 map of a unique formal vector field Y_j in $\tilde{V}_j$, vanishing on U_j up to order ≥ 2.*

Proof Let F_j be given by

$$\tilde{h}_j = h_j + \sum_{|\alpha|\geq 2} A_{j,\alpha}(h_j)v_j^\alpha, \quad \tilde{v}_j = v_j + \sum_{|\beta|\geq 2} B_{j,\beta}(h_j)v_j^\beta.$$

Drop the index j. We want to express it as the time-1 map of a tangent vector field

$$Y = \sum_{\ell\geq 2}\left\{\sum_{m=1}^{n} Y_m^{\ell,h}(h, v)\frac{\partial}{\partial h_m} + \sum_{r=1}^{d} Y_r^{\ell,v}(h, v)\frac{\partial}{\partial v_r}\right\},$$

where $Y_m^{\ell,h}(h, v)$, $Y_r^{\ell,v}(h, v)$ are homogeneous polynomials in v of degree ℓ. The flow of Y with time θ is given by

$$h_m^\theta = h_m + \sum_{|\alpha|\geq 2} A_{m,\alpha}^\theta(h)v^\alpha, \quad v_r^\theta = v_r + \sum_{|\alpha|\geq 2} B_{r,\alpha}^\theta(h)v^\alpha,$$

where A^θ, B^θ satisfy $A^0 = B^0 = 0$ and

$$\sum_{|\alpha|\geq 2} v_j^\alpha \frac{dA_{m,\alpha}^\theta(h_j)}{d\theta} = \sum_{\ell\geq 2} Y_m^{\ell,h}(h^\theta, v^\theta), \quad \sum_{|\alpha|\geq 2} v_j^\alpha \frac{dB_{r,\alpha}^\theta(h)}{d\theta} = Y_r^{\ell,v}(h^\theta, v^\theta).$$

Inductively, we can verify that $A_{m,\alpha}^1 - Y_{m,\alpha}^h$, $B_{m,\alpha}^1 - Y_{r,\alpha}^v$ are uniquely determined by $Y_{m',\beta}^{\ell,h}$, $Y_{r',\beta}^{\ell,v}$ with $\ell < |\alpha|$. □

Note that the formal time-1 mapping of $DN_{kj}(Y_j)$ on $\tilde{V}_k \cap \tilde{V}_j$ can also be defined and it equals $N_{kj}F_jN_{kj}^{-1}$ where F_j is the time-1 map of Y_j. Thus the uniqueness assertion in the lemma implies the following.

Proposition 2.14 *Any automorphism F of a formal neighborhood of C in N_C, which is tangent to identity, is the time-1 map of a unique vector field defined on a formal* *neighborhood of C in N_C and vanishing on C. Assume further that N_C is flat. Then any tangent vector field Y of N_C that vanishes on C to order two admits a decomposition*

$$Y=\sum_{\ell\geq 2}Y^{\ell},\quad Y^{\ell}\in H^0(C,T_CM\otimes S^{\ell}N_C^*).$$

We write $\delta_m=(\delta_m^h,\delta_m^v)$ corresponding to the open covering $\mathcal{U}$ and the splitting $T_CM\otimes S^mN_C^*=(TC\otimes S^m)\oplus(N_C\otimes S^mN_C^*)$. Let us set $\mathcal{G}_m:=\mathrm{Range}(\delta_m)$. We have a decomposition

$$Z^1(\mathcal{U},T_CM\otimes S^mN_C^*)=\mathcal{G}_m\oplus\mathcal{N}_m \tag{2.31}$$

where $\mathcal{N}_m\simeq H^1(\mathcal{U},TM_C\otimes S^mN_C^*)$. Let $C^0(\mathcal{U},TM_C\otimes S^mN_C^*)=\mathcal{R}_m\oplus\ker\delta_m$ with $\delta_m(\mathcal{R}_m)=\mathcal{G}_m$. We emphasize that the decomposition (2.31) is not unique. For our convergence result, a natural decomposition will be given via a possibly non-unique minimizing solution. Consequently, $\oplus$ is interpreted as merely a decomposition suitable for convergence proof.

Lemma 2.15 *Suppose that N_C is flat. Any formal transformation F_j of $\tilde{V}_j$ which is tangent to identity can be uniquely factorized as*

$$F_j=G_j^{-1}\circ H_j$$

where $H_j-I\in\sum_{m\geq 2}\mathcal{R}^m$, G_j is an automorphism of $\tilde{V}_j$, and terms of order m in G_j, H_j are uniquely determined by the terms of order at most m in F_j. Furthermore, $G_iN_{ik}=N_{ik}G_k$ for all i,k.

Proof We know that $F_j=\exp\sum_m C_j^m$ is the time-1 map of $\sum_{m\geq 2}C_j^m$.

We want to decompose

$$\exp\sum_m C_j^m=\left(\exp\sum_m A_j^m\right)\left(I+\sum_m H_j^m\right).$$

By Campbell–Hausdorff formula, we are led to the equation

$$H_j^m=C_j^m-A_j^m+E_j^m$$

where E_j^m depends only on C_j^{ℓ},A_j^{ℓ} for $\ell<m$. We determine A_j^m, B_j^m by decomposing C_j^m and E_j^m as follow : Let π be the (non-canonical) projection from $C^0(\mathcal{U},TM_C\otimes S^mN_C^*)$ onto $\ker\delta_m$. Let $\{A_j^m\}_j:=\pi(\{C_j^m+E_j^m\})$. Then $\{H_j^m\}\in\mathcal{R}_m$. □

Next, we study the dependence of cohomology classes of $[\phi_{kj}^h]^{\ell}$, $[\phi_{kj}^v]^{\ell}$ in coordinates. We first consider the full set of linear cohomological equations.

2.10 Formal Coordinates in the Absence of Formal Obstructions

For a power series $u(z_j, w_j)$, let $u^{\leq m}(z_j, w_j)$ be the Taylor polynomial of u about $w_j = 0$ with degree m. Thus we can define

$$u = u^{\leq m} + u^{>m}, \quad u^{>m}(z_j, w_j) = O(|w_j|^{m+1}), \quad [u]^m = u^{\leq m} - u^{<m},$$
$$[u]^m_\ell = u^{\leq m} - u^{<\ell}.$$

To describe the coboundary operator in next lemma, we define the linear operator $\widetilde{D}$ by

$$((\widetilde{D}u)f)(h_j, v_j) := \frac{\partial u}{\partial h_j}(h_j, 0) f^h(h_j, v_j) + \frac{\partial u}{\partial v_j}(h_j, 0) f^v(h_j, v_j),$$

for a function $u(h_j, v_j)$. The standard differential D is given by

$$((Du)f)(h_j, v_j) = \frac{\partial u}{\partial h_j}(h_j, v_j) f^h(h_j, v_j) + \frac{\partial u}{\partial v_j}(h_j, v_j) f^v(h_j, v_j).$$

Thus

$$(Du - \widetilde{D}u) f(h_j, v_j) = (Du(h_j, v_j) - Du(h_j, 0)) f(h_j, v_j). \tag{2.32}$$

For a multiindex $\alpha = (\alpha_h, \alpha_v)$, define

$$(\tilde{D}^\alpha u)(h_j) = \left\{ \frac{\partial^{|\alpha|} u}{\partial h_j^{\alpha_h} \partial v_j^{\alpha_v}} \right\} (h_j, 0).$$

Lemma 2.16 *Let $\Phi_{kj} = N_{kj} + \phi_{kj}$ satisfy condition L_m with $m \geq 1$. Suppose that $F_j(h_j, v_j) = (h_j, v_j) + f_j(h_j, v_j)$ with $f_j(h_j, v_j) = O(|v_j|^2)$ are formal mappings such that $\{F_k^{-1} \Phi_{kj} F_j\} \in L_m$. Then, on $U_j \cap U_k$, $l = 2, \ldots, m$,*

$$\begin{aligned}(\delta\{[f_j]^{\leq l}\})_{kj}(h_j, v_j) = &-\Big[N_{kj}((I + [f_j]^{\leq l-2})(h_j, v_j)) - N_{kj}(h_j, v_j) \\ &- DN_{kj}(h_j, v_j)[f_j]^{\leq l-2}(h_j, v_j)\Big]^{\leq l} - \Big(0, (Dt_{kj}(h_j)[f_j^h]^{\leq l-1}(h_j, v_j))v_j\Big).\end{aligned} \tag{2.33}$$

(a) *If $f_j(h_j, v_j) = O(|v_j|^{m+1})$ for all j, then $N_{kj} + \tilde{\phi}_{kj} = F_k^{-1} \Phi_{kj} F_j + O(|v_j|^{2m+1})$ hold if and only if on $U_j \cap U_k$*

$$(\delta\{[f_i]^{\leq 2m}\})_{kj} = [\tilde{\phi}_{kj} - \phi_{kj}]^{\leq 2m} - \Big(0, (Dt_{kj}(h_j)[f_j^h]^{\leq 2m-1})v_j\Big). \tag{2.34}$$

(b) *If* $\{F_j\}$ *defines a germ of biholomorphism of order* m *at the zero section of the normal bundle, i.e.*

$$F_k^{-1}N_{kj}F_j(h_j,v_j)=N_{kj}(h_j,v_j)+O(|v_j|^{m+1})$$

and if $f_j^h(h_j,v_j)=O(|v_j|^m)$*, then* $\mathcal{V}_j^{\le m}(h_j,v_j):=(h_j,v_j+[f_j^v]^{\le m})$ *preserves* $\{N_{kj}\}$.

(c) *Suppose* $F_k^{-1}\Phi_{kj}F_j\in L_{2m}$*. Assume further that either* N_C *is flat or*

$$H^0(C,TC\otimes S^pN_C^*)=0,\quad 2\le p\le 2m. \tag{2.35}$$

Then there exist $\hat{F}_j=I+O(|v_j|^{m+1})$ *where* $[\hat{F}_j^h]_{m+1}^{2m}$ *are uniquely determined by* $[\Phi_{kj}]_{m+1}^{2m}$ *such that* $\hat{F}_k^{-1}\Phi_{kj}\hat{F}_j\in L_{2m}$*. There exists a unique decomposition* $\{\hat{F}_j=\mathcal{H}_j\circ\mathcal{V}_j\circ\tilde{F}_j\}$ *in the form*

$$\mathcal{H}_j(h_j,v_j)=(h_j+H_j(h_j,v_j),v_j), \tag{2.36}$$
$$\mathcal{V}_j(h_j,v_j)=(h_j,v_j+V_j(h_j,v_j)), \tag{2.37}$$
$$[\tilde{F}_j]^i=0,\ \forall 2\le i\le 2m,\quad [H_j]^\ell=[V_j]^\ell=0,\ \forall \ell>2m. \tag{2.38}$$

Furthermore, $[H_j]^\ell=[V_j]^\ell=0$ *for* $\ell\le m$*, and* H_j *are uniquely determined by*

$$(\delta^h\{H_i\})_{kj}=-[\phi_{kj}^h]^{\le 2m}. \tag{2.39}$$

Moreover, $\tilde{\phi}_{kj}=\mathcal{H}_k^{-1}\Phi_{kj}\mathcal{H}_j-N_{kj}$ *satisfy* $\tilde{\phi}_{kj}^h(h_j,v_j)=O(|v_j|^{2m+1})$ *and* $\tilde{\phi}_{kj}^v(h_j,v_j)=O(|v_j|^{m+1})$*, and* V_i *satisfy*

$$(\delta^v\{V_i\})_{kj}=-[\tilde{\phi}_{kj}^v]^{\le 2m}. \tag{2.40}$$

Proof Let $\Phi_{kj}=N_{kj}+\phi_{kj}$ and $\tilde{\Phi}_{kj}=N_{kj}+\tilde{\phi}_{kj}$. Suppose that both ϕ_{kj} and $\tilde{\phi}_{kj}$ are of order $\ge m+1$ (i.e. $O(|v_j|^{m+1})$) and $F_k\Phi_{kj}=\tilde{\Phi}_{kj}F_j$. Recall that $F_k=I+f_k$. To use the coboundary operator, we write

$$\begin{aligned} f_k(N_{kj})-\widetilde{D}N_{kj}f_j+\phi_{kj}-\tilde{\phi}_{kj} &= \underbrace{\big(f_k(N_{kj}-f_k(N_{kj}+\phi_{kj}))\big)}_{A} \\ &\quad+\underbrace{\Big(\tilde{\phi}_{kj}(I+f_j)-\tilde{\phi}_{kj}\Big)}_{B} \\ &\quad+\underbrace{\big(N_{kj}(I+f_j)-N_{kj}-\widetilde{D}N_{kj}f_j\big)}_{C}. \end{aligned} \tag{2.41}$$

Since f_j has order ≥ 2 at $v_j = 0$, by the Taylor expansion at N_{kj} and at I, respectively, both A and B are of order $\geq m+2$ (w.r.t v_j) at the origin. For the same reason, the C is of order ≥ 4. We recall that, for each $\ell \in \mathbb{N}^*$, the coboundary operator δ sends $C^0(\mathcal{U}, T_C M \otimes S^\ell(N_C^*))$ into $C^1(\mathcal{U}, T_C M \otimes S^\ell(N_C^*))$ as sections. It is defined in coordinates by

$$(\delta f)_{kj} = \widetilde{D} N_{kj} f_j(h_j, v_j) - f_k(N_{kj}(h_j, v_j))$$

on $U_j \cap U_k$ when $f = \{f_j\} \in C^0(\mathcal{U}, T_C M \otimes S^\ell(N_C^*))$. As δ preserves the degree ℓ of f_j in v_j, we shall omit its dependence in ℓ. Truncating the Taylor expansion of (2.41) at $v_j = 0$ up to degree m will lead to the first point.

Since $f_j(h_j, v_j) = O(|v_j|^2)$, then A, B are of order $\geq m+1$. Using (2.32), we obtain

$$\begin{aligned} C &= N_{kj}(I + f_j(h_j, v_j)) - N_{kj}(h_j, v_j) - DN_{kj}(h_j, v_j) f_j(h_j, v_j) \\ &\quad + (DN_{kj}(h_j, v_j) - DN_{kj}(h_j, 0)) f_j(h_j, v_j). \end{aligned}$$

We have $(DN_{kj}(h_j, v_j) - DN_{kj}(h_j, 0)) f_j(h_j, v_j) = (0, Dt_{kj}(h_j) f_j^h(h_j, v_j) v_j)$. Thus,

$$C = (0, (Dt_{kj}(h_j) f_j^h(h_j, v_j) v_j) + a(1) - a(0) - a'(0)$$

with $a(\lambda) = N_{kj}(h_j + \lambda f_j^h, v_j + \lambda f_j^v)$. Note that

$$\begin{aligned} a(1) - a(0) - a'(0) &= \int_0^1 (1-\lambda) a''(\lambda)\, d\lambda \\ &= \sum_{|\alpha|=2} \frac{|\alpha|!}{\alpha!} \int_0^1 (1-\lambda) D^\alpha N_{kj}(I + \lambda f_j) f_j^\alpha \, d\lambda \\ &= \sum_{|\alpha|=2} \frac{|\alpha|!}{\alpha!} \int_0^1 (1-\lambda) D^\alpha N_{kj}(I + \lambda [f_j]^{\leq m-2})([f_j]^{\leq m-2})^\alpha \, d\lambda + O(|v_j|^{m+1}) \\ &= b(1) - b(0) - b'(0) + O(|v_j|^{m+1}) \end{aligned}$$

for $b(\lambda) = N_{kj}(I + \lambda [f_j]^{\leq m-2})$. This proves (2.33).

For point (a), we use (2.41) again. This time, we have $A(h_j, v_j) = O(|v_j|^{2m+1})$ and $B(h_j, v_j) = O(|v_j|^{2m+1})$, while $C = (0, DN_{kj}(h_j)[f_j^h]^{\leq 2m-1} v_j) + O(|v_j|^{2m+1})$. We have derived (2.34).

For point (b), note that $F_k^{-1} N_{kj} F_j = N_{kj} + O(|v_j|^{m+1})$ is equivalent to $F_k N_{kj} = N_{kj} F_j + O(|v_j|^{m+1})$. From the vertical components, we obtain

$$t_{kj}(h_j) v_j + f_k^v(\varphi_{kj}(h_j), t_{kj}(h_j) v_j) = t_{kj}(h_j + f_j^h)(v_j + f_j^v(h_j, v_j)) + O(|v_j|^{m+1}).$$

Since $f_j^h = O(|v_j|^m)$ and $f_j^v = O(|v_j|^2)$, the m-jet (w.r.t. v_j) above reads

$$t_{kj}(h_j)v_j + [f_k^v]^{\leq m}(\varphi_{kj}(h_j), t_{kj}(h_j)v_j) = t_{kj}(h_j)(v_j + [f_j^v]^{\leq m}(h_j, v_j)).$$

That is that $\mathcal{V}_k^{\leq m} N_{kj} = N_{kj}\mathcal{V}_j^{\leq m}$, as $\mathcal{V}_j^{\leq m}(h_j, v_j) = (h_j, v_j + [f_j]^{\leq m}(h_j, v_j))$.

The point (c) follows from Proposition 2.14 when N_C is flat. For the remaining case, it follows from points (a) and (b) as follows.

By (2.34) and $H^0(C, TC \otimes S^\ell N_C^*) = 0$, we obtain $[f_j^h]_2^m = 0$. By (b), we know that $[F_j]^{\leq m}$ preserve N_{kj}. Then $\hat{F}_j = F_j([F_j]^{\leq m})^{-1}$ meet the requirement. The uniqueness of $[\hat{F}_j^h]^\ell$ for $m < \ell \leq 2m$ follows from the assumption on H^0 too.

We are seeking a unique decomposition $F_j = \mathcal{H}_j \circ \mathcal{V}_j \circ \tilde{F}_j$. Let us write $F_k^{-1}\Phi_{kj}F_j = N_{kj} + \tilde{\phi}_{kj}$ with $\tilde{\phi}_{kj} = O(|v_j|^{2m+1})$. From the horizontal component of (2.34) in which $[\tilde{\phi}_{kj}^h]^{\leq 2m} = 0$ and condition (2.35), we *uniquely* determine $\{[f_j^h]^{\leq 2m}\}$. Take $\mathcal{H}_j(h_j, v_j) = (h_j + [f_j]^{\leq 2m}(h_j, v_j), v_j)$. Then

$$\mathcal{H}_k^{-1}\Phi_{kj}\mathcal{H}_j(h_j, v_j) = (\varphi_{kj}(h_j), t_{kj}(h_j)v_j + \tilde{\phi}_{kj}^v(h_j, v_j)) + O(|v_j|^{2m+1}). \tag{2.42}$$

We still have $(\mathcal{H}_k^{-1}F_k)^{-1}(\mathcal{H}_k^{-1}\Phi_{kj}\mathcal{H}_j)(\mathcal{H}_j^{-1}F_j) \in L_{2m}$. We have

$$\mathcal{H}_j^{-1}F_j(h_j, v_j) = \mathcal{V}_j(h_j, v_j) + O(|v_j|^{2m+1}), \quad \mathcal{V}_j(h_j, v_j) = (h_j, v_j + V_j(h_j, v_j)), \tag{2.43}$$

where $\tilde{\phi}_{kj}^v, V_j$ contain only terms of orders ℓ in v_j for $m + 1 \leq \ell \leq 2m$.

Since $F_j = \mathcal{H}_j\mathcal{V}_j + O(|v_j|^{2m+1})$, we have

$$\mathcal{V}_k^{-1}(\mathcal{H}_k^{-1}\Phi_{kj}\mathcal{H}_j)\mathcal{V}_j \in L_{2m}.$$

From the vertical components of (2.42)–(2.43), and (2.34) in which we take $Dt_{kj}[f_j^h]^{\leq 2m-1} = 0$, we see that (2.34) becomes (2.40), i.e. $(\delta^v[V]^\ell)_{kj} = -[\tilde{\phi}_{kj}^v]^\ell$ for $\ell = m+1, \ldots, 2m$. To show the uniqueness of $[F_j]^{\leq 2m}$, we may assume that $\Phi_{kj} = N_{kj} + O(|v_j|^{2m+1})$. Then the uniqueness follows from the above arguments. □

The following is in Ueda [42], when both the dimension and codimension of C are one.

Lemma 2.17 *Let Φ_{kj} satisfy condition V_m with $m \geq 1$. Suppose that N_C is flat and $H^0(C, N_C \otimes S^\ell(N_C^*)) = 0$ for $1 < \ell \leq m$. Then $[\phi_{kj}^v]^{m+1} \in H^1(\mathcal{U}, N_C \otimes S^{m+1}(N_C^*))$ is independent of coordinates of the neighborhoods of C. Furthermore, there are formal biholomorphic mappings $F_j = I + (f_j^h, f_j^v)$ with $f_j(h_j, v_j) = O(|v_j|^2)$ satisfy*

$$\{F_k^{-1}\Phi_{kj}F_j\} \in V_{m+1} \tag{2.44}$$

if and only if $[\phi_{kj}^v]^{m+1} = 0$ *in* $H^1(\mathcal{U}, N_C \otimes S^{m+1}(N_C^*))$. *When* (2.44) *holds,* $\{\tilde{F}_k^{-1}\Phi_{kj}\tilde{F}_j\}$ *is still in* V_{m+1}, *for*

$$\tilde{F}_j(h_j, v_j) = (h_j, v_j + [f_j^v]^{m+1}(h_j, v_j)).$$

Proof Let $\tilde{\Phi}_{kj} := F_k^{-1}\Phi_{kj}F_j$. We want to show that

$$[\tilde{\phi}_{kj}^v]^{m+1} = [\phi_{kj}^v]^{m+1} \quad \text{in } H^1(\mathcal{U}, N_C \otimes S^{m+1}(N_C^*)),$$

provided that $\tilde{\Phi}_{kj}(h_j, v_j) = N_{kj}(h_j, v_j) + (\tilde{\phi}_{kj}^h, \tilde{\phi}_{kj}^v)$, $\Phi_{kj}(h_j, v_j) = N_{kj}(h_j, v_j) + (\phi_{kj}^h, \phi_{kj}^v)$, and

$$\tilde{\phi}_{kj}^v(h_j, v_j) = O(|v_j|^{m+1}), \quad \phi_{kj}^v(h_j, v_j) = O(|v_j|^{m+1}). \tag{2.45}$$

First, we have $F_j(h_j, v_j) = (h_j, v_j) + O(|v_j|^2)$. Suppose that $[f_j^v]^{\leq m_*-1} = 0$ for $2 \leq m_* \leq m$. Comparing vertical components of $\Phi_{kj} \circ F_j = F_k \circ \tilde{\Phi}_{kj}$, we obtain

$$\begin{aligned}\left[t_{kj} \cdot \left(v_j + f_j^v(h_j, v_j)\right)\right]^{\leq m_*} &= (\Phi_{kj}^v \circ F_j)^{\leq m_*}(h_j, v_j)\\ &= (F_k^v \circ \tilde{\Phi}_{kj})^{\leq m_*}(h_j, v_j) = (F_k^v)^{\leq m_*} \circ N_{kj}(h_j, v_j).\end{aligned}$$

Here the last identity is obtained from $\tilde{\Phi}_{kj}(h_j, v_j) - N_{kj}(h_j, v_j) = O(|v_j|^2)$, $[F_j^v]^{\leq m_*}(h_j, v_j) = v_j + [f_j^v]^{m_*}$, and (2.45). Looking at terms of order m_* in w_j, we see that $\{[f_j^v]^\ell\}$ is a global section of $N_C \otimes S^\ell(N_C^*)$ for $\ell = m_*$. This shows that $[f_j^v]^{\leq m_*} = 0$ and we can take $m_* = m$, i.e. $[f_j^v]^{\leq m} = 0$.

We also have $[\Phi_{kj}^v F_j]^{m+1} = t_{kj}[f_j^v]^{m+1} + [\phi_{kj}^v]^{m+1}$ and $[F_k^v \tilde{\Phi}_{kj}]^{m+1} = [f_k^v]^{m+1} \circ N_{kj} + [\tilde{\phi}_{kj}^v]^{m+1}$. This shows that

$$[\tilde{\phi}_{kj}^v]^{m+1} - [\phi_{kj}^v]^{m+1} = t_{kj}[f_j^v]^{m+1} - [f_k^v]^{m+1} \circ N_{kj}. \tag{2.46}$$

The latter is equivalent to $[\tilde{\phi}_{kj}^v]^{m+1} = [\phi_{kj}^v]^{m+1}$ in $H^1(\mathcal{U}, N_C \otimes S^{m+1}(N_C^*))$, which follows from Lemma 2.7 (b). The last assertion is equivalent to (2.46) with $[\tilde{\phi}_{kj}^v]^{m+1} = 0$. □

3 A Majorant Method for the Vertical Linearization

Let C be an n-dimensional complex compact manifold embedded in an $(n + d)$-dimensional complex manifold. We assume that the normal bundle N_C is (flat and) *unitary*. Let $\{t_{kj}\}$ be its transition (constant) matrices in a suitable covering $\mathcal{U} = \{U_j\}$ of C, we have $t_{kj}t_{kj}^* = \mathrm{Id}$. Let $K(N_C \otimes S^m(N_C^*))$ be the "norm" of the cohomological operator acting on $C^0(\mathcal{U}, N_C \otimes S^m(N_C^*))$ as defined in Theorem A.12. Let us consider the sequence of numbers $\{\eta_m\}_{m\geq 1}$ with $\eta_1 = 1$ and

$$\eta_m = K(N_C \otimes S^m(N_C^*)) \max_{m_1+\cdots+m_p+s=m} \eta_{m_1}\cdots\eta_{m_p}, \quad m > 1, \tag{3.1}$$

where $1 \leq m_i < m$ for all i and $s \in \mathbb{N}$.

In this section, we shall prove the following

Theorem 3.1 *Let C be a compact complex submanifold in M with $T_C M = TC \oplus N_C$. Assume that the embedding is vertically linearizable by a formal holomorphic mapping which is tangent to the identity or that $H^1(C, N_C \otimes S^\ell(N_C^*)) = 0$ for all $\ell \geq 2$. We also assume that N_C is unitary flat and that $H^0(C, N_C \otimes S^\ell(N_C^*)) = 0$ for all $\ell \geq 2$. Assume that for the η_m defined above, there are positive constants L_0, L such that $\eta_m \leq L_0 L^m$ for all m. Then the embedding is actually holomorphically vertically linearizable.*

Remark 3.2 In the previous Theorem 3.1, if a neighborhood of C is formally vertically linearizable by a minimizing vertical mapping which is tangent to the identity and preserves the splitting of $T_C M$, then the assumption "$H^0(C, N_C \otimes S^\ell(N_C^*)) = 0$, $\ell > 1$" is not necessary. Here by a formal *minimizing vertical* mapping it means a map of the form $(h_j, v_j + f_j^v(h_j, v_j))$ with $\{f_j^v\} \in C^0(C, \bigoplus_{\ell\geq 2} N_C \otimes S^\ell(N_C^*))$ such that each $\{[f_j^v]^\ell\}_j$ is a possibly non-unique Donin (minimizing) solution of a suitable cohomology equation.

Corollary 3.3 *Under assumptions of Theorem 3.1, there exists, in a neighborhood of C in M, a smooth holomorphic d-dimensional foliation having C as a leaf.*

Proof According to Theorem 3.1, there is a neighborhood of the C in M with suitable holomorphic coordinates patches $(V_j, (h_j, v_j))$ with $(h_j, v_j) \in \mathbb{C}^n \times \mathbb{C}^d$ and $C \cap V_j = \{v_j = 0\}$, such that, on $V_j \cap V_k$, we have

$$v_k = t_{kj} v_j, \quad h_k = \tilde{\varphi}_{kj}(h_j, v_j).$$

We then define the foliation in chart V_j by $dv_j = 0$. □

The rest of the section is devoted to the proof of Theorem 3.1. We follow the method of majorant developed by Ueda [42] for 1-dimensional unitary normal bundle over compact complex curve.

3.1 Conjugacy Equations and Cohomological Equations

Let us first recall (2.23) and (2.22):

$$\mathcal{L}_{kj}^v(f_j^v) = \phi_{kj}^v(h_j, v_j + f_j^v) - \left(f_k^v(\hat{\Phi}_{kj}^h(h_j, v_j), t_{kj} v_j) - f_k^v(\varphi_{kj}(h_j), t_{kj} v_j)\right) \tag{3.2}$$

where

$$\hat{\Phi}^h_{kj}(h_j, v_j) = \varphi_{kj}(h_j) + \phi^h_{kj}(h_j, v_j + f^v_j),$$
$$\mathcal{L}^v_{kj}(f^v_j) = f^v_k(\varphi_{kj}(h_j), t_{kj}v_j) - t_{kj}f^v_j.$$

Let us expand $\phi^h_{kj}(h_j, v_j + f^v_j)$ in power of v_j by using

$$\phi^h_{kj}(h_j, w_j) =: \sum_{Q\in\mathbb{N}^d_2} \phi^h_{kj,Q}(h_j)w_j^Q$$
$$\phi^h_{kj}(h_j, v_j + f^v_j(h_j, v_j)) =: \sum_{Q\in\mathbb{N}^d_2} h'_{kj,Q}(h_j)v_j^Q =: h'_{kj}(h_j, v_j).$$

We have

$$\sum_{Q\in\mathbb{N}^d_2} h'_{kj,Q}(h_j)v_j^Q = \sum_{Q\in\mathbb{N}^d_2} \phi^h_{kj,Q}(h_j)(v_j + f^v_j(h_j, v_j))^Q. \tag{3.3}$$

Let us also set

$$\sum_{Q\in\mathbb{N}^d_2} h''_{kj,Q}(h_j)v_j^Q := f^v_k(\hat{\Phi}^h_{kj}(h_j, v_j), t_{kj}v_j) - f^v_k(\varphi_{kj}(h_j), t_{kj}v_j).$$

As we shall see below, the functions $[h']^m$ and $[h'']^m$ are defined by induction on $m \geq 2$ as they depend on $[f]^l$, $l = 2, \ldots, m-1$.

Therefore, the homogeneous polynomial of degree $m \geq 2$ of the Taylor expansion of solution of the conjugacy equation satisfies

$$\mathcal{L}^v_{kj}([f^v_j]^m) = [h'_{kj}]^m + [h''_{kj}]^m. \tag{3.4}$$

According to Lemma 2.17, there is a solution to the above equation either by the formal assumption or by the assumption that the cohomology class of $[h'_{kj}]^m + [h''_{kj}]^m$ is 0, i.e. it is a coboundary. Indeed, since the normal bundle is flat, this class is independent of the coordinates system and the neighborhood is formally vertically linearizable.

3.2 A Modified Fischer Norm for Symmetric Powers

We define a scaler product on the space of polynomials $\mathbb{C}[x_1, \ldots, x_d]$ as follows. First, we set

$$\left\langle x^R, x^Q \right\rangle_{\text{mf}} := \begin{cases} \frac{(r_1!)\cdots(r_d!)}{|R|!} & \text{if } R = Q \\ 0 & \text{otherwise} \end{cases}, \quad \left| \sum_Q C_Q x^Q \right|^2_{\text{mf}} := \sum_Q |C_Q|^2 \frac{Q!}{|Q|!}, \tag{3.5}$$

where $R = (r_1, \ldots, r_d)$ and $|R| = r_1 + \cdots + r_d$, and C_Q are constants. The subscript mf stands for "modified Fischer". The associated norm will be denoted by $|.|_k$. The Fischer (resp. modified Fischer) scalar product has been used in [10,24,40] (resp. [30]). Let ω be an open set on $\mathbb{C}^n$. For a vector of polynomials $g = (g_1, \ldots, g_k) \in \mathcal{O}^k(\omega) \otimes \mathbb{C}[x_1, \ldots, x_d]$, we set

$$|g|^2_{\mathrm{mf},\omega} := \sup_{z\in\omega} |g(z,\cdot)|^2_{\mathrm{mf}} := \sup_{z\in\omega} \sum_{j=1}^{k} \sum_{Q\in\mathbb{N}^d} \frac{Q!}{|Q|!} |g_{j,Q}(z)|^2. \tag{3.6}$$

We now apply the Fischer norm (resp. modified Fischer norm) to $f \in C^q(\mathcal{U}, E \otimes S^L N_C^*)$. Returning to notation in (2.27), we write

$$f_{i_0\ldots i_q}(p) = \sum_{\lambda=1}^{\mathrm{rank}\, E} \sum_{|Q|=L} f^{\lambda}_{i_0\ldots i_q;Q}(z_{i_q}(p)) e_{i_0,\lambda}(p) \otimes (w^*_{i_q}(p))^Q,$$

where e_{i_0} is the base of E over U_{i_0} and $w^*_{i_q}$ is the base of N_C^* on U_{i_q}. Define

$$|f|^2_{\mathrm{mf},\mathcal{U}} := \max_{(i_0,\ldots,i_q)\in\mathcal{I}^{q+1}} \sup_{z_{i_q}\in\varphi_{i_q}(U_{i_0\ldots i_q})} \sum_{\lambda=1}^{\mathrm{rank}\, E} \sum_{Q} \frac{Q!}{|Q|!} \left| f^{\lambda}_{i_0\cdots i_q;Q}(z_{i_q}) \right|^2. \tag{3.7}$$

When there is no confusion, we shall in the sequel write "f" instead of "mf". The following two propositions are a "version with parameters" of [30, Propositions 3.6–3.7] (see also [24]). We only give the proof of the last two points of next proposition.

Proposition 3.4 *Let $\mathcal{O}_n(\omega) \otimes \mathbb{C}[x_1, \ldots, x_d]$ be the set of polynomials $f(x, z)$ in x with coefficients holomorphic in $z \in \omega \subset \mathbb{C}^n$.*

(a) *Let $f, g \in \mathcal{O}_n(\omega) \otimes \mathbb{C}[x_1, \ldots, x_d]$ be homogeneous polynomials of degree k, k', respectively. Then*

$$|fg|_{f,\omega} \le |f|_{f,\omega} |g|_{f,\omega}.$$

(b) *Let $f \in \mathcal{O}_n(\omega) \otimes \mathbb{C}[x_1, \ldots, x_d]$ and let $\tilde{f}_P(z, x) = \frac{1}{P!}\partial_z^P f(z, x)$. Then*

$$|\tilde{f}_P|_{f,\omega'} \le \frac{|f|_{f,\omega}}{(\mathrm{dist}_*(\omega', \partial\omega))^{|P|}}, \quad \forall \omega' \subset \omega,\ \mathrm{dist}_*(\omega', \partial\omega) := \mathrm{dist}(\omega', \partial\omega)/\sqrt{n}.$$

(c) *Let T be a $d \times d$ unitary matrix. Let $f \in \mathcal{O}_n^d(\omega) \otimes \mathbb{C}[x_1, \ldots, x_d]$. Then,*

$$|Tf|_{f,\omega} = |f|_{f,\omega}.$$

(d) *Let T be a $d \times d$ unitary matrix. Let $f \in \mathcal{O}_n(\omega) \otimes \mathbb{C}[x_1, \ldots, x_d]$ and $f^T(z, x) := f(z, Tx)$. Then,*

$$|f^T|_{f,\omega} = |f|_{f,\omega}.$$

Proof We only prove the last two points. Fix $z \in \omega'$. The polydisc center at z with radius $\delta := \operatorname{dist}(\omega', \partial\omega)/\sqrt{n}$ is contained in ω.

By the Cauchy formula, we have

$$\begin{aligned}\tilde{f}_P(z,x) &= \frac{1}{\delta^{|P|}}\int_{[0,2\pi]^n} f(z+\delta(e^{i\theta_1},\ldots,e^{i\theta_n}),x)(e^{i\theta_1},\ldots,e^{i\theta_n})^{-P}\,\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_n}{2\pi}\\ &= \frac{1}{\delta^{|P|}}\sum_{Q\in\mathbf{N}^d} x^Q \int_{[0,2\pi]^n} f_Q(z+\delta(e^{i\theta_1},\ldots,e^{i\theta_n}))(e^{i\theta_1},\ldots,e^{i\theta_n})^{-P}\,\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_n}{2\pi}.\end{aligned}$$

We emphasize that the sum is finite. By the Cauchy–Schwarz inequality applied to the integral, we have

$$\begin{aligned}|\tilde{f}_P(z,\cdot)|_{\mathrm{mf}}^2 &= \frac{1}{\delta^{2|P|}}\sum_{Q\in\mathbf{N}^d}|x^Q|_{\mathrm{mf}}^2\\ &\quad\times\left|\int_{[0,2\pi]^n} f_Q(z+\delta(e^{i\theta_1},\ldots,e^{i\theta_n}))(e^{i\theta_1},\ldots,e^{i\theta_n})^{-P}\,\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_n}{2\pi}\right|^2\\ &\le \frac{1}{\delta^{2|P|}}\sum_{Q\in\mathbf{N}^d}|x^Q|_{\mathrm{mf}}^2\int_{[0,2\pi]^n}|f_Q(z+\delta(e^{i\theta_1},\ldots,e^{i\theta_n}))|^2\,\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_n}{2\pi}\\ &= \frac{1}{\delta^{2|P|}}\int_{[0,2\pi]^n}\sum_{Q\in\mathbf{N}^d}|x^Q|_{\mathrm{mf}}^2|f_Q(z+\delta(e^{i\theta_1},\ldots,e^{i\theta_n}))|^2\,\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_n}{2\pi}\\ &\le \frac{1}{\delta^{2|P|}}\int_{[0,2\pi]^n}|f|_\omega^2\,\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_n}{2\pi} = \frac{1}{\delta^{2|P|}}|f|_\omega^2.\end{aligned}$$

For the last point, we have, for a homogeneous polynomial f in x of degree m with holomorphic coefficients in ω the identity:

$$|f_m|_\omega^2 = \frac{1}{\pi^d m!}\sup_{z\in\omega}\int_{\mathbb{C}^d}|f(z,x)|^2 e^{-|x|^2}\,dV(x).$$

In particular, the integral is invariant under the transformation $x \to Tx$ when T is unitary (and constant). □

Proposition 3.5 *For a formal power series $f(h,v) = \sum_k f_k(z,v)$ with $f_k(z,v)$ being a homogeneous polynomial in v of degree k of which the coefficients are functions holomorphic in $z \in U$, the following properties are equivalent:*

(a) *f is uniformly convergent for v in a neighborhood of the origin, uniformly in U.*
(b) *There exist $M, R > 0$ such that for every k, $|f_k|_{mf,U} \le \frac{M}{R^k}$.*

For convenience, we will use the following orthonormal Fischer base of $S^L N_C^*$:

$$e_{j,Q}^* = \sqrt{\frac{|Q|!}{Q!}}(w_j^*)^Q, \quad |Q| = L, \quad Q \in \mathbf{N}^d.$$

The transition matrices t^L_{kj} of $S^L N^*_C$ is then determined in the following way : Let $F_k = \sum_{|P|=L} F_{k,P} e^*_{k,P}$. We have

$$(F_{k,P})_{|P|=L} = t^L_{kj}(F_{j,P})_{|P|=L}.$$

This can be computed from the transition matrices of N^*_C by expressing the basis $w^*_{k,1}, \dots, w^*_{k,d}$ in terms of $w^*_{j,1}, \dots, w^*_{j,d}$. Since t^L_{kj} maps orthonormal basis into orthonormal basis, by Proposition 3.4 we know that t^L_{kj} are unitary matrices, i.e. in operator norm defined in (A.4),

$$|t^L_{kj}| = 1, \quad L = 1, 2, \dots. \tag{3.8}$$

We will apply results in the appendix to the transition matrices t^L_{kj}.

3.3 A Majorization in the Modified Fischer Norm for the Vertical Linearization

Let $\{f^v_j\}$ be the formal solution of (3.2). We use notation (3.7). Let $\varphi_j(U_j) = \Delta_n$ and $U_{kj} := U_k \cap U_j$. Define $\hat{U}_{kj} = \varphi_j(U_{kj})$. Then, $\varphi_{kj}(\hat{U}_{kj}) = \hat{U}_{jk}$. Let us first assume that $H^0(C, N_C \otimes S^\ell(N^*_C)) = 0$ for all $\ell \geq 2$. We shall see later on how to get rid of this assumption to prove the general result.

Let us assume that there exists a vertical formal transformation $F := \{F_j\}$ fixing C, being tangent to identity on it, that linearizes vertically a neighborhood of C in M. Let us write

$$F_j(h_j, v_j) := (h_j, v_j + f_j), \quad f_j = \sum_{k \geq 2} [f_j]^k, \quad \{[f_j]^k\} \in C^0(C, N_C \otimes S^k(N^*_C)).$$

Assume that there is a sequence $\{A_k\}_{k \geq 2}$ of positive numbers such that

$$\forall k < m \quad |[f_j]^k|_{\hat{U}_j} \leq \eta_k A_k. \tag{3.9}$$

Let us set

$$A(t) = \sum_{k \geq 2} A_k t^k$$

with $t \in \mathbb{C}$. Let us first estimate both $|[h'_{kj}]^m|_{\hat{U}_{kj}}$ and $|[h''_{kj}]^m|_{\hat{U}_{kj}}$ as defined in (3.4) in term of $J^{m-1}A(t) := A_2 t^2 + \cdots + A_{m-1} t^{m-1}$.

Since ϕ^h_{kj} is holomorphic in $h_j \in \hat{U}_{kj}$ and v_j in a neighborhood of the origin, we can assume that there is a positive R such that

$$\sup_{h_j \in \hat{U}_{kj}} |\phi^h_{kj,Q}(h_j)| \leq R^{|Q|}$$

for all $Q \in \mathbb{N}_2^d$, where $\phi_{kj,Q}^h$ is defined by (3.3) and $\mathbb{N}_k^d := \{Q \in \mathbb{N}^d : |Q| \geq k\}$.

For $Q \in \mathbb{N}_2^d$, we have

$$\left[(v_j + f_j^v(h_j, v_j))^Q\right]^m = \sum_{\substack{(m_{1,1},\ldots,m_{1,q_1},\ldots,m_{d,1},\ldots,m_{d,q_d}) \\ \sum_{i=1}^d m_{i,1}+\cdots+m_{i,q_i}=m}} \prod_{i=1}^d [f_{j,i}]^{m_{i,1}} \cdots [f_{j,i}]^{m_{i,q_i}}$$

where we have set $f_j^v = (f_{j,1}, \ldots, f_{j,d})$, $[f_{j,i}]^1 = v_{j,i}$ and $[f_{j,i}]^0 = 0$. In the following, all $m_{i,j}$ are positive integers. Hence, by the first point of Proposition 3.4, we have

$$\left|\left[(v_j + f_j^v(h_j, v_j))^Q\right]^m\right|_{\hat{U}_{kj}} \leq \sum_{\substack{(m_{1,1},\ldots,m_{1,q_1},\ldots,m_{d,1},\ldots,m_{d,q_d}) \\ \sum_{i=1}^d m_{i,1}+\cdots+m_{i,q_i}=m}} \prod_{i=1}^d |[f_{j,i}]^{m_{i,1}}|_{\hat{U}_j} \cdots |[f_{j,i}]^{m_{i,q_i}}|_{\hat{U}_j}. \tag{3.10}$$

Let $m \geq 2$, for $Q \in \mathbb{N}_2^d$, $|Q| \leq m$, and let us set

$$E_{Q,m} = \left\{(m_{1,1}, \ldots, m_{1,q_1}, \ldots, m_{d,1}, \ldots, m_{d,q_d}) \in \mathbb{N}_1^{|Q|} : \sum_{i=1}^d m_{i,1} + \cdots + m_{i,q_i} = m\right\}.$$

Let $M_i = (m_{1,1}^{(i)}, \ldots, m_{1,q_1^{(i)}}^{(i)}, \ldots, m_{d,1}^{(i)}, \ldots, m_{d,q_d^{(i)}}^{(i)}) \in \mathbb{N}_1^{|Q^{(i)}|}$ with $|Q^{(i)}| \leq m_i$ and $m_i = \sum_{j=1}^d m_{j,1}^{(i)} + \cdots + m_{j,q_j^{(i)}}^{(i)}$, $i = 1, 2$. Define the concatenation $M_1 \sqcup M_2$ to be (M_1, M_2). We also have $\sum_{j=1}^2 \sum_{i=1}^d m_{i,1}^{(j)} + \cdots + m_{i,q_i^{(j)}}^{(j)} = m_1 + m_2$. Hence, we emphasize that the concatenation

$$\left(\bigcup_{2\leq|Q_1|\leq m_1} E_{Q_1,m_1}\right) \sqcup \left(\bigcup_{2\leq|Q_2|\leq m_2} E_{Q_2,m_2}\right) \subset \bigcup_{2\leq|Q|\leq m_1+m_2} E_{Q,m_1+m_2}. \tag{3.11}$$

As a consequence, according to (3.3) and (4.4), we have

$$\begin{aligned}\left|\left[\sum_{Q\in\mathbb{N}^d,|Q|=m} h'_{kj,Q}(h_j)v_j^Q\right]^m\right|_{\hat{U}_{kj}} &\leq \sum_{|Q|=2}^m R^{|Q|} \sum_{M\in E_{Q,m}} \prod_{i=1}^d |[f_{j,i}]^{m_{i,1}}|_{\hat{U}_j} \cdots |[f_{j,i}]^{m_{i,q_i}}|_{\hat{U}_j} \\ &\leq \sum_{|Q|=2}^m R^{|Q|} \sum_{M\in E_{Q,m}} \prod_{i=1}^d \eta_{m_{i,1}} A_{m_{i,1}} \cdots \eta_{m_{i,q_i}} A_{m_{i,q_i}}\end{aligned}$$

$$
\begin{aligned}
&\leq \left[\sum_{|Q|=2}^{m} \eta_{Q,m} R^{|Q|}(t + J^{m-1}(A(t))^{|Q|} \right]^m \\
&\leq E_m[g_m(t)]^m, \qquad (3.12)
\end{aligned}
$$

where we have set

$$
\eta_{Q,m} := \max_{M \in E_{Q,m}} \left(\prod_{i=1}^{d} \eta_{m_{i,1}} \cdots \eta_{m_{i,q_i}} \right), \quad E_m := \max_{\substack{Q \in \mathbb{N}^d \\ 2 \leq |Q| \leq m}} \eta_{Q,m},
$$

$$
g_m(t) := \sum_{|Q|=2}^{m} R^{|Q|}(t + J^{m-1}(A(t))^{|Q|}, \quad g(t) := \sum_{|Q| \geq 2} R^{|Q|}(t + A(t))^{|Q|}.
$$

Hence, as formal power series, we have

$$
g(t) = \left(\frac{1}{1 - R(t + A(t))} \right)^d - dR(t + A(t)) - 1. \qquad (3.13)
$$

Let $\mathcal{U}^* = \{U_i^*\}$ be an open covering of C such that U_i^* is relatively compact in U_i. We shall write $\hat{U}_k^* := \varphi_k(U_k^*)$. Let us consider the index j as fixed and let us estimate the Fischer norm of h''_{kj} on $\hat{U}_{kj}^* := \varphi_j(U_j \cap U_k^*)$. We have

$$
\begin{aligned}
&\left[\sum_{Q \in \mathbb{N}^d, |Q|=m} h''_{kj,Q}(h_j) v_j^Q \right]^m \\
&= \sum_{\substack{P \in \mathbb{N}_1^n \\ m_1 + m_2 = m}} \frac{1}{P!} \left[\partial_h^P f_k(\varphi_{kj}(h_j), t_{kj} v_j) \right]^{m_1} \left[\left(\phi_{kj}^h(h_j, v_j + f_j^v) \right)^P \right]^{m_2} \\
&= \sum_{\substack{P \in \mathbb{N}_1^n \\ m_1 + m_2 = m}} \frac{1}{P!} \left[\partial_h^P f_k(\varphi_{kj}(h_j), t_{kj} v_j) \right]^{m_1} \left[\left(h'_{kj}(h_j, v_j) \right)^P \right]^{m_2}.
\end{aligned}
$$

Here, both indices m_1 and m_2 are ≥ 2. Since the Fischer norm is submultiplicative, we have

$$
\left| \left[\left(h'_{kj}(h_j, v_j) \right)^P \right]^{m_2} \right|_{\hat{U}_{kj}^*} \leq E_{m_2} \left[\left(\sum_{|Q|=2}^{\frac{m}{2}} R^{|Q|}(t + J^{m-1}(A(t))^{|Q|} \right)^{|P|} \right]^{m_2}.
$$

Indeed,

$$\left[\left(h'_{kj}(h_j,v_j)\right)^P\right]^{m_2} = \left[\prod_{i=1}^n (h'_{kj,i})^{p_i}\right]^{m_2}$$
$$= \sum_{\sum_i (m_{i,1}+\cdots+m_{i,p_i})=m_2} \prod_{i=1}^n [h'_{kj,i}]^{m_{i,1}} \cdots [h'_{kj,i}]^{m_{i,p_i}}.$$

According to (3.11) and by (3.12), we have

$$\left|\prod_{i=1}^n [h'_{kj,i}]^{m_{i,1}} \cdots [h'_{kj,i}]^{m_{i,p_i}}\right|_{\hat{U}^*_{kj}} \le \prod_{i=1}^n E_{m_{i,1}} \left[g_{m_{i,1}}(t)\right]^{m_{i,1}} \cdots E_{m_{i,p_i}} \left[g_{m_{i,p_i}}(t)\right]^{m_{i,p_i}}$$
$$\le \max_{2\le |Q|\le m_2} \eta_{Q,m_2} \prod_{i=1}^n \left[g_{m_{i,1}}(t)\right]^{m_{i,1}} \cdots \left[g_{m_{i,p_i}}(t)\right]^{m_{i,p_i}}.$$

Hence, we have

$$\sum_{\sum_i (m_{i,1}+\cdots+m_{i,p_i})=m_2} \left|\prod_{i=1}^n [h'_{kj,i}]^{m_{i,1}} \cdots [h'_{kj,i}]^{m_{i,p_i}}\right|_{\hat{U}^*_{kj}} \le E_{m_2}[g(t)^{|P|}]^{m_2}.$$

We have, by definition $\left[\partial_h^P f_k(\varphi_{kj}(h_j), t_{kj}v_j)\right]^{m_1} = \partial_h^P [f_k]^{m_1}(\varphi_{kj}(h_j), t_{kj}v_j)$.

Recall that the Fischer norm is unitary invariant and by Proposition 3.4, we have

$$\left|\partial_h^P [f_k]^{m_1}(\varphi_{kj}(h_j), t_{kj}v_j)\right|^2_{\hat{U}^*_{kj}} = \left|\partial_h^P [f_k]^{m_1}(\varphi_{kj}(h_j), v_j)\right|^2_{\hat{U}^*_{kj}}$$
$$\le \left(\frac{P!}{\mathrm{dist}_*(\hat{U}^*_k, \partial \hat{U}_k)^{|P|}}\right)^2 |[f_k]^{m_1}|^2_{\hat{U}_k}.$$

Let us set $M := \inf_k \mathrm{dist}(\hat{U}^*_k, \partial\hat{U}_k)$. As a consequence, we have

$$\left|\left[\sum_{Q\in\mathbb{N}^d, |Q|=m} h''_{kj,Q}(h_j)v_j^Q\right]^m\right|_{\hat{U}^*_{kj}} \le \sum_{m_1+m_2=m} \sum_{\substack{P\in\mathbb{N}^n\\ |P|\ge 1}} \frac{1}{M^{|P|}} |[f_k]^{m_1}|_{\hat{U}_k} E_{m_2}[g(t)^{|P|}]^{m_2}$$
$$\le \sum_{m_1+m_2=m} |[f_k]^{m_1}|_{\hat{U}_k} \left[E_{m_2} \sum_{\substack{P\in\mathbb{N}^n\\ |P|\ge 1}} \left(\frac{g(t)}{M}\right)^{|P|}\right]^{m_2}$$
$$\le \left(\max_{m_1+m_2=m} \eta_{m_1} E_{m_2}\right) \left[A(t)\left(\left(\frac{M}{M-g(t)}\right)^n - 1\right)\right]^m. \tag{3.14}$$

Collecting estimates (3.12) and (3.14), we obtain

$$\left|\mathcal{L}^v_{kj}([f^v_j]^m)\right|_{\hat{U}^*_{kj}} \leq \left[E_m g(t) + \left(\max_{m_1+m_2=m} \eta_{m_1} E_{m_2}\right) A(t) \left(\left(\frac{M}{M-g(t)}\right)^n - 1\right)\right]^m.$$

Let us extend this to an estimate on $\hat{U}_{kj} = \varphi_j(U_j \cap U_k)$. Following again Ueda's argument [42] let us express the fact that $[h]^m := [h']^m + [h'']^m$ is a 1-cocycle with values in $N_C \otimes S^m(N_C^*)$. Let $p \in U_k \cap U_j$. Then $p \in U_k \cap U_j \cap U_i^*$ for some i. According to (3.4) and Lemma 2.7, at $p \in U_k \cap U_j \cap U_i^*$ we have

$$\begin{aligned} &t_{ki} \sum_{|Q|=m} h_{ik,Q}(z_k(p))(t_{kj}v_j)^Q - t_{ki} \sum_{|Q|=m} h_{ij,Q}(z_j(p))(v_j)^Q \\ &\quad + \sum_{|Q|=m} h_{kj,Q}(z_j(p))(v_j)^Q = 0. \end{aligned} \tag{3.15}$$

Here by (3.7) the Fischer norms of h_{kj} on all subdomains must be computed in the base e^v_k of N_C on U_k and the base w^*_j of N^*_C on U_j. We can apply the previous estimates (3.12) and (3.14) to the first two sums, respectively, on $\hat{U}^*_{ik}$ and $\hat{U}^*_{ij}$. To estimate the first sum, we need to change coordinates. From Sect. 2, t_{kj} (resp. s_{kj}) are transition matrices of N_C (resp. TC). Recall that $\{[h_{kj}]^m\} \in Z^1(\mathcal{U}^{r_*}, N_C \otimes S^m N^*_C)$ and

$$\begin{aligned} h_{ik}(p) &= \sum_{\lambda=1}^d \sum_{|Q|=m} h^\lambda_{ik;Q}(z_k(p)) e^v_{i,\lambda}(p) \otimes (w^*_k(p))^Q \\ &= \sum_{\lambda'=1}^d \sum_{\lambda=1}^d \sum_{|Q|=m} h^\lambda_{ik;Q}(z_k(p)) t^{\lambda'}_{ki,\lambda}(z_k(p)) e^v_{k,\lambda'}(p) \otimes (t_{kj} w^*_j(p))^Q =: \tilde{h}_{kj}(z_k(p), w^*_j). \end{aligned}$$

Thus, $\sum_{|Q|=m} h_{ik,Q}(z_k(p))(t_{kj}v_j)^Q = \tilde{h}_{kj}(z_k(p), v_j)$. By the unitary invariance by multiplication and composition of the Fischer norm and by definition (3.7), we have for fixed $z_k(p) \in \hat{U}^*_{ik}$,

$$\begin{aligned} |\tilde{h}_{kj}(z_k(p), v_j)|^2_{\mathrm{mf}} &= \sum_{\lambda'=1}^d \left| \sum_{|Q|=m} \left(\sum_{\lambda=1}^d t^{\lambda'}_{ki,\lambda}(z_k) h^\lambda_{ik;Q}(z_k) \right) (t_{kj}v_j)^Q \right|^2_{\mathrm{mf}} \\ &= \sum_{\lambda'=1}^d \left| \sum_{|Q|=m} \left(\sum_{\lambda=1}^d t^{\lambda'}_{ki,\lambda}(z_k) h^\lambda_{ik;Q}(z_k) \right) v_j^Q \right|^2_{\mathrm{mf}} \\ &= \sum_{\lambda'=1}^d \sum_{|Q|=m} \frac{Q!}{|Q|!} \left| \sum_{\lambda=1}^d t^{\lambda'}_{ki,\lambda}(z_k) h^\lambda_{ik;Q}(z_k) \right|^2 \\ &\leq \sum_{\lambda'} \sum_{|Q|=m} \frac{Q!}{|Q|!} \sum_{\lambda=1}^d \left| h^\lambda_{ik;Q}(z_k) \right|^2 \leq d |h_{ik}|^2_{\hat{U}^*_{ik}}, \end{aligned}$$

where the second last inequality is obtained by the Cauchy-Schwarz inequality. In a similar way, we have a similar estimate for the second sum in (3.15) on $\varphi_j(U_k \cap U_j \cap U_i^*)$. For the third sum in (3.15), we note that the entries of the unitary matrix t_{ki} have modulus at most one. Thus, there exist constants M', $\tilde{M}$ such that the third sum in (3.15) satisfies

$$\begin{aligned}|h_{kj}|_{\hat{U}_{kj}} &\leq M' \max_i (|h_{ik}|_{\hat{U}_{ik}^*} + |h_{ij}|_{\hat{U}_{ij}^*}) \\ &\leq \tilde{M} \max\left(E_m, \max_{\substack{m_1+m_2=m \\ m_1,m_2\geq 2}} \eta_{m_1} E_{m_2}\right)\left[g(t) + A(t)\left(\left(\frac{M}{M-g(t)}\right)^n - 1\right)\right]^m.\end{aligned}$$

We now adapt the estimate in Lemma A.2 (see also Theorem A.12). Recall that $[h_{kj}]^{\leq m}$ depends only on $[f]^{\leq m-1}$ and the hypothesis (3.9). By the formal assumption, we have a solution to (3.4):

$$\mathcal{L}_{kj}([f_j^v]^m) = [h_{kj}]^m.$$

By assumptions, $H^0(C, N_C \otimes S^\ell(N_C^*)) = 0$, for all $\ell \geq 2$. Hence, the solution of the previous equation is unique. By Lemma A.2, (A.5) and (3.8), the solution satisfies the estimate:

$$|\{[f_j^v]^m\}|_{\mathcal{U}} \leq C(1 + K_*(N_C \otimes S^m N_C^*)|)\{[h_{kj}]^m\}|_{\mathcal{U}}.$$

Here, C depends neither on N_C nor on $S^m N_C^*$. Therefore, we have

$$|[f_j^v]^m|_{\hat{U}_j} \leq K(N_C \otimes S^m(N_C^*)) \max_k \left|\mathcal{L}_{kj}^v([f_j^v]^m)\right|_{\hat{U}_{kj}^*}.$$

By definition (3.1), we have

$$K(N_C \otimes S^m(N_C^*)) \max\left(E_m, \max_{\substack{m_1+m_2=m \\ m_1,m_2\geq 2}} \eta_{m_1} E_{m_2}\right) \leq \eta_m.$$

Hence, we have

$$|[\{f^v\}]^m|_{\hat{U}} \leq \tilde{M}\eta_m \left[g(t) + A(t)\left(\left(\frac{M}{M-g(t)}\right)^n - 1\right)\right]^m. \tag{3.16}$$

Let us consider the functional equation

$$A(t) = \mathcal{F}(t, A(t)) := \tilde{M}\left(g(t) + A(t)\left(\left(\frac{M}{M-g(t)}\right)^n - 1\right)\right),$$

where $g(t)$ is a function of A by (3.13). This equation has a unique analytic solution vanishing at the origin at order 2.

We now can prove the theorem. Indeed by assumption, there are positive constants M, L such that $\eta_m \leq ML^m$ for all $m \geq 2$. Since $A(t)$ converges at the origin, then $A_m \leq D^m$ for some positive D. According to (3.16), we have also proved

$$|[\{f^v\}]^m|_{\hat{U}} \leq \eta_m A_m,$$

so that, finally, $|[\{f^v\}]^m|_{\hat{U}} \leq M(DL)^m$ for all $m \geq 2$. Hence, $f^v = \sum_{m\geq 2}[\{f^v\}]^m$ converges at the origin and this proves the theorem.

Let us see how we can prove Remark 3.2. The issue is that, when considering a solution $[f_j^v]^m$ of the cohomological equation $\mathcal{L}_{kj}([f_j^v]^m) = R^m$, the estimate given by Lemma A.2 and Proposition A.4 might be obtained by another solution. Hence, the formal solution might not be the good one for the estimate. Furthermore, we cannot replace a solution at degree m as we wish to ensure that higher order terms in the vertical component can be eliminated formally. We now explain the general result as formulated in the theorem. We will assume that there are formal mappings

$$\tilde{F}_j(h_j, v_j) = (h_j, v_j) + \left(0, \sum_{\ell>2} \tilde{f}^v_{j,\ell}(h_j, v_j)\right)$$

satisfying the following

(a) $\{\tilde{F}_k^{-1}\Phi_{kj}\tilde{F}_j - N_{kj}\}^v = 0$ for all k, j. In other words, $\{\tilde{F}_j\}$ formally linearizes Φ_{kj} vertically. In particular,

$$\{(\tilde{F}_k^m)^{-1}\Phi_{kj}\tilde{F}_j^m - N_{kj}\}^v = [\phi_{kj}^v]^m + R_{kj}^m(\{[\phi_{kj}]^\ell, [\tilde{f}_k^v]^\ell\}_{2\leq\ell<m}) + O(|v_j|^{m+1})$$

for

$$\tilde{F}_j^m = (h_j, v_j) + \left(0, \sum_{2\leq\ell\leq m} \tilde{f}^v_{j,\ell}(h_j, v_j)\right).$$

(The last assertion can be check easily since $(\tilde{F}_j^m)^{-1}\tilde{F}_j(h_j, v_j) = (h_j, v_j) + O(|v_j|^{m+1})$).

(b) Each $\{\tilde{f}^v_{j,m}\}_j$ is a "minimizer" in the sense that it satisfies the equation

$$\{\delta^v \tilde{f}_m^v\}_{kj} = [\phi_{kj}^v]^m + [R^m(\{[\phi_{kj}]^\ell, [\tilde{f}_k^v]^\ell\}_{2\leq\ell<m})]^m$$

and the estimate

$$|\tilde{f}_m^v| \leq K(N_C \otimes S^m(N_C^*))|[\phi^v]^m + [R_{kj}^m(\{[\phi_{kj}]^\ell, [\tilde{f}_k^v]^\ell\}_{2\leq\ell<m})]^m|.$$

As a consequence, the scheme of convergence applies to that formal solution $\{\tilde{F}_j\}$ and we are done.

4 A Majorant Method for the Full Linearization with a Unitary Normal Bundle

In this section, we shall devise a proof of Theorem 1.4, that is of the linearization of the neighborhood problem in the case N_C is unitary (and flat) following a majorant method scheme.

Let us recall the *horizontal cohomological operator*

$$\mathcal{L}^h_{kj}(f^h_j) := f^h_k(\varphi_{kj}(h_j), t_{kj}v_j) - s_{kj}(h_j)f^h_j(h_j, v_j),$$

where $s_{kj}(h_j) = D\varphi_{kj}(h_j)$. We then have the horizontal equation (2.19)

$$\begin{aligned}\mathcal{L}^h_{kj}(f^h_j) = \phi^h_{kj}(h_j + f^h_j, v_j + f^v_j)\\ + \varphi_{kj}(h_j + f^h_j(h_j, v_j)) - \varphi_{kj}(h_j) - D\varphi_{kj}(h_j)f^h_j(h_j, v_j).\end{aligned} \tag{4.1}$$

Let us recall the *vertical cohomological operator*

$$\mathcal{L}^v_{kj}(f^v_j) := f^v_k(\varphi_{kj}(h_j), t_{kj}v_j) - t_{kj}f^v_j,$$

and vertical Eq. (2.21) (recall that N_C is flat)

$$\mathcal{L}^v_{kj}(f^v_j) = \varphi^v_{kj}(h_j + f^h_j, v_j + f^v_j). \tag{4.2}$$

By assumption, there exists a formal solution $f_j = (f^h_j, f^v_j) = \sum_{k\geq 2}[f_j]^k$ with $\{[f_j]^k\} \in C^0(C, T_CM\otimes S^k(N^*_C))$. In case we assume $H^1(C, T_CM\otimes S^k(N^*_C)) = 0$, for all $k \geq 2$, this follows from Lemma 2.10. We now use the "norm" of the cohomological operator acting on $C^0(\mathcal{U}, T_CM \otimes S^m(N^*_C))$ as defined by Theorem A.12. We have, for $m \geq 2$

$$\tilde{K}_m := \max\left(K(N_C \otimes S^m(N^*_C)), K(T_C \otimes S^m(N^*_C))\right).$$

As in the foliation problem, we consider the sequence of numbers $\{\eta_m\}_{m\geq 1}$ with $\eta_1 = 1$ and, if $m \geq 2$

$$\eta_m := \tilde{K}_m \max_{m_1+\cdots+m_p+s=m} \eta_{m_1}\cdots\eta_{m_p}, \tag{4.3}$$

where, in the maximum, $1 \leq m_i < m$ for all i and $s \in \mathbb{N}$. In what follows, $f^\bullet_j$ (resp. $\phi^\bullet_{kj}$) stands for either f^h_j or f^v_j (resp. ϕ^h_{kj} or ϕ^v_{kj}). As in the previous section, let us expand $\phi^\bullet_{kj}(h_j + f^h_j, v_j + f^v_j)$ appeared in (4.1) and (4.2) in power series of v_j and let us define

$$\phi^\bullet_{kj}(z_j, w_j) =: \sum_{Q\in\mathbb{N}^d_2} \phi^\bullet_{kj,Q}(z_j)w^Q_j$$

$$\phi_{kj}^{\bullet}(h_j + f_j^h(h_j, v_j), v_j + f_j^v(h_j, v_j)) =: \sum_{Q\in\mathbb{N}_2^d} h_{kj,Q}^{\bullet}(h_j)v_j^Q =: h_{kj}^{\bullet}(h_j, v_j).$$

Then we obtain

$$\sum_{Q\in\mathbb{N}_2^d} h_{kj,Q}^{\bullet}(h_j)v_j^Q = \sum_{Q\in\mathbb{N}_2^d} \phi_{kj,Q}^{\bullet}(h_j + f_j^h(h_j, v_j))(v_j + f_j^v(h_j, v_j))^Q.$$

We further expand the first expression on the right-hand side as

$$\tilde{h}_{kj,Q}^{\bullet} := \phi_{kj,Q}^{\bullet}(h_j + f_j^h(h_j, v_j)) = \sum_{P\in\mathbb{N}^n} \frac{1}{P!}\partial_h^P \phi_{kj,Q}^{\bullet}(h_j)(f_j^h(h_j, v_j))^P.$$

Hence, for any $m \geq 2$,

$$[h_{kj}^{\bullet}]^m = \sum_{m_1+m_2=m}\sum_{Q\in\mathbb{N}_2^d}\sum_{P\in\mathbb{N}^n} \frac{1}{P!}\partial_h^P \phi_{kj,Q}^{\bullet}(h_j)\left[(f_j^h(h_j, v_j))^P\right]^{m_1}\left[(v_j + f_j^v(h_j, v_j))^Q\right]^{m_2}.$$

Let $\{f_j^{\bullet}\}$ be the formal solution of (4.1) and (4.2). Let us first assume that $H^0(C, T_CM \otimes S^{\ell}(N_C^*)) = 0$ for all $\ell \geq 2$. We shall see later on how to replace this assumption with suitable minimizing solutions. Assume that there is a sequence $\{A_k\}_{k\geq 2}$ of positive numbers such that

$$\forall k < m \quad |[f_j]^k|_{\hat{U}_j} \leq \eta_k A_k.$$

Let us set

$$A(t) = \sum_{k\geq 2} A_k t^k$$

with $t \in \mathbb{C}$.

Since $\phi_{kj}^{\bullet}$ is holomorphic in $h_j \in \hat{U}_{kj}$ and v_j in a neighborhood of the origin, we can assume that there is a positive R such that

$$\sup_{h_j\in\hat{U}_{kj}} |\phi_{kj,Q}^{\bullet}(h_j)| \leq R^{|Q|}. \tag{4.4}$$

According to (3.10) and the proof of (3.12), we obtain

$$\left|\left[(v_j + f_j^v(h_j, v_j))^Q\right]^{m_2}\right|_{\hat{U}_{kj}} \leq \sum_{\substack{(m_{1,1},\ldots,m_{1,q_1},\ldots,m_{d,1},\ldots,m_{d,q_d})\\ \sum_{i=1}^d m_{i,1}+\cdots+m_{i,q_i}=m_2}} \prod_{i=1}^d |[f_{j,i}]^{m_{i,1}}|_{\hat{U}_j}\cdots|[f_{j,i}]^{m_{i,q_i}}|_{\hat{U}_j}$$

$$\leq \sum_{M\in E_{Q,m_2}} \prod_{i=1}^{d} \eta_{m_{i,1}} A_{m_{i,1}} \cdots \eta_{m_{i,q_i}} A_{m_{i,q_i}}$$
$$\leq \eta_{Q,m_2} \left[\left(t + J^{m_2-1}A(t)\right)^{|Q|} \right]^{m_2}.$$

On the other hand, let $\mathcal{U}^* = \{U_i^*\}$ be an open covering of C such that U_i^* is relatively compact in U_i. We shall write $\hat{U}_k^* := \varphi_k(U_k^*)$. Let us set

$$M := \min_k \operatorname{dist}(\hat{U}_k^*, \partial \hat{U}_k).$$

Let us consider the index j as fixed and let us estimate the Fischer norm of $[\tilde{h}_{kj}^{\bullet}]^{m_1}$ on $\hat{U}_{kj}^* := \varphi_j(U_j \cap U_k^*)$. We get

$$\left|[\tilde{h}_{kj}^{\bullet}]^{m_1}\right|_{\hat{U}_{kj}^*} = \sum_{P\in\mathbb{N}^n} \frac{1}{P!} \left| \partial_h^P \phi_{kj,Q}^{\bullet}(h_j) \left[(f_j^h(h_j, v_j))^P \right]^{m_1} \right|_{\hat{U}_{kj}^*}$$
$$\leq \sum_{P\in\mathbb{N}^n} \left(\frac{1}{\operatorname{dist}(\hat{U}_k^*, \partial \hat{U}_k)} \right)^{|P|} \left| \phi_{kj,Q}^{\bullet} \right|_{\hat{U}_{kj}} \left| \left[(f_j^h(h_j, v_j))^P \right]^{m_1} \right|_{\hat{U}_{kj}^*}$$
$$\leq \sum_{P\in\mathbb{N}^n} \left(\frac{1}{M} \right)^{|P|} R^{|Q|} \left| \left[(f_j^h(h_j, v_j))^P \right]^{m_1} \right|_{\hat{U}_{kj}^*}.$$

Since f_j is of order ≥ 2 at $v_j = 0$, we have $|P| \leq \frac{m_1}{2}$ in the above sum. According to estimate (3.10) and following the proof of (3.12), we obtain

$$\left|[\tilde{h}_{kj,Q}^{\bullet}]^{m_1}\right|_{\hat{U}_{kj}^*} \leq \sum_{P\in\mathbb{N}^n, |P|=0}^{\frac{m_1}{2}} \left(\frac{1}{\operatorname{dist}(\hat{U}_k^*, \partial \hat{U}_k)} \right)^{|P|} R^{|Q|} \eta_{P,m_1} \left[A(t)^{|P|} \right]^{m_1}. \tag{4.5}$$

Combining inequalities (4.5) and (4.5), we obtain

$$\left|[h_{kj}^{\bullet}]^{m}\right|_{\hat{U}_{kj}^*} \leq \sum_{m_1+m_2=m} \sum_{Q\in\mathbb{N}_2^d} \sum_{P\in\mathbb{N}^n} \frac{1}{P!} \left| \partial_h^P \phi_{kj,Q}^{\bullet}(h_j) \left[(f_j^h(h_j, v_j))^P \right]^{m_1} \right.$$
$$\left. \left[(v_j + f_j^v(h_j, v_j))^Q \right]^{m_2} \right|_{\hat{U}_{kj}^*}$$
$$\leq \sum_{m_1+m_2=m} \sum_{\substack{Q\in\mathbb{N}^d \\ |Q|=2}}^{m_2} \sum_{\substack{P\in\mathbb{N}^n \\ |P|=0}}^{\frac{m_1}{2}} \left(\frac{1}{M} \right)^{|P|} R^{|Q|} \eta_{P,m_1}$$
$$\left[A(t)^{|P|} \right]^{m_1} \eta_{Q,m_2} \left[\left(t + J^{m_2-1} A(t) \right)^{|Q|} \right]^{m_2}$$

$$\leq \sum_{m_1+m_2=m} \sum_{\substack{Q\in\mathbb{N}^d\\|Q|=2}}^{m_2} \sum_{\substack{P\in\mathbb{N}^n\\|P|=0}}^{\frac{m_1}{2}} \left[\left(\frac{A(t)}{M}\right)^{|P|}\right]^{m_1} \eta_{P,m_1}\eta_{Q,m_2}$$
$$\left[\left(Rt + RJ^{m_2-1}A(t)\right)^{|Q|}\right]^{m_2}$$
$$\leq \tilde{E}_m \left[\left(\frac{1}{1-\frac{A(t)}{M}}\right)^n \left(\left(\frac{1}{1-(Rt+RA(t))}\right)^d - 1 - d(Rt+RA(t))\right)\right]^m.$$

Here, we have set

$$\tilde{E}_m = \max_{m_1+m_2=m} \max_{\substack{P\in\mathbb{N}^n, Q\in\mathbb{N}^d\\|P|\leq\frac{m_1}{2}, 2\leq|Q|\leq m_2,}} \eta_{P,m_1}\eta_{Q,m_2}.$$

It remains to estimate the rest of terms in (4.1). We define

$$B_m := \left[\varphi_{kj}(h_j + f_j^h(h_j, v_j)) - \varphi_{kj}(h_j) - D\varphi_{kj}(h_j)f_j^h(h_j, v_j)\right]^m$$
$$= \sum_{l=2}^{\frac{m}{2}} \sum_{|P|=l} \frac{1}{P!}\partial_h^P \varphi_{kj}(h_j)\left[(f_j^h)^P\right]^m.$$

Hence, as above, we have

$$|B_m|_{\hat{U}^*_{kj}} \leq |\varphi_{kj}|_{\hat{U}_{kj}} \sum_{l=2}^{\frac{m}{2}} \sum_{|P|=l} \left(\frac{1}{M}\right)^{|P|} \left[(A(t))^{|P|}\right]^m$$
$$\leq |\varphi_{kj}|_{\hat{U}_{kj}} \left[\left(\frac{1}{1-\frac{A(t)}{M}}\right)^n - 1 - n\frac{A(t)}{M}\right]^m.$$

By the same reasoning as in the foliation section, the previous estimates on $\hat{U}^*_{kj}$ extend to estimates on $\hat{U}_{kj}$, by multiplication by a constant $\tilde{M}$.

Let us define constant $C_0 := \max_{kj} |\varphi_{kj}|_{\hat{U}_{kj}}$ Since we have

$$|[f_j^\bullet]^m|_{\hat{U}_j} \leq \tilde{K}_m \max_k \left|\mathcal{L}_{kj}([f_j^\bullet]^m)\right|_{\hat{U}_{kj}},$$

then

$$|[f_j^\bullet]^m|_{\hat{U}_j} \leq \tilde{M}\tilde{K}_m \left(C_0\left[\left(\frac{1}{1-\frac{A(t)}{M}}\right)^n - 1 - n\frac{A(t)}{M}\right]^m\right.$$
$$\left. + \tilde{E}_m \left[\left(\frac{1}{1-\frac{A(t)}{M}}\right)^n \left(\left(\frac{1}{1-(Rt+RA(t))}\right)^d - 1 - d(Rt+RA(t))\right)\right]^m\right).$$

We emphasize that due to the vanishing assumption of the spaces $H^0(\mathcal{U}, T_C M \otimes S^m(N_C^*)), m \geq 2$, the solution of cohomological equation $\mathcal{L}_{kj}([f_j^\bullet]^m) = R^m$ is unique and is equal to the minimizing solution obtained in Lemma A.2 and Proposition A.4. Consider the following analytic functional equation :

$$A(t) = \tilde{M}\left(C_0\left[\left(\frac{1}{1-\frac{A(t)}{M}}\right)^n - 1 - n\frac{A(t)}{M}\right] + \left(\frac{1}{1-\frac{A(t)}{M}}\right)^n\left(\left(\frac{1}{1-(Rt+RA(t))}\right)^d - 1 - d(Rt+RA(t))\right)\right).$$

It has a unique analytic solution A of order ≥ 2 at the origin. Since we have

$$\tilde{K}_m \max(1, \tilde{E}_m) \leq \eta_m, \quad |[f_j^\bullet]^m|_{\hat{U}_j} \leq A_m \eta_m, \quad m \geq 2$$

then $\sum_{m\geq 2}[f_j^\bullet]^m$ converges in a neighborhood of the origin.

Let us see how the general case is reduced to the previous one. The issue is that, when considering a solution $[f_j^\bullet]^m$ of the cohomological equation $\mathcal{L}_{kj}([f_j^\bullet]^m) = R^m$, the estimate given by Lemma A.2 and Proposition A.4 might be obtained by another solution. Hence, the formal solution might not be the good one for the estimates. So we will need to correct it. As we already emphasized, Eqs. (4.1) and (4.2) read

$$\mathcal{L}_{kj}(\{[f]_i^\ell\}) = \mathcal{R}_{kj,\ell}([f]^{\ell'}, \ell' < \ell; [\Phi]^l, l \leq \ell)$$

where $\mathcal{R}_{kj,\ell}$ is an analytic function of its arguments. Let us start at $\ell = 2$.

(1) $\mathcal{R}_{kj,2}$ is just a function of the $[\Phi_{kj}]^2$'s and we have $\mathcal{L}_{kj}([f]^2) = \mathcal{R}_{kj,2}$. Let $\{[\tilde{f}_{j,2}]^2\}$ be the minimizer solution of this equation obtained by Lemma A.2 and Proposition A.4 and let $[k_j]^2 := [f_j]^2 - [\tilde{f}_{j,2}]^2$. We have $\{[k_j]^2\} \in H^0(\mathcal{U}, T_C M \otimes S^2(N_C^*))$.

(2) According to Lemma 2.14, $F_{j,2} := F_j \exp(-[k_j]^2)$ linearizes Φ_{kj} since

$$F_{j,2}^{-1}\Phi_{kj}F_{j,2} = \exp(-[k_j]^2)^{-1}N_{kj}\exp(-[k_j]^2) = N_{kj}.$$

$F_{j,2}$ is tangent to identity and its 2nd order term is the minimizer $[\tilde{f}_j]^2$.

(3) Assume that $F_{j,\ell}$ linearizes Φ_{kj}, is tangent to identity at the origin and has the minimizers solution up to degree ℓ as Taylor expansion at 0. This means that $F_{j,\ell} = Id + \sum_{l=2}^{\ell}[\tilde{f}_{j,l}]^l + \sum_{l\geq \ell+1}[f_{j,\ell}]^l$. Let us write the conjugacy equation. By induction we have, for all $2 \leq l \leq \ell$,

$$\mathcal{L}_{kj}(\{[\tilde{f}_{i,l}]^l\}) = \mathcal{R}_{kj,l}(\{[\tilde{f}_{i,l'}]^{l'}\}_i, l' < l; [\Phi]^m, m \leq l).$$

Furthermore, it satisfies at degree $\ell + 1$

$$\mathcal{L}_{kj}(\{[f_{i,\ell+1}]^{\ell+1}\}) = \mathcal{R}_{kj,\ell+1}(\{[\tilde{f}_{i,\ell'}]^{\ell'}\}_i, \ell' \leq \ell; [\Phi]^m, m \leq \ell + 1).$$

Let $[\tilde{f}_{i,\ell+1}]^{\ell+1}$ be the minimizer solution of the above cohomological equation. Let $[k_{i,\ell+1}]^{\ell+1} = [f_{i,\ell+1}]^{\ell+1} - [\tilde{f}_{i,\ell+1}]^{\ell+1}$. As above, it defines an element of $H^0(\mathcal{U}, T_C M \otimes S^{\ell+1}(N_C^*))$. Let us set $F_{j,\ell+1} = F_{j,\ell} \exp([k_{j,\ell+1}]^{\ell+1})^{-1}$. Then it linearizes Φ_{kj} and has the minimizers solution up to degree $\ell+1$ as Taylor expansion at 0: $F_{j,\ell+1} = Id + \sum_{l=2}^{\ell+1}[\tilde{f}_{j,l}]^l + \sum_{l\geq \ell+2}[f_{j,\ell+1}]^l$.

(4) Since $F_{j,\ell+1}F_{j,\ell}^{-1} = I + O(\ell+1)$, the sequence $\{F_{j,\ell}\}_\ell$ converges in the space of formal power series to $\tilde{F}_j$. Furthermore, $\{\tilde{F}_j\}$ linearizes $\{\Phi_{kj}\}$ as each $\{F_{j,\ell}\}_j$ does. The Taylor expansion of $\tilde{F}_j$ at the origin is

$$\tilde{F}_j = Id + \sum_{l\geq 2}[\tilde{f}_{j,l}]^l.$$

(5) We can estimate the $[\tilde{f}_{j,l}]^l$ as we did above in the case of vanishing cohomology since the Taylor coefficient are minimizer solutions of the same equations.

Hence, we are done.

In summary, we have proved the following theorem.

Theorem 4.1 *Let C be an embedded compact manifold in M. Assume that the embedding is linearizable by a formal holomorphic mapping which is tangent to the identity and N_C is unitary. Suppose that $\{\eta_m\}_{m\geq 1}$ defined by (4.3) satisfy $\eta_m \leq L_0 L^m$, for some positive numbers L_0, L and for all m. Then the embedding is actually holomorphically linearizable.*

We remark that in general there is a rigidity theory on deformations in an analytic family of complex manifolds due to Kodaira [25]. Strengthening Corollary 3.3, we finish the section with the following corollary. This may be regarded as a rigidity for a simply connected manifold.

Corollary 4.2 *Keep the assumptions in Theorem 4.1. Assume further that C is simply connected. Then a neighborhood of C in M is biholomorphic to $C \times B^d$ where B^d is the unit ball in $\mathbf{C}^d$.*

Proof We already know that M admits a horizontal foliation by Corollary 3.3. To show that each leaf is biholomorphic to C, we may assume that $M = N_C$ and we will use the projection $\pi\colon N_C \to C$. We fix $x_0 \in C$. We take a point $p \in \pi^{-1}(x_0)$ close to C. Let L be the (connected) leaf of the foliation containing p. Then L intersects each fiber of N_C at a unique point. To verify this, we connect a point in $x \in C$ to x_0 by a continuous path γ in C with $\gamma(0) = x_0$ and $\gamma(1) = x$. By continuation along leaves, we can find a lifted continuous path $\tilde{\gamma}$ and the germ $L_\gamma^*(t)$ at $\tilde{\gamma}(t)$ of a leaf $L_\gamma(t)$ such that $\pi(\tilde{\gamma}(t)) = \gamma(t)$. Note that $L_\gamma^*(t')$, $L_\gamma^*(t)$ are contained in the same leaf on which π is injective, when t' is sufficiently close to t. The lifting $\tilde{\gamma}(1)$ is independent of γ. Indeed if $\gamma^\theta (a \leq \theta \leq b)$ is a continuous family of paths connecting x_0 to x. Let L_{γ^θ} be the leaf associated to γ^θ. Then $\widetilde{\gamma^\theta}(t) \in L_{\gamma^{\theta_0}}(t)$ when θ is sufficiently close to θ_0, as $L_{\gamma^\theta}(0) = L_{\gamma^a}(0)$ as a leaf near p.

Obviously, $x \mapsto \tilde{\gamma}(1)$ gives a biholomorphism from C onto the leaf through p. And $(x, v) \to \tilde{\gamma}(1)$ defines a biholomorphisms from $C \times B$ into N_C, where B is a small neighborhood of $0 \in \pi^{-1}(x_0)$. □

5 The Full Linearization

The main purpose of this section is to solve the linearization problem in the general setting (i.e. N_C not necessarily being flat) under general hypotheses on the existence of bounds to the cohomology equations. At the end of the section we will illustrate the results with Arnold's examples [2], following computations by Arnol'd [3].

We shall devise a Newton scheme to solve the linearization of the neighborhood problem. Let us recall the condition.

(L_m) : The neighborhood of C agrees with the neighborhood of the zero section of the normal bundle up to order m.

That embedding of C has property (L_m) means that the order of $(\phi^h_{kj}(h_j, v_j), \phi^v_{kj}(h_j, v_j))$ along $v_j = 0$ as defined in (5.16) is $\geq m+1$.

Assume that (L_m) holds. We shall assume either that $H^0(C, TC \otimes S^p N_C^*) = 0$, $2 \leq p \leq 2m$ or that N_C is flat. According to Lemma 2.16 (c) and (d), the following linearization step in the Newton method is fulfilled :

(N_m) : If $\{\Phi_{kj}\} \in L_m$, then $\{F_k^{-1}\Phi_{kj}F_j\} \in L_{2m}$ for some $\{F_j = I + f_j\}$ with $f_j(h_j, v_j) = O(|v_j|^{m+1})$.

5.1 Domains for Iteration and the Donin Condition

Following Lemma A.6 and Proposition A.19, we shall consider a family of nested coverings $\mathcal{U}^r = \{U_i^r\}_{i\in I}$ of C with $r_* \leq r \leq r^*$. Let us fix a trivialization of N_C^* (resp. TC) over $U_i^{r^*}$ by fixing a holomorphic basis $e_i = (e_{i,1}, \dots, e_{i,n+d})$ of $T_C M$ on $U_i^{r^*}$.

We first define various domains. Let $\hat{U}_j^r := \varphi_j(U_j^r) = \Delta_n^r$ and $U_{kj}^r := U_k^r \cap U_j^r$. We have $U_{kj}^r = U_{jk}^r$. Define $\hat{U}_{kj}^r = \varphi_j(U_{kj}^r)$. Then

$$\varphi_{kj}(U_{kj}^r) = \hat{U}_{jk}^r.$$

Donin Condition. Let $\mathcal{U}^r$ be a family of nested covering of C for $r_* < r < r^*$. Let $E' = TC$ or N_C. Suppose that there are constants $D(E' \otimes S^m N_C^*)$ for $m = 2, 3, \dots$ such that for all r', r'' with $r_* < r'' < r' < r < r^*$ and $r' - r'' \leq r^* - r$, and all $f \in Z^1(\mathcal{U}^{r'}, E' \otimes S^m N_C^*)$ with $f = 0$ in $H^1(\mathcal{U}^{r'}, E' \otimes S^m N_C^*)$, there is a solution $u \in C^0(\mathcal{U}^{r''}, E' \otimes S^m N_C^*)$ to $\delta u = f$ such that

$$\max_j \sup |u_j|_{L^\infty(\hat{U}_j^{r''})} \leq \frac{D(E' \otimes S^m N_C^*)}{(r' - r'')^\tau} \max_{k,j} |f_{kj}|_{L^\infty(\hat{U}_{kj}^{r'})}, \tag{5.1}$$

where $D(E' \otimes S^m N_C^*)$ is independent of r', r'' and f and $\tau = \tau(N_C^*)$ is independent of m.

In what follows, we shall express sections of bundles in coordinates. For the purpose of estimates, we need to choose suitable domains for trivialization of the vector bundle N_C. Recall that the N_C has trivializations N_j and transition functions N_{kj}. Let B_d^r be the ball of radius r in $\mathbf{C}^d$ centered at the origin. Thus, we define

$$\hat{V}_j^r = N_j(V_j^r) = \hat{U}_j^r \times B_d^r, \quad V_{i_0\cdots i_q}^r := V_{i_0}^r \cap \cdots \cap V_{i_q}^r,$$

$$\hat{V}_{i_0\cdots i_q}^r := N_{i_q}(V_{i_0\cdots i_q}^r) \subset \varphi_{i_q}(U_{i_0\cdots i_q}^r) \times \mathbf{C}^d, \tag{5.2}$$

$$\hat{V}_{jk}^r = N_{kj}(\hat{V}_{kj}^r), \quad N_{kj} = N_{jk}^{-1} \quad \text{on } \hat{V}_{kj}^r, \tag{5.3}$$

$$N_{ki}N_{ij} = N_{kj} \quad \text{on } \hat{V}_{kij}^r. \tag{5.4}$$

Denote the corresponding domains by $\tilde{V}_j^r$, $\tilde{V}_{kj}^r$ when N_j are replaced by Φ_j. Then we still have the above relations when N_j, N_{kj} are replaced by Φ_j, Φ_{kj}. We know that Φ_{kj} are perturbations of the transition functions N_{kj} of the normal bundle of C in M, which are defined on different domains but in the same space. We will, however, work on domains $\hat{V}_{kj}^r$ for Φ_{kj}, instead of $\tilde{V}_{kj}^r$.

With notation of Sect. 2.7, for $L \geq 1$ and for $r_* \leq r \leq r^*$, we consider a cochain $\{f_I\} \in C^{q+1}(\mathcal{U}^r, \mathcal{O}(T_C M \otimes S^L(N_C^*)))$, given by

$$f_I := f_{i_0\cdots i_q}(p) = \sum_{\lambda=1}^{n+d} \sum_{|Q|=L} f_{i_0\cdots i_q;Q}^{\lambda}(z_{i_q}(p)) e_{i_0,\lambda}(p) \otimes (w_{i_q}^*(p))^Q$$

where $I = (i_0, \ldots, i_q) \in \mathcal{I}^{q+1}$. Recall that $\hat{V}_I^r = N_{i_q}(V_{i_0}^r \cap \cdots \cap V_{i_q}^r)$. Define

$$|f_I|_r = \sup_{(h_{i_q}, v_{i_q}) \in \hat{V}_I^r} |\sum_Q f_{I,Q}(h_{i_q}) v_{i_q}^Q|.$$

We also set $|\{f_I\}|_r = \max_I |f_I|_r$.

Note that $\hat{V}_j^r = \hat{U}_j \times B_d^r$ are product domains. Also,

$$\hat{U}_{kj}^r \times B_d^{c_* r} \subset \hat{V}_{kj}^r \subset \hat{U}_{kj}^r \times B_d^{c^* r}, \quad c_* \leq 1 \leq c^*.$$

Define $B_{kj}^r(h_j)$ to be $\{v_j \in B_d^r \colon t_{kj}(h_j) v_j \in B_d^r\}$. The skewed domain $\hat{V}_{kj}^r$ can be described as follows:

$$(h_j, v_j) \in \hat{V}_{kj}^r \quad \text{if and only if} \quad h_j \in \hat{U}_{kj}^r, \ v_j \in B_{kj}^r(h_j).$$

Next, we note that the d-torus action $(h_j, v_j) \to (h_j, (\zeta_1 v_1, \ldots, \zeta_d v_d))$ with $\zeta \in (S^1)^d$ does not preserve $\hat{V}_{kj}^r$ when $t_{kj}(h_j)$ is not diagonal. Nevertheless, the $\hat{V}_{kj}^r$ has a *disc structure*:

$$(h_j, \zeta v_j) \in \hat{V}_{kj}^r, \qquad \forall (h_j, v_j) \in \hat{V}_{kj}^r, \quad \forall \zeta \in \Delta.$$

Indeed, suppose that $(h_j, v_j) \in \hat{V}_{kj}^r$. Then $h_j \in \hat{U}_{kj}^r$ and $(h_j, v_j) = N_j(p)$ with $p \in V_k^r \cap V_j^r$ and $N_k(p) = (h_k, v_k) \in \hat{V}_k^r$. By definition, $\hat{V}_j^r = \hat{U}_j \times B_d^r$. Take

$\tilde{p} = N_j^{-1}(h_j, \zeta v_j)$. We have $\tilde{p} \in V_j^r$ and $N_k(\tilde{p}) = (h_k, t_{kj}(\zeta v_j)) = (h_k, \zeta t_{kj}(v_j)) \in \hat{U}_{kj}^r \times B_d^r$.

Throughout this section, we use

$$|u_j|_\rho = \sup_{(h_j,v_j)\in\hat{V}_j^\rho} |u_j(h_j, v_j)|, \quad |u_{kj}|_\rho = \sup_{(h_j,v_j)\in\hat{V}_{kj}^\rho} |u_{kj}(h_j, v_j)|$$

where u_j, u_{kj} are functions on $\hat{V}_j^r$ and $\hat{V}_{kj}^r$, respectively. We also define $|\{u_I\}|_\rho = \max_I |u_I|_\rho$.

With the above disc structure, we now prove the following.

Lemma 5.1 *Let u_{kj} be a holomorphic function on $\hat{V}_{kj}^r$ with $r_* < r < \tilde{r} < r^*$. Suppose that*

$$\hat{V}_{kj}^{r_*} \neq \emptyset. \tag{5.5}$$

For $0 < \theta < 1$ with $\theta r > r^*$, we have

$$|u_{kj}|_{\theta r} \leq \theta^m |u_{kj}|_r, \quad \text{if } u_{kj}(h_j, v_j) = O(|v_j|^m); \quad |[u_{kj}]^\ell|_r \leq |u_{kj}|_r;$$
$$\sum_{\ell=i}^{\infty} |[u_{kj}]^\ell|_{\theta r} \leq \frac{\theta^i}{1-\theta} |u_{kj}|_r.$$

Proof Let $u = u_{kj}$. The first inequality follows from the Schwarz lemma applied to the holomorphic function $\zeta \to u(h_j, \zeta v_j)$ on the unit disk for fixed $(h_j, v_j) \in \hat{V}_{kj}^r$. Note that $[u]^i(h_j, \zeta v_j) = \zeta^i [u]^i(h_j, v_j)$. Thus the second inequality follows directly by averaging,

$$[u]^\ell(h_j, v_j) = \frac{1}{2\pi i} \int_{\zeta\in\partial\Delta} u(h_j, \zeta v_j) \frac{d\zeta}{\zeta^{\ell+1}}, \quad (h_j, v_j) \in \hat{V}_{kj}^r.$$

The last inequality follows from the first two inequalities. □

For the rest of this section, we rename r in the Donin Condition by $\tilde{r}$ which is fixed now. We will let r vary in $(r_*, \tilde{r})$.

Lemma 5.2 *Let $r_* < \theta r < r < \tilde{r} < r^* < 1$. Fix $k, j \in \mathcal{I}$. Suppose that $(1-\theta^4)r < r^* - \tilde{r}$ and (5.5) holds.*

(a) *We have*

$$\text{dist}(\hat{V}_j^{\theta r}, \partial \hat{V}_j^r) \geq r(1-\theta)/C_0, \quad \text{dist}(\hat{V}_{kj}^{\theta r}, \partial \hat{V}_{kj}^r) \geq r(1-\theta)/C_0, \tag{5.6}$$

for some constant C_0.

(b) *Assume further that $\theta^4 r > r_*$. There exists a constant C_0^* such that if $F_j = I + f_j$ satisfy*

$$|f_j|_{\theta^2 r} \leq (1-\theta)r/C_0^*, \tag{5.7}$$

then we have

$$F_j(\hat{V}_j^{\theta^2 r}) \subset \hat{V}_j^{\theta r}, \quad F_j(\hat{V}_{kj}^{\theta^2 r}) \subset \hat{V}_{kj}^{\theta r}, \tag{5.8}$$

$$F_j^{-1}(\hat{V}_j^{\theta^4 r}) \subset \hat{V}_j^{\theta^3 r}, \quad F_j F_j^{-1} = I \text{ on } \hat{V}_j^{\theta^4 r}. \tag{5.9}$$

Proof (a) The $\hat{V}_j^r$ is the product domain $\hat{U}_j^r \times B_d^r$. Thus the first inequality in (5.6) holds trivially since $\hat{U}_j^r$ is a polydisc. Note that $\hat{V}_{kj}^r$ are open sets. Then $\delta := \mathrm{dist}((h, v), (\tilde{h}, \tilde{v})) = \mathrm{dist}(\hat{V}_{kj}^{\theta r}, \partial \hat{V}_{kj}^r)$ is attained by

$$(h, v) \in \partial \hat{V}_{kj}^{\theta r}, \quad (\tilde{h}, \tilde{v}) \in \partial \hat{V}_{kj}^r. \tag{5.10}$$

If $\tilde{h} \in \partial \hat{U}_{kj}^r$, we immediately get $\delta \geq \mathrm{dist}(\hat{U}_{kj}^{\theta r}, \partial \hat{U}_{kj}^r) \geq (1-\theta)r/C$ by Lemma A.6. Assume that $\tilde{h} \in \hat{U}_{kj}^r$. Then by the continuity of the function t_{kj}, $\tilde{v}$ must be in $\partial B_{kj}^r(\tilde{h})$. Otherwise, both $\tilde{h} \in \hat{U}_{kj}^r$ and $\tilde{v} \in B_{kj}^r(\tilde{h})$ are interior points of the two sets, then any small perturbation of $(\tilde{h}, \tilde{v})$ still satisfies the second condition in (5.10). The last assertion implies that $(\tilde{h}, \tilde{v})$ cannot be a boundary point and we get a contradiction. Therefore, we have

$$\tilde{v} \in \partial B_d^r \quad \text{or} \quad t_{kj}(\tilde{h})\tilde{v} \in \partial B_d^r.$$

The first case yields $|\tilde{v} - v| \geq \mathrm{dist}(B_d^{\theta r}, \partial B_d^r) = (1-\theta)r$. We now consider the second case. By assumption t_{kj} is holomorphic in $\overline{\omega}$ for a neighborhood ω of $\hat{U}_{kj}$. Thus there is $\delta_* > 0$ depending only on $\hat{U}_{kj}$ such that if $h \in \hat{U}_{kj}$ and $|\tilde{h} - h| < \delta_*$, then the line segment γ connecting h to $\tilde{h}$ is contained in ω. Suppose that $|\tilde{h} - h| < (1-\theta)r/C_1$ for C_1 to be determined so that $(1-\theta)r/C_1 < \delta_*$. Applying the mean-value-theorem to $t_{kj}(\gamma)$ and using $t_{kj}(h)v \in B_d^{\theta r}$, we get

$$\begin{aligned} C_4|\tilde{v} - v| \geq |t_{kj}(\tilde{h})(\tilde{v} - v)| &\geq \Big| |t_{kj}(\tilde{h})\tilde{v} - t_{kj}(h)v)| - |(t_{kj}(\tilde{h}) - t_{kj}(h))v| \Big| \\ &\geq (1-\theta)r - C_5|\tilde{h} - h||v| \geq (1-\theta)r/2, \end{aligned}$$

when C_1 is sufficiently large. Thus we get $\mathrm{dist}(\hat{U}_{kj}^{\theta r}, \partial \hat{U}_{kj}^r) \geq (1-\theta)r/C$ as in the first case. If $|\tilde{h} - h| \geq (1-\theta)r/C_1$, the required estimate is immediate.

(b) Note that $\theta > r_*$. By choosing a larger C_0^*, (5.8) follows from (5.6) immediately. We want to find F^{-1}. By (5.7) and the Cauchy estimate, we know that

$$|\partial_{h_j} f_j(h_j, v_j)| + |\partial_{v_j} f_j(h_j, v_j)| \leq C_6/C_0^*, \quad \forall (h_j, v_j) \in \hat{V}_j^{\theta^3 r}. \tag{5.11}$$

Note that $\overline{V_j^r} = \overline{\hat{U}_j^r} \times \overline{B_d^r}$ is convex. By (5.11) and the fundamental theorem of calculus, we have

$$|f_j(p_1) - f_j(p_0)| \leq C_7|p_1 - p_0|/C_0^*, \quad \forall p_0, p_1 \in \hat{V}_j^{\theta^3 r}.$$

Suppose that $C_0^* > 2C_7$. Then $F_j\colon \hat V_j^{\theta^3 r} \to \hat V_j^{\theta^2 r}$ is injective, and $T(h_j, v_j) = (\tilde h_j, \tilde v_j) - f_j(h_j, v_j)$ defines a contraction mapping on $\hat V_j^{\theta^3 r}$, if $(\tilde h_j, \tilde v_j) \in \hat V_j^{\theta^4 r}$ and C_0^* is sufficiently large. This gives us (5.9). □

In this section, we change notation and let

$$f_j^\bullet = (f_j^h, f_j^v), \quad \phi_{kj}^\bullet = (\phi_{kj}^h, \phi_{kj}^v).$$

Lemma 5.3 *Let $r_* < \theta r < r < \tilde r < r^* < 1$. Suppose that $\hat V_{kj}$ satisfies (5.5). There exists a constant $C_1^* > 1$ such that if*

$$|\phi_{kj}^\bullet|_r \leq (1-\theta) r / C_1^* \tag{5.12}$$

then we have

$$\Phi_{kj}(\hat V_{kj}^{\theta r}) \subset \hat V_{jk}^r.$$

Proof Note that $\theta > r_*$. Since $\Phi_{kj} - N_{kj} = \phi_{kj}^\bullet$ and $N_{kj}(\hat V_{kj}^{\theta r}) = \hat V_{jk}^{\theta r}$, the assertion follows from (5.6) and (5.12) for sufficiently large C_1^*. □

Proposition 5.4 *Let $r_* < \theta^7 r < r < \tilde r < r^* < 1$. Assume that $\hat V_{kj}$ satisfies (5.5). Suppose that $\Phi_{kj} = N_{kj} + \phi_{kj}^\bullet$ satisfy (5.12). Let $F_j = I + f_j$ satisfy $f_j(h_j, v_j) = O(|v_j|^2)$.*

Suppose $\tilde\Phi_{kj} = F_k^{-1}\Phi_{kj} F_j = N_{kj} + \tilde\phi_{kj}^\bullet$. There exists a constant C_2^* such that if

$$|\{f_j\}|_{\theta^2 r} \leq (1-\theta) r / C_2^*, \tag{5.13}$$

and $\tilde\phi_{kj}^\bullet(h_j, v_j) = O(|v_j|^{\widetilde m})$, then

$$|\{\tilde\phi_{kj}^\bullet\}|_{\theta^7 r} \leq C_2 \theta^{\widetilde m}(|\{f_j\}|_{\theta^2 r} + |\{\phi_{kj}^\bullet\}|_r, \tag{5.14}$$

$$|\{\tilde\phi_{kj}^\bullet\}|_{\theta^7 r} \leq C_2 \theta^{\widetilde m}(1-\theta) r. \tag{5.15}$$

Proof Let us write $\tilde\Phi_{kj} = N_{kj} + \tilde\phi_{kj}^\bullet$ and $F_k^{-1} = I + g_k$. Thus

$$\begin{aligned}\tilde\phi_{kj}^h &= g_k^h \circ \Phi_{kj} \circ F_j + \phi_{kj}^h \circ F_j + (\varphi_{kj}(I + f_j^h) - \varphi_{kj}),\\ \tilde\phi_{kj}^v &= g_k^v \circ \Phi_{kj} \circ F_j + \phi_{kj}^v \circ F_j\\ &\quad + (t_{kj}(h_j + f_j^h) - t_{kj}(h_j)) \times (v_j + f_j^v) + t_{kj}(h_j) \times f_j^v(h_j, v_j).\end{aligned}$$

According to (5.9), we have $F_k(I + g_k) = I$ on $\hat V_k^{\theta^4 r}$. Thus $g_k = -f_k \circ F_k^{-1}$ implies that

$$|g_k|_{\theta^4 r} \leq |f_k|_{\theta^3 r}.$$

For $(h_j, v_j) \in \hat{V}_{kj}^{\theta^6 r}$, using $\mathrm{dist}(\hat{U}_{kj}^{\theta^6 r}, \partial \hat{U}_{kj}^{\theta^5 r}) \geq (1-\theta)\theta^5 r/C_0$, we can obtain $|t_{kj}(h_j + f_j^h(h_j, v_j)) - t_{kj}(h_j)| \leq C_3 |f^h(h_j, v_j)|$ and $|\varphi_{kj}(h_j + f_j^h(h_j, v_j)) - \varphi_{kj}(h_j, v_j)| \leq C_3 |f_j(h_j, v_j)|$. Nesting domains and using (5.12), (5.13) and hence (5.7), we obtain by Lemma 5.2 in which r is replaced by $\theta^5 r$:

$$
\begin{aligned}
|\{\tilde{\phi}_{kj}^{\bullet}\}|_{\theta^6 r} &\leq C_4(|\{f_j\}|_{\theta r} + |\{\phi_{kj}^{\bullet}\}|_r, \\
|\{\tilde{\phi}_{kj}^{\bullet}\}|_{\theta^6 r} &\leq C_4(1-\theta) r.
\end{aligned}
$$

Applying Schwarz inequality, we get (5.14)–(5.15). □

When we apply the above to iteration, the new Φ_{kj} in the sequence of iteration is defined by

$$
(F_k^{(m)})^{-1}(\cdots((F_k^{(1)})^{-1}\Phi_{kj}F_j^{(1)})\cdots)F_j^{(m)}
$$

on $\hat{V}_{kj}^{r_{m+1}}$ with $F_j^{(m)}(\hat{V}_{kj}^{r_{m+1}}) \subset \hat{V}_{kj}^{r_m}$.

Let us find $[f_j]_{m+1}^{2m}(h_j, v_j)$, a polynomial of order $\geq m+1$ and of degree $\leq 2m$ in v_j (holomorphic in h_j), such that $\{F_k^{-1}\Phi_{kj}F_j\} \in L_{2m}$ holds for some $\{F_j = I + [f_j]_{m+1}^{2m}\}$.

Let us consider the neighborhood written in the new coordinates $\{F_j\}$. We obtain for $(h_k, v_k) = \hat{\Phi}_{kj}(h_j, v_j)$:

$$
\begin{aligned}
h_k &= \hat{\Phi}_{kj}^h(h_j, v_j) := \varphi_{kj}(h_j) + \hat{\phi}_{kj}^h(h_j, v_j), \\
v_k &= \hat{\Phi}_{kj}^v(h_j, v_j) := t_{kj}(h_j)v_j + \hat{\phi}_{kj}^v(h_j, v_j).
\end{aligned}
\tag{5.16}
$$

We assume that $\hat{\phi}_{kj}^{\bullet} := (\hat{\phi}_{kj}^h, \hat{\phi}_{kj}^v)$ has order $\geq 2m+1$ at $v_j = 0$.

Let us write down the horizontal and vertical equations for the linearization problem: $F_k\hat{\Phi}_{kj} = \Phi_{kj}F_j$. We obtain the horizontal equation

$$
\begin{aligned}
&\varphi_{kj}(h_j) + \hat{\phi}_{kj}^h(h_j, v_j) + f_k^h(\varphi_{kj} + \hat{\phi}_{kj}^h, t_{kj}(h_j)v_j + \hat{\phi}_{kj}^v) \\
&\quad = \varphi_{kj}(h_j + f_j^h(h_j, v_j)) + \phi_{kj}^h(h_j + f_j^h, v_j + f_j^v).
\end{aligned}
$$

The vertical equation reads

$$
\begin{aligned}
&t_{kj}(h_j)v_j + \hat{\phi}_{kj}^v(h_j, v_j) + f_k^v(\varphi_{kj} + \hat{\phi}_{kj}^h, t_{kj}(h_j)v_j + \hat{\phi}_{kj}^v) \\
&\quad = t_{kj}(h_j + f_j^h)(v_j + f_j^v) + \phi_{kj}^v(h_j + f_j^h, v_j + f_j^v).
\end{aligned}
$$

We will interpret the above identity as power series in v_j with coefficients being holomorphic in $\varphi_j(U_k \cap U_j)$. In what follows, degrees or orders of sections are considered w.r.t. v_j at $v_j = 0$.

5.2 A Newton Method for the Full Linearization

For this problem, the two previous equations can be written as

$$\mathcal{L}_{kj}(f_j) = \Big(0, Dt_{kj}(h_j)f_j^h v_j\Big) + \mathcal{F}_{kj}(f_j), \tag{5.17}$$

where $\mathcal{L}_{kj}(f_j)$ stands for $(\mathcal{L}_{kj}^h(f_j^h), \mathcal{L}_{kj}^v(f_j^v))$ as defined by (2.18), (2.20):

$$\mathcal{L}_{kj}^h(f_j^h) := f_k^h(\varphi_{kj}(h_j), t_{kj}(h_j)v_j) - s_{kj}(h_j)f_j^h(h_j, v_j), \tag{5.18}$$
$$\mathcal{L}_{kj}^v(f_j^v) := f_k^v(\varphi_{kj}(h_j), t_{kj}(h_j)v_j) - t_{kj}(h_j)f_j^v(h_j, v_j). \tag{5.19}$$

Recall that $s_{kj}(h_j) = D\varphi_{kj}(h_j)$ is the Jacobian matrix of φ_{kj}. Furthermore, we have the horizontal error term

$$\begin{aligned}\mathcal{F}_{kj}^h(f_j) := {} & \phi_{kj}^h(h_j + f_j^h, v_j + f_j^v) - \hat{\phi}_{kj}^h \\ & + \Big(f_k^h(\varphi_{kj}, t_{kj}(h_j)v_j) - f_k^h(\varphi_{kj} + \hat{\phi}_{kj}^h, t_{kj}(h_j)v_j + \hat{\phi}_{kj}^v)\Big) \\ & + \varphi_{kj}(h_j + f_j^h(h_j, v_j)) - \varphi_{kj}(h_j) - D\varphi_{kj}(h_j)f_j^h(h_j, v_j),\end{aligned} \tag{5.20}$$

as well as the vertical error term

$$\begin{aligned}\mathcal{F}_{kj}^v(f_j) := {} & \phi_{kj}^v(h_j + f_j^h, v_j + f_j^v) - \hat{\phi}_{kj}^v + Dt_{kj}(h_j)f_j^h f_j^v \\ & + \Big(f_k^v(\varphi_{kj}, t_{kj}(h_j)v_j) - f_k^v(\varphi_{kj} + \hat{\phi}_{kj}^h, t_{kj}(h_j)v_j + \hat{\phi}_{kj}^v)\Big) \\ & + \Big(t_{kj}(h_j + f_j^h(h_j, v_j)) - t_{kj}(h_j) - Dt_{kj}(h_j)f_j^h\Big)(v_j + f_j^v).\end{aligned} \tag{5.21}$$

We collect $2m$ jets from (5.17), (5.20), (5.21). Since $f_j = O(m+1)$ and $\hat{\phi}_{kj}^{\bullet} = O(2m+1)$, this gives us

$$[(\delta^h f^h)_{kj}]^{\leq 2m} = -[\phi_{kj}^h]^{\leq 2m}, \tag{5.22}$$
$$[(\delta^v f^v)_{kj}]^{\leq 2m} = -Dt_{kj}(h_j)[f_j^h]^{\leq 2m-1}v_j - [\phi_{kj}^v]^{\leq 2m}. \tag{5.23}$$

Under formal assumptions, according to Lemma 2.16 (c), Eqs. (5.22)–(5.23) have a solution $([f_j^h]_{m+1}^{2m}, [f_j^v]_{m+1}^{2m})$.

We first consider the case that $H^0(C, \oplus_{k=2}^{2m} TC \otimes S^k(N_C^*)) = 0$. Then, for any $r_* < r'' < r' < \tilde{r} < r^*$ with

$$r'' = \theta r' = \theta^2 r, \quad r' - r'' < r^* - \tilde{r},$$

the solution to (5.22) is unique and by Theorem A.12 that unique solution satisfies the estimate

$$|\{[f_k^h]^l\}|_{r'} \leq \frac{D(TC \otimes S^l(N_C^*))}{(r-r')^\tau}|\{[\phi_{kj}^h]^l\}|_r, \quad l = m+1, \ldots, 2m. \tag{5.24}$$

In particular, $\{[f_k^h]_{m+1}^{2m}\}$ has been determined. The solvability of (5.23) and Theorem A.12 imply that we can find a solution $\{[f_k^v]_{m+1}^{2m}\}$ such that for $l = m+1, \ldots, 2m$,

$$|\{[f_k^v]^l\}|_{r''} \leq \frac{D(N_C \otimes S^l(N_C^*))}{(r'-r'')^\tau}\left\{c\frac{D(TC \otimes S^{l-1}(N_C^*))}{(r-r')^\tau}|\{[\phi_{kj}^h]^{l-1}\}|_r + |\{[\phi_{kj}^v]^l\}|_r\right\}. \tag{5.25}$$

Here c depends only on the Dt_{kj} over the initial covering.

If $H^0(C, \oplus_{k=m+1}^{2m} TC \otimes S^k(N_C^*)) \neq 0$, we are in the flat case, that is $Dt_{kj} = 0$. Thus, we can find a solution $\{[f_k^v]_{m+1}^{2m}\}$ such that for $l = m+1, \ldots, 2m$,

$$|\{[f_k^v]^l\}|_{r''} \leq \frac{D(N_C \otimes S^l(N_C^*))}{(r'-r'')^\tau}|\{[\phi_{kj}^v]^l\}|_r. \tag{5.26}$$

Let us set

$$D_*(2m) := 1 + \max_{2\leq l\leq 2m}\left\{(1 + cK(TC \otimes S^{l-1}(N_C^*)))D(N_C \otimes S^l(N_C^*)\right\}. \tag{5.27}$$

Hence, in any case, estimates (5.24)-(5.26) lead to

$$|\{[f_k^\bullet]^l\}|_{\theta^2 r} \leq \frac{C_1 D_*(2m)}{(r-\theta^2 r)^{2\tau}}|\{[\phi_{kj}^\bullet]^l\}|_r$$

for all θ and r satisfying $r_* \leq \theta^2 r < r < \tilde{r} < r^*$ and all $m+1 \leq l \leq 2m$. Assume further that $\theta^6 r > r_*$ and $(1-\theta^7)r < r^* - \tilde{r}$. We obtain, by Proposition 5.4 with $\tilde{m} = 2m+1$

$$|\hat{\phi}_{kj}^\bullet|_{\theta^7 r} \leq \frac{C_1 D_*(2m)\theta^{2m+1}}{(r-\theta^2 r)^{2\tau}}|\phi_{kj}^\bullet|_r \leq \theta^{2m+1}(1-\theta)r/C_0,$$

provided

$$|\{\phi_{kj}^\bullet\}|_r \leq (1-\theta)r/C_0, \tag{5.28}$$

$$\frac{D_*(2m)}{(r-\theta^2 r)^{2\tau}}|\{\phi_{kj}^\bullet\}|_r \leq (1-\theta)r/C_0. \tag{5.29}$$

Note that condition (5.28) follows from (5.29) as $D_*(\ell) \geq 1$.

Rename $\Phi_{kj}, \phi^{\bullet}_{kj}, F_j, f^{\bullet}_j, \hat{\Phi}_{kj}, \hat{\phi}^{\bullet}_{kj}$, respectively as $\Phi^{(0)}_{kj}, \phi^{(0)}_{kj}, F^{(0)}_j, f^{(0)}_j, \Phi^{(1)}_{kj}, \phi^{(1)}_{kj}$. Thus $\Phi^{(1)}_{kj} = (F^{(0)}_k)^{-1}\Phi^{(0)}_{kj}F^{(0)}_j$. Repeating this formally, we obtain

$$\Phi^{(\ell+1)}_{kj} = (F^{(\ell)}_k)^{-1}\Phi^{(\ell)}_{kj}F^{(\ell)}_j, \quad F^{(\ell)}_j = I + f^{(\ell)}_j, \quad \Phi^{(\ell+1)}_{kj} = N_{kj} + \phi^{(\ell+1)}_{kj}.$$

Set $r_{\ell+1} = \theta_\ell^7 r_\ell$ and $m_\ell = 2^\ell$. We also have

$$F^{(\ell)}_j(\hat{V}^{r_{\ell+1}}_j) \subset \hat{V}^{r_\ell}_j, \tag{5.30}$$

$$|\phi^{(\ell+1)}_{kj}|_{r_{\ell+1}} \leq \theta_\ell^{2m_\ell+1}(1-\theta_\ell)r_\ell/C_0 \tag{5.31}$$

provided

$$r_* \leq \theta_\ell^7 r_k < 1, \quad 0 < \theta_k < 1; \tag{5.32}$$

$$\frac{C_1 D_*(2m_\ell)}{(r_\ell - \theta_\ell^2 r_\ell)^{2\tau}}|\{\phi^{(\ell)}_{kj}\}|_{r_\ell} \leq (1-\theta_\ell)r_\ell/C_0. \tag{5.33}$$

To set parameters, we follow Russmann [37]; see [4,5,41] for different choices of parameters. As in [37], we now use an addition assumption that

$$D^*_\ell \geq \ell, \quad \ell \geq 1. \tag{5.34}$$

Indeed, when $\tilde{D}_*(k) = \max(D_*(k), k)$ replaces with $D_*(k)$, the sequence $D_*(k)$ still increases and $\sum 2^{-k}\log D_*(2^k)$ converges. For a constant $C_* \geq 1$ to be determined later, define

$$m_\ell = 2^{\ell_0+\ell}, \quad r_{\ell+1} = \theta_\ell^7 r_\ell, \quad r_0 = 1,$$
$$1-\theta_\ell = \delta_\ell, \quad \delta_\ell = C_*\frac{\log D_*(m_{\ell+2})}{m_{\ell+2}}.$$

Note that in [37, Lemma 6.2] and [4,5,41], $\omega(m_{\ell+1})$ is used to define δ_ℓ. Shifting the index by 1, we use $D_*(m_{\ell+2})$ to simplify the argument. We can find $\ell_0 = \ell_0(C_*)$ such that $0 < \theta_\ell < 1$ for all ℓ and furthermore

$$\prod_{\ell=0}^{\infty}\theta_\ell^7 = \prod_{\ell=0}^{\infty}(1-\delta_\ell)^7 \geq \exp\left\{-\sum_{\ell=0}^{\infty}\frac{7C_*}{2}\frac{\log D_*(m_{\ell+2})}{m_{\ell+2}}\right\}.$$

Since $\sum 2^{-k}\log D_*(2^k) < \infty$, the latter is larger than r_*, provided $\ell_0 > \ell_0(C_*)$. Inductively, we want to show that if $(5.33)_\ell$ holds, then $(5.33)_{\ell+1}$ also holds. Indeed, with $(5.33)_\ell$, we can use $(5.31)_{\ell+1}$ to obtain

$$\frac{C_1 D_*(m_{\ell+2})}{(r_{\ell+1} - \theta_{\ell+1}^2 r_{\ell+1})^{2\tau}}|\{\phi^{\ell+1}_{kj}\}|_{r_{\ell+1}} \times \frac{C_0}{(1-\theta_{\ell+1})r_{\ell+1}}$$

$$\leq \frac{D_*(m_{\ell+2})\theta_\ell^{2m_\ell-6}}{(r_{\ell+1}-\theta_{\ell+1}^2 r_{\ell+1})^{2\tau}} \times \frac{1-\theta_\ell}{1-\theta_{\ell+1}} \qquad \text{(by (5.31))}$$
$$\leq \frac{C_2 D_*(m_{\ell+2})\theta_\ell^{2m_\ell-6}}{(1-\theta_{\ell+1})^{2\tau+1}} = \frac{C_2 D_*(m_{\ell+2})(1-\delta_\ell)^{2m_\ell-6}}{\delta_{\ell+1}^{2\tau+1}}.$$

We need to check that the last expression is less than one using logarithm. Note that

$$\log(1-\delta) < -\delta, \quad \forall \delta \in (0,1).$$

Therefore,

$$\log \frac{C_2 D_*(m_{\ell+2})(1-\delta_\ell)^{2m_\ell-6}}{\delta_{\ell+1}^{2\tau+1}} < \log C_2 - (2m_\ell-6)\delta_\ell + \log D_*(m_{\ell+2}) - (2\tau+1)\log \delta_{\ell+3}$$
$$= \log C_2 - (2m_\ell-6)C_* \frac{\log D_*(m_{\ell+2})}{m_{\ell+2}} + \log D_*(m_{\ell+2}) - (2\tau+1)\log\left(C_* \frac{\log D_*(m_{\ell+3})}{m_{\ell+3}}\right)$$
$$= \left\{\log C_2 - \frac{(2m_\ell-6)C_*}{3}\frac{\log D_*(m_{\ell+2})}{m_{\ell+2}}\right\} + \left\{\log D_*(m_{\ell+2}) - \frac{(2m_\ell-6)C_*}{3}\frac{\log D_*(m_{\ell+2})}{m_{\ell+2}}\right\}$$
$$+ \left\{-\frac{(2m_\ell-6)C_*}{3}\frac{\log D_*(m_{\ell+2})}{m_{\ell+2}} - (2\tau+1)\log\left(C_*\frac{\log D_*(m_{\ell+3})}{m_{\ell+3}}\right)\right\}.$$

When ℓ_0 is sufficiently large, then $m_{\ell+2} > 24$. This implies that if $C_* > 12$, the sum in each of first two braces is negative. Since log increases, we have by (5.34)

$$-\log D_*(m_{\ell+3}) \leq \log \frac{1}{m_{\ell+3}},$$
$$-\log\left(C_* \frac{\log D_*(m_{\ell+3})}{m_{\ell+3}}\right) \leq -\log\left(-C_* \frac{\log \frac{1}{m_{\ell+3}}}{m_{\ell+3}}\right).$$

With $m_\ell > 6$, the difference in the last brace is bounded above by

$$\frac{(2m_\ell-6)C_*}{3}\frac{\log \frac{1}{m_{\ell+2}}}{m_{\ell+2}} - (2\tau+1)\log\left(C_* \frac{\log m_{\ell+3}}{m_{\ell+3}}\right) \leq \left(-\frac{1}{12}C_* + 2\tau + 1\right)\log m_{\ell+2},$$

which is negative when $C_* > 24\tau + 12$. We have determined C_*. This allows us to determine $\ell_0(C_*)$ so that $0 < \theta_\ell < 1$ and $\prod_{\ell=0}^\infty \theta_\ell^7 > r_*$. Therefore, $(5.33)_\ell$ holds if it holds for initial value $\ell = 0$. Using a dilation $v_j \to \epsilon v_j$ for $\epsilon > 0$, we may replace $\Phi_{kj}(h_j, v_j)$ by $(\varphi_{kj}(h_j) + \phi_{kj}^h(h_j, \epsilon v_j), t_{kj}(h_j)v_j + \epsilon^{-1}\phi_{kj}^v(h_j, \epsilon v_j))$. This yields $(5.33)_0$ when ϵ is sufficiently small, as $\phi_{kj}^\bullet(h_j, v_j) = O(|v_j|^2)$.

To finish the proof, we set $\Psi_j^{(\ell)} := F_j^{(0)} \circ \cdots \circ F_j^{(\ell)}$. We have

$$\Psi_j^{(\ell)}(\hat{V}_j^{r_{\ell+1}}) \subset \hat{V}_j^{r_\ell}, \quad \Psi_j^{(\ell+1)}(h_j, v_j) - \Psi_j^{(\ell)}(h_j, v_j) = O(|v_j|^\ell).$$

Consequently, the sequence $\Psi_j^{(\ell)}$ is bounded in $\hat{V}_j^{r\infty}$. Fix $0 < \theta < 1$. By the Schwarz lemma, we get

$$\sup_{\hat{U}_j^{r\infty} \times B_d^{\theta r\infty}} |\Psi_j^{(\ell+1)} - \Psi_j^{(\ell)}| \leq C\theta^{\ell}.$$

Therefore, of $\Psi_j^{(\ell)}$ converges uniformly on $\hat{U}_j^{r\infty} \times B_d^{\theta r\infty}$ to a holomorphic mapping Ψ_j^{∞}. Then $F := N_j^{-1}\Psi_j^{\infty}\Phi_j$ is well defined. Indeed, $N_k^{-1}\Psi_k^{\infty}\Phi_k = N_j^{-1}\Psi_j^{\infty}\Phi_j$ is equivalent to $\Psi_k^{\infty}(\Phi_k\Phi_j^{-1}) = (N_k N_j^{-1})\Psi_j^{\infty}$. Since Ψ_j^{∞} are tangent to the identity, they are germs of biholomorphisms. Therefore, F linearizes a small neighborhood of C in M.

Therefore, we have proved the following full linearization result.

Theorem 5.5 *Let a neighborhood of the compact manifold C in M be equivalent to a neighborhood of the zero section of normal bundle N_C of C in M by a formal holomorphic mapping which is tangent to the identity. Assume that $H^0(C, TC \otimes S^{\ell}(N_C^*)) = 0$ for all $\ell > 1$ or that the normal bundle N_C is flat. If $\{D_*(2^k)\}$ defined by (5.1) and (5.27) satisfies*

$$\sum_{k\geq 1} \frac{\log D_*(2^{k+1})}{2^k} < +\infty, \tag{5.35}$$

there is a neighborhood of the compact manifold C in M that is biholomorphic to a neighborhood of the zero section of normal bundle of C in M.

When the transition functions of C are affine and N_C is flat, the formal equivalence assumption can be relaxed by assuming that the neighborhoods are equivalent under a formal biholomorphisms fixing C pointwise. This follows from Lemma 2.4 (c).

We now present two examples to illustrate the results in this paper.

5.3 An Example of Arnol'd

This is originally studied by Arnold [2], [3, §27] for linearization of a neighborhood. See also Ilyashenko–Pyartli [23] for linearization for flat tori in higher dimensions.

Example 5.6 [3, §27]. Let C be defined by identifying points in $\mathbf{C}$ via

$$h = 0 \mod (2\pi, 2\omega), \quad h \in \mathbf{C},$$

where $\omega = a + ib$ with $b > 0$ and $a \geq 0$. Consider domains in C defined by parallelograms

$$U_1 = P(-r\pi - r\omega, (1+r)\pi - r\omega, (1+r)\pi + (1+r)\omega, -r\pi + (1+r)\omega)$$
$$U_4 = U_1 + \pi, \quad U_3 = U_4 + \omega, \quad U_2 = U_3 - \pi.$$

Suppose that $r > 0$ is sufficiently small. Then $U_i \cap U_j$ has two connected components $U_{ij,0}$ and $U_{ij,1}$ with

$$U_{14,1} = U_{14,0} - \pi, \quad U_{34,1} = U_{34,0} - \omega, \quad U_{23,1} = U_{23,0} - \pi, \quad U_{12,1} = U_{12,0} - \omega.$$

Let $\hat{U}_j = U_j$ and $\hat{V}_j = \hat{U}_j \times \Delta_\delta$. Define $M = \cup \hat{V}_j / \sim$, $V_j = \{[x] \colon x \in \hat{V}_j\}$, $\Phi_j \colon V_j \to \hat{V}_j$ and the transition functions Φ_{kj} on $V_{kj} = V_k \cap V_j$ of M as follows. Let

$$f(h, v) = (h + 2\omega + vb(h, v), \lambda v(1 + va(h, v))), \quad |\operatorname{Im} h| < \delta$$

where a, b are 2π periodic holomorphic functions in h. Define

$$\Phi_{12,0} = \mathrm{I}, \quad \Phi_{43,0} = \mathrm{I}, \quad \Phi_{12,1} = f|_{\hat{V}_{12}}, \quad \Phi_{43,1} = f|_{\hat{V}_{43}}, \tag{5.36}$$

$$\Phi_{14} = \mathrm{I}, \quad \Phi_{23} = \mathrm{I}, \tag{5.37}$$

$$\Phi_{13,0} = \mathrm{I}, \quad \Phi_{13,1} = f|_{\hat{V}_{13,1}}, \quad \Phi_{42,0} = \mathrm{I}, \quad \Phi_{42,1} = f|_{\hat{V}_{42,1}}. \tag{5.38}$$

The linearization of a neighborhood of C in M is equivalent to $G_k^{-1} \Phi_{kj} G_j = \hat{\Phi}_{kj}$ where $\hat{\Phi}_{kj}$ are constructed as above by replacing f with $\hat{f}$ defined by

$$\hat{f}(h, v) = (h + 2\omega, \lambda v).$$

Thus TM has transition functions:

$$\hat{\Phi}_{14} = \mathrm{I}, \quad \hat{\Phi}_{23} = \mathrm{I}, \quad \hat{\Phi}_{12,0} = \mathrm{I}, \quad \hat{\Phi}_{43,0} = \mathrm{I}, \quad \hat{\Phi}_{12,1} = \hat{f}|_{\hat{V}_{12}}, \quad \hat{\Phi}_{43,1} = \hat{f}|_{\hat{V}_{43}}.$$

Then we have $g := G_1 = G_4$ on $\hat{V}_1 \cap \hat{V}_4$, $g := G_2 = G_3$ on $\hat{V}_2 \cap \hat{V}_3$, $g := G_1 = G_2$ on $\hat{V}_{12,0}$ and $g := G_3 = G_4$ on $\hat{V}_{34,0}$. In other words, g is 2π periodic and defined on $-\delta \operatorname{Im} \omega < \operatorname{Im} h < 2(1 + \delta) \operatorname{Im} \omega$. The cohomology equation is reduced to $G_1^{-1} \Phi_{12} G_2 = \hat{\Phi}_{12}$ and $G_4^{-1} \Phi_{43} G_3 = \hat{\Phi}_{43}$. Equivalently, we need to solve

$$g^{-1} f g = \hat{f}. \tag{5.39}$$

Assume that f has been normalized so that

$$va(h, v) = v^n a_n(h) + O(n + 1), \quad vb(h, v) = v^n b_n(h) + O(n + 1), \quad n = 1, 2, \ldots.$$

For the purpose of illustration, we will only restrict to a *special* unitary line bundle case where $|\lambda| = 1$. Then by the non-resonance condition that λ is not a root of unity, we may assume that as in [3, p. 211]

$$g(h, v) = (h + v^n B_n(h), v(1 + v^n A_n(h)) + O(n + 1).$$

This leads to decoupled equations of the form

$$\lambda^n A_n(h+2\omega) - A_n(h) = -a_n(h),$$
$$\lambda^n B_n(h+2\omega) - B_n(h) = -b_n(h). \tag{5.40}$$

Note that a_n, b_n are holomorphic in $|\operatorname{Im} h| < \delta$ and we are seeking a solution on a large strip

$$-\delta' < \operatorname{Im} h < \operatorname{Im} \omega + \delta'.$$

In Fourier coefficients $a_{n,\ell}$ and a non-resonant condition, the Fourier coefficients of A_n are given by

$$A_{n,j} = \frac{a_{n,j}}{\lambda^n e^{2\omega j\sqrt{-1}} - 1}.$$

Assume that a_n are holomorphic and 2π periodic in h for $S_\delta \colon |\operatorname{Im} h| < \delta$. Suppose that

$$|\lambda^n e^{2j\omega\sqrt{-1}} - 1| \geq c|\lambda^n - 1|.$$

Then

$$|A_{n,j}| \leq \frac{C}{|\lambda^n - 1|}|a_n|_{L^2(S_\delta)} e^{-|j|\delta},$$
$$|A_{n,j}e^{jh}| \leq \frac{C}{|\lambda^n - 1|}|a_n|_{L^2(S_\delta)} e^{-|j|(\delta-\delta')}, \quad -\delta' < \operatorname{Im} h < \operatorname{Im} \omega + \delta'.$$

Furthermore, we can verify that

$$|A_n|_{L^2(S_{\delta'})} \leq \frac{C}{(\delta - \delta')|\lambda^n - 1|}|a_n|_{L^2(S_\delta)}.$$

Note that t_{kj} are locally constant with values $1, \lambda, \lambda^{-1}$.

Therefore, we have verified

$$D((T_C \oplus N_C) \otimes S^n N_C^*) \leq \frac{C}{|\lambda^n - 1|}.$$

By Lemma A.2, we get an estimate with equivalent bounds (up to a scalar) but in the original domain, i.e. without shrinking domains.

Strictly speaking, the above covering $\{U_j^r\}$ has non smooth boundary. The intersection is non-transversal either. However, this covering can be easily modified to get a generic covering defined early, replacing $\hat{U}_j$ by smooth strictly convex domains $\hat{U}_j$ and then replacing $\hat{U}_j$ by $\hat{U}_j + c_j$ for suitable small constants.

5.4 Counter-Examples

We now show that a certain small-condition is necessary to ensure the vertical and full linearizations. We will achieve this by establishing a connection between the classical linearization problem for germs of one-dimensional holomorphic mappings and the vertical linearization of foliated neighborhood of an elliptic curve.

We keep the notation in subsecton 5.3. Let us start with a power series

$$a(h, v) = \sum_{n\geq 2} a_n v^n := a(v). \tag{5.41}$$

Set $b(h, v) = 0$. Then we have a neighborhood of C associated to

$$f(h, v) = (h + 2\omega, \lambda v + a(v)). \tag{5.42}$$

Since the vertical part of the transition functions depends only on v, then M already admits a horizontal foliation with center C being compact.

Proposition 5.7 *Let $M_{\lambda,\omega,a}$ be neighborhood of C defined by transition functions Φ_{kj} given by (5.36)–(5.38) where f is given by (5.41)–(5.42). Suppose that λ, ω satisfy the nonresonance condition*

$$\lambda^n e^{2j\omega\sqrt{-1}} - 1 \neq 0, \qquad n = 2, 3, \ldots, j \in \mathbf{Z}. \tag{5.43}$$

Then $M_{\lambda,\omega,a}$ is vertically (resp. formally) linearizable by a mapping tangent to the identity if and only if the germ of holomorphic mapping $\varphi(v) = \lambda v + a(v)$ is holomorphically (resp. formally) linearizable.

Proof Suppose that M is vertically linearizable by a holomorphic mapping that is tangent to the identity. By Proposition 2.6, it is vertically linearization by a mapping G_j such that

$$G_j(h_j, v_j) = (h_j, v_j + O(|v_j|^2)).$$

By the non-resonance condition (5.43), we can verify that (5.39) is equivalent to that the g in (5.39) has the form $g(h, v) = (h, \psi(v))$ and φ is linearized by ψ. □

The existence of non-holomorphically linearizable φ is well-known. By theorems of Bruno [4,5] and Yoccoz [43], Proposition 5.7 shows that $M_{\lambda,\omega,\hat{a}}$ with $\hat{a}(v) = v^2$ is vertically linearizable and hence linearizable if and only if λ is a Bruno number, that is

$$\sum_{k\geq 1} \frac{\log \max_{2\leq j\leq 2^k} |\lambda^j - 1|^{-1}}{2^k} < +\infty.$$

5.5 A Foliation Example

Here we specialize Ueda's theory for elliptic curves. Let us first discuss the Fischer norms and Bergman norm when the N_C is unitary. Let us recall two formulae from Zhu [44, p. 22]:

$$\int_{\partial B_r^d} |z^Q|^2\, d\sigma_d = \frac{(d-1)!Q!}{(|Q|+d-1)!} r^{2d-1+2|Q|},$$
$$\int_{B_r^d} |z^Q|^2\, dV_d = \frac{d!Q!}{(|Q|+d)!} r^{2|Q|+d}.$$

Therefore, there is a precise asymptotic behavior of Fischer norm and the Bergman norm:

$$c_d\|g\|^2_{L^2(B_r^d)} \leq |g|^2_{f,r} \leq C_d\|g\|^2_{L^2(B_r^d)}, \quad 1/4 < r < 4. \tag{5.44}$$

We also have Bergman's inequality for L^2 holomorphic functions [15, p. 189]:

$$|f|_{\infty,\hat{V}_j^{(1-\theta)r}} \leq \frac{C_d}{(\theta r)^d} \sup_{h_j} |f(h_j,\cdot)|_{L^2(B_r^d)}, \tag{5.45}$$

$$\sup_{h_j} |f(h_j,\cdot)|_{L^2(B_r^d)} \leq C_d|f|_{\infty,B_r^d}, \quad 1/4 < r < 4. \tag{5.46}$$

In general, we get

$$|\phi^{\bullet}_{kj}|_{L^\infty(\hat{V}_{kj,(1-\theta)r})} \leq \frac{C_d}{(\theta r)^d} \sup_{h_j} |\phi^{\bullet}_{kj}(h_j,\cdot)|_{L^2(B^d_{kj,r}(h_j))}, \tag{5.47}$$

$$\sup_{h_j} |\phi^{\bullet}_{kj}|_{L^2(B^d_{kj,r}(h_j))} \leq C_d|\phi^{\bullet}_{kj}|_{L^\infty(\hat{V}^r_{kj})}, \quad 1/4 < r < 4. \tag{5.48}$$

Note that when t_{kj} are unitary, the skewed domain $\hat{V}^r_{kj}$ defined in (5.2) are actually product domains

$$\hat{V}^r_{kj} = \hat{U}^r_{kj} \times B^r_d.$$

Therefore, the Fischer norm and Bergman norm bound each other with constants depending only on θ and d. We can fix θ too by applying Lemma A.2 as we did in Sects. 3 and 4. Therefore, any estimate of cohomology equations in Fischer norms has a counter part in super norm on the unit ball in $\mathbf{C}^d$ and vice versa.

Note that the small divisors condition

$$|\lambda^n - 1| \geq Cn^{-\tau}, \quad n = 1, 2, \ldots \tag{5.49}$$

for some constants C, τ is equivalent to Ueda's condition in terms of $\mathrm{dist}(N_C^n, 1)$ for the foliation problem when C is an elliptic curve of type zero. In this case the corresponding linearized equation is equation (5.40) for which the small divisor $1/K_*(N_C \otimes S^n N_C^*)$ can be chosen to be $|\lambda^n - 1|$.

Finally, we should mention that the assumption $\eta_m \leq L_0 L^m$ is satisfied under Siegel's small divisor condition $|\lambda^n - 1| \geq C n^{-\tau}$ by a method of Siegel; see Ueda [42] for the vertical linearization problem. It is also satisfied under the Bruno condition [4,5] which is a condition weaker than (5.49). For the details, we refer to [4,5,36].

Appendix A. L^2 Bounds of Cohomology Solutions and Small Divisors

A.1 A Question of Donin

Let E be a holomorphic vector bundle on a compact complex manifold C. The main purpose of this section is to obtain L^2 and sup-norm bounds for the cohomology equation

$$\delta u = f \tag{A.1}$$

where $f \in Z^1(\mathcal{U}, \mathcal{O}(E))$ and $\mathcal{U}$ is a suitable covering of C. Our goal is to show that if $f = 0$ in $H^1(C, \mathcal{O}(E))$, then there is a solution u such that

$$\|u\|_{\mathcal{U}} \leq K(E)\|f\|_{\mathcal{U}}. \tag{A.2}$$

Here $\|\cdot\|_{\mathcal{U}}$ is the L^2-norm for cochains of the covering $\mathcal{U}$. The main assertion is that the solution u admits estimate on the *original* covering $\mathcal{U}$ without any refinement, which is important to the application in this paper. For this purpose, we will choose the covering $\mathcal{U}$ which consists of biholomorphic images of the unit polydisc and which are in the general position. The question on the existence of such an estimate and solutions was raised by Donin who asked the general question if $\mathcal{O}(E)$ is replaced by a coherent analytic sheaf $\mathcal{F}$ on C and f is any p-cocycle, with $p > 0$, of a covering $\mathcal{U}$ [9]. The result in this appendix provides an affirmative answer to Donin's question for $p = 1$ and the sheaf of holomorphic sections of a holomorphic vector bundle. Furthermore, we will introduce the small divisor for (A.1) in (A.2). Although some of results in this appendix can be further developed for a general setting, we limit to the case of $H^1(C, \mathcal{O}(E' \otimes E''))$; this suffices applications in this paper. One may take E'' to be the trivial bundle to deal with a general vector bundle E. In the applications we have in mind, C is embedded into a complex manifold M and we will take E'' to be symmetric powers $\mathrm{Sym}^\ell N_C^*$ of N_C^*, the dual of the normal bundle of C in M. In this paper, $S^\ell E$ denotes the symmetric power $\mathrm{Sym}^\ell E$ of a vector bundle E over C. We are mainly concerned with how various bounds depend on ℓ as $\ell \to \infty$ when we employ the important Fisher metric on $S^\ell N_C^*$ for unitary the normal bundle N_C. This will be crucial in our applications.

To prove (A.2), we will first use the original estimate of Donin [9], without solving the cohomology equation. This serves as a *smoothing decomposition* in the sense of

Grauert [15] by expressing

$$f = g + \delta u \tag{A.3}$$

where g is defined on a larger covering while u is defined on a shrinking covering. We will then combine it with the proof of finiteness theorem of cohomology groups from Grauert–Remmert [15] to refine the decomposition (A.3) by expressing g in a base of cocycles. Finally, we will obtain (A.2) by avoiding shrinking of covering. This last step is motivated by a method of Kodaira–Spencer and Ueda [42]. We take a different approach by an essential use of the uniqueness theorem. This allows us to introduce the *small divisors* in (A.2) to the cohomology equation (A.1).

A.2 Bounds of Solutions of Cohomology Equations

We now start to introduce nested coverings of C. This will be an essential ingredient of the small divisors for the cohomology equation. We cover C by finitely many open sets $U_i, i \in \mathcal{I}$ such that there are open sets V_i in M with $V_i \cap C = U_i$. We also assume that there are biholomorphic mappings Φ_i from V_i onto the polydisc $\Delta_{n+d}^{r^*}$ of radius r^*, where n is the dimension of C and $n+d$ is the dimension of M. Assume further that $\Phi_i(U_i^{r^*}) = \Delta_n^{r^*} \times \{0\}$ for $\varphi_i \times \{0\} = \Phi_i|_{U_i}$. Set $\mathcal{U}^r = \{U_i^r : i \in \mathcal{I}\}$ with $U_i^r = \varphi_i^{-1}(\Delta_n^r)$. We assume that $r^* < 1$ and $\mathcal{U}^{r_*}$ with $r_* < r^*$, remains a covering of C. When $U_I^r := U_{i_0}^r \cap \cdots U_{i_q}^r$ is non-empty, it is still Stein [15, p. 127].

Definition A.1 Let $\{U_j^r\}$ be an open covering of C for each $r \in [r_*, r^*]$. We say that the family of coverings $\{U_j^r\}$ is *nested*, if each connected component of $U_k^\rho \cap U_j^{r_*}$ intersects $U_k^{r_*} \cap U_j^{r_*}$ when $r_* \leq \rho \leq r^*$. In particular, $U_k^{r_*} \cap U_j^{r_*}$ is non-empty if and only if $U_k^\rho \cap U_j^{r_*}$ is non-empty.

Let $N(U_i^{r^*})$ be the union of all $\overline{U_k^{r^*}}$ that intersect $\overline{U_i^{r^*}}$; as in [9] we will call the union the *star* of $U_i^{r^*}$. Refining $\mathcal{U}^{r^*}$ if necessary, we may assume that there is a biholomorphism φ_i from a neighborhood of the star onto an open set in $\mathbf{C}^n$. If E', E'' are holomorphic vector bundles over C, we will fix a trivialization of E' over U_i by fixing a holomorphic basis $e'_k = \{e'_{k,1}, \dots, e'_{k,m}\}$ in $\overline{U_k^{r^*}}$. We also fix a holomorphic base $e''_j = \{e''_{j,1}, \dots, e''_{j,d}\}$ of E'' in $\overline{U_j^{r^*}}$. On $U_I^{r^*} = U_{i_0}^{r^*} \cap \cdots \cap U_{i_q}^{r^*}$, it will be convenient to use the base

$$e_{i_0 \dots i_q} := e'_{i_0} \otimes e''_{i_q} := \{e'_{i_0,k} \otimes e''_{i_q,j} \colon 1 \leq k \leq m, 1 \leq j \leq d\}.$$

Throughout the paper $\|\cdot\|_D$ and $|\cdot|_D$ denote, respectively, the L^2 and sup norms of a function in D, when D is a domain in $\mathbf{C}^n$. If $f = (f_1, \dots, f_d)$ is a vector of functions, we define the L^2 norm, metric, and sup norms as follows:

$$\|f\|_D^2 := \|f\|_{L^2(D)}^2 := \|f_1\|_D^2 + \cdots + \|f_d\|_D^2,$$

$$|f|_D^2 := \sup_{z\in D} |f_1(z)|^2 + \cdots + |f_d(z)|^2,$$
$$|f|_{\infty,D} := \sup_{z\in D} \max\{|f_1(z)|, \dots, |f_d(z)|\}.$$

For a $d \times d$ matrix t of functions on D, denote by $|t|_D$, $\|t\|_D$, $|t|_{\infty,D}$, respectively, the operator norms defined by

$$|t|_D = \sup_{|f|_D=1} |tf|_D, \quad \|t\|_D = \sup_{\|f\|_D=1} \|tf\|_D, \quad |t|_{\infty,D} = \sup_{|f|_{\infty,D}=1} |tf|_{\infty,D}.$$

Therefore, $\|t\|_D \leq |t|_D$ as $\|tf\|_D \leq (\sup_{z\in D} |t(z)|)\|f\|_D = |t|_D\|f\|_D$.

Then we define the L^2 norm for $f \in C^q(\mathcal{U}^r, \mathcal{O}(E' \otimes E''))$ by

$$a_I e_I := \sum_{\mu=1}^{md} a_I^\mu e_{I,\mu},$$
$$\|f\|_{\mathcal{U}^r} := \max_{I=(i_0,\dots,i_q)\in\mathcal{I}^{q+1}} \left\{\|a_I \circ \varphi_{i_q}^{-1}\|_{\varphi_{i_q}(U_I)} \colon f_i = a_I e_I \text{ in } U_I\right\}.$$

Sometimes we denote $\|f\|_{\mathcal{U}^{r_*}}$ by $\|f\|$ for abbreviation. We define similarly the metric norm $|f|_{\mathcal{U}^{r_*}}$, or $|f|$, and the sup-norm $|f|_{\infty,\mathcal{U}^{r_*}}$ or $\sup|f|$. It is obvious that

$$||f|| \leq C|f|, \quad \sup|f| \leq \|f\| \leq C\sqrt{\operatorname{rank}(E' \otimes E'')}\sup|f|,$$
$$|t|_\infty \leq |t| \leq C \operatorname{rank}(E' \otimes E'')|t|_\infty,$$

where C does not depend on E', E''.

The first result of this appendix is to find a way to obtain solutions to (A.1) that have certain bounds on the original covering, if a solution with a bound exists on a shrinking covering. This relies on the nested coverings defined above. We first study the L^2 norms case.

Lemma A.2 *Let $\mathcal{U}^r = \{U_i^r : i \in \mathcal{I}\}$ with $r_* \leq r \leq r^*$ be a family of nested finite coverings of C. Suppose that $f \in C^1(\mathcal{U}^{r^*}, E' \otimes E'')$ and $f = 0$ in $H^1(\mathcal{U}^{r^*}, E' \otimes E'')$. Assume that there is a solution $v \in C^0(\mathcal{U}^{r_*})$ such that*

$$\delta v = f, \quad \|v\|_{\mathcal{U}^{r_*}} \leq K\|f\|_{\mathcal{U}^{r^*}}. \tag{A.4}$$

Then there exists a solution $u \in C^0(\mathcal{U}^{r^*})$ such that $\delta u = f$ on $\mathcal{U}^{r^*}$ and

$$\|u\|_{\mathcal{U}^{r^*}} \leq C(|\{t'_{kj}\}|_{\mathcal{U}^{r^*}} + K|\{t'_{kj}\}|_{\mathcal{U}^{r^*}}|\{t''_{kj}\}|_{\mathcal{U}^{r_*}})\|f\|_{\mathcal{U}^{r^*}}, \tag{A.5}$$

where t'_{kj}, t''_{kj} are the transition matrices of E', E'', respectively, and C depends only on the number $|\mathcal{I}|$ of open sets in $\mathcal{U}^{r^*}$ and transition functions of C. In particular, C does not depend on E', E''.

Proof By assumptions, we have

$$f_{jk} = (\delta v)_{jk}, \quad U_j^{r_*} \cap U_k^{r_*}, \tag{A.6}$$

$$\|v\|_{\mathcal{U}^{r_*}} \leq K\|f\|_{\mathcal{U}^{r_*}}. \tag{A.7}$$

Take any $v^* \in C^0(\mathcal{U}^{r^*}, E' \otimes E'')$ such that $\delta v^* = f$. Then $(\delta v^* - \delta v)_{jk} = 0$ in $U_j^{r_*} \cap U_k^{r_*}$, because $(\delta v^*)_{jk} = f_{jk}$ on the larger set $U_j^{r^*} \cap U_k^{r^*}$. Since $\{U_j^{r_*}\}$ is a covering of C then $w := v_j - v_j^*$ is a global section of $E' \otimes E''$. This shows that v_j, via v_j^*, extends to a holomorphic section in $U_j^{r^*}$. In fact, v_j is the restriction of $u_j = v_j^* + w$ defined on $U_j^{r_*}$.

We now derive the bound for u_j. Suppose that $U_j^{r^*} \cap U_k^{r_*}$ is non-empty. By the assumptions, each component of $U_j^{r^*} \cap U_k^{r_*}$ intersects $U_j^{r_*} \cap U_k^{r_*}$. We have $u_j = u_k + f_{jk}$ on $U_j^{r_*} \cap U_k^{r_*}$ and hence the uniqueness theorem implies that it holds on $U_j^{r^*} \cap U_k^{r_*}$ too. And on $U_j^{r^*} \cap U_k^{r_*}$, we have $u_k = v_k$ and $u_j = v_k - f_{kj}$. We express the identity in coordinates

$$u_j = \tilde{u}_j e_j, \quad v_k = \tilde{v}_k e_k = \hat{v}_{kj} e_j, \quad f_{kj} = \tilde{f}_{kj} e_{kj} = \hat{f}_{kj} e_{jj}.$$

Let t'_{kj}, t''_{kj}, respectively, be the transition matrices of e'_j, e''_j for E', E''. Then $\tilde{t}_{kj} = t'_{kj} \otimes t''_{kj}$ are the transition matrices of e_{kj} for $E' \otimes E''$. Then we have

$$\hat{v}_{kj} = t'_{jk} \otimes t''_{jk} \tilde{v}_k, \quad \hat{f}_{kj} = t'_{jk} \otimes I_d \tilde{f}_{kj}.$$

Thus, $\tilde{u}_j = \hat{v}_{kj} - \hat{f}_{kj} = t'_{jk} \otimes t''_{jk} \tilde{v}_k - t'_{jk} \otimes I_d \tilde{f}_{kj}$. We have

$$\begin{aligned} \|\tilde{u}_j\|_{L^2(U_j^{r^*} \cap U_k^{r_*})} &= \|\tilde{u}_j \circ \varphi_j^{-1}\|_{L^2(\varphi_j(U_j^{r^*} \cap U_k^{r_*}))} \\ &\leq \|(t'_{jk} \otimes t''_{jk} \tilde{v}_k) \circ \varphi_j^{-1}\|_{L^2(\varphi_j(U_j^{r^*} \cap U_k^{r_*}))} \\ &\quad + \|(t'_{jk} \otimes I_d \tilde{f}_{kj}) \circ \varphi_j^{-1}\|_{L^2(\varphi_j(U_j^{r^*} \cap U_k^{r^*}))}. \end{aligned}$$

Here $t_{jk} \circ \varphi_j^{-1} = t_{jk} \circ \varphi_k^{-1} \circ \varphi_{kj}$. By the properties of operator norm and $\|t'_{kj} \otimes t''_{kj}\|_D \leq |t'_{kj} \otimes t''_{kj}|_D \leq |t'_{kj}|_D |t''_{kj}|_D$ for $D = \varphi_j(U_j^{r^*} \cap U_k^{r_*})$, we have

$$\|(t'_{jk} \otimes t''_{jk} \tilde{v}_k) \circ \varphi_j^{-1}\|_D^2 \leq C_* |t'_{jk}|_D^2 \times |t''_{jk}|_D^2 \times \|\tilde{v}_k\|^2_{\varphi_k(U_j^{r^*} \cap U_k^{r_*})},$$

where the constant C_* comes from the Jacobian of $z_k = \varphi_{kj}(z_j)$. By (A.7), we have

$$\|\tilde{v}_k \circ \varphi_k^{-1}\|_{L^2}^2 \leq K^2 \|f\|_{L^2}^2.$$

We also have

$$\|(t'_{jk}\otimes I_d\tilde f_{kj})\circ\varphi_j^{-1}\|_{\varphi_j(U_j^{r^*}\cap U_k^{r_*})}\leq |t'_{jk}\circ\varphi_j^{-1}|_{\varphi_j(U_j^{r^*}\cap U_k^{r_*})}\times\|f\|_{\varphi_j(U_j^{r^*}\cap U_k^{r_*})}.$$

Since $U_j^{r^*}$ is covered by $\{U_j^{r^*}\cap U_k^{r_*}\}$, we get the desired bound from

$$\|\tilde u_j\|_{L^2(U_j^{r^*})}\leq\sum_k\|\tilde u_j\|_{L^2(U_j^{r^*}\cap U_k^{r_*})}.$$

□

The argument for the norm $|\cdot|$ is verbatim and we can take the above constant C_* to be one.

Corollary A.3 *With notations and assumptions in Lemma A.2, the solution u also satisfies*

$$|u|_{\infty,\mathcal{U}^{r_*}}\leq C(|\{t'_{kj}\}|_{\mathcal{U}^{r_*}}+K|\{t'_{kj}\}|_{\mathcal{U}^{r_*}}|\{t''_{kj}\}|_{\mathcal{U}^{r_*}})\sqrt{\operatorname{rank}(E'\otimes E'')}|f|_{\infty,\mathcal{U}^{r^*}},$$

where C does not depend on E', E''.

The above lemma leads us to the following proposition and definition.

Proposition A.4 *Let $\mathcal{U}^r=\{U_i^r : i\in\mathcal{I}\}$ with $r_*\leq r\leq r^*$ be a family of nested coverings of a compact complex manifold C. Let E' (resp. E'') be a holomorphic vector bundle over C with bases $\{e'_j\}$ (resp. $\{e''_j\}$) and transition matrices t'_{kj} (resp. $\{t''_{kj}\}$). Suppose that there is a finite number K such that for any $f\in C^1(\mathcal{U}^{r^*},E'\otimes E'')$ with $f=0$ in $H^1(\mathcal{U}^{r^*},E'\otimes E'')$, there is a solution $v\in C^0(\mathcal{U}^{r_*},E'\otimes E'')$ satisfying (A.4). Then there is a possible different solution $v\in C^0(\mathcal{U}^{r_*},E'\otimes E'')$ satisfying (A.4) in which K is replaced by*

$$\begin{aligned}K_*(E'\otimes E'')=\sup_{u_1}\inf_{u_0}\big\{&\|u_0\|_{\mathcal{U}^{r_*}}:\delta u_0=\delta u_1\ on\,\mathcal{U}^{r_*},\\ &\|\delta u_1\|_{\mathcal{U}^{r^*}}=1,u_i\in C^0(\mathcal{U}^{r_i},E'\otimes E'')\big\}.\end{aligned}\tag{A.8}$$

Proof By the assumption, $K_*=K_*(E'\otimes E'')$ is well-defined and $K_*\leq K$. Fix $u_1\in C^0(\mathcal{U}^{r_i},E'\otimes E'')$. Suppose that $\delta u_1=f$ and $\|f\|_{\mathcal{U}^{r^*}}=1$. By the definition (A.8), there exists u_0^j such that $\delta u_0^m=f$ on $\mathcal{U}^{r_*}$ and $\|u_0^m\|_{\mathcal{U}^{r_*}}\leq K_*+1/m$. By the Cauchy formula on polydiscs, $(u_0^m)_j\circ\varphi_j^{-1}$ is locally bounded in $\varphi_j(U_j)$ in sup-norm. We may assume that as $m\to\infty$, $(u_0^m)_j$ converges uniformly to u_0^∞ on each compact subset of U_j for all j. This shows that $\|(u_0^\infty)_j\circ\varphi_j^{-1}\|_{L^2(E)}\leq K_*$ for any compact subset E of $\varphi_j(U_j)$. Since E is arbitrary, we obtain $\|u_0^\infty\|_{U^{r_*}}\leq K_*$. By the uniform convergence, we also have $\delta u_0^\infty=f$ on $\mathcal{U}^{r_*}$. □

Definition A.5 Let E', E'', e'_j, e''_j, t'_{kj}, t''_{kj} be as in Proposition A.4. Let $t''_{kj}(S^m E'')$ be the transition matrices of the symmetric power $S^m E''$ induced by t''_{kj}. For $m = 2, 3, \dots,$ we shall call

$$
\begin{aligned}
K(E' \otimes S^m E'') = |\{t'_{kj}(E')\}|_{\mathcal{U}^{r^*}} \\
+ K_*(E' \otimes S^m E'')|\{t'_{kj}(E')\}|_{\mathcal{U}^{r^*}}|\{t''_{kj}(S^m E'')\}|_{\mathcal{U}^{r^*}}
\end{aligned}
$$

the *generalized small divisors* of $E' \otimes E''$ with respect to e''_j, t''_{kj}.

A.3 Donin's Smoothing Decomposition

Grauert's smoothing decomposition for cochains of analytic sheaves is an important tool. Here we will follow an approach of Donin [9], by specializing for vector bundles.

We first need to introduce coverings by analytic polydiscs.

Lemma A.6 *Let C be a compact complex manifold. Let $\{U_i^{r_*} : i \in \mathcal{I}\}$ be a finite open covering of C, and let φ_j map U_j^r biholomorphically onto Δ_r^n for $r_* < r < r^* < 1$. Assume further that φ_i is a biholomorphism defined in a neighborhood of the star $N(U_i^{r^*})$ onto a domain in $\mathbf{C}^n$. Suppose that $r_* < r'_i < r_i < r^*$, and*

$$
U_I^{r'} := U_{i_0}^{r'_0} \cap \cdots \cap U_{i_q}^{r'_q} \neq \emptyset.
$$

Then for constant $c_n \in (0, 1)$ depending only on n,

$$
\operatorname{dist}\left(\partial(\varphi_{i_q}(U_I^r)), \partial(\varphi_{i_q}(U_I^{r'}))\right) \geq c_n \kappa \min_j (r_j - r'_j), \tag{A.9}
$$

$$
\kappa := \inf\left\{1, \frac{|\varphi_{i_q} \circ \varphi_{i_\ell}^{-1}(z') - \varphi_{i_q} \circ \varphi_{i_\ell}^{-1}(z)|}{|z' - z|} : z, z' \in \Delta_{r^*}^n, \forall U_{i_0 \dots i_q}^{r^*} \neq \emptyset\right\}. \tag{A.10}
$$

Proof Note that for sets in $\mathbf{C}^n$, if $A \subset A'$, $B \subset B'$, and A, B are non-empty, then

$$
\operatorname{dist}(A, B) \geq \operatorname{dist}(A', B').
$$

Recall that φ_{i_q} is a diffeomorphism from a neighborhood V of the star $N(U_{i_q})$ onto a subset $\hat{V}$ of $\mathbf{C}^n$. We have $\partial\varphi_{i_q}(U_I^r) \subset \cup_j \partial\varphi_{i_q}(U_{i_j}^r)$. Thus

$$
\begin{aligned}
\operatorname{dist}(\partial\varphi_{i_q}(U_I^r), \varphi_{i_q}(U_I^{r'})) &\geq \min_j \operatorname{dist}(\partial\, \varphi_{i_q}(U_{i_j}^r), \varphi_{i_q}(U_I^{r'})) \\
&\geq \min_j \operatorname{dist}(\partial\varphi_{i_q}(U_{i_j}^r), \varphi_{i_q}(U_{i_j}^{r'})).
\end{aligned}
$$

We have $\operatorname{dist}(\partial(\varphi_{i_q}(U^r_{i_j}), \varphi_{i_q}(U^{r'}_{i_j})) = \operatorname{dist}(\partial(\varphi_{i_q} \circ \varphi^{-1}_{i_j}(\Delta^n_r)), \varphi_{i_q} \circ \varphi^{-1}_{i_j}(\Delta^n_{r'}))$. Recall that φ_{i_q} is defined on $N(U_{i_q}) \supset U^{r_*}_{i_j}$. Then the distance is attained for some $z' \in \partial\Delta^n_{r'}$ and $z \in \partial\Delta^n_r$. By the definition of κ, we get the desired estimate. □

We will recall the following smoothing decomposition of Donin [9]. Here we restrict to the case of H^1 and the holomorphic vector bundle to indicate the specific bounds in the estimates.

Theorem A.7 (Donin [9]). *Let C be a compact complex manifold and let $\mathcal{U}^r$ $(r_* < r < r^* < 1)$ be a family of open coverings of C as in Lemma A.6. Let $E' \otimes E''$ be a holomorphic vector bundle of rank m over C and fix a holomorphic base e'_j (resp. e''_j) for E' (resp. E'') over U_j. Let $r_* < r'' < r' < r < r^*$, and*

$$r' - r'' \le r^* - r.$$

Assume that

$$U^{r_*}_{kj} \neq \emptyset, \quad \textit{whenever}\, U^{r^*}_{kj} \neq \emptyset. \tag{A.11}$$

Let $\{f_{jk}\} \in Z^1(\mathcal{U}^{r'}, \mathcal{O}(E' \otimes E''))$. Then there exist $g \in Z^1(\mathcal{U}^r, \mathcal{O}(E' \otimes E''))$ and $u \in C^0(\mathcal{U}^{r''}, \mathcal{O}(E' \otimes E''))$ such that

$$f = g + \delta u, \quad \textit{in}\, C^1(\mathcal{U}^{r''}, \mathcal{O}(E' \otimes E'')), \tag{A.12}$$

$$\|u\|_{\mathcal{U}^{r''}} + \|g\|_{\mathcal{U}^r} \le \frac{C_n |\{t'_{kj}\}||\{t''_{kj}\}|}{(r' - r'')\kappa} \|f\|_{\mathcal{U}^{r'}}, \tag{A.13}$$

where κ is defined (A.10). The constant C_n is independent of E', E''. Furthermore, $f \mapsto g = Lf$ and $f \mapsto u = Sf$ are $\mathbf{C}$-linear.

Proof With $f^{r'}_{ij} = f_{ij}$ we are given a cocycle $\{f^{r'}_{ij}\}$ of holomorphic sections of $E' \otimes E''$ over the covering $\mathcal{U}^{r'}$. Recall that $r_* < r'' < r' < r < r^*$ and $\mathcal{U}^{r''}$ is an open covering of C.

As in [9], we will apply L^2-theory for (0, 1)-forms on a bounded pseudoconvex domain in $\mathbf{C}^n$. In our case the domain is actually a polydisc. Fix a holomorphic base $e'_k = (e'_{k,1}, \dots, e'_{k,m})$ for the vector bundle E' in $U^{r^*}_k$ with transition functions $t'_{kj}(z_j)$. Analogously, let $t''_{kj}(z_j)$ be the transition matrices for basis e''_k of E'' for $\mathcal{U}^{r^*}$. For brevity, we write t_{kj} for $t_{kj}(z_j)$.

We can write

$$f^{r'}_{ij} = \tilde{f}^{r'}_{ij} e_{ij} = t'_{ki} \otimes t''_{kj} \tilde{f}^{r'}_{ij} e_{kk} := \hat{f}^{r';r^*}_{ij;k} e_{kk}, \quad \text{on}\, U^{r'}_i \cap U^{r'}_j \cap U^{r^*}_k. \tag{A.14}$$

The $U^{r^*}_k$ is covered by $\mathcal{U}^{r';r^*}_k := \{U^{r'}_i \cap U^{r^*}_k\}_i$, while $\{\hat{f}^{r';r^*}_{ij;k}\} \in Z^1(\mathcal{U}^{r';r^*}_k, \mathcal{O}^{md})$. Now $\{\hat{f}^{r';r^*}_{ij;k} \circ \varphi^{-1}_k\} \in Z^1(\varphi_k(\mathcal{U}^{r';r^*}_k), \mathcal{O}^{md})$, where $\varphi_k(\mathcal{U}^{r';r^*}_k)$ is a covering of the polydisc $\Delta^n_{r^*}$. By Lemma A.6, we have

$$c_{i;k} := \operatorname{dist}(\partial(\varphi_k(U_i^{r'} \cap U_k^{r^*})), \varphi_k(U_i^{r''} \cap U_k^r)) \geq c_n\kappa(r'-r''). \tag{A.15}$$

Let $d_{i;k}(z)$ be the distance to $\varphi_k(U_i^{r''} \cap U_k^r)$ from $z \in \mathbf{C}^n$. Let χ be a non-negative smooth function in $\mathbf{R}$ so that $\chi(t) = 1$ for $t < 3/4$ and $\chi(t) = 0$ for $t > 7/8$. By smoothing the Lipschitz function $\chi(\frac{1}{c_{i;k}}d_{i;k}(z))$, we obtain a non-negative smooth function $z \to \tilde{\phi}_{i;k}^{r'';r'}(z)$ that equals 1 when $d_{i;k}(z) \leq \frac{1}{2}c_{i;k}$ and by (A.15) it has compact support in $\varphi_k(U_i^{r'} \cap U_k^{r^*})$. Note that we can achieve

$$|\nabla\tilde{\phi}_{i;k}^{r'';r'}| < C_n c_{i;k}^{-1} \leq c_n C_n \kappa^{-1}/(r'-r''). \tag{A.16}$$

Then $\tilde{\phi}_{i;k}^{r'';r'} \circ \varphi_k$ is a non negative function with compact support in $U_i^{r'} \cap U_k^{r^*}$ such that for $\tilde{\phi}_k^{r'';r'} := \sum \tilde{\phi}_{i;k}^{r'';r'}$, we have $\tilde{\phi}_k^{r'';r'} \circ \varphi_k > 1/2$ in $U_k^r = \bigcup_i(U_i^{r''} \cap U_k^r)$ since $\chi(\frac{1}{c_{i;k}}d_{i;k}) = 1$ on $\varphi_k(U_i^{r''} \cap U_k^r)$. Then by the mean-value theorem and the first inequality of (A.16), we get

$$\tilde{\phi}_k^{r'';r'}(\varphi_k(x)) > 1/4, \quad \text{if } \operatorname{dist}(\varphi_k(x), \varphi_k(U_k^r)) < \min_i c_{i,k}/C_*, \tag{A.17}$$

for some suitable C_*. Recall that $c_n \leq 1$ and $\kappa_n \leq 1$. Since $\operatorname{dist}(\varphi_k(U_k^r), \varphi_k(\partial U_k^{r^*})) = r^* - r' \geq c_n\kappa(r'-r'')$, there is a smooth function $\hat{\phi}_k^{r;r^*}: \varphi_k(U_k^{r^*}) \to [0,1]$ with compact support such that $\hat{\phi}_k^{r;r^*} = 1$ in $\varphi_k(U_k^r)$, and

$$\hat{\phi}_k^{r;r^*}(x) < 3/4, \quad \text{if } \operatorname{dist}(\varphi_k(x), \varphi_k(U_k^r)) > \min_i c_{i,k}/C_*. \tag{A.18}$$

Note that the latter can be achieved with

$$|\nabla\hat{\phi}_k^{r;r^*}| < \tilde{C}_1/\min_i c_{i,k} \leq C_2\kappa^{-1}/(r'-r'').$$

In $U_k^{r^*}$, define a non-negative smooth function

$$\phi_{i;k}^{r'';r'} = \left\{\frac{\tilde{\phi}_{i;k}^{r'';r'}}{1-\hat{\phi}_k^{r;r^*}+\tilde{\phi}_k^{r'';r'}}\right\} \circ \varphi_k,$$

where the smoothness follows from the denominator being bigger than 1/4 by (A.17) and (A.18). Thus, $\phi_{i;k}^{r'';r'}$ has compact support in $U_i^{r'} \cap U_k^{r^*}$ and $\sum_i \phi_{i;k}^{r'';r'} = 1$ in $U_k^r = \bigcup_i(U_i^{r''} \cap U_k^r)$, as $\hat{\phi}_k^{r;r^*} = 1$ on U_k^r. We can verify that

$$|\nabla(\phi_{i;k}^{r'';r'} \circ \varphi_k^{-1})| < C'\kappa^{-1}/(r'-r''). \tag{A.19}$$

Consider the expression

$$w_{j;k}=\sum_{\ell}\phi_{\ell;k}^{r'';r'}\hat{f}_{\ell j;k}^{r';r^*}. \tag{A.20}$$

Recall that $\phi_{\ell;k}^{r'';r'}$ has compact support in $U_\ell^{r'}\cap U_k^{r^*}$. Thus it is smooth on $\omega:=U_j^{r'}\cap U_k^{r^*}\cap U_\ell^{r'}$ and vanishes on an open set D containing $U_j^{r'}\cap U_k^{r^*}\setminus\omega$. On the other hand, $\hat{f}_{\ell j;k}^{r';r^*}$ is holomorphic in ω. Hence the product $\phi_{\ell;k}^{r'';r'}\hat{f}_{\ell j;k}^{r';r^*}$ is smooth in $U_j^{r'}\cap U_k^{r^*}$. Then $v_{j;k}=\overline{\partial}w_{j;k}$ is a smooth $(0,1)$ form in $U_j^{r'}\cap U_k^{r^*}$.

Let $\mathcal{A}$ denote the sheaf of smooth functions on C. We now pull back the forms from the polydisc Δ^n via φ_k. For each fixed k, we have $\{w_{j;k}\}_j\in C^0(\mathcal{U}_k^{r';r^*},\mathcal{A}^m)$. Let us denote $t'_{kj}\otimes I$ by t'_{kj}. By $f_{ij}=f_{ik}-f_{jk}$ and (A.14), we have

$$t'_{ki}\otimes t''_{kj}\tilde{f}_{ij}^{r'}=t'_{ki}\tilde{f}_{ik}^{r'}-t'_{kj}\tilde{f}_{jk}^{r'}.$$

Since $\sum_i\phi_{i;k}^{r'';r'}=1=\hat{\phi}_k^{r;r^*}\circ\varphi_k$ on U_k^r, then by $\delta f=0$ and (A.14), we get on $U_i^{r'}\cap U_k^r\cap U_j^{r'}$

$$\begin{aligned}w_{i;k}-w_{j;k}&=\sum_\ell\phi_{\ell;k}^{r'';r'}(\hat{f}_{\ell i;k}^{r';r^*}-\hat{f}_{\ell j;k}^{r';r^*})=\sum_\ell\phi_{\ell;k}^{r'';r'}(t'_{k\ell}\otimes t''_{ki}\tilde{f}_{\ell i}^{r'}-t'_{k\ell}\otimes t''_{kj}\tilde{f}_{\ell j}^{r'})\\&=\sum_\ell\phi_{\ell;k}^{r'';r'}(t'_{kj}\tilde{f}_{jk}^{r'}-t'_{ki}\tilde{f}_{ik}^{r'})=t'_{kj}\tilde{f}_{jk}^{r'}-t'_{ki}\tilde{f}_{ik}^{r'}.\end{aligned}$$

The latter is holomorphic. Thus $(\delta v)_{ij;k}=\overline{\partial}(\delta w)_{ij;k}=0$ on $U_i^{r'}\cap U_k^{r^*}\cap U_j^{r'}$. This shows that

$$v_k:=v_{j;k}$$

is actually a $\overline{\partial}$-closed $(0,1)$ form in $U_k^{r^*}$. Thus $(\varphi_k^{-1})^*v_k$ is a $\overline{\partial}$-closed $(0,1)$-form on the polydisk $\Delta_{r^*}^n$. By the L^2 theory [21, Thm. 4.4.3] applied to each component of $v_k=\sum_{\ell=1}^m\tilde{v}_k^\ell e_{kk,\ell}$, we have a bounded linear operator $S\colon v_k\to u_k$ such that $\overline{\partial}((\varphi_k^{-1})^*u_k)=(\varphi_k^{-1})^*v_k$. Returning to the complex manifold via φ_k, we have

$$\begin{aligned}\|u_k\|_{U_k^{r^*}}&=\|u_k\circ\varphi_k^{-1}\|_{L^2(\Delta_{r^*}^n)}\leq C\|v_k\circ\varphi_k^{-1}\|_{L^2(\Delta_{r^*}^n)}\\&\leq\frac{\tilde{C}\kappa^{-1}|\{t'_{kj}\}||\{t''_{kj}\}|}{r'-r''}\|f\|_{L^2(\mathcal{U}^{r^*})}.\end{aligned}$$

Here we have used (A.20), estimate (A.19) and the definition of norm (A.4). Note that the $\tilde{C}$ is independent of the rank since we applied the L^2 norm componentwise. Set $\hat{g}_{j;k}^{r';r}=w_{j;k}-u_k$ on $U_j^{r'}\cap U_k^r$. We obtain

$$\hat{g}_{i;k}^{r',r}-\hat{g}_{j;k}^{r',r}=\hat{f}_{ij;k}^{r';r^*},\quad U_i^{r'}\cap U_k^r\cap U_j^{r'}, \tag{A.21}$$

$$\max_j \|\hat{g}^{r';r}_{j;k}\|_{U^{r'}_j \cap U^r_k} \leq \frac{C\kappa^{-1}|\{t'_{kj}\}||\{t''_{kj}\}|}{r' - r''}\|f\|_{\mathcal{U}^{r'}}. \tag{A.22}$$

We have obtained (A.13).

To verify (A.12), we will use the same base e_k and take the product of (A.21) with e_k to obtain on $U^{r''}_i \cap U^{r''}_j \cap U^r_k \cap U^r_\ell$

$$g^{r';r}_{i;k} - g^{r';r}_{j;k} = \hat{f}^{r';r*}_{ij;k} e_k = f^{r'}_{ij} = \hat{f}^{r';r*}_{ij;\ell} e_\ell = g^{r';r}_{i;\ell} - g^{r';r}_{j;\ell}$$

and thus

$$g^{r';r}_{j;\ell} - g^{r';r}_{j;k} = g^{r';r}_{i;\ell} - g^{r';r}_{i;k}, \quad \text{on } U^{r''}_i \cap U^{r''}_j \cap U^r_k \cap U^r_\ell. \tag{A.23}$$

Then we have a (well-defined) holomorphic section

$$g^r_{k\ell} := g^{r';r}_{i;\ell} - g^{r';r}_{i;k}, \quad U^r_k \cap U^r_\ell.$$

We verify that $\{g^r_{k\ell}\} \in Z^1(\mathcal{U}^r, \mathcal{O}^m)$. Set $u^{r''}_i := g^{r'';r}_{i;i}$. Since $r' \leq r$ we actually have $\{u^{r''}_i\} \in C^0(\mathcal{U}^{r'}, E' \otimes E'')$. However, only on $U^{r''}_i \cap U^{r''}_j$, we can verify via (A.23) that

$$g^r_{ij} - f^{r'}_{ij} = (g^{r'';r}_{i;j} - g^{r'';r}_{j;j}) - (g^{r'';r}_{i;j} - g^{r'';r}_{i;i}) = u^{r''}_i - u^{r''}_j.$$

□

The above result is a type of Grauert's smoothing decomposition, which can also be obtained by open mapping theorem. See for instance [15, p. 200]. However, this yields an unknown bound in the estimates.

A.4 Finiteness Theorem with Bounds

The above smoothing decomposition does not provide a solution to the cohomology equations, i.e. if $f = 0$ in $H^1(\mathcal{U}^{r'}, \mathcal{O}(E' \otimes E''))$, then there exists $u \in C^0(\mathcal{U}^{r''}, \mathcal{O}(E' \otimes E''))$ such that $\delta u = f$ on $\mathcal{U}^{r''}$, for some $r'' \leq r'$. We will follow [15] to derive the finiteness theorem with explicit bounds. In particular, this provides solutions of first cohomology equations with bounds on shrinking domains.

We first recall the resolution atlases from [15, p. 194], specializing them for the vector bundles. Assume that we have coordinate charts

$$\varphi_k \colon U^{r^*}_k \to P_k := \varphi_k(U^{r^*}_k) = \Delta^{r^*}_n.$$

Define $U^{r^*}_I = U^{r^*}_{i_0} \cap \cdots \cap U^{r^*}_{i_q}$ for $I \in \mathcal{I}^{q+1}$. Then $\varphi_I = (\varphi_{i_0}, \ldots, \varphi_{i_q})$ is defined on $U^{r^*}_I$ with range $\hat{U}^{r^*}_I$. Unless otherwise stated, we omit the superscript r^* in $U^{r^*}_I$. We can define a *proper* embedding

$$\varphi_I : U_I \to \hat{U}_I \hookrightarrow P_I := \Delta^{r^*}_{n_q}, \quad n_q = n(q+1).$$

Then the push-forward of the vector bundle $E' \otimes E''|_{U_I}$ defines a coherent analytic sheaf $(\varphi_I)_*(E' \otimes E'')$ over P_I by trivial zero extension; see [15, p. 5, p. 195] and [14, p. 239]. A section $f \in \Gamma(U_I, E' \otimes E'')$ yields a section $\hat{f}_I$ of $(\varphi_I)_*(E' \otimes E'')$ over P_I by

$$\hat{f}_I \circ \varphi_I(x) = (f_I(x), \dots, f_I(x)), \quad \hat{f}_I|_{P_I \setminus \hat{U}_I} = 0.$$

Note that $\overline{U^{r^*}}$ has a Stein neighborhood. Then following notation in [15, p. 196] we have an epimorphism by Cartan's Theorem A:

$$\epsilon_I : \mathcal{O}^\ell|_{\Delta^{r^*}_{n_q}} \to (\varphi_I)_*(E' \otimes E'')|_{U_I}, \quad \ell \geq \operatorname{rank}(E' \otimes E''),$$

where ϵ_I is defined by finitely many global sections defined in a neighborhood of $\overline{P_I}$. When $E' \otimes E''$ is a vector bundle, we take ℓ to be the minimal value, the rank of $E' \otimes E''$, and specify the above ϵ_I by taking

$$\epsilon_I : g_I \to \tilde{g}_I := (\varphi_I)_*\{g_I \circ \varphi_I e_I\}.$$

Here we want to obtain a more general description without restricting to a vector bundle. Define

$$C^q(\mathcal{U}) := \prod_{I \in \mathcal{I}^{q+1}} \mathcal{O}^\ell(P_I).$$

(Set $\mathcal{O}^\ell(P_I) = 0$ when $U_I^{r^*}$ is empty.) We recall that $P_I = \Delta^{r^*}_{n_q}$ is independent of the order of multi-indices. Thus

$$C^q(\mathcal{U}) \cong (\mathcal{O}(\Delta^{r^*}_{n_q}))^L := \mathcal{O}^L(\Delta^{r^*}_{n_q}).$$

Here $L \leq |\mathcal{I}^{q+1}|\ell$. Let $\mathcal{O}_h(\Delta^r_{n_q})$ be the space of holomorphic functions on $\Delta^r_{n_q}$ with finite L^2 norm on $\Delta^r_{n_q}$. Set $P_I^r = \Delta^r_{n_q}$ for $I \in \mathcal{I}^{q+1}$. We define a Hilbert space

$$C^q_h(\mathcal{U}^r) := \prod_{I \in \mathcal{I}^{q+1}} \mathcal{O}^\ell_h(P_I^r) := \mathcal{O}^L_h(\Delta^r_{n_q}),$$

which is a subspace of $C^q(\mathcal{U}^r)$.

Using the collection $\epsilon = \{\epsilon_I : I \in \mathcal{I}^{q+1}\}$, we define

$$\mathcal{C}^q_h(\mathcal{U}^r, E' \otimes E'') := \epsilon(C^q_h(\mathcal{U}^r)) \cong C^q_h(\mathcal{U}^r)/(\ker \epsilon \cap C^q_h(\mathcal{U}^r)),$$

which is the vector space of q-cochains, equipped with the standard coboundary operator δ.

Remark A.8 Our cochains are not necessary alternating. As in [15, p. 35], we let $\mathcal{C}_a^q(\mathcal{U}, E' \otimes E'')$ denote alternating cochains. For the isomorphism of the two kinds of Cečh cohomology groups; see [15, p. 35] and Serre [39]. Since we are interested in the cohomological solutions with bounds, we fix our notation without requiring that the cochains be alternating.

Let $\|\cdot\|_{\Delta^r_{n_q}}$ be the Hilbert space norm on $\mathcal{C}_h^q(\mathcal{U}^r)$ and set

$$\|\zeta\|^{\bullet}_{\mathcal{U}^r} = \inf\{\|v\|_{\Delta^r_{n_q}} : v \in C_h^q(\mathcal{U}^r), \epsilon(v) = \zeta\}, \quad \zeta \in C_h^q(\mathcal{U}^r, E' \otimes E'').$$

The inclusion $\mathcal{C}_h^q(\mathcal{U}^r, E' \otimes E'') \hookrightarrow \mathcal{C}^q(\mathcal{U}^r, E' \otimes E'')$ is continuous and compact ([15, Thm. 3, p. 197]). We also define

$$\begin{aligned}
&Z_h^q(\mathcal{U}^r) := \epsilon^{-1}(Z_h^q(\mathcal{U}^r, E' \otimes E'')),\\
&\|\zeta\|_{\mathcal{U}^r} := \inf\{\|v\|_{\Delta_r^{n_q}} : v \in Z_h^q(\mathcal{U}^r), \epsilon(v) = \zeta\}, \quad \forall \zeta \in Z_h^q(\mathcal{U}^r, E' \otimes E''),\\
&\overline{v} := \epsilon(v).
\end{aligned}$$

Then $Z_h^q(\mathcal{U}^r, E' \otimes E'')$ is an isometric subspace of $\mathcal{C}_h^q(\mathcal{U}^r, E' \otimes E'')$ via inclusion. Let $\{g_0, g_1, \dots\}$ be a monotone orthogonal base of $Z_h^1(\mathcal{U}^r)$ ([15, p. 141, p. 201]). An important feature of the monotone base is that the vanishing orders of g_j at the origin satisfy

$$\operatorname{ord}_0 g_0 \leq \operatorname{ord}_0 g_1 \leq \cdots, \quad \lim_{i\to\infty} \operatorname{ord}_0 g_i = \infty.$$

By [15, Thm. 1, p. 192 and p. 201], for a given ν there is an μ such that

$$g_i(Z) = O(|Z|^{\nu}), \quad i > \mu, \quad Z \in \Delta_r^{n_q}. \tag{A.24}$$

In fact, let the index set be $\mathcal{I} = \{1, \dots, L\}$. Set $\omega((f_1, \dots, f_L)) = \min\{(\alpha, Q) \colon f_{\alpha,Q} \neq 0\}$ by using order $<$ on $\mathcal{I} \times \mathbf{N}^m$ defined by $(\alpha, P) < (\beta, Q)$ if $|P| < |Q|$, or if $|P| = |Q|$ and there is an ℓ such that $p_\ell < q_\ell$ and $p_{\ell'} = q_{\ell'}$ for all $\ell' > \ell$, or if $P = Q$ and $\alpha < \beta$. Then the basis $\{g_j\}$ satisfies

$$\omega(g_j) < \omega(g_{j+1}).$$

We now return to the case $q = 1$ with $n_q = 2n$. In the sequel, $\{|t'_{kj}|\} = \{|t'_{kj}|\}_{\mathcal{U}^{r*}}$ and $\{|t''_{kj}\}| = \{|t''_{kj}|\}_{\mathcal{U}^{r*}}$.

Theorem A.9 (Donin-Grauert-Remmert). *Let C be a compact complex manifold and let $\mathcal{U}^r$ $(r_* < r < r^* < 1)$ be a family of open coverings of C as in Lemma A.6 such that (A.11) holds for all k, j. Let $E = E' \otimes E''$ be a holomorphic vector bundle of positive rank m over C and fix a holomorphic base e'_j (resp. e''_j) for E' (resp. E'') over*

$U_j^{r^*}$. Suppose that $r_* < r'' < r' < r < r^*$ and $r' - r'' \leq r^* - r$. Let $\theta = r'/r$. Let $\{g_0, g_1, \dots\}$ be a monotone orthogonal base of $Z_h^1(\mathcal{U}^r)$ as above. Assume that μ, ν satisfy (A.24) and

$$t := \frac{C_n \kappa^{-1}}{(r' - r'')(r - r')^{2n}} \theta^{\nu} < 1/2. \tag{A.25}$$

There exist $\overline{g_{m_0}}, \dots, \overline{g_{m_{\mu^*}}}$ such that their equivalence classes in $H^1(\mathcal{U}^r, E)$ form a $\mathbf{C}$-linear basis of subspace spanned by $\overline{g_0}, \cdots, \overline{g_\mu}$ in $H^1(\mathcal{U}^r, E)$. For any $f \in Z_h^1(\mathcal{U}^{r'}, E)$ there exists $v \in C_h^0(\mathcal{U}^{r''}, E)$ satisfying $f = \delta v + \sum_0^{\mu^*} c_i \overline{g_{m_i}}$ with

$$|c_i| \leq \frac{C_n \kappa^{-1} A_r(E)}{r - r'} \|f\|_{\mathcal{U}^{r'}}, \tag{A.26}$$

$$\|v\|_{\mathcal{U}^{r''}} \leq \frac{C_n \kappa^{-1} B_{r_-}(E)}{r - r'} \|f\|_{\mathcal{U}^{r'}}, \quad \forall r_- \in [r', r), \tag{A.27}$$

$$g_j = \sum_{i=0}^{\mu^*} c_{ji} \overline{g_{m_i}} + \delta \eta_j^*, \quad \eta_j^* \in C^0(\mathcal{U}^r, E), \tag{A.28}$$

$$A_r(E) = |\{t'_{kj}\}||\{t''_{kj}\}| \max_{0 \leq i \leq \mu_*} \sum_{j=0}^{\mu} |c_{ji}|, \quad B_{r_-}(E) = |\{t'_{kj}\}||\{t''_{kj}\}| \sum_{j=0}^{\mu} \|\{\eta_j^*\}\|_{\mathcal{U}^{r_-}}. \tag{A.29}$$

Furthermore, all $c_j = 0$ when $f = 0$ in $H^1(C, E)$.

Remark A.10 The solution operator $f \to v$ may not be linear. See a proof by Donin [9] to get a linear solution operator for which the constant C_* results from a lemma of Schwartz.

Remark A.11 The previous theorem gives a solution v, defined on a smaller domain, to the equation $f = \delta v$ (i.e cohomological equations) whenever f is 0 in the first cohomology group. It also provides a bound of the solution in terms of the data. We emphasize that this bound depends on the bundle $E' \otimes E''$. In the applications we have in mind, we will have to consider a sequence of bundles $\{S^m E''\}_m$, and we will need to control the growth of these bounds as m goes to infinite, similarly to the *small divisors* appearing in local dynamical systems.

Proof Recall that $q = 1$ and $n_1 = 2n$. We may assume that $\|g_j\|_{\Delta_{2n}^r} = 1$. By the definition of μ, ν and the monotone basis, we have for any $v \in Z_h^1(\mathcal{U}^r)$,

$$\|v - \sum_{j=0}^{\mu} (v, g_j) g_j\|_{\Delta_{2n}^{r'}} \leq \frac{C_n}{(r - r')^{2n}} (r'/r)^{\nu} \|v\|_{\Delta_{2n}^r} \tag{A.30}$$

where $C_n (r - r')^{-2n}$ is the constant M in [15, Thm. 6, p. 191].

Replacing the smoothing lemma in [15, p. 200] by Theorem A.7, we derive some estimates following the proof of the finiteness lemma in [15, p. 201]. By assumption, we have

$$t = \frac{C_n \kappa^{-1}}{(r'-r'')(r-r')^{2n}}\theta^\nu < 1/2, \quad \theta = \frac{r'}{r} < 1.$$

Let $\zeta_0 := f \in Z_h^1(\mathcal{U}^{r'}, E' \otimes E'')$. By Theorem A.7, we have for some $\xi_0 \in Z_h^1(\mathcal{U}^r, E' \otimes E'')$

$$\begin{aligned}
&\zeta_0 = \xi_0 + \delta\eta_0,\\
&\|\xi_0\|_{\mathcal{U}^r} \leq t'\|\zeta_0\|_{\mathcal{U}^{r'}}, \quad \|\eta_0\|_{\mathcal{U}^{r''}} \leq t'\|\zeta_0\|_{\mathcal{U}^{r'}},
\end{aligned}$$

with $t' := \frac{C_n|\{t'_{kj}\}||\{t''_{kj}\}|}{\kappa(r'-r'')}$. Let $\overline{v}$ denote $\epsilon(v)$. Then $\xi_0 = \overline{v}_0$ for some v_0 satisfying $\|v_0\|_{\Delta^r_{2n}} = \|\xi_0\|_{\mathcal{U}^r}$; see [15, p. 198]. Consider

$$w_1 = v_0 - \sum_{j=0}^{\mu}(v_0, g_j)_{\Delta^r_{2n}} g_j, \quad \zeta_1 = \overline{w}_1.$$

According to (A.30), we have

$$\|\zeta_1\|_{\mathcal{U}^{r'}} \leq \|w_1\|_{\mathcal{U}^{r'}} \leq \frac{C_n}{(r-r')^{2n}}(r'/r)^\nu \|v_0\|_{\Delta^r_{2n}} \leq t\|\zeta_0\|_{\mathcal{U}^{r'}}.$$

Therefore,

$$\zeta_0 = \sum_{j=0}^{\mu}(v_0, g_j)_{\Delta^r_{2n}} \overline{g}_j + \delta\eta_0 + \zeta_1.$$

In general, we have

$$\begin{aligned}
&\zeta_\ell = \sum_{j=0}^{\mu}(v_\ell, g_j)_{\Delta^r_{2n}} \overline{g}_j + \delta\eta_\ell + \zeta_{\ell+1},\\
&\|v_\ell\|_{\Delta^r_{2n}} = \|\xi_\ell\|_{\mathcal{U}^r} \leq t't^\ell\|\zeta_0\|_{\mathcal{U}^{r'}},\\
&\|\zeta_{\ell+1}\|_{\mathcal{U}^{r'}} \leq t\|\zeta_\ell\|_{\mathcal{U}^{r'}} \leq t^{\ell+1}\|\zeta_0\|_{\mathcal{U}^{r'}},\\
&\|\eta_\ell\|_{\mathcal{U}^{r''}} \leq t't^\ell\|\zeta_0\|_{\mathcal{U}^{r'}}.
\end{aligned}$$

Then we have

$$f = \zeta_0 = \sum_{j=0}^{\mu}\sum_{\ell=0}^{\infty}(v_\ell, g_j)_{\Delta^r_{2n}} \overline{g}_j + \delta\sum_{\ell=0}^{\infty}\eta_\ell,$$

$$\sum_{\ell=0}^{\infty}|(v_\ell, g_j)| \leq \sum_{\ell=0}^{\infty}\|v_\ell\|_{\Delta_{2n}^r} \leq \frac{t'}{1-t}\|\zeta_0\|_{\mathcal{U}^{r'}},$$
$$\sum_{\ell=0}^{\infty}\|\eta_\ell\|_{\mathcal{U}^{r''}} \leq \frac{t'}{1-t}\|\zeta_0\|_{\mathcal{U}^{r'}}.$$

So far we have followed the proof of the finiteness lemma in [15, p. 201]. We now finish the proof of the theorem. Let us first find the linearly independent elements $\overline{g_{i_0}}, \dots, \overline{g_{i_{\mu*}}}$. Assume first that all $\overline{g_i} = 0$ in $H^1 := H^1(\mathcal{U}^r, E' \otimes E'')$. Then $\delta\eta_j = \overline{g_j}$ with $\eta_j \in C^0(\mathcal{U}^r, E)$. Assume now that $\overline{g_{m_0}} \neq 0$ in H^1 for some m_0. Then we have two cases again: either $\overline{g_i} = c_{i0}\overline{g_{m_0}} + \delta\eta_i$ on $\mathcal{U}^r$ for all $i \in \{0, \dots, \mu\} \setminus m_0$, or it fails for some m_1. We repeat this to exhaust all elements so that

$$\overline{g_j} = \delta\eta_j^* + \sum_{i=0}^{\mu_*} c_{ji}\overline{g_{m_i}}, \quad \eta_j^* \in C^0(\mathcal{U}^r, E), \quad 0 \leq j \leq \mu \tag{A.31}$$

while $\overline{g_{m_0}}, \dots, \overline{g_{m_{\mu*}}}$ are linearly independent in H^1. (Note that the above expression means the trivial identity $\overline{g_j} = \overline{g_j}$ when j is not in $\{m_0, \dots, m_{\mu^*}\}$.) We have obtained (A.28) with the decomposition

$$f = \sum_{j=0}^{\mu^*} c_j\overline{g_{m_j}} + \delta v,$$
$$c_j = \sum_{\ell=0}^{\infty}(v_\ell, g_j)_{\Delta_{2n}^r} + \sum_{i=0}^{\mu} c_{ij}\sum_{\ell=0}^{\infty}(v_\ell, g_i)_{\Delta_{2n}^r},$$
$$v = \sum_{i=0}^{\mu}\sum_{\ell=0}^{\infty}(v_\ell, g_i)_{\Delta_{2n}^r}\eta_i^* + \sum_{\ell=0}^{\infty}\eta_\ell.$$

The solution η_j^* in (A.31) can be bounded in U^{r_-} for any $r_- < r$. Of course we need to estimate η_j^* on $\mathcal{U}^{r'}$. Thus, $r_- \geq r'$. We have

$$\sum_{j=0}^{\mu}\sum_{\ell=0}^{\infty}|(v_\ell, g_j)_{\Delta_{2n}^r}c_{ji}| \leq \frac{t'}{1-t}\sum_{j=0}^{\mu}|c_{ji}|\|\zeta_0\|_{\mathcal{U}^{r'}},$$
$$\left\|\left\{\sum_{\ell=0}^{\infty}\eta_\ell + \sum_{\ell=0}^{\infty}\sum_{j=1}^{\mu}(v_\ell, g_j)_{\Delta_{2n}^r}\eta_j^*\right\}\right\|_{\mathcal{U}^{r_-}} \leq \frac{t'}{1-t}\left\{1 + \sum_{j=0}^{\mu}\|\eta_j^*\|_{\mathcal{U}^{r_-}}\right\}\|\zeta_0\|_{\mathcal{U}^{r'}}.$$

Set $A_r(E) = |\{t'_{kj}\}||\{t''_{kj}\}|\max_{i=0}^{\mu^*}\sum_{j=0}^{\mu}|c_{ji}|$ and $B_{r_-}(E) = |\{t'_{kj}\}||\{t''_{kj}\}|(1 + \sum_{j=0}^{\mu}\|\eta_j^*\|_{\mathcal{U}^{r_-}})$. We have obtained the required estimates.

Finally, let us assume that $f = 0$ in $H^1(C, E)$ to show that all $c_j = 0$ and thus $f = \delta v$. Since each $U^{r''}$ is Stein, we also have $f = 0$ in $H^1(\mathcal{U}^r, E)$. Thus $f = \delta\tilde{v}$

with $\tilde{v} \in \mathcal{C}^0(U^{r''}, E)$. We get $\delta(\tilde{v} - v) = \sum_{j=0}^{\mu^*} c_j \overline{g_{m_j}}$. By the linear independence, we conclude that $c_j = 0$. We are done. □

Theorem A.12 *Let C be a compact complex manifold and let $\mathcal{U}^r$ $(r_* \leq r \leq r^* < 1)$ be nested coverings of C as in Proposition A.19. Let $\mu, \nu, r, r', r'', r_*, r^*$ be given in Theorem A.9, which satisfy (A.25). Let $f \in Z^1(\mathcal{U}^{r'}, E' \otimes E'')$. Suppose that $f = 0$ in $H^1(C, E' \otimes E'')$. Then there exists a solution $\{u_j\} \in C^0(\mathcal{U}^{r'}, E' \otimes E'')$ such that $\delta u = f$ and*

$$\|u\|_{\mathcal{U}^{r'}} \leq K(E' \otimes E'')\|f\|_{\mathcal{U}^{r'}}, \tag{A.32}$$
$$K(E' \otimes E'') := C(|\{t'_{kj}\}|_{\mathcal{U}^{r'}} + K_*(E' \otimes E'')|\{t'_{kj}\}|_{\mathcal{U}^{r'}}|\{t''_{kj}\}|_{\mathcal{U}^{r'}}), \tag{A.33}$$

where $K_*(E' \otimes E'')$, defined by (A.8), satisfies

$$K_*(E' \otimes E'') \leq \frac{C_n B_{r_-}(E' \otimes E'')}{(r - r')\kappa}, \tag{A.34}$$

where κ and B_{r_-} are defined by (A.10) and (A.29). The same conclusion holds if both sides are in sup norms $|\cdot|_{\mathcal{U}^{r'}}$, when $(r - r')\kappa$ is replaced by $((r - r')\kappa)^n$.

Remark A.13 The main conclusion is that (A.32) holds without shrinking the covering $\{U_i^{r'}\}$ on which f is defined. The solution operator $f \mapsto u$ may not be linear. The small divisor conditions are carried by B_{r_-} which is determined by (A.25) and (A.29), while the bounds in Theorem A.7 as smoothing lemma do not involve small divisors.

Proof By the Leray theorem, we know that $[f] = 0$ in $H^1(\mathcal{U}^{r'}, E)$. By Theorem A.9, we have a solution $u \in C^0(\mathcal{U}^{r''}, E)$ so that

$$f_{jk} = (\delta u)_{jk}, \quad U_j^{r''} \cap U_k^{r''},$$
$$\|u\|_{\mathcal{U}^{r''}} \leq K\|f\|_{\mathcal{U}^{r'}}.$$

Then the conclusion follows from Lemma A.2.

When the super norm is used, we first obtain a solution $u = \{u_k\}$ for $\mathcal{U}^{r^*}$ for $r^* = (r'' + r')/2$, while (A.34) takes the form

$$\|u\|_{\mathcal{U}^{r^*}} \leq K\|f\|_{\mathcal{U}^{r'}} \leq (\sqrt{\pi} r')^n K |f|_{\mathcal{U}^{r'}}.$$

By $\operatorname{dist}(\varphi_k(U_k^{r''}), \partial\varphi_k(U_k^{r^*})) = r^* - r''$ and power series expansion, we have $|u|_{\mathcal{U}^{r''}} \leq (\sqrt{\pi}(r^* - r''))^{-n}\|u\|_{\mathcal{U}^{r^*}}$. Then the conclusion follows from Lemma A.2 again. □

A.5 Existence of Nested Coverings

In this subsection, our main goal is to construct nested coverings using transversality theorems and analytic polyhedrons. We recall that C_n is an n-dimensional compact complex manifold. We shall omit to mention its dimension in what follows.

We first deal with the transversality for a piecewise smooth boundary of an analytic polyhedron and we then define the general position property of several analytic polyhedrons.

Definition A.14 (a) Let M_j be a C^1 real hypersurface defined by $r_j = 0$, where r_j is a C^1 function in an open set ω_j of a complex manifold C and $dr_j \neq 0$ on M_j. We say that $M_1, \dots, M_N$ are in the *general* position, if $dr_{i_0} \wedge \cdots \wedge dr_{i_q} \neq 0$ at each point of $M_{i_0} \cap \cdots \cap M_{i_q}$ for any $1 \leq i_0 < \cdots < i_q \leq N$.

(b) Let ω be a *proper* open set of a complex manifold C and let $f \in \mathcal{O}^N(\omega)$. We say that

$$Q := Q_N(f, \omega) := \{z \in \omega \mid |f(z)| := \max\{|f_1(z)|, \dots, |f_N(z)|\} < 1\} \tag{A.35}$$

is an analytic N-polyhedron in ω if Q is non-empty and relatively compact in ω, and Q does not contain any compact connected component. We say that Q is *generic*, if

$$(d|f_{i_1}| \wedge \cdots \wedge d|f_{i_\ell}|)(x) \neq 0 \quad \forall x \in \{|f_{i_1}| = \cdots = |f_{i_\ell}| = 1\} \cap \partial Q \tag{A.36}$$

for all $i_1 < \cdots < i_\ell$ and $1 \leq \ell \leq N$.

We will apply transversality theorems. This requires us to use open submanifolds in $\mathbf{C}^n$ which may not be closed in $\mathbf{C}^n$. Since $Q_N = Q_N(f, \omega)$ does not contain compact connected component, the closure of each connected component of Q_N must intersect some $Q_N^i := \{|f_i| = 1\} \cap \omega$. We will call Q_N^i a *face* of Q_N. Removing each Q_N^i from ω if it does not intersect $\overline{Q}_N$, we get a new ω such that $\overline{Q}_N$ intersects each Q_N^i. Applying the same procedure to $Q_N^{i_1 \dots i_k} := Q_N^{i_1} \cap \cdots \cap Q_N^{i_k}$, we may assume that the non-empty intersection of any number of $Q_N^1, \dots Q_N^N$ intersects $\overline{Q}_N$. By (A.36), the closed set $\overline{Q}_N$ does not intersect the closed subset of ω defined by

$$(d|f_{i_1}| \wedge \cdots \wedge d|f_{i_\ell}|)(x) = 0 \quad |f_{i_1}|(x) = \cdots = |f_{i_\ell}|(x) = 1.$$

Removing the above sets from ω, we find a neighborhood ω^* of $\overline{Q}_N$ such that if $Q_N^{i_1 \dots i_k}$ with $i_1 < i_2 < \cdots < i_k$ intersects ω^*, then it intersects $\overline{Q}_N$ and it is a codimension k smooth submanifold in ω^*. For brevity we will call ω^* a *neat* neighborhood of Q. We will take $\omega = \omega^*$ without specifying ω^*.

Definition A.15 Let ω_i be open sets in C. For $i = 0, \dots, p$, assume that $\phi_i \in \mathcal{O}^{N_i}(\omega_i)$ and $Q_{N_i} := Q_{N_i}(\phi_i, \omega_i)$ is an analytic polyhedron in ω_i. We say that they are in the *general position*, if all faces $Q_{N_i}^j$ of Q_{N_i} for $1 \leq j \leq N_i$ and $0 \leq i \leq p$ are in general position. More precisely, $\omega_{N_i}^* \cap Q_{N_i}^j$ are in the general position, where each ω_i^* is a neat neighborhood of $\overline{Q_{N_i}}$.

Let us describe some elementary properties of generic analytic polyhedrons. If $Q_N(f, \omega)$ is defined in ω by (A.35), we denote for $\rho = (\rho_1, \dots, \rho_N)$

$$Q_N^{\rho}(f, \omega) := \{z \in \omega \colon |f_j(z)| < \rho_j,\ j = 1, \dots, N\}.$$

Lemma A.16 *Let $Q_{N_i} = Q_{N_i}(\phi_i, \omega_i)$ be generic polyhedrons in C for $0 \leq i \leq p$. Suppose that $Q_{N_0}, \cdots, Q_{N_p}$ are in the general position. Then*

$$Q_{N_0+\cdots+N_p}((\phi_0, \dots, \phi_p), \omega_0 \cap \cdots \cap \omega_p) = Q_{N_0} \cap \cdots \cap Q_{N_p},$$

if non-empty, is a generic $N_0+\cdots+N_p$ analytic polyhedron in $\omega_{0\cdots p} := \omega_0 \cap \cdots \cap \omega_p$.

Proof Let $N = N_0 + \cdots + N_p$. It is clear that $Q := Q_{N_0} \cap \cdots \cap Q_{N_p} = Q_N((\phi_0, \dots, \phi_p), \omega_{i_0\cdots i_p})$. Since $\overline{Q} \subset \cap \overline{Q_{N_i}}$, then $\overline{Q}$ is compact in $\omega_{0\cdots p}$. Write $(\phi_0, \dots, \phi_p) = (\psi_1, \cdots, \psi_N)$. Suppose that $x \in \partial Q$. Since $\overline{Q}$ is compact in ω, then there exist $\mu_1 < \cdots < \mu_m$ with $m \geq 1$ such that $|\psi_{\mu_i}(x)| = 1$ and $|\psi_j(x)| < 1$ for $j \neq \mu_\ell$. By the assumption of the general position, we see that the faces of Q are in the general position. □

Let X, Y be smooth real manifolds without boundary and W a smooth submanifold of Y. Following [11, p. 50], we say that a smooth mapping $h \colon X \to Y$ is *transversal* to W at $x \in X$, denoted by $h \pitchfork W$ at x, if either $h(x) \notin W$ or

$$T_{h(x)}W + dh(T_x X) = T_{h(x)}Y.$$

Denote $h \pitchfork W$ on A if $h \pitchfork W$ at each $x \in A \subset X$. When h is the inclusion, we denote $h \pitchfork W$ on A by $X \pitchfork W$ on A. Finally, extending Definition A.14 (a), we say that smooth real submanifolds $W_0, \dots, W_k$ in Y are in *the general position* if for any $0 \leq i_1 < \cdots < i_m \leq k$ we have

$$\bigwedge_{\ell=1}^{k} \bigwedge_{j=1}^{d_{i_\ell}} dr_{i_\ell, j}(y) \neq 0, \quad \forall y \in W_{i_1} \cap \cdots \cap W_{i_m}, \tag{A.37}$$

where $W_i \subset \omega_i$ is defined by $r_{i,1} = \cdots = r_{i,d_i} = 0$ with $dr_{i,1} \wedge \cdots \wedge dr_{i,d_i} \neq 0$ at each point of W_i. Thus d_i is the codimension of W_i in ω_i. It is clear that (A.37) holds if and only if

$$W_{i_j} \pitchfork (W_{i_1} \cap \cdots \cap W_{i_{j-1}}) \text{ at } y, \quad \forall y \in W_{i_1} \cap \cdots \cap W_{i_k},\ 0 < j \leq m. \tag{A.38}$$

For an analytic N-polyhedron Q_N in ω with faces $Q_N^1, \dots, Q_N^N$, we call $Q_N^{i_1\cdots i_k} = Q_N^{i_1} \cap \cdots \cap Q_N^{i_k}$ with $i_1 < \cdots < i_k$ and $k \geq 1$ an *edge* of Q. When Q_N is generic, a nonempty edge $Q_N^{i_1\cdots i_k}$ is a codimension k submanifold in ω. Let $\{Q_N^1 \cdots, Q_N^{N'}\}$ be the set of all edges, with the first N edges being the faces.

Proposition A.17 *Let $Q_{N_i} = Q_{N_i}(\phi_i, \omega_i)$ be generic polyhedrons in C for $0 \leq i \leq p$ with ω_i being a neat neighborhood of $\overline{Q}_{N_i}$. Then $Q_{N_0}, \ldots, Q_{N_p}$ are in the general position if and only if for all $0 \leq i_1 < \cdots < i_k \leq p$ and $1 \leq j_\ell \leq N'_{i_\ell}$, the edges $Q^{j_1}_{N_{i_1}}, \cdots, Q^{j_k}_{N_{i_k}}$ are in the general position. Equivalently, each edge $Q^s_{N_\ell}$ intersects transversally with each edge of the intersection of any number of $Q_{N_0}, \ldots, Q_{N_{\ell-1}}$, for $\ell = 1, \ldots, p$.*

Proof Since each edge of a polyhedron is the intersection of its faces, it is clear that if $Q_{N_0}, \ldots, Q_{N_p}$ are in the general position, then the edges $Q^{j_1}_{N_{i_1}}, \cdots, Q^{j_k}_{N_{i_k}}$ are in the general position for $0 \leq i_1 < \cdots < i_k \leq p$.

Conversely, let $\phi_i = (\phi_{i,1}, \ldots, \phi_{i,N_i})$ and let $\psi_1, \ldots, \psi_m$ be a subset of $\phi_{0,1}, \ldots, \phi_{0,N_0}, \ldots, \phi_{p,1}, \ldots, \phi_{p,N_p}$. We emphasize that we do not assume that the latter are distinct functions, although $\phi_{i,1}, \ldots, \phi_{i,N_i}$ are distinct by the general position property of the faces of Q_{N_i}. Suppose that ψ_ℓ is in $\{\phi_{i_\ell,1}, \ldots, \phi_{i_\ell,N_{i_\ell}}\}$. We need to show that

$$d|\psi_1| \wedge \cdots \wedge d|\psi_m|(x) \neq 0 \tag{A.39}$$

if for all ℓ, $|\psi_\ell|(x) = 1$ and $x \in \overline{Q}_{N_{i_\ell}}$. Without loss of generality, we may assume that $i_1 \leq i_2 \leq \cdots \leq i_m$. Thus

$$(\psi_1, \ldots, \psi_m) = (\tilde{\psi}_{\alpha_1}, \ldots, \tilde{\psi}_{\alpha_\ell}), \quad \alpha_1 < \alpha_2 < \cdots < \alpha_\ell$$

with $\tilde{\psi}_{\alpha_\beta}$ being a non-empty subset of components of ϕ_{α_β}. Without loss of generality, we may assume that $\tilde{\psi}_{\alpha_\beta} = (\phi_{\alpha_\beta,1}, \ldots, \phi_{\alpha_\beta,\gamma_\beta})$ with $\gamma_\beta > 0$. Thus $|\phi_{\alpha_\beta,1}| = \cdots = |\phi_{\alpha_\beta,\gamma_\beta}| = 1$ define an edge W_{α_β} of Q_{α_β}. Then (A.39) is equivalent to

$$\left(\bigwedge_{\delta=1}^{\gamma_\ell} d|\phi_{\alpha_\ell,\delta}|\right) \wedge \left(\bigwedge_{\ell'=1}^{\ell-1} \bigwedge_{\delta=1}^{\gamma_{\ell'}} d|\phi_{\alpha_{\ell'},\delta}|\right)(x) \neq 0.$$

The equivalence of (A.37) and (A.38) implies that (A.39) follows from the assumption that $W_{\alpha_\ell} \pitchfork (W_{\alpha_1} \cap \cdots \cap W_{\alpha_{\ell-1}})$, for $\alpha_1 < \alpha_2 < \cdots < \alpha_\ell$. □

Lemma A.18 (Golubitsky-Guillemin [11, p. 53]). *Let X, B, and Y be smooth manifolds with W a submanifold of Y. Let $\psi \colon B \to C^\infty(X, Y)$ be a mapping (not necessarily continuous) and define $\Psi \colon X \times B \to Y$ by $\Psi(x, b) = \psi(b)(x)$. Assume that Ψ is smooth and that $\Psi \pitchfork W$. Then the set $\{b \in B \mid \psi(b) \pitchfork W\}$ is dense in B.*

Proposition A.19 *Let C be a compact complex manifold of dimension n. Let $\{U_i : i = 1, \ldots, m\}$ be a finite open covering of C. Assume that φ_j is a biholomorphism from a neighborhood ω_j of the star $N(U_j)$ of U_j onto $\hat{\omega}_j \subset \mathbf{C}^n$ such that $U_j = \varphi_j^{-1}(\Delta_n) = Q_n(\varphi_j, \omega_j)$. There exists $\delta > 0$ satisfying the following:*

(a) *For each j, there are a relatively compact open set $\tilde{\omega}_j$ (resp. $\tilde{U}_j$) in ω_j (resp. $\tilde{\omega}_j$) and a dense open set A_j of Δ_n^δ such that if $c_j \in A_j$, then $\tilde{\varphi}_j := \varphi_j - c_j$ is a biholomorphic mapping from $\tilde{U}_j$ onto Δ_n, and $\tilde{U}_1 := Q_n(\tilde{\varphi}_1, \tilde{\omega}_1), \ldots,$*

$\tilde U_m := Q_n(\tilde\varphi_m, \tilde\omega_m)$ are generic n-polyhedrons in the general position, where $\{\tilde U_1, \dots \tilde U_m\}$ remains an open covering of C and $\tilde\omega_j$ is a neighborhood of $N(\tilde U_j)$. In particular each $\tilde\varphi_j$, a translation of φ_j, is injective on $\tilde\omega_j$.

(b) *There is $0 < r_* < 1$ such that if $r_* \leq \rho_i \leq 1$, then $\tilde U_{i_0}^{\rho_0}, \dots, \tilde U_{i_q}^{\rho_q}$ are generic n-polyhedrons in the general position, where $\tilde U_i^\rho := \tilde\varphi_i^{-1}(\Delta_n^\rho)$.*

Proof (a) We will apply the transversality theorem for real submanifolds in $\mathbf{C}^n$. Therefore, we will use old coordinate charts φ_j to map edges of polyhedrons $Q_j(\varphi_j, \omega_j)$ into $\mathbf{C}^n$. Set $c_1 = 0$, $\tilde\varphi_1 = \varphi_1$, $\tilde U_1 = U_1$. Let $\hat W_1, \dots, \hat W_{L_0}$ be all edges of Δ_n. Let $\tilde U_1^1, \dots, \tilde U_1^{N'}$ be all edges of $\tilde U_1$. Set $\widetilde W_1^\ell = \varphi_2(\omega_2 \cap \tilde U_1^\ell)$. Define

$$\Psi\colon \mathbf{C}^n \times \Delta_n^\delta \to Y := \mathbf{C}^n$$

with $\Psi(x, b) = x + b$ and $\psi^b(x) = \Psi(x, b)$. Let $\psi^b|_{\widehat W_{\ell'}}$ be the restriction of ψ^b to $\hat W_{\ell'}$. Applying Lemma A.18, mainly the density assertion in the lemma, finitely many times in which $W = \widetilde W_1^\ell$, we can find $b_2 \in \Delta_n^\delta$ such that

$$\psi^{b_2}|_{\widehat W_{\ell'}} \pitchfork \widetilde W_1^\ell \quad \text{on}\, \varphi_2(\overline{\tilde U_1} \cap \overline{\omega_2'}), \quad \forall \ell, \ell'$$

where ω_2' is a relatively compact open subset of ω_2 which is independent of δ, and $\overline{U_2} \subset \omega_2'$. We also remark that (A.18) can be applied for finitely many times since $\varphi_2(\overline{\tilde U_1} \cap \overline{\omega_2'})$ is compact. Since $\overline{\tilde U_1} \cap \overline{U_2}$ is compact, then

$$\psi^{c_2}|_{\widehat W_{\ell'}} \pitchfork \widetilde W_1^\ell \quad \text{on}\, \varphi_2(\overline{\tilde U_1} \cap \overline{\omega_2'}), \quad \forall \ell, \ell' \tag{A.40}$$

when $|c_2 - b_2|$ is sufficiently small. Applying φ_2^{-1} to (A.40) yields

$$\varphi_2^{-1}(\psi^{c_2}|_{\widehat W_{\ell'}}) \pitchfork (\omega_2 \cap \tilde U_1^\ell) \quad \text{on}\, \overline{\tilde U_1} \cap \overline{\omega_2'}, \quad \forall \ell, \ell'. \tag{A.41}$$

With c_2 being determined, set

$$\tilde\varphi_2^{-1} = \varphi_2^{-1}(\mathrm{I} + c_2).$$

Thus $\tilde\varphi_2 = \varphi_2 - c_2$. When δ and $|c_2 - b_2|$ are sufficiently small, we have $\overline{\tilde U_2} = \tilde\varphi_2^{-1}(\overline{\Delta_n}) \subset \omega_2'$. Therefore, (A.41) implies that every edge of $\tilde U_2$ intersects each edge of $\tilde U_1$. We have determined $\tilde U_2 = \tilde\varphi_2^{-1}(\Delta_n)$.

We have verified (a) when $m = 2$. Let us assume that it also holds for $m \geq j$. By Lemma A.16, each edge of a non-empty intersection of any number of $\tilde U_1, \dots, \tilde U_j$ is a smooth submanifold. We remark the above transversality argument mainly uses the fact that φ_2 is a biholomorphism, while each edge of $\tilde U_1$ is a smooth submanifold.

To repeat the above argument for $m = 2$ in details, we list all edges of all possible intersections of $\tilde U_1, \dots, \tilde U_j$ as $W_1', \dots, W_L'$ so that each W_j is an edge of some analytic polyhedron U_j', where U_j' is the intersection of some of $\tilde U_1, \dots, \tilde U_{j'}$ which are in general position by the induction hypothesis as mentioned above. Therefore, by Lemma A.16, each U'_ℓ is generic. Now we are in the situation of $m = 2$ by considering the sets of two analytic polyhedrons $\{U'_\ell, U_{j+1}\}$ one by one for $\ell = 1, \dots, j'$. Here $U_{j+1} = \varphi_{j+1}^{-1}(\Delta_n)$ with φ_{j+1} being biholomorphic in a neighborhood $N(U_{j+1})$ of U_{j+1}. Therefore, we can find $\tilde\varphi_{j+1} = \varphi_{j+1} - c_{j+1}$ such that each edge of $\tilde U_{j+1}$ intersects each W'_ℓ transversally on $\overline{\tilde U_{j+1}} \cap \overline{U'_\ell}$.

The above argument shows the existence of $c_1, \dots, c_N$ in Δ_n^δ when δ is sufficiently small. The openness property on A_j is clear, since by shrinking $\tilde\omega_j$ slightly the general position and generic properties are preserved under small perturbation of c_j. Then density of A_j when δ is sufficiently small can also be achieved; indeed when c_j is sufficiently small, we may shrink ω_j slightly and apply the above argument by replacing $\varphi_j - c_j$ with φ_j. Finally, $\{\tilde U_1, \dots, \tilde U_N\}$ still covers C when δ is sufficiently small. We have verified (a).

The assertion (b) follows from (a) and Proposition A.17. Indeed, we first note that when r_* is less than 1, but it is sufficiently close to 1, the $\partial Q^\rho(\tilde\varphi_j)$ is in a given neighborhood of $\partial Q(\tilde\varphi_j, \tilde\omega_j)$, as $Q^\rho(\tilde\varphi_j, \tilde\omega_j)$ does not have any compact connected component. By the relative compactness of $Q_n(\tilde\varphi_i, \tilde\omega_i)$, the condition (A.36) with f_j being replaced by f_j/ρ_j and the general position condition remain true when ρ_j are in $[r_*, 1]$ when $r_* < 1$ is sufficiently close to 1. The proof is complete. □

The following is a basic property of a generic analytic polyhedron.

Proposition A.20 *Let C be a compact complex manifold of dimension n. Let $Q_N(f, \omega)$ be a generic analytic N-polyhedron C defined by (A.35) and (A.36). There exists $r_* \in (0, 1)$ satisfying the following.*

(a) *If $\rho = (\rho_1, \dots, \rho_N)$ and $\rho' = (\rho'_1, \dots, \rho'_N)$ satisfy $r_* \le \rho'_i \le \rho_i \le 1$, every connected component of $Q_N^\rho(f, \omega)$ intersects $Q_N^{\rho'}(f, \omega)$ and the latter is non-empty.*

(b) *There are finitely many open sets ω''_j in C and smooth diffeomorphisms ϕ_j sending ω''_j onto $\hat\omega''_j$ in $\mathbf{R}^{2n}$ such that $\{\omega''_j\}$ covers $\partial Q_N(f, \omega)$, and for any $p_0, p_1 \in \phi_j(\omega''_j \cap Q_N^\rho(f, \omega))$ there is a smooth curve γ in $\phi_j(\omega''_j \cap Q_N^\rho(f, \omega))$ connecting p_0 and p_1 with length $|\gamma| \le C|p_1 - p_0|$, where C depends only on ϕ_j and ω''_j.*

Proof (a) Set $Q = Q_N(f, \omega)$ and $Q^\rho = Q_N^\rho(f, \omega)$. For each $x \in \partial Q$, we find $\mu_1 < \dots < \mu_m$ with $m \le N$ such that

$$|f_{\mu_i}(x)| = 1, \quad i \le m; \quad |f_j(x)| < 1, \quad j \ne \mu_1, \dots, \mu_m. \tag{A.42}$$

Note that $\{\mu_1, \dots, \mu_m\}$ is uniquely determined by x. By the transversality condition (A.36), we have $m \le 2n$. Choose an open set ω' such that $x \in \omega' \subset \omega$ and

$$|f_i(z)| < 1, \quad \forall z \in \overline{\omega'}, \ i \ne \mu_1, \dots, \mu_m.$$

In particular, we have

$$Q \cap \omega' = \{z \in \omega' \colon |f_{\mu_i}(z)| < 1, \quad i = 1, \dots, m\}.$$

By (A.36), we can take $(|f_{\mu_1}|, \dots, |f_{\mu_m}|)$ to be the first m components of a smooth diffeomorphism $\varphi\colon \omega' \to \hat{\omega}$, shrinking ω' if necessary. Taking a smaller open subset ω'' of ω' with $x \in \omega''$, we may assume that

$$t\zeta \in \hat{\omega}, \quad \forall \zeta \in \hat{\omega}'' := \varphi(\omega''), \quad 1-\delta \le t \le 1,$$

for some $\delta \in (0, 1]$.

Since ∂Q is compact, there exists $\{x_j, \omega_j'', \omega_j'\colon j = 1, \dots, k\}$ satisfying the following:

(a) The k is finite. For each j, we have that $x_j \in \omega_j'' \subset \omega_j' \subset \omega$, $x_j \in \partial Q$, and ω_j' is an open subset of ω. For each j, we have m_j and $\mu_{j,1} < \dots < \mu_{j,m_j}$, which are the numbers associated to x_j, so that (A.42) holds for $x = x_j$. $\{\omega_1'', \dots \omega_k''\}$ is an open covering of ∂Q.

(b) $|f_{\mu_{j,\ell}}(x_j)| = 1$ for $\ell = 1, \dots, m_j$ and

$$M_j := \sup_{z\in\overline{\omega_j'}} \{|f_i(z)|\colon i \neq \mu_{j,1}, \dots, \mu_{j,m_j}\} < 1,$$
$$\omega_j' \cap Q = \{z \in \omega_j'\colon |f_{\mu_{j,\ell}}(z)| < 1, \ell = 1, \dots, m_j\}.$$

Here we set $M_j = 0$ if $m_j = N$.

(c) The $(|f_{\mu_{j,1}}|, \dots, |f_{\mu_{j,m_j}}|)$ are the first m_j components of a smooth diffeomorphism ϕ_j from ω_j onto a subset $\hat{\omega}_j$ of $\mathbf{C}^n$. There exists $\delta^* > 0$ such that $\hat{\omega}_j'' := \phi_j(\omega_j'')$ satisfies

$$\{t\zeta\colon \zeta \in \hat{\omega}_j''\} \subset \hat{\omega}_j, \quad \forall j, \forall t \in [1-\delta^*, 1]. \tag{A.43}$$

Indeed, let $\phi_j(x_j) = (1, \dots, 1, \tilde{x}_j)$ with $\tilde{x}_j \in \mathbf{R}^{2n-m_j}$. We can take

$$\hat{\omega}_j'' = (1-\delta^*, 1+\delta^*)^{m_j} \times B_{2n-m_j}^{\delta''}(\tilde{x}_j) \tag{A.44}$$

where $B_{2n-m_j}^{\delta''}(\tilde{x}_j)$ is the ball in $\mathbf{R}^{2n-m_j}$ centered at $\tilde{x}_j$ with a sufficiently small radius δ''. Note that

$$\phi_j(Q^\rho \cap \omega_j'') = (1-\delta^*, \rho_1) \times \cdots \times (1-\delta^*, \rho_{m_j}) \times B_{2n-m_j}^{\delta''}(\tilde{x}_j). \tag{A.45}$$

Define

$$M^* = \sup\{|f(z)|\colon z \in Q \setminus \cup_{j=1}^k \omega_j''\}.$$

Then $M^* < 1$. By the maximum principle, we have $|f| \le M^*$ on $Q \setminus \cup_{j=1}^k \omega_j''$. Fix r_* so that

$$1 > r_* > \max\{1-\delta^*, M^*, M_1, \dots, M_k\}.$$

Suppose that $r_* \leq \rho_i' \leq \rho_i \leq 1$ for $i = 1, \dots, N$. Let Ω be a connected component of Q_N^ρ. Since Ω does not have a compact connected component, there exists $z^* \in \partial\Omega$ satisfying $|f_i(z^*)| = \rho_i$ for some i. Since $\rho_i > M^*$, then $z^* \in \omega_j''$ for some j. Let us assume that $z^* \in \omega_1''$, and $(\mu_{1,1}, \dots, \mu_{1,m_1}) = (1, \dots, m_1)$. Thus $\phi_1 = (|f_1|, \dots, |f_{m_1}|, \tilde{f}_{m_1+1}, \dots, \tilde{f}_{2n})$. We now replace z^* by some $z_* \in \Omega \cap \omega_1''$. We consider a path defined by

$$t \to \gamma(t) := \phi_1^{-1}(t\phi_1(z_*)), \quad 1 - \delta^* \leq t \leq 1.$$

Note that by (A.43), γ is well defined and is contained in ω_1. We now have

$$|f_\ell(\gamma(t))| = t|f_\ell(z_*)| \leq t\rho_\ell, \quad \ell \leq m_1. \tag{A.46}$$

Since $\gamma(t) \in \omega_1$, we also have

$$|f_\ell(\gamma(t))| \leq M_1 < r_*, \quad \ell > m_1. \tag{A.47}$$

This shows that $\gamma(t) \in Q_N^\rho$. Since Ω is a connected component of Q_N^ρ and $\gamma(1) = z_* \in \Omega$, we must have $\gamma(t) \in \Omega$. By the definition of M_j, at $t = 1 - \delta^*$ we have $t\rho_\ell \leq 1 - \delta^* < \rho_\ell'$. Combining with (A.46)–(A.47), we get $\gamma(1 - \delta^*) \in Q_N^{\rho'}$.

(b) Since p_0, p_1 are in the same $\hat{\omega}_j''$, the assertion also follows from the above construction of $\hat{\omega}_j''$ via (A.44)–(A.45) and the convexity of $\hat{\omega}_j''$. □

In summary, by Proposition A.19 we cover C by generic analytic n-polyhedrons $U_i = \varphi_i^{-1}(\Delta_n)$ $(i = 1, \dots, m)$, which are in the general position. By Lemma A.16, each $U_i \cap U_j$, if non-empty, is a generic analytic polyhedron. Applying Proposition A.20 (a) to all non-empty $U_i \cap U_j$, we know that $\{U_i^r = \varphi_i^{-1}(\Delta_n^r) : i = 1, \dots, m\}$ for $r_* \leq r \leq 1$ is a family of nested coverings. Therefore, we can apply Theorem A.9 and Theorem A.12.

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