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RESEARCH CONTRIBUTION

Dynamical Moduli Spaces and Polynomial Endomorphisms of Configurations

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Abstract

A portrait is a combinatorial model for a discrete dynamical system on a finite set. We study the geometry of portrait moduli spaces, whose points correspond to equivalence classes of point configurations on the affine line for which there exist polynomials realizing the dynamics of a given portrait. We present results and pose questions inspired by a large-scale computational survey of intersections of portrait moduli spaces for polynomials in low degree.

Keywords Algebraic dynamics · Configuration spaces · Dynamical moduli spaces

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1 Introduction

The study of preperiodic points of polynomials is of central importance in complex and arithmetic dynamics. Typically, one starts with a polynomial $f(x) \in \mathbb{C}[x]$ and then tries to understand the preperiodic points of $f(x)$; that is, the points $\alpha \in \mathbb{C}$ for which the *orbit* $\{\alpha, f(\alpha), f(f(\alpha)), f(f(f(\alpha))), \ldots\}$ is finite. We take a different perspective: given a configuration q of finitely many points in $\mathbb{C}$, we seek to understand the collection of all polynomials $f(x) \in \mathbb{C}[x]$ such that $f(q) \subseteq q$. To that end, let

$$\mathrm{Conf}^n := \{(q_1, q_2, \ldots, q_n) \in \mathbb{C}^n : q_i \neq q_j \text{ for all } i \neq j\}$$

denote the configuration space of n distinct points on the affine line over $\mathbb{C}$, and define

$$\mathrm{End}(q) := \{f(x) \in \mathbb{C}[x] : f(q) \subseteq q\}$$

to be the semigroup (with respect to composition) of all polynomials which stabilize $q \in \mathrm{Conf}^n$ as a set. Note that Conf^n naturally carries the structure of a complex algebraic variety.

Remark We have chosen, for the sake of concreteness, to work exclusively over the complex field $\mathbb{C}$. However, the main results of this article may be extended to arbitrary algebraically closed fields of characteristic 0 by appealing to the Lefschetz principle.

Our general goal is to begin addressing the following question.

Question 1.1 How does the semigroup $\mathrm{End}(q)$ vary with $q \in \mathrm{Conf}^n$?

Lagrange interpolation implies the existence of a unique polynomial of degree at most $n-1$ interpolating any set-theoretic endomorphism of $q \in \mathrm{Conf}^n$. For this reason, we focus on the elements of $\mathrm{End}(q)$ with degree less than n. Let $\mathrm{End}_d(q)$ denote the degree-d graded component of $\mathrm{End}(q)$.

Question 1.2 For $0 \leq d \leq n-1$, how does $\mathrm{End}_d(q)$ vary with $q \in \mathrm{Conf}^n$? What is the maximum cardinality $E_{n,d}$ of $\mathrm{End}_d(q)$ as q ranges over Conf^n, and which configurations achieve this maximum?

A *portrait* on $[n] := \{1, \ldots, n\}$ is a function $\mathcal{P} : [n] \to [n]$. The space $\mathrm{Conf}_{\mathcal{P},d}$ of *degree-d realizations* of a portrait $\mathcal{P}$ on $[n]$ is the subspace of Conf^n defined by

$$\begin{aligned}\mathrm{Conf}_{\mathcal{P},d} := \{q \in \mathrm{Conf}^n &: \text{there exists } f(x) \in \mathbb{C}[x] \\ &\text{of degree } d \text{ such that } f(q_i) = q_{\mathcal{P}(i)} \text{ for all } i\}.\end{aligned}$$

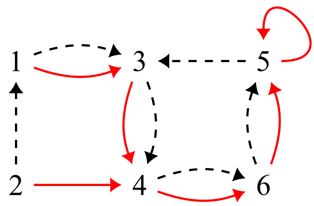


Fig. 1 An illustration of portraits $\mathcal{P}$ (red, solid) and $\mathcal{Q}$ (black, dashed) acting on $\{1, 2, 3, 4, 5, 6\}$. For example, the red, solid arrow from 2 to 4 indicates that $\mathcal{P}(2) = 4$ (color figure online)

The affine group Aff_1 of linear polynomials $\ell(x) = ax + b \in \mathbb{C}[x]$ acts naturally on Conf^n by

$$\ell(q) := (\ell(q_1), \ldots, \ell(q_n)).$$

Observe that the action of Aff_1 stabilizes $\text{Conf}_{\mathcal{P},d}$: Indeed, if there exists a degree-d polynomial f such that $f(q_i) = q_{\mathcal{P}(i)}$ for all $i \in [n]$, then $\widetilde{f} := \ell \circ f \circ \ell^{-1}(x)$ is also a degree-d polynomial such that $\widetilde{f}(\ell(q_i)) = \ell(q_{\mathcal{P}(i)})$ for all $i \in [n]$, so $\ell(q) \in \text{Conf}_{\mathcal{P},d}$. The *degree-d portrait moduli space* of a portrait $\mathcal{P}$ is defined to be the quotient

$$\mathcal{M}_{\mathcal{P},d} := \text{Conf}_{\mathcal{P},d}/\text{Aff}_1.$$

Since Aff_1 acts sharply 2-transitively on $\mathbb{A}^1$, $\mathcal{M}_{\mathcal{P},d}$ has the following simple model as a complex algebraic variety:

$$\mathcal{M}_{\mathcal{P},d} = \{(q_1, \ldots, q_n) \in \text{Conf}_{\mathcal{P},d} : q_1 = 0 \text{ and } q_2 = 1\}.$$

Observe that a configuration q has several degree-d endomorphisms precisely when q lies in the intersection of several portrait realization spaces. Thus, Question 1.2 leads us to consider the spaces

$$\text{Conf}_{\mathcal{P},\mathcal{Q},d} := \text{Conf}_{\mathcal{P},d} \cap \text{Conf}_{\mathcal{Q},d} \quad \text{and} \quad \mathcal{M}_{\mathcal{P},\mathcal{Q},d} := \text{Conf}_{\mathcal{P},\mathcal{Q},d}/\text{Aff}_1.$$

We refer to $\text{Conf}_{\mathcal{P},d}$ and $\text{Conf}_{\mathcal{P},\mathcal{Q},d}$ collectively as *portrait realization spaces*, and to $\mathcal{M}_{\mathcal{P},d}$ and $\mathcal{M}_{\mathcal{P},\mathcal{Q},d}$ as *portrait moduli spaces*.

A portrait $\mathcal{P}$ (resp., a pair of portraits $\{\mathcal{P}, \mathcal{Q}\}$) on $[n]$ naturally yields a functional graph (resp., a pair of functional graphs) on the vertex set $[n]$; see Fig. 1 for an example. The isomorphism classes of the moduli spaces $\mathcal{M}_{\mathcal{P},d}$ and $\mathcal{M}_{\mathcal{P},\mathcal{Q},d}$ depend only on the combinatorial type of the portrait $\mathcal{P}$ and the portrait pair $\{\mathcal{P}, \mathcal{Q}\}$, respectively; see Proposition 2.4 for a more precise statement. We therefore ask the following:

Question 1.3 How do combinatorial properties of portraits $\mathcal{P}$ and $\mathcal{Q}$ determine geometric properties of the moduli spaces $\mathcal{M}_{\mathcal{P},d}$ and $\mathcal{M}_{\mathcal{P},\mathcal{Q},d}$?

In this paper, we initiate the study of these questions guided by computational results in low degrees.

1.1 Results

We briefly summarize the main results of this article.

1.1.1 The Geometry of $\mathcal{M}_{\mathcal{P},d}$ and $\mathcal{M}_{\mathcal{P},\mathcal{Q},d}$

Our first result provides a combinatorial characterization of portraits $\mathcal{P}$ for which the moduli space $\mathcal{M}_{\mathcal{P},d}$ is nonempty and achieves the expected dimension.

Theorem 1.4 *Let $n, d \geq 2$ be integers and let $\mathcal{P} : [n] \to [n]$ be a portrait. Then, $\mathcal{M}_{\mathcal{P},d} \neq \emptyset$ if and only if*

(1) *every element of $[n]$ has at most d preimages under $\mathcal{P}$, and*
(2) *for every integer $k \geq 1$, $\mathcal{P}$ has at most $\frac{1}{k}\sum_{j|k}\mu(k/j)d^j$ periodic cycles of length k.*

In this case, $\dim \mathcal{M}_{\mathcal{P},d} = \min\{d-1, n-2\}$.

Remark One direction of Theorem 1.4 follows almost immediately from Proposition 15.1 and Theorem 15.8 of [5]. More precisely, the cited results of [5] are used in the proof of Theorem 1.4 to show that if $n \geq d+1$ and $\mathcal{M}_{\mathcal{P},d}$ is nonempty, then conditions (1) and (2) are satisfied and $\dim \mathcal{M}_{\mathcal{P},d} = d-1$; see the proof of Proposition 2.8. The other direction of the statement does not follow directly from the results of [5], nor does the statement in the case that $n \leq d$.

We will call a portrait $\mathcal{P}$ satisfying the two conditions in Theorem 1.4 an *admissible degree-d portrait.* If $\mathcal{P}$ and $\mathcal{Q}$ are admissible degree-d portraits on n points, then a simple count of parameters and constraints suggests that the dimension of $\mathcal{M}_{\mathcal{P},\mathcal{Q},d}$ should be $2d-n$ (see Sect. 3). There are finitely many admissible degree-d portraits—hence finitely many pairs $\{\mathcal{P}, \mathcal{Q}\}$ of such portraits—on a set with n elements, so in principle one can survey all of the portrait moduli spaces $\mathcal{M}_{\mathcal{P},\mathcal{Q},d}$ for any fixed n and d. We have conducted this survey for $(n, d) = (4, 2)$ and $(n, d) = (6, 3)$; in these instances, the expected dimension is zero.

For computational reasons, it is simpler to work with the moduli spaces

$$\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d} := \bigcup_{e\leq d}\mathcal{M}_{\mathcal{P},\mathcal{Q},e} \subseteq \mathrm{Conf}^n/\mathrm{Aff}_1$$

of all affine equivalence classes of degree-at-most-d realizations of $\mathcal{P}$ and $\mathcal{Q}$. Since we are considering the case $n = 2d$, and since $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d} \subseteq \widehat{\mathcal{M}}_{\mathcal{P},d}$ by definition, it follows from Theorem 1.4 that the dimension of $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ is bounded above by $\min\{d-1, n-2\} = d-1$. Table 1 (which we also include as Table 3 in Sect. 3.3 for convenience) lists, for $d = 2, 3$ and $-1 \leq e \leq d-1$, the number of pairs of admissible degree-d portraits on $2d$ points up to combinatorial equivalence for which we have $\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d} = e$. Here and throughout the article, we use the convention that the empty variety has dimension -1.

Table 1 For each $d \in \{2, 3\}$ and $e \in \{-1, 0, 1, 2\}$, the number of equivalence classes of pairs $\{\mathcal{P}, \mathcal{Q}\}$ of admissible degree-d portraits on $[2d]$ satisfying $\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d} = e$

	$\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$				
d \ e	−1	0	1	2	Total
2	198	568	14		780
3	52310	1297349	1065	18	1350742

While the dimension of $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ typically agrees with our expectation that generically $\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d} = 0$, there are portrait pairs $\{\mathcal{P}, \mathcal{Q}\}$ achieving the full range of possible dimensions for $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$.

Theorem 1.5 identifies an interesting family of extreme examples of portraits on $n = 2d$ points for which $\mathcal{M}_{\mathcal{P},\mathcal{Q},d}$ is either as small as possible (i.e., $\mathcal{M}_{\mathcal{P},\mathcal{Q},d} = \emptyset$) or as large as possible (i.e., $(d-1)$-dimensional). Theorem 1.5 accounts for all but one of the 14 equivalence classes of portrait pairs $\{\mathcal{P}, \mathcal{Q}\}$ for which $\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},2} = 1$ and all of the 18 pairs for which $\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},3} = 2$.

First, some terminology: The *fiber partition* of a portrait $\mathcal{P}$ on n points is the partition of $[n]$ given by $\Pi_{\mathcal{P}} := \{\mathcal{P}^{-1}(i) : i \in \mathcal{P}([n])\}$. A *two-image portrait* is a portrait $\mathcal{P} : [2d] \to [2d]$ such that $\Pi_{\mathcal{P}}$ consists of two sets with d elements each.

Theorem 1.5 *Let $d \geq 1$ and let $\Pi := \{A, B\}$ be a partition of $[2d]$ into two sets with d elements each.*

(1) *If $\mathcal{P}$ and $\mathcal{Q}$ are two-image portraits with $\Pi_{\mathcal{P}} = \Pi_{\mathcal{Q}} = \Pi$, then*

$$\mathrm{Conf}_{\mathcal{P},d} = \mathrm{Conf}_{\mathcal{Q},d}.$$

Hence, $\mathrm{Conf}_{\mathcal{P},d}$ depends only on the partition Π; let $\mathrm{Conf}_{\Pi,d} := \mathrm{Conf}_{\mathcal{P},d}$ and $\mathcal{M}_{\Pi,d} := \mathrm{Conf}_{\Pi,d}/\mathrm{Aff}_1$. Thus

$$\mathcal{M}_{\mathcal{P},\mathcal{Q},d} = \mathcal{M}_{\Pi,d}.$$

(2) *$\mathrm{Conf}_{\Pi,d}$ is the $(d+1)$-dimensional Zariski closed subspace of Conf^{2d} defined by the equations*

$$e_k(x_A) = e_k(x_B),$$

for $1 \leq k < d$, where e_k is the kth elementary symmetric function in d variables and $x_A := \{x_a : a \in A\}$ is the subset of $\{x_1, x_2, \ldots, x_{2d}\}$ of variables indexed by A, likewise for x_B.

(3) *If $\Pi' \neq \Pi$ is any other partition of $[2d]$ into two sets of d elements each, then*

$$\mathrm{Conf}_{\Pi,d} \cap \mathrm{Conf}_{\Pi',d} = \emptyset.$$

Equivalently, if $\mathcal{P}$ and $\mathcal{Q}$ are two-image portraits with distinct fiber partitions, then

$$\mathcal{M}_{\mathcal{P},\mathcal{Q},d} = \emptyset.$$

(4) *For each* $q \in \mathrm{Conf}_{\Pi,d}$, $\mathrm{End}(q)$ *contains at least* $2d(2d-1)$ *degree-d endomorphisms of* q. *Hence*

$$E_{2d,d} := \max_{q \in \mathrm{Conf}^{2d}} |\mathrm{End}(q)| \geq 2d(2d-1).$$

We deduce Theorem 1.5(3) as a corollary of the following result of possible independent interest.

Theorem 1.6 *Let* K *be an algebraically closed field, and suppose that* $f(x), g(x) \in K[x]$ *are polynomials such that for some distinct* $a, b \in K$,

$$f^{-1}(\{a,b\}) = g^{-1}(\{a,b\})$$

as sets with multiplicity. If either $\mathrm{char}(K) \neq 2$ *or* f *and* g *have odd degree, then* $f(x) = g(x)$ *or* $f(x) = a + b - g(x)$.

A close analysis of portrait pairs $\{\mathcal{P}, \mathcal{Q}\}$ for which the dimension of $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ differs from expectation revealed the following result as the most common source of deviations.

Theorem 1.7 *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be admissible degree-d portraits on n points and suppose that either*

(1) $d = 2$ *and there is at least one pair* $i, j \in [n]$ *such that* $\mathcal{P}(i) = \mathcal{P}(j)$ *and* $\mathcal{Q}(i) = \mathcal{Q}(j)$,
(2) $d = 3$ *and there are at least two pairs* $i, j \in [n]$ *such that* $\mathcal{P}(i) = \mathcal{P}(j)$ *and* $\mathcal{Q}(i) = \mathcal{Q}(j)$, *or*
(3) d *is arbitrary and the fiber partitions* $\Pi_{\mathcal{P}}$, $\Pi_{\mathcal{Q}}$ *have a common fiber with* d *elements.*

If q *is a degree-at-most-d realization of both* $\mathcal{P}$ *and* $\mathcal{Q}$ *via the polynomials* f *and* g, *respectively, then* $f(x) = \ell(g(x))$ *for some linear polynomial* $\ell(x)$.

The condition $f(x) = \ell(g(x))$ for any pair of realizations $f(x), g(x)$ is highly restrictive. In Sect. 3.4.1 we identify a number of ways in which Theorem 1.7 can be used to explain why $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d} = \emptyset$ for $\mathcal{P}$ and $\mathcal{Q}$ satisfying one of the conditions of the theorem. Furthermore, in Sect. 3.4.2 we show how Theorem 1.7 leads to a conditional revised expected dimension for $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ which heuristically accounts for the majority of the deviations towards higher dimensions.

1.1.2 Endomorphisms of Symmetric Configurations

Our survey of $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},2}$ for portraits on 4 points allows us to answer Question 1.2 for $(n, d) = (4, 2)$ in characteristic 0: the maximum cardinality of $\mathrm{End}_2(q)$ with $q \in \mathrm{Conf}^4(\mathbb{C})$ is $E_{4,2} = 28$ and is realized by the highly symmetric configuration $q = \mu_4 = (1, i, -1, -i)$ of 4th roots of unity. Note that this more than doubles the lower bound of $E_{4,2} \geq 12$ given by Theorem 1.5. This suggests that the sequence of

Table 2 The cardinality of $\mathrm{End}_d(\mu_n)$ for $3 \le n \le d$ and $0 \le d \le n-1$

n \ d	0	1	2	3	4	5	6	7
3	3	3	21					
4	4	4	28	220				
5	5	5	5	105	3005			
6	6	6	6	30	1992	44616		
7	7	7	7	7	105	4907	818503	
8	8	8	8	8	280	2968	186840	16587096

configurations $q = \mu_n$ of nth roots of unity may be good candidates for understanding the extremal behavior of $\mathrm{End}(q)$. Table 2 lists the number of degree-d endomorphisms of μ_n for $3 \le n \le 8$ and $0 \le d < n$.

Note that for each degree d, the n monomials $\zeta_n^k x^d$ belong to $\mathrm{End}_d(\mu_n)$. In Sect. 4, we prove that, as suggested by the table, these are the only endomorphisms of μ_n with degree less than $n/2$.

Theorem 1.8 *If $n > 2d \ge 1$, then the only degree-d polynomial endomorphisms of μ_n in $\mathbb{C}[x]$ are of the form $\zeta_n^k x^d$ for some $k \ge 0$.*

Theorem 1.8 is an immediate consequence of a result of Cargo and Schneider [2, Theorem 1]; however, we provide a self-contained proof in Sect. 4.

1.2 Related Work

Questions 1.1 and 1.2 relate to several conjectures on preperiodic points in arithmetic dynamics. We note two examples below.

1.2.1 Uniform Boundedness Conjecture

The Morton–Silverman uniform boundedness conjecture [6, p. 100] asserts, in part, that for any number field K of degree e over $\mathbb{Q}$ and any polynomial $f(x) \in K[x]$ with degree d, the number of K-rational preperiodic points of $f(x)$ is uniformly bounded by a constant $B_{d,e}$ depending only on d and e. See [7, Sect. 3.3] for the precise statement of the uniform boundedness conjecture and a list of further references.

The uniform boundedness conjecture is equivalent to the statement that $\mathrm{End}_d(q) = \emptyset$ for any configuration q consisting of more than $B_{d,e}$ points in $\mathbb{A}^1(K)$. We now briefly justify this equivalence, which relies on the fact that if $q_1, \ldots, q_n$ is the full set of K-rational preperiodic points for a polynomial $f(x) \in K[x]$ of degree d, then for $q = (q_1, \ldots, q_n)$ we have $f \in \mathrm{End}_d(q)$. In fact, from this observation it follows immediately that if $\mathrm{End}_d(q) = \emptyset$ for all configurations q of more than $B_{d,e}$ points in $\mathbb{A}^1(K)$, then no degree-d polynomial can have more than $B_{d,e}$ K-rational preperiodic points.

Now suppose the uniform boundedness conjecture holds, and let q be a configuration of points in $\mathbb{A}^1(K)$ with more than $B_{d,e}$ points. Since $B_{d,e}$ depends on d, we may as well assume[1] that $B_{d,e} \geq d$. Suppose for contradiction that there exists $f(x) \in \mathrm{End}_d(q)$. Then q is a set of preperiodic points for f, so it remains to show that f has coefficients in K, which would then contradict the uniform boundedness conjecture. But the fact that f has coefficients in K follows immediately from Lagrange interpolation (see also the proof of Proposition 2.3) since q has more than $d = \deg f$ points.

1.2.2 Common Preperiodic Point Conjecture

De Marco et al. [4] conjecture that for any degree d, there exists a uniform bound C_d such that if $f(x), g(x) \in \mathbb{C}[x]$ are polynomials of degree d, then either

$$|\mathrm{PrePer}(f) \cap \mathrm{PrePer}(g)| \leq C_d \quad \text{or} \quad \mathrm{PrePer}(f) = \mathrm{PrePer}(g),$$

where $\mathrm{PrePer}(f)$ denotes the set of all complex preperiodic points for f. Hence, if q is a configuration of more than C_d complex points, then this conjecture implies that all the elements of $\mathrm{End}_d(q)$ have identical sets of preperiodic points.

1.2.3 Dynamical Moduli Spaces

The portrait moduli spaces $\mathcal{M}_{\mathcal{P},d}$ are closely related, but not identical, to dynamical moduli spaces that have been studied since at least the 1980s. Typically one begins with a family $\mathcal{F}$ of endomorphisms of the projective line $\mathbb{P}^1$ (e.g., degree-d rational functions, degree-d polynomials, or degree-d polynomials with a single critical point) and a portrait $\mathcal{P}$ on $[n]$, then constructs the space

$$\mathcal{F}[\mathcal{P}] := \{(f, q_1, \ldots, q_n) : f \in \mathcal{F},\ f(q_i) = q_{\mathcal{P}(i)} \text{ for all } i, \text{ and } q_i \neq q_j \text{ for } i \neq j\}.$$

If $G \subseteq \mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}_2$ is a subgroup stabilizing $\mathcal{F}$ under conjugation, then $\phi \in G$ acts on $\mathcal{F}[\mathcal{P}]$ by

$$(f, q_1, \ldots, q_n)^\phi = (\phi \circ f \circ \phi^{-1}, \phi(q_1), \ldots, \phi(q_n)).$$

Then, we get a moduli space of dynamical systems of type $\mathcal{F}$ with *level structure* $\mathcal{P}$ by taking the quotient $\mathcal{M}[\mathcal{P}] := \mathcal{F}[\mathcal{P}]/G$.

Our spaces $\mathrm{Conf}_{\mathcal{P},d}$ and $\mathcal{M}_{\mathcal{P},d}$ are the projections of $\mathcal{F}[\mathcal{P}]$ and $\mathcal{M}[\mathcal{P}]$, respectively, onto the q-coordinates, where we have taken $\mathcal{F}$ to be the family of degree-d polynomials and $G = \mathrm{Aff}_1$ the group of affine linear transformations. The spaces $\mathrm{Conf}_{\mathcal{P},d}$ are more natural than $\mathcal{F}[\mathcal{P}]$ in the context of Questions 1.1 and 1.2, as configurations q with exceptional endomorphism semigroups arise as points of intersection of $\mathrm{Conf}_{\mathcal{P},d}$ for several $\mathcal{P}$ and d. Furthermore, when $n > d$, Lagrange interpolation implies that the degree-d polynomial $f(x)$ witnessing $q \in \mathrm{Conf}_{\mathcal{P},d}$ is uniquely determined by q, hence no information is lost in the projection $\pi : \mathcal{F}[\mathcal{P}] \to \mathrm{Conf}_{\mathcal{P},d}$ (see Theorem 1.4).

[1] In fact, if $B_{d,e}$ exists, it is not too difficult to show that the inequality $B_{d,e} \geq d$ must necessarily hold. For the purposes of this discussion, though, we do not need to prove this.

There is a large and growing literature on dynamical moduli spaces. We recommend the article [5], which constructs the parameter spaces $\mathrm{End}_d^N[\mathcal{P}]$ and moduli spaces $\mathcal{M}_d^N[\mathcal{P}] := \mathrm{End}_d^N[\mathcal{P}]/\mathrm{PGL}_{N+1}$, where End_d^N denotes the family of all degree-d endomorphisms of $\mathbb{P}^N$, and develops the basic theory of such spaces. See [5, Sect. 2] for a brief survey of prior work on dynamical moduli spaces and [5, Theorem 17.5] for a statement showing that the general Morton-Silverman uniform boundedness conjecture can be rephrased in terms of K-rational points on dynamical moduli spaces.

1.3 Organization

The remainder of the paper is divided into three sections. The parameter and moduli spaces $\mathrm{Conf}_{\mathcal{P},d}$ and $\mathcal{M}_{\mathcal{P},d}$ are constructed in Sect. 2. Theorem 1.4 is proved as Corollary 2.9. In Sect. 3, we study intersections of the portrait realization spaces and discuss the results of computational surveys in low degrees. Theorem 1.5 appears in parts as Theorem 3.20 and Proposition 3.24; Theorem 1.6 is proved as Theorem 3.25; and Theorem 1.7 is proved as Theorem 3.15. Finally, in Sect. 4, we prove that $E_{4,2} = 28$ (Proposition 4.1) and prove Theorem 1.8 as Theorem 4.2.

1.4 Supplementary Code

A Sage notebook containing functions and calculations related to the contents of this paper is available at:

https://github.com/tghyde/portrait_moduli_supplement.

2 Portrait Realization Spaces and Moduli Spaces

Let $\mathcal{P}$ be a portrait on n points, that is, $\mathcal{P} : [n] \to [n]$ is a set-theoretic endomorphism of $[n] := \{1, 2, \ldots, n\}$. Recall from the introduction that $(q_1, \ldots, q_n) \in \mathrm{Conf}^n$ is a *degree-d realization* of $\mathcal{P}$ if there exists a degree-d polynomial $f(x) \in \mathbb{C}[x]$ such that $f(q_i) = q_{\mathcal{P}(i)}$ for all $1 \le i \le n$. For the purposes of this paper, we consider the zero polynomial to have degree 0.

Example 2.1 It is convenient to represent portraits on $[n]$ as directed graphs with vertices labeled $1, 2, \ldots, n$. Figure 2 illustrates three such diagrams: On the left is the portrait $\mathcal{P}$ that maps 1, 2, 3, and 4 to 1, 1, 2, and 4, respectively. In the middle is the portrait $\mathcal{Q}$ that maps 1, 2, 3, and 4 to 1, 3, 3, and 1, respectively. On the right is the pair of portraits $\{\mathcal{P}, \mathcal{Q}\}$: $\mathcal{P}$ is drawn with solid red arrows and $\mathcal{Q}$ is drawn with dashed black arrows.

To conclude this example, note that $q = (q_1, q_2, q_3, q_4) = (0, 1, 2, 3)$ is a degree-2 realization of the pair of portraits $\{\mathcal{P}, \mathcal{Q}\}$. Indeed, if $f(x) := \frac{1}{2}x(x-1)$ and $g(x) := -x(x-3)$, then

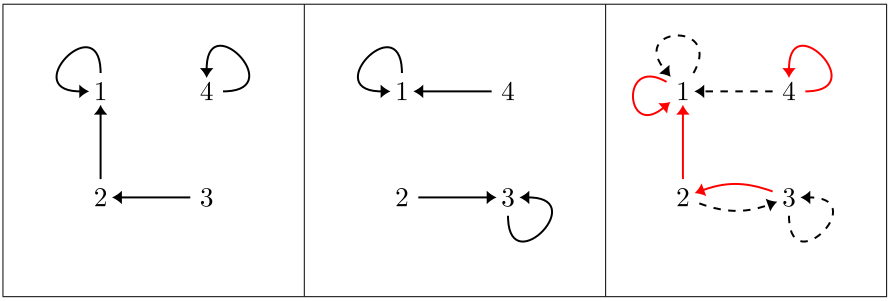


Fig. 2 An illustration of $\mathcal{P}$, $\mathcal{Q}$, and $\{\mathcal{P}, \mathcal{Q}\}$, respectively

$$\begin{aligned} f(q_1) &= f(0) = 0 = q_1 & g(q_1) &= g(0) = 0 = q_1 \\ f(q_2) &= f(1) = 0 = q_1 & g(q_2) &= g(1) = 2 = q_3 \\ f(q_3) &= f(2) = 1 = q_2 & g(q_3) &= g(2) = 2 = q_3 \\ f(q_4) &= f(3) = 3 = q_4 & g(q_4) &= g(3) = 0 = q_1, \end{aligned}$$

so q realizes $\mathcal{P}$ and $\mathcal{Q}$ via the polynomials f and g, respectively.

We are interested in the *portrait realization spaces*

$$\begin{aligned} \mathrm{Conf}_{\mathcal{P},d} &:= \{q \in \mathrm{Conf}^n : q \text{ is a degree-}d \text{ realization of } \mathcal{P}\}, \\ \widehat{\mathrm{Conf}}_{\mathcal{P},d} &:= \bigcup_{0 \le e \le d} \mathrm{Conf}_{\mathcal{P},e} = \{q \in \mathrm{Conf}^n : q \text{ is a degree-at-most-}d \text{ realization of } \mathcal{P}\}. \end{aligned}$$

In this section, we study portrait realization spaces, culminating in a proof of Theorem 1.4.

2.1 $\mathrm{Conf}_{\mathcal{P},d}$ as a Complex Algebraic Variety

The following lemma is used in the proofs of Propositions 2.3 and 2.8.

Lemma 2.2 *Let* $(q_1, q_2, \ldots, q_n) \in \mathrm{Conf}^n$ *be a configuration of n distinct points in* $\mathbb{C}$*, and let* $d \geq n$*. Then for any portrait* $\mathcal{P} : [n] \to [n]$*, there exists a monic polynomial* $f(x) \in \mathbb{C}[x]$ *of degree d such that* $f(q_i) = q_{\mathcal{P}(i)}$ *for all* $1 \leq i \leq n$*.*

Proof Lagrange interpolation implies there exists a polynomial $g(x) \in \mathbb{C}[x]$ with degree at most $d-1$ such that $g(q_i) = q_{\mathcal{P}(i)} - q_i^d$ for each i. Hence $f(x) := x^d + g(x)$ is a monic degree-d polynomial such that $f(q_i) = q_{\mathcal{P}(i)}$ for each $1 \leq i \leq n$. □

Proposition 2.3 *For any portrait* $\mathcal{P} : [n] \to [n]$ *and any degree* $d \geq 0$*,* $\widehat{\mathrm{Conf}}_{\mathcal{P},d}$ *is a Zariski closed subset of* Conf^n*, and* $\mathrm{Conf}_{\mathcal{P},d}$ *is a Zariski open subset of* $\widehat{\mathrm{Conf}}_{\mathcal{P},d}$*.*

Proof If $d \geq n$, then it follows from Lemma 2.2 that for any configuration $q \in \mathrm{Conf}^n$, there exists a *monic* degree-d polynomial $f(x)$ such that $f(q_i) = q_{\mathcal{P}(i)}$ for all $i \in [n]$. Therefore, in this case

$$\mathrm{Conf}_{\mathcal{P},d} = \widehat{\mathrm{Conf}}_{\mathcal{P},d} = \mathrm{Conf}^n,$$

so the conclusion holds trivially.

Now, suppose $d \leq n-1$. The configuration space Conf^n is, by definition, the complement of the hyperplane arrangement $\{x_i = x_j : i \neq j\}$ in $\mathbb{A}^n$. Let

$$R_n := \mathbb{Z}[x_i, (x_j - x_k)^{-1} : 1 \leq i, j, k \leq n, j \neq k]$$

be the coordinate ring of Conf^n. Lagrange interpolation tells us that

$$f(x) := \sum_{i=1}^{n} x_{\mathcal{P}(i)} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0 \in R_n[x]$$

is the unique polynomial of degree at most $n-1$ with coefficients in R_n which realizes the portrait $\mathcal{P}$ on the indeterminates x_i. Thus, a configuration $q = (q_1, q_2, \ldots, q_n)$ is a degree-at-most-d realization of $\mathcal{P}$ if and only if $b_{n-1}(q) = b_{n-2}(q) = \cdots = b_{d+1}(q) = 0$. In other words, $\widehat{\mathrm{Conf}}_{\mathcal{P},d}$ is the Zariski closed subset of Conf^n defined by the vanishing of b_k with $d < k \leq n-1$. Moreover, the space $\mathrm{Conf}_{\mathcal{P},d}$ of degree-d realizations of $\mathcal{P}$ is the open subset of $\widehat{\mathrm{Conf}}_{\mathcal{P},d}$ defined by $b_d(q) \neq 0$. □

If $\mathcal{P} : [n] \to [n]$ is a portrait and σ is a permutation of $[n]$, then we let $\mathcal{P}^\sigma := \sigma^{-1} \circ \mathcal{P} \circ \sigma$ denote the conjugate of $\mathcal{P}$ by σ, which amounts to the relabeling of $\mathcal{P}$ induced by σ. We end this section by recording the fact that relabeled portraits have isomorphic realization spaces. We omit the proof, which is a straightforward application of the definitions.

Proposition 2.4 *Let $\mathcal{P} : [n] \to [n]$ be a portrait and let σ be a permutation of $[n]$. The map*

$$\begin{aligned} \Phi_\sigma : \mathrm{Conf}^n &\longrightarrow \mathrm{Conf}^n \\ (q_1, \ldots, q_n) &\longmapsto (q_{\sigma(1)}, \ldots, q_{\sigma(n)}) \end{aligned} \tag{2.1}$$

induces isomorphisms $\mathrm{Conf}_{\mathcal{P},d} \xrightarrow{\sim} \mathrm{Conf}_{\mathcal{P}^\sigma,d}$ and $\widehat{\mathrm{Conf}}_{\mathcal{P},d} \xrightarrow{\sim} \widehat{\mathrm{Conf}}_{\mathcal{P}^\sigma,d}$ for all $d \geq 0$.

2.2 Dimension of $\mathrm{Conf}_{\mathcal{P},d}$

We now state necessary and sufficient conditions on a portrait $\mathcal{P}$ for the degree-d realization space $\mathrm{Conf}_{\mathcal{P},d}$ to be nonempty and, in that case, compute the dimension of $\mathrm{Conf}_{\mathcal{P},d}$.

The most dynamically interesting situation is when $d \geq 2$, but for completeness we also briefly discuss the cases $d = 0$ and $d = 1$. Certainly for a portrait $\mathcal{P}$ on $[n]$ to have degree-0 realizations, $\mathcal{P}$ must be a constant portrait, and conversely any configuration in Conf^n can realize a constant portrait. We now record Proposition 2.5, which is a consequence of the following elementary facts about degree-1 polynomials $\ell(x)$:

(i) ℓ is determined by how it acts on any two points;

(ii) If $\ell(x) \neq x$, then ℓ has at most one fixed point; and
(iii) If ℓ has a periodic point of exact period $k > 1$, then $\ell^k(x) = x$ and $\ell^m(x) \neq x$ for all $1 \leq m < k$, where $\ell^k(x)$ denotes the kth iterate of ℓ. In particular, every point except the unique fixed point has period equal to k.

Proposition 2.5 *Let $n \geq 1$ be an integer and let $\mathcal{P} : [n] \to [n]$ be a portrait.*

(1) $\mathrm{Conf}_{\mathcal{P},0} \neq \emptyset$ *if and only if $\mathcal{P}$ is constant. In this case,* $\mathrm{Conf}_{\mathcal{P},0} = \mathrm{Conf}^n$ *and* $\dim \mathrm{Conf}_{\mathcal{P},0} = n$.
(2) $\mathrm{Conf}_{\mathcal{P},1} \neq \emptyset$ *if and only if the following three conditions hold,*

   (a) *$\mathcal{P}$ is a bijection,*
   (b) *$\mathcal{P}$ is either the identity function or has at most one fixed point, and*
   (c) *If $\mathcal{P}$ has a k-cycle for some $k > 1$, then all but at most one point in $[n]$ belongs to a k-cycle.*

   In this case, if $\gamma(\mathcal{P})$ denotes the number of orbits of $\mathcal{P}$, then $\dim \mathrm{Conf}_{\mathcal{P},1}$ *is equal to $\gamma(\mathcal{P})$ when $\mathcal{P}$ has a fixed point and is equal to $\gamma(\mathcal{P}) + 1$ otherwise.*

Now suppose that $d \geq 2$. There are two natural combinatorial obstructions to $\mathcal{P}$ admitting degree-d realizations: First, if $\mathcal{P}$ is to be realized by a polynomial of degree d, then no element of $[n]$ should have more than d preimages under $\mathcal{P}$. Second, for a given integer $k \geq 1$, a polynomial of degree $d \geq 2$ can have at most $M_k(d)$ periodic cycles of length k, where

$$M_k(x) := \frac{1}{k} \sum_{j|k} \mu(k/j) x^j$$

is the *kth necklace polynomial*; this follows from the fact that a point of period k for a degree-d polynomial f is a root of the *kth dynatomic polynomial*

$$\Phi_k(x) := \prod_{j|k} \left(f^j(x) - x\right)^{\mu(k/j)},$$

which has degree $kM_k(d)$, so that f has at most $M_k(d)$ cycles of length k. Here f^j denotes the jth iterate $f \circ \cdots \circ f$ of f.

Definition 2.6 A portrait $\mathcal{P} : [n] \to [n]$ is *admissible in degree d* for $d \geq 2$ if

(1) Every element of $[n]$ has at most d preimages under $\mathcal{P}$, and
(2) For every integer $k \geq 1$, $\mathcal{P}$ has at most $M_k(d)$ periodic cycles of length k.

Our discussion above implies that admissibility in degree d is a necessary condition for $\mathrm{Conf}_{\mathcal{P},d}$ to be nonempty. We prove that admissibility is also sufficient; one should compare this to [5, Theorem 14.2].

Lemma 2.7 *Let $d \geq 2$, and let $\mathcal{P}$ be a degree-d admissible portrait. Then* $\mathrm{Conf}_{\mathcal{P},d} \neq \emptyset$.

Proof It suffices to show that for each $d \geq 2$ there exists at least one degree-d polynomial $f(x) \in \mathbb{C}[x]$ with the following properties:

(1) For all $k \geq 1$, f has exactly $M_k(d)$ periodic cycles of length k, and
(2) Every preperiodic point for f has exactly d preimages in $\mathbb{C}$; equivalently, f has no preperiodic critical points.

We claim that $f(x) = x^d + 1$ satisfies these properties. Indeed, 0 is the only critical point for f, and a simple induction argument shows that $f^n(0) \geq 2^{2^{n-2}}$ for all $n \geq 2$, so it follows that the sequence of iterates $0, f(0), f^2(0), \ldots$ tends to ∞. Hence, f has no preperiodic critical points. Thus, (2) holds.

Suppose for contradiction that (1) fails—that is, suppose f has fewer than $M_k(d)$ periodic cycles of length k. Then, the polynomial $g(x) := f^k(x) - x$ has fewer zeros than expected; hence, g and g' have a common root α. Since $g(\alpha) = f^k(\alpha) - \alpha = 0$, α is periodic with period dividing k, and $(f^k)'(\alpha) = g'(\alpha) + 1 = 1$. This means that α belongs to a *rationally indifferent* cycle. By [1, Theorem 9.3.2], this cycle must attract a critical point; since 0 is the only critical point, the orbit of 0 must approach the cycle containing α, contradicting the fact that 0 has orbit tending to ∞. □

Proposition 2.8 *Let $n \geq 1$ and $d \geq 2$ be integers, and let $\mathcal{P} : [n] \to [n]$ be a portrait. The following are equivalent:*

(1) *$\mathcal{P}$ is admissible in degree d.*
(2) $\mathrm{Conf}_{\mathcal{P},d} \neq \emptyset$.
(3) $\dim \mathrm{Conf}_{\mathcal{P},d} = \min\{d+1, n\}$.

Proof First, suppose $n \leq d$. If $q = (q_1, q_2, \ldots, q_n) \in \mathrm{Conf}^n$ is any configuration, then by Lemma 2.2 there exists a monic degree-d polynomial $f(x)$ such that $f(q_i) = q_{\mathcal{P}(i)}$. As in the proof of Proposition 2.3, it follows that $\mathrm{Conf}_{\mathcal{P},d} = \mathrm{Conf}^n$. Thus, when $n \leq d$, (2) and (3) are true, and (1) is true since a portrait on at most d points is automatically admissible in degree d.

Now suppose $n \geq d + 1$. Consider the augmented realization space

$$\mathrm{Poly}_{\mathcal{P},d} := \{(f, q_1, \ldots, q_n) : \deg f = d \text{ and } f(q_i) = q_{\mathcal{P}(i)} \text{ for all } 1 \leq i \leq n\}.$$

Thinking of a polynomial as a rational map for which ∞ is a totally ramified fixed point, the dimension counting results[2] of [5, Theorem 15.8 and Remark 15.9] imply that

$$\mathcal{P} \text{ is a degree-}d \text{ admissible portrait} \Longleftarrow \mathrm{Poly}_{\mathcal{P},d} \neq \emptyset \iff \dim \mathrm{Poly}_{\mathcal{P},d} = d + 1.$$

Since we have already shown that (1) implies (2) in Lemma 2.7, it therefore suffices to show that $\mathrm{Poly}_{\mathcal{P},d} \cong \mathrm{Conf}_{\mathcal{P},d}$. Consider the projection map

[2] Let $\mathcal{P}'$ be the *weighted* portrait, in the terminology of [5], obtained by adding to $\mathcal{P}$ a single fixed point of weight d, corresponding to a totally ramified fixed point. The cited results of [5] show that the parameter space associated to $\mathcal{P}'$—denoted $\mathrm{End}_d^1[\mathcal{P}']$ in [5]—has dimension $d+2$, but we lose a dimension by moving the totally ramified fixed point to ∞.

$$\Psi_d : \mathrm{Poly}_{\mathcal{P},d} \longrightarrow \mathrm{Conf}_{\mathcal{P},d}$$
$$(f, q_1, \dots, q_n) \longmapsto (q_1, \dots, q_n).$$

Certainly Ψ_d is a morphism, and it is surjective by the definition of $\mathrm{Conf}_{\mathcal{P},d}$. Moreover, since $n \geq d+1$, Lagrange interpolation implies that for each configuration $q \in \mathrm{Conf}_{\mathcal{P},d}$ there is a *unique* polynomial f for which $\Psi_d(f, q) = q$, so Ψ_d is bijective as a map on sets. The coefficients of f may be expressed as regular functions on Conf^n (see the proof of Proposition 2.3) so the inverse Ψ_d^{-1} is also a morphism. Therefore, Ψ_d is an isomorphism, completing the proof. □

2.3 Remarks on the Moduli Spaces $\mathcal{M}_{\mathcal{P},d}$ and $\widehat{\mathcal{M}}_{\mathcal{P},d}$

Let Aff_1 denote the group of linear polynomials with respect to composition.[3] If $\ell(x) \in \mathrm{Aff}_1$ and $f(x)$ is the degree-d polynomial witnessing the realization $q \in \mathrm{Conf}_{\mathcal{P},d}$, then $\widetilde{f} := \ell \circ f \circ \ell^{-1}$ has degree d and

$$\widetilde{f}(\ell(q_i)) = \ell(f(q_i)) = \ell(q_{\mathcal{P}(i)}).$$

Hence $\ell(q) = (\ell(q_1), \ell(q_2), \dots, \ell(q_n)) \in \mathrm{Conf}_{\mathcal{P},d}$. Thus we define the *portrait moduli spaces* $\mathcal{M}_{\mathcal{P},d}$ and $\widehat{\mathcal{M}}_{\mathcal{P},d}$ as the quotients

$$\mathcal{M}_{\mathcal{P},d} := \mathrm{Conf}_{\mathcal{P},d}/\mathrm{Aff}_1 \quad \text{and} \quad \widehat{\mathcal{M}}_{\mathcal{P},d} := \widehat{\mathrm{Conf}}_{\mathcal{P},d}/\mathrm{Aff}_1.$$

For $n \geq 2$, $\mathcal{M}_{\mathcal{P},d}$ (resp., $\widehat{\mathcal{M}}_{\mathcal{P},d}$) is a fine moduli space for the moduli problem of degree-d realizations (resp., degree-at-most-d realizations) of $\mathcal{P}$. Indeed, $\mathcal{M}_{\mathcal{P},d}$ and $\widehat{\mathcal{M}}_{\mathcal{P},d}$ are coarse moduli spaces by construction; hence, it suffices to show that no point on $\widehat{\mathrm{Conf}}_{\mathcal{P},d}$ (thus no point on $\mathrm{Conf}_{\mathcal{P},d}$) has a nontrivial stabilizer. Recall that Aff_1 acts sharply 2-transitively on $\mathbb{A}^1$. Thus, if $q = (q_1, q_2, \dots, q_n) \in \widehat{\mathrm{Conf}}_{\mathcal{P},d}$ and $\ell(x) \in \mathrm{Aff}_1$ fixes q_1 and q_2, then $\ell(x) = x$.

Moreover, for $n \geq 2$, $\dim \mathcal{M}_{\mathcal{P},d} = \dim \mathrm{Conf}_{\mathcal{P},d} - 2$, since Aff_1 is a 2-dimensional algebraic group acting faithfully on $\mathrm{Conf}_{\mathcal{P},d}$. Thus Corollary 2.9 is an immediate consequence of Proposition 2.8.

Corollary 2.9 *Let $n, d \geq 2$, and let $\mathcal{P} : [n] \to [n]$ be a portrait. The following are equivalent in characteristic 0:*

(1) *$\mathcal{P}$ is a degree-d admissible portrait.*
(2) $\mathcal{M}_{\mathcal{P},d} \neq \emptyset$.
(3) $\dim \mathcal{M}_{\mathcal{P},d} = \min\{d-1, n-2\}$.

While $\mathrm{Conf}_{\mathcal{P},d}$ is a Zariski open subset of $\widehat{\mathrm{Conf}}_{\mathcal{P},d}$, it need not be dense in $\widehat{\mathrm{Conf}}_{\mathcal{P},d}$. In fact, $\widehat{\mathrm{Conf}}_{\mathcal{P},d}$ may have components with dimension strictly larger than the dimension of $\mathrm{Conf}_{\mathcal{P},d}$. We illustrate with two examples:

[3] From the perspective of PGL_2 acting on $\widehat{\mathbb{C}}$ by Möbius transformations, the group Aff_1 is the Borel subgroup of upper triangular matrices in PGL_2.

(1) If $\mathcal{P}$ is the portrait consisting of two 2-cycles, then $\mathcal{P}$ is not degree-2 admissible, hence $\mathrm{Conf}_{\mathcal{P},2} = \emptyset$. On the other hand, $\widehat{\mathrm{Conf}}_{\mathcal{P},2} = \mathrm{Conf}_{\mathcal{P},1}$ has dimension 3 by Proposition 2.5(2).
(2) If $\mathcal{P}$ is the portrait consisting of three 4-cycles, then $\mathcal{P}$ is degree-2 admissible, so $\mathrm{Conf}_{\mathcal{P},2}$ is nonempty; in fact, the dimension of $\mathrm{Conf}_{\mathcal{P},2}$ is 3 by Proposition 2.8. On the other hand, by Proposition 2.5(2), we have $\dim \mathrm{Conf}_{\mathcal{P},1} = 4$. Since $\widehat{\mathrm{Conf}}_{\mathcal{P},2}$ is the (disjoint) union of $\mathrm{Conf}_{\mathcal{P},1}$ and $\mathrm{Conf}_{\mathcal{P},2}$, we have $\dim \widehat{\mathrm{Conf}}_{\mathcal{P},2} = 4 > \dim \mathrm{Conf}_{\mathcal{P},2}$.

On the other hand, if $n \le 2d$ and $\mathcal{P}$ is not the identity portrait[4], then $\dim \mathrm{Conf}_{\mathcal{P},1} \le d+1$ by Proposition 2.5, with equality when $\mathcal{P}$ consists of d 2-cycles. Hence if $\mathcal{P}$ is admissible in degree d, then

$$\dim \widehat{\mathrm{Conf}}_{\mathcal{P},d} = \dim \mathrm{Conf}_{\mathcal{P},d} = d+1.$$

However, if $\dim \mathrm{Conf}_{\mathcal{P},1} = \dim \mathrm{Conf}_{\mathcal{P},d} = d+1$, then the space $\mathrm{Conf}_{\mathcal{P},d}$ will still not be Zariski dense in $\widehat{\mathrm{Conf}}_{\mathcal{P},d}$. Taking quotients, the above remarks are also valid for $\mathcal{M}_{\mathcal{P},d} \subset \widehat{\mathcal{M}}_{\mathcal{P},d}$. In particular, if $n \le 2d$ and $\mathcal{P}$ is not the identity portrait, then

$$\dim \widehat{\mathcal{M}}_{\mathcal{P},d} = \dim \mathcal{M}_{\mathcal{P},d} = d-1. \tag{2.2}$$

Finally, note that Theorem 1.4 is equivalent to Corollary 2.9.

3 Intersections of Realization Spaces

In this section, we study the intersections of portrait realization spaces. If $\mathcal{P}, \mathcal{Q} : [n] \to [n]$ are portraits, then $\mathrm{Conf}_{\mathcal{P},d}$ and $\mathrm{Conf}_{\mathcal{Q},d}$ live in the same ambient space Conf^n. Let

$$\mathcal{M}_{\mathcal{P},\mathcal{Q},d} := (\mathrm{Conf}_{\mathcal{P},d} \cap \mathrm{Conf}_{\mathcal{Q},d})/\mathrm{Aff}_1 \quad \text{and} \quad \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d} := (\widehat{\mathrm{Conf}}_{\mathcal{P},d} \cap \widehat{\mathrm{Conf}}_{\mathcal{Q},d})/\mathrm{Aff}_1$$

denote the moduli space of affine equivalence classes of configurations $q \in \mathrm{Conf}^n$ which have degree-d (respectively, degree-at-most-d) realizations of both $\mathcal{P}$ and $\mathcal{Q}$.

Just as we defined in Definition 2.6 what it means for a single portrait to be admissible (in degree d), we now do the same for pairs of portraits:

Definition 3.1 A pair $\{\mathcal{P}, \mathcal{Q}\}$ of portraits is *admissible in degree d* if $\mathcal{P}$ and $\mathcal{Q}$ are *distinct* portraits, each of which is admissible in degree d.

For $(n, d) = (4, 2)$ and $(6, 3)$ we have conducted a computational survey of all the spaces $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ with $\{\mathcal{P}, \mathcal{Q}\}$ an admissible degree-d pair of portraits. An analysis of the findings and results inspired by the data are discussed below.

[4] If $\mathcal{P}$ is the identity portrait on n elements, then $\dim_{\mathcal{P},1} = n$ by Proposition 2.5. Moreover, by Proposition 2.8, $\mathcal{P}$ is admissible in degree d if and only if $n \le d$. In this case, $\dim \mathrm{Conf}_{\mathcal{P},d} = d+1 > n = \mathrm{Conf}_{\mathcal{P},1}$, so the subsequent discussion is still valid for the identity portrait $\mathcal{P}$.

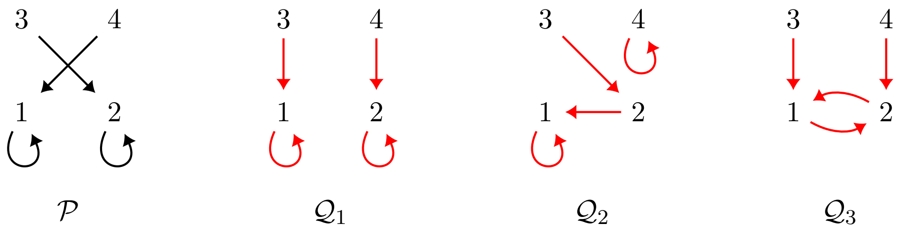


Fig. 3 Some admissible degree-2 portraits on $\{1, 2, 3, 4\}$

3.1 Dimension Heuristic

Suppose for simplicity that $n > d \geq 1$. Let $q \in \mathrm{Conf}^n$ be a configuration of n points. If $q \in \mathrm{Conf}_{\mathcal{P},d}$, then the existence of a degree-d endomorphism $f(x)$ of q imposes $n - d - 1$ algebraic constraints on the coordinates of q. (See the proof of Proposition 2.3.) Thus, the expected dimension of $\mathrm{Conf}_{\mathcal{P},d} \cap \mathrm{Conf}_{\mathcal{Q},d}$ is $n - 2(n - d - 1) = 2d - n + 2$, where we interpret a negative dimension to mean that the space is empty.

The two-dimensional group Aff_1 acts freely on Conf^n, hence the expected dimension of the quotient $\mathcal{M}_{\mathcal{P},\mathcal{Q},d}$ is $2d - n$. The same dimension heuristic applies to $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$. We emphasize that this is only a heuristic: the algebraic conditions imposed by the two portraits may not be independent, or the system of equations may only have degenerate solutions in $\mathbb{A}^n \backslash \mathrm{Conf}^n$. Since $\mathcal{M}_{\mathcal{P},\mathcal{Q},d} \subseteq \mathcal{M}_{\mathcal{P},d}$, Corollary 2.9 implies that in characteristic 0 we have $\dim \mathcal{M}_{\mathcal{P},\mathcal{Q},d} \leq d - 1$, and if we restrict to $n \leq 2d$, (2.2) implies that $\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d} \leq d - 1$.

Example 3.2 As noted above, if $n = 4$ and $d = 2$, then the expected dimension of $\mathcal{M}_{\mathcal{P},\mathcal{Q},2}$ is 0 and the maximum possible dimension is 1. Consider the portraits $\mathcal{P}$, $\mathcal{Q}_1$, $\mathcal{Q}_2$, and $\mathcal{Q}_3$ illustrated in Fig. 3. For $i = 1, 2, 3$, a quick computation shows that the spaces $\mathcal{M}_{\mathcal{P},\mathcal{Q}_i,2}$ have dimensions -1, 0 and 1, respectively. More precisely, the space $\mathcal{M}_{\mathcal{P},\mathcal{Q}_1,2} = \emptyset$, $\mathcal{M}_{\mathcal{P},\mathcal{Q}_2,2} = \{(0, 1, 2, 3)\}$ consists of exactly one point, and $\mathcal{M}_{\mathcal{P},\mathcal{Q}_3,2}$ is a genus-0 curve.

3.2 Combinatorial Equivalence

Recall that if $\mathcal{P} : [n] \to [n]$ is a portrait and σ is a permutation of $[n]$, then $\mathcal{P}^\sigma := \sigma^{-1} \circ \mathcal{P} \circ \sigma$ is the relabeling of $\mathcal{P}$ by σ.

Definition 3.3 Two unordered pairs $\{\mathcal{P}, \mathcal{Q}\}$ and $\{\mathcal{P}', \mathcal{Q}'\}$ of distinct portraits on $[n]$ are *combinatorially equivalent* if there exists a permutation σ of $[n]$ such that $\{\mathcal{P}', \mathcal{Q}'\} = \{\mathcal{P}^\sigma, \mathcal{Q}^\sigma\}$. The combinatorial equivalence class of $\{\mathcal{P}, \mathcal{Q}\}$ is denoted $\langle \mathcal{P}, \mathcal{Q} \rangle$. We say that the equivalence class $\langle \mathcal{P}, \mathcal{Q} \rangle$ is *admissible in degree d* if one (hence every) representative of the equivalence class is admissible in degree d.

A combinatorial equivalence class of portraits $\{\mathcal{P}, \mathcal{Q}\}$ may be visualized as a pair of phase portraits acting on an unlabeled set; see Fig. 4. Note that if $\{\mathcal{P}, \mathcal{Q}\}$ and $\{\mathcal{P}', \mathcal{Q}'\}$

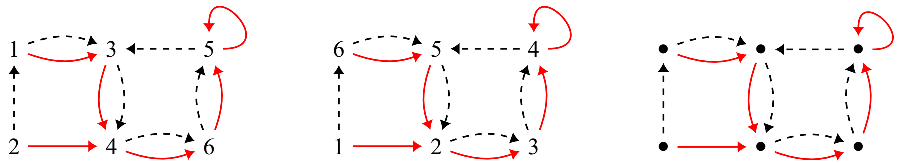


Fig. 4 The first two diagrams illustrate distinct but equivalent portrait pairs, and the third diagram illustrates their equivalence class

Table 3 For each $e \in \{-1, 0, 1, 2\}$, the number of quadratic and cubic portrait pairs $\langle \mathcal{P}, \mathcal{Q} \rangle$ satisfying $\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d} = e$

Dimension	-1	0	1	2	Total
Quadratic pairs	198	568	14		780
Cubic pairs	52310	1297349	1065	18	1350742

are combinatorially equivalent, then applying Proposition 2.4 and taking intersections implies that $\mathcal{M}_{\mathcal{P},\mathcal{Q},d} \cong \mathcal{M}_{\mathcal{P}',\mathcal{Q}',d}$ and $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d} \cong \widehat{\mathcal{M}}_{\mathcal{P}',\mathcal{Q}',d}$.

3.3 Computational Results

We refer to a degree-2 (resp., degree-3) admissible combinatorial equivalence class $\langle \mathcal{P}, \mathcal{Q} \rangle$ on four (resp., six) points as a *quadratic portrait pair* (resp., *cubic portrait pair*). Note that the definition of admissibility requires $\mathcal{P} \neq \mathcal{Q}$. There are 780 quadratic portrait pairs and 1,350,742 cubic portrait pairs. For all such portrait pairs we computed basic invariants of the moduli spaces $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ of degree-at-most-d realizations of $\mathcal{P}$ and $\mathcal{Q}$.

The first invariant of $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ we consider is dimension. Note that

$$-1 \leq \dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d} \leq d - 1,$$

where the upper bound holds because $n = 2d$ for $(n, d) = (4, 2)$ or $(6, 3)$ (see Sect. 3.1). Recall that we say the dimension of a space $\mathcal{M}$ is -1 when $\mathcal{M} = \emptyset$. The dimension heuristic derived in Sect. 3.1 suggests that $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ should typically have dimension 0, hence that $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ should be a finite set. The dimensions of $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ for each quadratic and cubic portrait pair are tabulated in Table 3 (labeled Table 1 in the introduction).

In both degrees, the most common dimension is 0, matching our expectation. However, there are many portrait pairs for which the dimension of $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ takes an unexpected value.

Question 3.4 What combinatorial properties of $\langle \mathcal{P}, \mathcal{Q} \rangle$ imply that the dimension $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ will differ from the expected dimension?

In the remainder of Sect. 3.3 we discuss our findings on the cases when $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ is zero-dimensional (Sect. 3.3.1), when $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ achieves the maximum dimension (Sect. 3.3.2), and when $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d} = \emptyset$ (Sect. 3.3.3).

Table 4 For each degree $e \in \{1, \ldots, 7\}$, the number of quadratic portrait pairs $\langle \mathcal{P}, \mathcal{Q} \rangle$ with $\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},2} = 0$ and $\deg \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},2} = e$

Degree	1	2	3	4	5	6	7	Total
Quadratic pairs	65	166	121	116	62	29	9	568

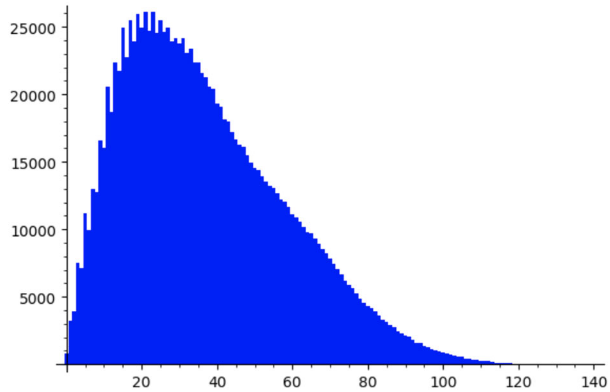


Fig. 5 Degree of $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},3}$ for cubic pairs $\langle \mathcal{P}, \mathcal{Q} \rangle$ with $\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},3} = 0$. Degrees are listed along the horizontal axis; the vertical axis lists the number of cubic portrait pairs for which $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},3}$ has the given degree

3.3.1 Dimension 0

In this section, we discuss fields of definition of points on the spaces $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ as well as multiplicities of points on those schemes $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ which are zero-dimensional. Thus, for this section only, we consider $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ as a scheme over $\mathbb{Q}$ rather than as a variety over $\mathbb{C}$. We justify this by noting that $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ is the subvariety of $\widehat{\mathrm{Conf}}_{\mathcal{P},\mathcal{Q},d}$ obtained by setting $q_1 = 0$ and $q_2 = 1$, and the equations from Proposition 2.3 cutting $\widehat{\mathrm{Conf}}_{\mathcal{P},\mathcal{Q},d}$ out of affine space have rational coefficients.

Recall that the *degree* of a zero-dimensional scheme is the (finite) number of points on that scheme counted with multiplicity. For every quadratic portrait pair $\langle \mathcal{P}, \mathcal{Q} \rangle$ with $\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},2} = 0$, the degree of $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},2}$ was at most 7, and in Table 4 we list the number of quadratic portrait pairs with $\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},2} = 0$ of each given degree.

Since $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ is defined over $\mathbb{Q}$, it is natural to ask, when $\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d} = 0$, for the extension of $\mathbb{Q}$ over which the finitely many points on $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ are defined. Of the 568 quadratic portrait pairs with zero-dimensional moduli spaces, the only pairs with any $\mathbb{Q}$-rational realizations are the 65 with degree 1; note that since the spaces $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ are defined over $\mathbb{Z}$, degree 1 forces the unique point to be rational.

Question 3.5 If $\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d} = 0$, how do properties of the splitting fields of $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ reflect combinatorial properties of $\langle \mathcal{P}, \mathcal{Q} \rangle$?

Over 96% of the cubic portrait pairs $\langle \mathcal{P}, \mathcal{Q} \rangle$ have $\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},3} = 0$. The histogram in Fig. 5 summarizes the frequency with which each degree occurs for these portraits.

The maximum degree is 144, achieved by the unique cubic portrait pair $\langle \mathcal{P}, \mathcal{Q} \rangle$ illustrated in Fig. 6. We note that for this pair $\langle \mathcal{P}, \mathcal{Q} \rangle$, the space $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},3}$ is irreducible

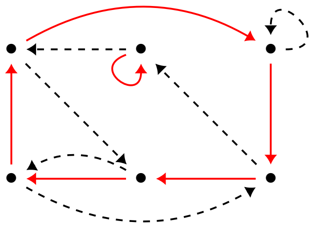


Fig. 6 The unique cubic portrait pair $\langle\mathcal{P},\mathcal{Q}\rangle$ for which $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},3}$ is zero-dimensional and achieves the maximum degree of 144

over $\mathbb{Q}$, hence consists of 144 distinct, Galois conjugate points. The most common degree is 22, followed closely by 24, realized by 26,083 and 26,071 cubic portrait pairs respectively.

Remark When $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ is zero-dimensional, one can typically apply Bézout's theorem to the defining equations for $\widehat{\mathcal{M}}_{\mathcal{P},d}$ and $\widehat{\mathcal{M}}_{\mathcal{Q},d}$ to get an upper bound on the degree of $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$. (See the proof of Proposition 2.3 for the equations, and recall that to construct $\widehat{\mathcal{M}}$ from $\widehat{\mathrm{Conf}}$ one may set $q_1 = 0$ and $q_2 = 1$.) This bound depends on the portraits $\mathcal{P}$ and $\mathcal{Q}$, but in any case the upper bound coming from Bézout's theorem appears to be quite a bit higher than the degree of $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$. One reason for this seems to be that a significant number of common "realizations" q of $\mathcal{P}$ and $\mathcal{Q}$ are degenerate, in the sense that $q_i = q_j$ for some pair $i \neq j$.

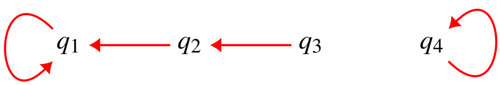
We also observe an apparent bias towards even degrees in Fig. 5, most easily seen in the interlaced spikes near and to the left of the mean. In fact, there are 28,911 more cubic portrait pairs for which $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},3}$ has even degree than odd degree. This bias also occurs in the quadratic case, although it is less pronounced given the smaller data set.

Question 3.6 Is it the case for all degrees $d \geq 2$ that if $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ has dimension 0, then the degree of $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ is more likely to be even? If so, why does this bias occur?

3.3.2 Maximum Dimension

There are 14 quadratic portrait pairs and 18 cubic portrait pairs $\langle\mathcal{P},\mathcal{Q}\rangle$ for which $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ achieves the largest possible dimension of $d-1$. The 14 quadratic portrait pairs are illustrated in Fig. 7. All but one of these 14 quadratic portrait pairs and all 18 of these cubic portrait pairs are examples of *two-image portraits with the same fiber partition* (see Sect. 3.5). In Theorem 3.20 we prove that $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ achieves the maximum dimension of $d-1$ for all such portrait pairs.

Example 3.7 The bottom rightmost portrait in Fig. 7 is the one example not explained by Theorem 3.20. In this case, both portraits have the combinatorial type of the portrait $\mathcal{P}$ shown below:



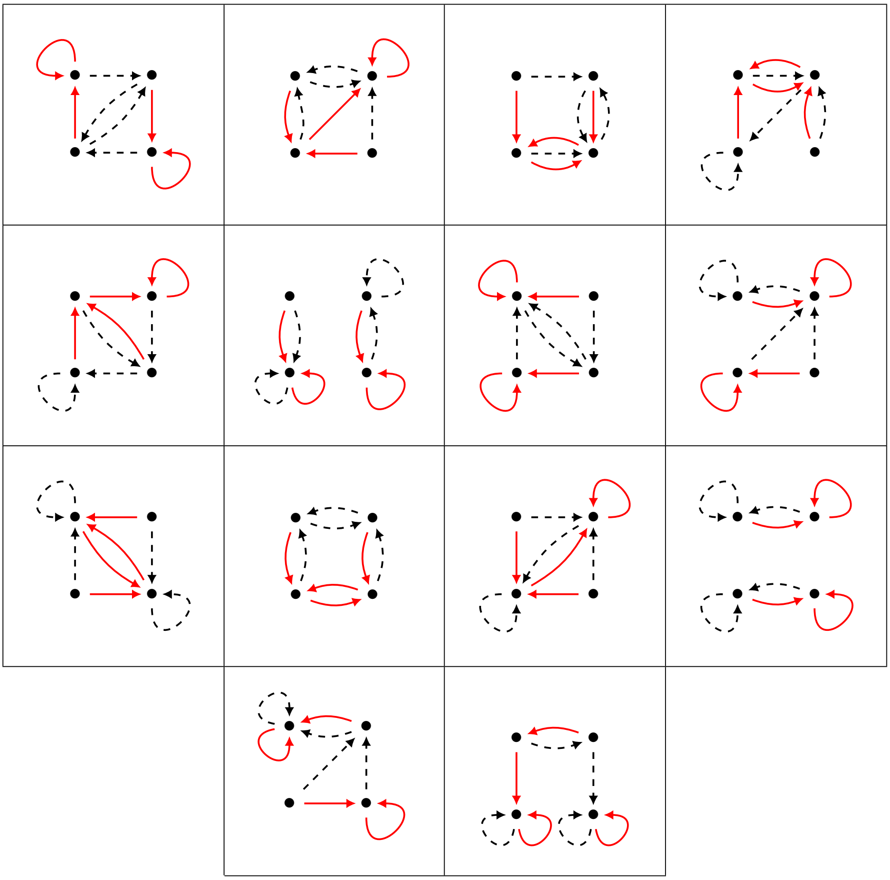


Fig. 7 Quadratic pairs with moduli spaces of maximal dimension. For each pair, we indicate one portrait with solid red arrows and the other with dashed black arrows (color figure online)

The proof of Proposition 2.3 implies that $\mathrm{Conf}_{\mathcal{P},2}$ is the hypersurface in Conf^4 defined by

$$q_2^2 - q_2q_3 + q_3^2 + q_1q_4 - q_1q_3 - q_2q_4 = 0.$$

Observe that $q_i \mapsto q_{5-i}$ is an automorphism of this hypersurface. Hence if (q_1, q_2, q_3, q_4) is a realization of $\mathcal{P}$, then so is (q_4, q_3, q_2, q_1). Thus, if σ is the permutation of $\{1, 2, 3, 4\}$ defined by $i \mapsto 5 - i$, then $\mathrm{Conf}_{\mathcal{P},2} = \mathrm{Conf}_{\mathcal{P}^\sigma,2}$ and consequently $\mathcal{M}_{\mathcal{P},\mathcal{P}^\sigma,2} = \mathcal{M}_{\mathcal{P},2}$ has dimension 1. Note that $\mathcal{P}$ is not admissible in degree less than 2, hence $\widehat{\mathcal{M}}_{\mathcal{P},2} = \mathcal{M}_{\mathcal{P},2}$.

3.3.3 Impossible Portraits

We say a degree-d admissible equivalence class $\langle \mathcal{P}, \mathcal{Q} \rangle$ is *impossible in degree d* (resp., *impossible in degree at most d*) if $\mathcal{M}_{\mathcal{P},\mathcal{Q},d} = \emptyset$ (resp., $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d} = \emptyset$).

Fig. 8 Two quadratic portrait pairs

Example 3.8 The quadratic portrait pair $\langle \mathcal{P}, \mathcal{Q} \rangle$ on the left of Fig. 8 is impossible in degree 2 but has realizations in degree 1; for example, $q = (1, i, -1, -i)$ realizes $\mathcal{P}$ and $\mathcal{Q}$ via the degree-1 polynomials $f(x) = ix$ and $g(x) = -ix$, respectively. The quadratic portrait pair on the right is impossible in degree at most 2.

There are 198 quadratic portrait pairs and 52,310 cubic portrait pairs which are impossible in degree at most 2 and 3, respectively. We now turn to the following natural problem:

Obstruction Problem 3.9 Determine whether an admissible degree-d combinatorial class $\langle \mathcal{P}, \mathcal{Q} \rangle$ is impossible based on the combinatorics of the portraits $\mathcal{P}$ and $\mathcal{Q}$.

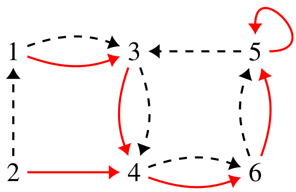
The simplest obstruction stems from Lagrange interpolation.

Proposition 3.10 (Interpolation Obstruction) *Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are admissible degree-d portraits on $[n]$. If $\mathcal{P}(i) = \mathcal{Q}(i)$ for at least $d + 1$ elements $i \in [n]$ but $\mathcal{P} \neq \mathcal{Q}$, then $\langle \mathcal{P}, \mathcal{Q} \rangle$ is impossible in degree at most d.*

Proof Lagrange interpolation implies that a degree-at-most-d polynomial is uniquely determined by its values on $d+1$ distinct points. If $f(x)$ and $g(x)$ are degree-at-most-d polynomials realizing $\mathcal{P}$ and $\mathcal{Q}$, respectively, on a configuration q, then our assumption implies that $f(q_i) = g(q_i)$ for at least $d + 1$ distinct i, hence that $f(x) = g(x)$ as polynomials. But the assumption that $\mathcal{P} \neq \mathcal{Q}$ implies that $f(q_j) \neq g(q_j)$ for some j, a contradiction. Hence $\langle \mathcal{P}, \mathcal{Q} \rangle$ is impossible in degree at most d. □

There are 39 quadratic portrait pairs and 12,773 cubic portrait pairs obstructed by Proposition 3.10; approximately 20% and 24% of the impossible portraits in degree 2 and 3, respectively.

Example 3.11 The following pair of admissible degree-3 portraits is obstructed by Proposition 3.10 since the functions agree on $\{1, 3, 4, 6\}$ but disagree on $\{2, 5\}$.



The obstruction problem is discussed further in Sects. 3.4 and 3.5.

3.4 Left Associate Realizations

In this section, we introduce the restrictive notion of left associate polynomials, show how left associate realizations of portrait pairs may be detected combinatorially, and demonstrate their relation to deviations in the dimension of $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ from the generic expectation.

Definition 3.12 Polynomials $f(x)$ and $g(x)$ are said to be *left associates* if there exists a linear polynomial $\ell(x)$ such that $f(x) = \ell(g(x))$.

Left association is a degree-respecting equivalence relation. One immediate and essential property of left associate polynomials is that they have the same fiber partition as self-maps of the affine line.

Definition 3.13 If $F : X \to X$ is an self-map of a set X, then the *fiber partition* of F is the set partition Π_F of X defined by

$$\Pi_F := \{F^{-1}(x) : x \in X\}.$$

There are several ways to detect combinatorially when realizations of a portrait pair must be left associates. The most robust method, especially in low degree, is via common collisions.

Definition 3.14 A *collision* for a function $F : X \to Y$ is a pair $x_1, x_2 \in X$ of distinct elements such that $F(x_1) = F(x_2)$.

We now restate Theorem 1.7 using this terminology.

Theorem 3.15 *Let $\mathcal{P}$ and $\mathcal{Q}$ be admissible degree-d portraits on n points such that at least one of the following conditions holds:*

(1) *$d = 2$ and $\mathcal{P}$, $\mathcal{Q}$ have a common collision,*
(2) *$d = 3$ and $\mathcal{P}$, $\mathcal{Q}$ have two common collisions, or*
(3) *d is arbitrary and the fiber partitions $\Pi_{\mathcal{P}}$, $\Pi_{\mathcal{Q}}$ share a set with d elements.*

If q is a degree-at-most-d realization of both $\mathcal{P}$ and $\mathcal{Q}$ via the polynomials f and g, respectively, then f and g are left associates.

Remark Note that condition (1) in the statement of Theorem 3.15 is a special case of condition (3); however, we state (1) separately to emphasize the theme of portraits having common collisions.

We first establish a lemma which provides a general test for left associates. Given a polynomial $f(x)$, let $\delta f(x, y)$ be the two-variable symmetric polynomial defined by

$$\delta f(x, y) := \frac{f(x) - f(y)}{x - y}.$$

For each integer $d \geq 1$, let $\rho_d : \mathbb{C}^2 \to \mathbb{C}^d$ be the function defined by

$$\rho_d(x, y) := \left(\frac{x - y}{x - y}, \frac{x^2 - y^2}{x - y}, \ldots, \frac{x^d - y^d}{x - y}\right),$$

where $\frac{x^k - y^k}{x-y}$ should be interpreted as $x^{k-1} + x^{k-2}y + \ldots + y^{k-1}$ so that ρ_d is defined on all of $\mathbb{C}^2$.

Lemma 3.16 *Let $f(x), g(x) \in \mathbb{C}[x]$ be degree-at-most-d polynomials. Suppose that $\{x_i, y_i\}$ for $1 \leq i < d$ are $d-1$ pairs of distinct points in $\mathbb{C}$ such that*

(1) *$\delta f(x_i, y_i) = \delta g(x_i, y_i) = 0$ for each i, and*
(2) *the $d-1$ vectors $\rho_d(x_i, y_i)$ with $1 \leq i < d$ are linearly independent.*

Then f and g are left associates.

Proof If $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d$, then

$$\delta f(x, y) = \frac{f(x) - f(y)}{x - y} = a_1 \frac{x - y}{x - y} + a_2 \frac{x^2 - y^2}{x - y} + \cdots + a_d \frac{x^d - y^d}{x - y}.$$

Hence each point (x_i, y_i) on the curve $\delta f(x, y) = 0$ imposes a linear condition on the vector of coefficients $(a_1, a_2, \ldots, a_d)$. From the definition of $\rho_d(x, y)$ and the assumed independence of $\rho_d(x_i, y_i)$ for $1 \leq i < d$, it follows that the points (x_i, y_i) determine the vector $(a_1, a_2, \ldots, a_d)$ up to a scalar multiple. Thus if $g(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_d x^d$, then there is some nonzero scalar c such that

$$(a_1, a_2, \ldots, a_d) = c(b_1, b_2, \ldots, b_d).$$

Let $\ell(x) = cx + (a_0 - cb_0)$. Then $f(x) = \ell(g(x))$, as we wished to show. □

Now we turn to the proof of Theorem 3.15.

Proof of Theorem 3.15 Suppose that $f(x)$ and $g(x)$ are degree-at-most-d polynomials realizing $\langle \mathcal{P}, \mathcal{Q} \rangle$ on some configuration q. We show that in each of the three listed cases, $f(x)$ and $g(x)$ are left associates. As we remarked following the statement of Theorem 3.15, condition (1) is a special case of (3), so we omit the case of condition (1).

Assume that condition (2) holds. First, suppose that $q_1, q_2, q_3, q_4 \in \mathbb{A}^1$ are points such that $\rho_3(q_1, q_2) = c\rho_3(q_3, q_4)$ for some scalar c. Comparing first components, we see that $c = 1$, hence

$$q_1 + q_2 = q_3 + q_4$$
$$q_1^2 + q_1 q_2 + q_2^2 = q_3^2 + q_3 q_4 + q_4^2.$$

Since $x^2 + xy + y^2 = (x + y)^2 - xy$, it follows that $q_1 q_2 = q_3 q_4$. The first two elementary symmetric functions of a pair of numbers uniquely determines the pair as a set, hence $\{q_1, q_2\} = \{q_3, q_4\}$. Therefore if $\{q_1, q_2\} \neq \{q_3, q_4\}$ are common collisions for cubic polynomials $f(x)$ and $g(x)$, then $\rho_3(q_1, q_2)$ and $\rho_3(q_3, q_4)$ are linearly independent and thus $f(x)$ is left associate to $g(x)$ by Lemma 3.16.

We now assume that (3) holds. If $\Pi_{\mathcal{P}}$ and $\Pi_{\mathcal{Q}}$ share a part with d elements, then $f^{-1}(q_i) = g^{-1}(q_j)$ for some i and j. Thus $f(x) - q_i$ and $g(x) - q_j$ have the same roots with multiplicity. Unique factorization implies the existence of some $c \neq 0$ such that $f(x)-q_i = c(g(x)-q_j)$. If we let $\ell(x) := cx+(q_i-cq_j)$, then $f(x) = \ell(g(x))$. □

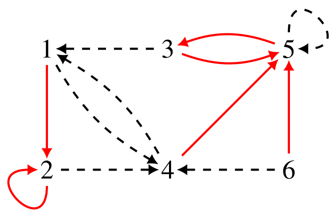
We apply Theorem 3.15 in Sect. 3.4.1 to identify several obstructions to the realizations of quadratic and cubic portraits, and in Sect. 3.4.2 we show how Theorem 3.15 leads to a conditional dimension heuristic that explains most of the unexpectedly large dimensions of $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ for degrees $d = 2, 3$.

3.4.1 Obstructions

Suppose $\{\mathcal{P}, \mathcal{Q}\}$ is a pair of admissible degree-d portraits with distinct fiber partitions $\Pi_{\mathcal{P}} \neq \Pi_{\mathcal{Q}}$ and which satisfies any of the conditions in Theorem 3.15. We claim that $\langle \mathcal{P}, \mathcal{Q} \rangle$ is an impossible portrait pair. Indeed, if $f(x)$, $g(x)$ is a pair of degree-at-most-d polynomials realizing $\mathcal{P}$ and $\mathcal{Q}$ on some configuration q, then Theorem 3.15 tells us that f and g are left associates, and it follows that $f(x)$ and $g(x)$ must have the same fiber partition. In particular, this implies that $\mathcal{P}$ and $\mathcal{Q}$ must have the same fiber partition, a contradiction. Hence, $\langle \mathcal{P}, \mathcal{Q} \rangle$ is impossible in degree at most d. We call this the *collision obstruction*.

The collision obstruction accounts for 133 (approximately 67%) of the 198 quadratic portrait pairs which are impossible in degree at most 2 and accounts for 39,519 (approximately 75.5%) of the 52310 cubic portrait pairs which are impossible in degree at most 3. This is the most common known obstruction in degrees 2 and 3.

Example 3.17 The following cubic portrait pair $\langle \mathcal{P}, \mathcal{Q} \rangle$ illustrates the collision obstruction. Let $\mathcal{P}$ be the solid red portrait and let $\mathcal{Q}$ be the dashed black portrait.



Then $\{1, 2\}$ and $\{3, 4\}$ are common collisions for $\mathcal{P}$ and $\mathcal{Q}$, but

$$\Pi_{\mathcal{P}} = \{\{1, 2\}, \{3, 4, 6\}, \{5\}\} \neq \{\{1, 2, 6\}, \{3, 4\}, \{5\}\} = \Pi_{\mathcal{Q}}.$$

Hence, Theorem 3.15 implies that $\langle \mathcal{P}, \mathcal{Q} \rangle$ is impossible in degree at most 3.

In Sect. 3.5 we discuss one more obstruction related to the special family of two-image portraits.

Table 5 For $2 \leq m \leq 4$ and $-1 \leq n \leq 2$, the number of cubic portrait pairs $\langle \mathcal{P}, \mathcal{Q} \rangle$ with at least 2 collisions and m image points satisfying $\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},3} = n$

	$\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},3}$			
m \ n	−1	0	1	2
2	0	0	0	**18**
3	98	0	**818**	0
4	1536	**4421**	73	0

The bold entries denote the expected dimension

3.4.2 Conditional Dimension Heuristic

Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are admissible degree-d portraits on $n > d$ points with the same fiber partition for which we know that any pair of degree-at-most-d polynomials realizing $\mathcal{P}$ and $\mathcal{Q}$, respectively, must be left associates. In this situation, a different dimension heuristic for $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ applies.

The data of a pair of degree-at-most-d polynomials $f(x)$ and $g(x)$ realizing $\mathcal{P}$ and $\mathcal{Q}$ are, in this case, equivalent to the data of a degree-at-most-d polynomial $f(x)$ and a linear polynomial $\ell(x)$ such that $g(x) = \ell(f(x))$. Let m be the number of points in the image of $\mathcal{P}$, which is necessarily the same as that of $\mathcal{Q}$. Thus, a point in $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ is specified by n parameters subject to $n - d - 1$ constraints from interpolating $f(x)$ in degree d and $m - 2$ constraints from interpolating $\ell(x)$ in degree 1, modulo the action of the two-dimensional group Aff_1. Hence, the expected dimension of $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d}$ is

$$n - (n - d - 1) - (m - 2) - 2 = d - m + 1.$$

Example 3.18 Suppose that $\langle \mathcal{P}, \mathcal{Q} \rangle$ is a cubic portrait pair with at least 2 common collisions and the same fiber partition. Then Theorem 3.15 implies that any pair of realizations of $\langle \mathcal{P}, \mathcal{Q} \rangle$ must be left associates. Thus we expect the dimension of $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},3}$ to be $4 - m$. There are 6964 cubic portrait pairs satisfying these conditions; Table 5 shows the number of such cubic portrait pairs with a given dimension and image size m. The bold entries denote the expected dimension.

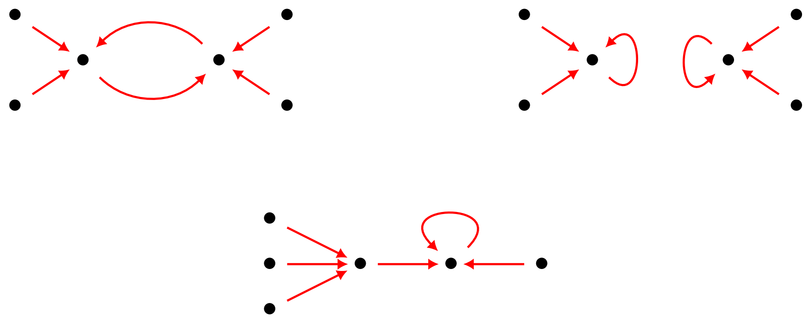
Comparing Tables 3 and 5, we see that the conditional dimension heuristic accounts for the majority of cubic portrait pairs for which $\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},3} = 1$ and all of those for which $\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},3} = 2$.

3.5 Two-Image Portraits

In this section, we study another exceptional family of portraits.

Definition 3.19 A *two-image portrait* $\mathcal{P}$ is an admissible degree-d portrait on $2d$ points such that $|\mathcal{P}([2d])| = 2$. That is, $\Pi_{\mathcal{P}}$ consists of two sets with d elements each.

There are three combinatorial types of two-image portraits $\mathcal{P}$ in each degree d, determined by their action on the points in the image of $\mathcal{P}$. Examples of these three types in the degree 3 case are illustrated below.



Theorem 3.20 shows that the degree-d realization space of a two-image portrait $\mathcal{P}$ depends only on the fiber partition $\Pi_{\mathcal{P}}$ of $\mathcal{P}$. As we will see, the special properties of *two*-image portraits all stem from the fact that Aff_1 acts sharply *two*-transitively on the affine line. Note that if h is a symmetric function and q is a multiset of complex numbers, then $h(q)$ has a well-defined value.

Theorem 3.20 *Let $d \geq 1$ and let $\Pi = \{A, B\}$ be a partition of $[2d]$ into two sets with d elements.*

(1) *If $\mathcal{P}$ and $\mathcal{Q}$ are two-image portraits on $[2d]$ with the same fiber partition $\Pi_{\mathcal{P}} = \Pi_{\mathcal{Q}} = \Pi$, then*

$$\widehat{\mathrm{Conf}}_{\mathcal{P},d} = \mathrm{Conf}_{\mathcal{P},d} = \mathrm{Conf}_{\mathcal{Q},d} = \widehat{\mathrm{Conf}}_{\mathcal{Q},d}.$$

Hence, $\mathrm{Conf}_{\mathcal{P},d}$ depends only on the partition Π, and we let $\mathrm{Conf}_{\Pi} := \mathrm{Conf}_{\mathcal{P},d} = \mathrm{Conf}_{\mathcal{Q},d}$ denote this common subspace of Conf^{2d}.

(2) *Conf_{Π} is the $(d+1)$-dimensional Zariski closed subspace of Conf^{2d} defined by the equations*

$$e_k(x_A) = e_k(x_B)$$

for $1 \leq k < d$, where e_k is the kth elementary symmetric function in d variables, and x_A, x_B are the subsets of the set of indeterminates $\{x_1, x_2, \ldots, x_{2d}\}$ indexed by the elements of A and B, respectively.

(3) *For each configuration $q \in \mathrm{Conf}_{\Pi}$, there exist at least $2d(2d-1)$ distinct degree-d polynomials $f(x)$ such that $f(q) \subseteq q$.*

Proof (1) First note that since $\mathcal{P}$ contains a fiber with d elements, $\mathcal{P}$ is not admissible for any degree less than d. Hence $\widehat{\mathrm{Conf}}_{\mathcal{P},d} = \mathrm{Conf}_{\mathcal{P},d}$. Now suppose that $q \in \mathrm{Conf}_{\mathcal{P},d}$, and let $f(x)$ be the degree-d polynomial realizing $\mathcal{P}$. Without loss of generality, suppose that $\mathrm{im}(\mathcal{P}) = \{1, 2\}$ and $\mathrm{im}(\mathcal{Q}) = \{i, j\}$. Furthermore, since $\Pi_{\mathcal{P}} = \Pi_{\mathcal{Q}} = \{A, B\}$ we may suppose that $\mathcal{P}^{-1}(1) = \mathcal{Q}^{-1}(i) = A$ and $\mathcal{P}^{-1}(2) = \mathcal{Q}^{-1}(j) = B$. Let $\ell(x)$ be the unique linear polynomial such that $\ell(q_1) = q_i$ and $\ell(q_2) = q_j$. Then, $g := \ell \circ f$ is a degree-d polynomial such that for each $a \in A$,

$$g(q_a) = \ell(f(q_a)) = \ell(q_1) = q_i,$$

and similarly, $g(q_b) = q_j$ for each $b \in B$. Hence $g(x)$ realizes $\mathcal{Q}$ on the configuration q, which is to say that $q \in \mathrm{Conf}_{\mathcal{Q},d}$. Thus, $\mathrm{Conf}_{\mathcal{P},d} \subseteq \mathrm{Conf}_{\mathcal{Q},d}$, and the reverse inclusion follows by symmetry.

(2) Let $\mathcal{P} : [2d] \to [2d]$ be a portrait with fiber partition $\Pi = \{A, B\}$, and let $i := \mathcal{P}(A)$ and $j := \mathcal{P}(B)$. By construction, $\mathcal{P}$ is admissible in degree d. If $q \in \mathrm{Conf}_\Pi = \mathrm{Conf}_{\mathcal{P},d}$, and f is a degree-d polynomial realizing $\mathcal{P}$, then $f(x) - q_i$ and $f(x) - q_j$ vanish at the elements of q_A and q_B respectively. Comparing coefficients, we conclude that $e_k(q_A) = e_k(q_B)$ for all $1 \le k < d$.

Conversely, if $q = (q_A, q_B)$ is a configuration such that $e_k(q_A) = e_k(q_B)$ for $1 \le k < d$, then the polynomial

$$f(x) := x^d - e_1(q_A)x^{d-1} + e_2(q_A)x^{d-2} + \ldots + (-1)^{d-1}e_{d-1}(q_A)x$$

is constant on each of q_A and q_B. Let a and b be the images of q_A and q_B under f, respectively, and let ℓ be the unique linear polynomial such that $\ell(a) = q_i$ and $\ell(b) = q_j$. Then $\ell \circ f$ maps q_A to q_i and q_B to q_j, thus q is in $\mathrm{Conf}_{\mathcal{P},d} = \mathrm{Conf}_\Pi$. Therefore, Conf_Π is equal to the space defined by the equations $e_k(x_A) = e_k(x_B)$ for $1 \le k < d$.

Given $q := (q_A, q_B) \in \mathrm{Conf}_\Pi$, let $f(x) = b_d x^d + \ldots + b_1 x + b_0$ be the unique degree-d polynomial such that $f(q_A) = 0$ and $f(q_B) = 1$. (The first condition determines the roots, hence determines the coefficients up to scaling; the second condition determines the leading coefficient.) Now consider the map $\iota : \mathrm{Conf}_\Pi \to \mathbb{C}^{d+1}$ defined by $\iota(q) = (b_0, b_1, \ldots, b_d)$. Lagrange interpolation implies that ι is a well-defined finite-to-one morphism onto a dense open subset of $\mathbb{C}^{d+1}$ corresponding to all degree-d polynomials for which 0 and 1 are not critical values. Therefore, $\dim \mathrm{Conf}_\Pi = d+1$.

(3) Given the partition Π, a two-image portrait $\mathcal{P}$ with fiber partition Π is determined by choosing the ordered pair of values $(\mathcal{P}(A), \mathcal{P}(B))$. Hence there are $2d(2d-1)$ such portraits. Part (1) shows that $\mathrm{Conf}_{\mathcal{P},d} = \mathrm{Conf}_\Pi$ for any such portrait $\mathcal{P}$. Therefore, given a configuration $q \in \mathrm{Conf}_\Pi$, there exist at least $2d(2d-1)$ distinct degree-d polynomials $f(x)$ such that $f(q) \subseteq q$. □

Remark The proof of part (1) of Theorem 3.20 generalizes directly to rational functions and three-image portraits on $3d$ points: For a portrait $\mathcal{P} : [n] \to [n]$, let $\mathrm{Conf}_{\mathcal{P},d}(\widehat{\mathbb{C}})$ denote the space of all configurations q of n distinct points on the projective line $\widehat{\mathbb{C}} := \mathbb{P}^1(\mathbb{C})$ for which there is a degree-d rational function f realizing the portrait $\mathcal{P}$ on q. Then, if $\mathcal{P}$, $\mathcal{Q}$ are three-image portraits on $3d$ points with the same fiber partition, we have that

$$\mathrm{Conf}_{\mathcal{P},d}(\widehat{\mathbb{C}}) = \mathrm{Conf}_{\mathcal{Q},d}(\widehat{\mathbb{C}}).$$

The main point is that the automorphism group of the affine line is sharply 2-transitive, whereas the automorphism group of the projective line is sharply 3-transitive.

Continuing with the notation of Theorem 3.20, let $\mathcal{M}_{\Pi,d}$ be the space defined by

$$\mathcal{M}_{\Pi,d} := \mathrm{Conf}_{\Pi,d}/\mathrm{Aff}_1.$$

The following corollary is an immediate consequence of Theorem 3.20.

Corollary 3.21 *Let $\mathcal{P}$ and $\mathcal{Q}$ be two-image portraits on* $[2d]$ *with the same fiber partition. Then*

$$\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},d} = \mathcal{M}_{\mathcal{P},\mathcal{Q},d} = \mathcal{M}_{\Pi,d} \quad \textit{and} \quad \dim \mathcal{M}_{\Pi,d} = d - 1.$$

Since any two partitions of $[2d]$ into two d-element sets are related by a permutation, it follows from Proposition 2.4 and Corollary 3.21 that the isomorphism class of $\mathcal{M}_{\Pi,d}$ depends only on the degree d and not the partition Π.

Example 3.22 If $d = 2$ and $\Pi := \{\{1, 2\}, \{3, 4\}\}$, then Theorem 3.20 (2) implies that

$$\mathcal{M}_{\Pi,2} = \left\{(0, 1, q_3, q_4) \in \mathrm{Conf}^4 : q_3 + q_4 = 1\right\},$$

so $\mathcal{M}_{\Pi,2}$ is a line minus the points at which 0, 1, q_3, and q_4 are not pairwise distinct. In particular, $\mathcal{M}_{\Pi,2}$ has infinitely many rational points. Of the 14 quadratic portrait pairs $\langle\mathcal{P}, \mathcal{Q}\rangle$ with $\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},2} = 1$, all but one is isomorphic to $\mathcal{M}_{\Pi,2}$ (see Example 3.7).

Example 3.23 If $d = 3$ and $\Pi := \{\{1, 2, 3\}, \{4, 5, 6\}\}$, then Theorem 3.20 (2) tells us that

$$\mathcal{M}_{\Pi,3} = \Big\{(0, 1, q_3, q_4, q_5, q_6) \in \mathrm{Conf}^6 : 1 + q_3 = q_4 + q_5 + q_6 \quad \text{and } q_3 = q_4q_5 + q_4q_6 + q_5q_6\Big\}. \tag{3.1}$$

Since (3.1) defines a quadric surface, $\mathcal{M}_{\Pi,3}$ is birational over $\overline{\mathbb{Q}}$ to $\mathbb{P}^2$. Furthermore, since this model of $\mathcal{M}_{\Pi,3}$ has a $\mathbb{Q}$-rational point—take $(q_1, q_2, q_3, q_4, q_5, q_6) = (0, 1, -4, -2, 2, -3)$, for example—it follows that $\mathcal{M}_{\Pi,3}$ is birational over $\mathbb{Q}$ to $\mathbb{P}^2$. In particular, $\mathcal{M}_{\Pi,3}$ has infinitely many rational points.

For all 18 of the cubic portrait pairs $\langle\mathcal{P}, \mathcal{Q}\rangle$ with $\dim \widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},3} = 2$, the space $\widehat{\mathcal{M}}_{\mathcal{P},\mathcal{Q},3}$ is isomorphic to the surface $\mathcal{M}_{\Pi,3}$.

Two-image portraits exhibit a remarkable rigidity: Theorem 3.20 tells us that two-image portraits $\mathcal{P}$ and $\mathcal{Q}$ with the same fiber partition have identical realization spaces. On the other hand, we now show that two-image portraits with different fiber partitions have disjoint realization spaces.

Proposition 3.24 (Two-Image Obstruction) *Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are two-image portraits on* $2d$ *points. If $\Pi_{\mathcal{P}} \neq \Pi_{\mathcal{Q}}$, then $\langle\mathcal{P}, \mathcal{Q}\rangle$ is impossible in degree at most d.*

Remark Note that Theorem 3.20 and Proposition 3.24 combine to give Theorem 1.5.

The two-image obstruction is relatively rare: there are 24 quadratic portrait pairs and 91 cubic portrait pairs obstructed by Proposition 3.24. We deduce Proposition 3.24 from the following result.

Theorem 3.25 *Let K be an algebraically closed field, and suppose that $f(x), g(x) \in K[x]$ are polynomials such that for some distinct $a, b \in K$*

$$f^{-1}(\{a, b\}) = g^{-1}(\{a, b\}), \tag{3.2}$$

as sets with multiplicity. If either $\mathrm{char}(K) \neq 2$ *or f and g have odd degree, then $f(x) = g(x)$ or $f(x) = a + b - g(x)$.*

Remark We thank the anonymous referee for suggesting the more general hypotheses and the following argument.

Proof of Theorem 3.25 Note that our hypothesis (3.2) implies that f and g must have the same degree d, and we may assume that $d \geq 1$ as the conclusion is immediate for f and g constant.

First suppose that $\mathrm{char}(K) \neq 2$. Then, there exists a linear polynomial $\ell(x) \in K[x]$ such that $\ell(a) = 1$ and $\ell(b) = -1$, as $\mathrm{char}(K) \neq 2$ implies that ± 1 are distinct in K. Replacing f and g by $\ell \circ f$ and $\ell \circ g$, it suffices to show that $f(x) = \pm g(x)$. Our hypothesis implies that

$$f(x)^2 - 1 = c^2(g(x)^2 - 1)$$

for some $c \in K$. Rearranging this identity, we have

$$1 - c^2 = f(x)^2 - c^2 g(x)^2 = (f(x) - cg(x))(f(x) + cg(x)).$$

Unique factorization in $K[x]$ implies that $c^2 = 1$, whence $f(x) = \pm g(x)$.

Now suppose that K is an arbitrary field and that f and g have odd degree d. Let

$$r_a := |f^{-1}(a) \cap g^{-1}(a)|, \quad r_b := |f^{-1}(a) \cap g^{-1}(b)|,$$
$$s_a := |f^{-1}(b) \cap g^{-1}(a)|, \quad s_b := |f^{-1}(b) \cap g^{-1}(b)|,$$

where the intersections are considered as sets with multiplicity. By hypothesis we have

$$d = r_a + r_b = s_a + s_b = r_a + s_a = r_b + s_b.$$

After possibly replacing $g(x)$ by $a + b - g(x)$, we may assume that $r_a > d/2$. Here, we use the assumption that d is odd to eliminate the possibility that $r_a = d/2$. Thus, $s_a < d/2$, which in turn implies that $s_b > d/2$. Therefore, $f(x)$ and $g(x)$ are degree-d polynomials which agree on $r_a + s_b > d$ points with multiplicity, hence $f(x) = g(x)$. □

Remark A *unique range set* for a family $\mathcal{F}$ of meromorphic functions is a set S such that if $f^{-1}(S) = g^{-1}(S)$ as sets with multiplicity for $f, g \in \mathcal{F}$, then $f = g$. Theorem 3.25 implies that $\{a, b\}$ is as close to being a unique range set for the family $\mathcal{F} = \mathbb{C}[x]$ as a set with two elements can be. See Chen [3, Question 1.2] and the discussion that follows for a survey of results related to unique range sets.

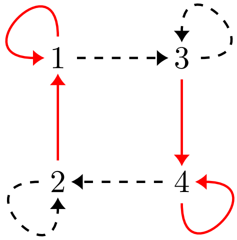


Fig. 9 A pair $\{\mathcal{P}, \mathcal{Q}\}$ of two-image portraits with degree-at-most-2 realizations in characteristic 2 but not in characteristic 0. The portraits $\mathcal{P}$ and $\mathcal{Q}$ are indicated by solid red and dashed black arrows, respectively (color figure online)

Remark Theorem 3.25 may fail in even characteristic if f and g have even degree. Let $\mathbb{F}_4 = \mathbb{F}_2(\omega)$ where $\omega^2 + \omega + 1 = 0$. Then, the polynomials $f(x) = x^2 + x$ and $g(x) = \omega x^2 + (\omega + 1)x$ in $\mathbb{F}_4[x]$ satisfy the condition

$$f^{-1}(\{0, 1\}) = g^{-1}(\{0, 1\}),$$

but clearly $f(x)$ is not equal to $g(x)$ or $1 - g(x)$.

Proof of Proposition 3.24 We prove the contrapositive. Suppose that $q \in \widehat{\mathrm{Conf}}_{\mathcal{P},d} \cap \widehat{\mathrm{Conf}}_{\mathcal{Q},d}$, and let $f(x), g(x) \in \mathbb{C}[x]$ be degree-at-most-d polynomials realizing $\mathcal{P}$ and $\mathcal{Q}$ on q. Since $\mathcal{P}$ and $\mathcal{Q}$ are two-image portraits, we have $f(q) = g(q) = \{a, b\}$ for some distinct $a, b \in \mathbb{C}$. Therefore, $f(x) = g(x)$ or $f(x) = a + b - g(x)$ by Theorem 3.25. In either case, f is left associate to g. Thus f and g have the same fiber partition on $\mathbb{C}$; in particular, $\Pi_{\mathcal{P}} = \Pi_{\mathcal{Q}}$. □

Remark Unlike Theorem 3.20, Proposition 3.24 cannot be directly generalized to rational realizations of three-image portraits. Consider the family of rational functions $f_t(x)$ with $t \in \mathbb{C}$ defined by

$$f_t(x) = -\frac{x^2 - tx}{tx - 1}.$$

If $t \neq \pm 1$, then f_t is a degree-2 rational map with fixed points at 0, 1, and ∞. Each fixed point has an additional preimage: the non-fixed preimages of 0, 1, and ∞ are t, -1, and $1/t$, respectively. Thus, if $t \notin \{0, \pm 1\}$, then f_t is an endomorphism of the configuration of six distinct points given by

$$q_t := (0, 1, \infty, t, -1, 1/t) \in \mathrm{Conf}^6,$$

with fiber partition $\Pi_{f_t} = \big\{\{0, t\}, \{-1, 1\}, \{\infty, 1/t\}\big\}$. If $t \neq \pm 1$, then $q_{1/t}$ and q_t are different as configurations but equal as sets; hence $f_{1/t}$ is a degree-2 rational endomorphism of q_t with a fiber partition different from that of f_t. Thus if $\mathcal{P}$ and $\mathcal{Q}$ are the three-image portraits corresponding to the action of f_t and $f_{1/t}$ on q_t, then $\mathcal{P}$ and $\mathcal{Q}$ have distinct fiber partitions, yet

$$q_t \in \mathrm{Conf}_{\mathcal{P},2}(\widehat{\mathbb{C}}) \cap \mathrm{Conf}_{\mathcal{Q},2}(\widehat{\mathbb{C}}).$$

4 Endomorphism Semigroups of Roots of Unity

Recall from the introduction that $\mathrm{End}(q)$ is the semigroup of polynomial endomorphisms of a configuration $q \in \mathrm{Conf}^n$; that is, polynomials $f(x)$ such that $f(q) \subseteq q$. If $d \geq 0$, then $\mathrm{End}_d(q)$ is the degree-d graded component of $\mathrm{End}(q)$. In Question 1.2 we asked for the maximum cardinality $E_{n,d}$ of $\mathrm{End}_d(q)$ as q varies over Conf^n and $0 \leq d < n-1$. Theorem 3.20(3) gives us the first nontrivial lower bound of

$$E_{2d,d} \geq 2d(2d-1)$$

coming from any configuration supporting a two-image portrait. Our survey of all quadratic pairs of portraits on 4 points gives the following determination of $E_{4,2}$.

Proposition 4.1 *The maximum cardinality of* $\mathrm{End}_2(q)$ *as* q *varies over* Conf^4 *is* $E_{4,2} = 28$. *This cardinality is achieved by a unique affine equivalence class of configurations, namely the class represented by* $q = \mu_4 := (1, i, -1, -i)$, *the 4th roots of unity.*

Proof Let $q \in \mathrm{Conf}^4$ be a configuration with $m := |\mathrm{End}_2(q)|$ quadratic endomorphisms. Then there are portraits $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$ on [4] such that $q \in \bigcap_{i=1}^m \mathrm{Conf}_{\mathcal{P}_i,2}$. If m is sufficiently large, then $\left(\bigcap_{i=1}^m \mathrm{Conf}_{\mathcal{P}_i,2}\right)/\mathrm{Aff}_1$ is 0-dimensional, and a computation shows that $m > 12$ suffices. Specifically, there are finitely many portraits on [4], and we may check by exhaustion that if $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ are three distinct portraits for which $\left(\bigcap_{i=1}^3 \mathrm{Conf}_{\mathcal{P}_i,2}\right)/\mathrm{Aff}_1$ is one-dimensional, then the $\mathcal{P}_i$ are all two-image portraits with the same fiber partition; there are 12 admissible degree-2 two-image portraits on [4]. In this case, the affine equivalence class of q belongs to one of the finitely many portrait moduli spaces $\mathcal{M}_{\mathcal{P},\mathcal{Q},2}$ of dimension 0, and hence we have finitely many candidates for configurations q maximizing $|\mathrm{End}_2(q)|$. For each of these candidates q, one may compute the degree of the unique degree-at-most-2 realization (if it exists) of each of the 4^4 portraits on q. This procedure yields the configuration $q = \mu_4 = (1, i, -1, -i)$ as the unique affine equivalence class of complex configurations maximizing $|\mathrm{End}_2(q)|$, with a total of 28 quadratic endomorphisms, which we construct explicitly below. We note that the configuration $q = (0, 1, 1+\sqrt{2}, 2+\sqrt{2})$ comes in second place with 20 quadratic endomorphisms.[5]

One may check that the three polynomials

$$g_1(x) := x^2, \quad g_2(x) := \frac{i+1}{2}\left(x^2 + i\right), \quad \text{and} \quad g_3(x) := \frac{1}{2}\left(x^2 + (i-1)x + i\right)$$

are endomorphisms of μ_4. Furthermore, since ix is a linear automorphism of μ_4, each of the polynomials $i^j g_k(i^\ell x)$ is an endomorphism of μ_4. There are 28 such maps, hence these are all quadratic endomorphisms of μ_4. □

Remark Note that both $(1, i, -1, -i)$ and $(0, 1, 1+\sqrt{2}, 2+\sqrt{2})$ support two-image portraits, hence Theorem 3.20(3) delivers $2d(2d-1) = 12$ of their quadratic endomorphisms. Moreover, each of these configurations has nontrivial automorphisms: the

[5] Details of the calculations described here may be found in the Sage notebook https://github.com/tghyde/portrait_moduli_supplement.

configuration $(1, i, -1, -i)$ has the nontrivial automorphisms $i^k x$ for $1 \leq k \leq 3$, and $(0, 1, 1+\sqrt{2}, 2+\sqrt{2})$ has the nontrivial automorphism $-x+2+\sqrt{2}$.

In retrospect, it is clear that configurations with an exceptional number of linear automorphisms should have more than the average number of degree-d endomorphisms, since any one such endomorphism gives rise to more by pre- and post-composing with automorphisms as in Proposition 4.1. The automorphism group of any configuration is finite cyclic—typically trivial—so in the interest of finding configurations in $\mathrm{Conf}^n(\mathbb{C})$ with many low degree endomorphisms, a natural starting point is to consider the family μ_n of configurations of nth roots of unity in $\mathbb{C}$, since μ_n is the unique smallest configuration (up to affine conjugation) with an automorphism group of order n. Note that the endomorphism semigroup of q is independent of the ordering of the points in the configuration, hence the ordering of μ_n is immaterial.

By Lagrange interpolation, each of the n^n portraits on n points is realized on μ_n by some polynomial of degree at most $n-1$. In Table 2 we listed the cardinality of $\mathrm{End}_d(\mu_n)$ for $3 \leq n \leq 8$ and $0 \leq d \leq n-1$. Let $\zeta_n \in \mu_n$ be a primitive nth root of unity, and note that for each degree d, the n monomials $\zeta_n^k x^d$ belong to $\mathrm{End}_d(\mu_n)$. Table 2 suggests that these are the only degree-d endomorphisms of μ_n for $d < n/2$. We prove this in Theorem 4.2 below.

Theorem 4.2 *If $n > 2d \geq 1$, then the only degree-d polynomial endomorphisms of μ_n in $\mathbb{C}[x]$ are of the form $\zeta_n^k x^d$ for some $k \geq 0$.*

Proof Let $\mathcal{C}$ denote the unit circle in $\mathbb{C}$ and let $f(x) \in \mathbb{C}[x]$ be a degree-d polynomial. We claim that if there are more than $2d$ points $\zeta \in \mathcal{C}$ for which $f(\zeta) \in \mathcal{C}$, then $f(x) = \xi x^d$ for some $\xi \in \mathcal{C}$. The result then follows: if $f(\mu_n) \subseteq \mu_n$ with $n > 2d$, then $f(x) = \xi x^d$ and $\xi = f(1) \in f(\mu_n) \subseteq \mu_n$.

We now prove the claim. This is a special case of a result due to Cargo and Schneider [2, Theorem 1]; we recreate their argument for the reader's convenience.

Let $f(x) := b_d x^d + \ldots + b_1 x + b_0 \in \mathbb{C}[x]$ with $b_d \neq 0$ and let $\overline{f}(x)$ be the result of applying complex conjugation to the coefficients of $f(x)$. If $\zeta \in \mathcal{C}$ is a point on the unit circle such that $f(\zeta) \in \mathcal{C}$, then

$$1 = f(\zeta)\overline{f(\zeta)} = f(\zeta)\overline{f}(\zeta^{-1}).$$

Hence ζ is a root of the degree-$2d$ polynomial $x^d(f(x)\overline{f}(x^{-1}) - 1)$. Thus if $f(\mu_n) \subseteq \mu_n$ for $n > 2d$, it follows that

$$\begin{aligned} 1 &= f(x)\overline{f}(x^{-1}) \\ &= (b_d\overline{b_0})x^d + (b_d\overline{b_1} + b_{d-1}\overline{b_0})x^{d-1} + \ldots + b_d\overline{b_d} + \ldots + (b_0\overline{b_d})x^{-d}. \end{aligned}$$

Comparing coefficients we conclude that $f(x) = b_d x^d$ and $b_d\overline{b_d} = 1$, which is to say that $b_d \in \mathcal{C}$. □

Theorem 4.2 is sharp in the sense that for each $d \geq 2$, there exist degree-d endomorphisms of μ_{2d} that are not monomials. The polynomial x^d realizes a two-image portrait on μ_{2d}, since if $\zeta \in \mu_{2d}$, then $\zeta^d \in \{\pm 1\} \subseteq \mu_{2d}$. Thus, Theorem 3.20(3) implies that μ_{2d} has at least $2d(2d-1)$ degree-d endomorphisms. These may be explicitly constructed as $\ell(x^d)$ where $\ell(x)$ is any linear polynomial such that $\ell(\pm 1) \in \mu_{2d}$, and only $2d$ of these endomorphisms are monomials. Note that Table 2 implies that these two-image portrait realizations are the only cubic endomorphisms of μ_6, but far from the only degree-4 endomorphisms of μ_8.

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