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RESEARCH CONTRIBUTION

Properness of Polynomial Maps with Newton Polyhedra

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Abstract
We discuss the notion of properness of a polynomial map $\boldsymbol{f} : \mathbb{K}^m \to \mathbb{K}^n$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, at a point of the target. We present a method to describe the set of non-proper points of $\boldsymbol{f}$ with respect to Newton polyhedra of $\boldsymbol{f}$. We obtain an explicit precise description of such a set of $\boldsymbol{f}$ when $\boldsymbol{f}$ satisfies certain condition (1.5). A relative version is also given in Sect. 3. Several tricks to describe the set of non-proper points of $\boldsymbol{f}$ without the condition (1.5) is also given in Sect. 5.

Keywords Polynomial map · Proper map · Newton polyhedra

Mathematics Subject Classification 14E15 · 58K05 · 58K30

We consider a polynomial map $\boldsymbol{f} = (f^1, \ldots, f^n) : \mathbb{K}^m \to \mathbb{K}^n$, defined by

$$f^j = \sum_{\boldsymbol{\nu}} c^j_{\boldsymbol{\nu}} \boldsymbol{x}^{\boldsymbol{\nu}}, \ c^j_{\boldsymbol{\nu}} \in \mathbb{K}, \ \boldsymbol{x}^{\boldsymbol{\nu}} = (x_1)^{\nu_1} \cdots (x_m)^{\nu_m},$$
$$\boldsymbol{x} = (x_1, \ldots, x_m), \boldsymbol{\nu} = (\nu_1, \ldots, \nu_m), \qquad (0.1)$$

where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We say that a point $\boldsymbol{y}_0 \in \mathbb{K}^n$ is **proper** for $\boldsymbol{f}$ (or a **proper point** of $\boldsymbol{f}$) if, for any (algebraic) arc $\boldsymbol{x}(t) : \mathbb{K}^*, 0 \to \mathbb{K}^m$, $\mathbb{K}^* = \mathbb{K}\backslash\{0\}$, the following condition holds:

$$\lim_{t\to 0} \boldsymbol{f}(\boldsymbol{x}(t)) = \boldsymbol{y}_0 \implies \lim_{t\to 0} \boldsymbol{x}(t) \text{ exists in } \mathbb{K}^m.$$

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Let $S_{\boldsymbol{f}}$ denote the set of points $\boldsymbol{y}_0$ in $\mathbb{K}^n$ which are not proper points of $\boldsymbol{f}$. We say that $\boldsymbol{f} : \mathbb{K}^m \to \mathbb{K}^n$ is **proper** if $S_{\boldsymbol{f}} = \emptyset$.

In this paper, we are looking for a method to determine whether a point $\boldsymbol{y}_0$ in $\mathbb{K}^n$ is proper or not. The first statement is Theorem 1.3, which gives a complete description of $S_{\boldsymbol{f}}$ when $\boldsymbol{f}$ satisfies certain non-degeneracy condition with respect to the Newton polyhedron of $\boldsymbol{f}$ (see (1.5) in Theorem 1.3). Our approach is based on simple and careful analysis of $\boldsymbol{f}$ along arcs $\boldsymbol{x}(t)$, which suggests us usefulness of using arcs to describe the set $S_{\boldsymbol{f}}$, even though $\boldsymbol{f}$ is degenerate (Remark 2.6). In Sect. 3, we describe a relative version of our discussion. We present several examples to show how our method works in Sect. 4.

The set $S_{\boldsymbol{f}}$ was introduced by Jelonek [4], [5] and showed that it is empty or a uniruled hypersurface of $\mathbb{K}^n$ when $\mathbb{K} = \mathbb{C}$ and $m = n$. It is thus an interesting problem to seek a method to describe $S_{\boldsymbol{f}}$ in several concrete examples. Chen et al. [2] have investigated the bifurcation locus of a polynomial map $\mathbb{K}^m \to \mathbb{K}^n, m \geq n$, with respect to Newton polyhedron. The bifurcation locus is the minimal locus in the target where the map is not locally trivial, and they show a supset of the bifurcation locus under their non-degeneracy condition. Jelonek and Lasoń [6] called $S_{\boldsymbol{f}}$ as the non-properness set of $\boldsymbol{f}$ and showed that it is covered by parametric curves of degree at most $d - 1$ where d is the algebraic degree of $\boldsymbol{f}$ for $\mathbb{K} = \mathbb{C}$. Their words "covered by parametric curves" mean that the set $S_{\boldsymbol{f}}$ has a "$\mathbb{C}$-ruling". They also discuss real counterpart of their results. Recently, El Hilany [3] has investigated to describe the set $S_{\boldsymbol{f}}$ via the Newton polyhedra of $\boldsymbol{f}$. He calls $S_{\boldsymbol{f}}$ as Jelonek set. He has introduced the notion of T-BG maps and claimed that $S_{\boldsymbol{f}}$ is described using only the data of $\boldsymbol{f}$ at several faces of its Newton polyhedra. Comparing with these results, our method provides much precise information on the set $S_{\boldsymbol{f}}$ with simple description. For example, Theorem 1.3 shows an explicit decomposition of $S_{\boldsymbol{f}}$ providing an explicit ruling of each component in many cases. In Sect. 3, we present a relative version of our theorem. Namely, we consider the non-properness set $S_{\boldsymbol{f}|_X}$ for $\boldsymbol{f}|_X : X \to \mathbb{K}^n$ where $\boldsymbol{f} = (f^1, \dots, f^n) : \mathbb{K}^m \to \mathbb{K}^n$ is a certain polynomial map and $X = (f^{n-k+1}, \dots, f^n)^{-1}(c), c \in \mathbb{K}^k$. In Sect. 5, we present tricks to describe $S_{\boldsymbol{f}}$ for certain degenerate $\boldsymbol{f}$.

We say some words for the definition of $S_{\boldsymbol{f}}$ here. We compactify $\boldsymbol{f}$ as $\bar{\boldsymbol{f}} : X \to Y$ where X and Y are suitable projective manifolds. We set $X_\infty = X \backslash \mathbb{K}^m$, $Y_\infty = Y \backslash \mathbb{K}^n$ and we can assume that X_∞ and Y_∞ are simple normal crossing divisors. Then, the condition $\boldsymbol{y}_0 \in S_{\boldsymbol{f}}$ is equivalent to one of the following conditions.

- There exists an algebraic arc $\boldsymbol{x}(t) : \mathbb{K}, 0 \to X$, such that

$$\lim_{t \to 0} \boldsymbol{x}(t) \in X_\infty, \quad \text{and} \quad \lim_{t \to 0} \boldsymbol{f}(\boldsymbol{x}(t)) = \boldsymbol{y}_0. \tag{0.2}$$

- There exists an analytic arc $\boldsymbol{x}(t) : \mathbb{K}, 0 \to X$ defined near 0 with (0.2).
- There exists a sequence $\{\boldsymbol{x}_k\}$ in X, such that $\lim_{k \to \infty} \boldsymbol{x}_k \in X_\infty$ and $\lim_{k \to \infty} \boldsymbol{f}(\boldsymbol{x}_k) = \boldsymbol{y}_0$.

The last condition is equivalent to the condition that $\boldsymbol{y}_0$ is not a proper point of f as a continuous map between metric spaces. We also have

$$S_{\boldsymbol{f}} = \bar{\boldsymbol{f}}(X_\infty) \cap \mathbb{K}^n = \bar{\boldsymbol{f}}(X_\infty) \cap (Y \backslash Y_\infty).$$

Since $\bar{\boldsymbol{f}}$ is proper, the set $\bar{\boldsymbol{f}}(X_\infty)$ is closed in Y and we obtain that $S_{\boldsymbol{f}}$ is closed.

When $\mathbb{K} = \mathbb{C}$ and $m > n$, Noether's normalization asserts that, for any $\boldsymbol{y} \in \mathbb{C}^n$, there is a linear surjection $p : \boldsymbol{f}^{-1}(\boldsymbol{y}) \to \mathbb{C}^d$, $0 \le d \le m$, where $d = \dim_{\mathbb{C}} \boldsymbol{f}^{-1}(\boldsymbol{y})$. If $\boldsymbol{f}^{-1}(\boldsymbol{y})$ is compact, then $p(\boldsymbol{f}^{-1}(\boldsymbol{y})) = \mathbb{C}^d$ is compact and we obtain $d = 0$. Since $d \ge m - n$, we conclude that $m \le n$. This implies that $S_{\boldsymbol{f}}$ is the closure of the image of $\boldsymbol{f}$ whenever $m > n$. Therefore, we assume $m \le n$ when $\mathbb{K} = \mathbb{C}$.

When $\mathbb{K} = \mathbb{C}$, Jelonek's result asserts that $S_{\boldsymbol{f}}$ is Zariski closed. However, if $\mathbb{K} = \mathbb{R}$, $S_{\boldsymbol{f}}$ may not be Zariski closed (for example, $S_{\boldsymbol{f}} = \{(0, y_2) \in \mathbb{R}^2 : y_2 \ge 0\}$ for $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2$, $(x_1, x_2) \mapsto (x_1, x_1^2 x_2^2)$).

Throughout the paper, we use the following notational convention:

$$\mathbb{K}^J = \{(x_1, \dots, x_n) \in \mathbb{K}^n : x_i = 0, i \notin J\}, \ \mathbb{Z}^J = \{(\nu_1, \dots, \nu_n) \in \mathbb{Z}^n : \nu_i = 0, i \notin J\}.$$

for a subset J of $\{1, \dots, n\}$. We set $\mathbb{Z}^J_{\ge 0} = \{(\nu_1, \dots, \nu_n) \in \mathbb{Z}^J : \nu_i \ge 0, i \in J\}$. We also set $(\mathbb{Z}_{\ge 0})^n = \{(\nu_1, \dots, \nu_n) \in \mathbb{Z}^n : \nu_i \ge 0, i = 1, \dots, n\}$. We often abbreviate $\mathbb{Z}_{\ge 0}$ as $\mathbb{Z}_{\ge}$ following custom. We identify $\mathbb{K}^n$ with $\mathbb{K}^J \times \mathbb{K}^{J^c}$ where $J^c = \{1, \dots, n\} \setminus J$ without notice.

1 Newton Polyhedra

Let $\Delta(f^j)$ denote Newton polyhedron of f^j, the convex hull of the set $\{\boldsymbol{\nu} : c^j_{\boldsymbol{\nu}} \ne 0\}$, under the notation in (0.1). For $\boldsymbol{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m$, we define

$$d_j(\boldsymbol{p}) = -\min\{\langle \boldsymbol{p}, \boldsymbol{\nu} \rangle : \boldsymbol{\nu} \in \Delta(f^j)\}, \tag{1.1}$$

$$\gamma_j(\boldsymbol{p}) = \{\boldsymbol{\nu} \in \Delta(f^j) : \langle \boldsymbol{p}, \boldsymbol{\nu} \rangle = -d_j(\boldsymbol{p})\}. \tag{1.2}$$

We call $\gamma_j(\boldsymbol{p})$ the face of $\Delta(f^j)$ supported by $\boldsymbol{p}$.

We say $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)$ which is a face of $\boldsymbol{\Delta}(\boldsymbol{f}) = (\Delta(f^1), \dots, \Delta(f^n))$ if there exist $\boldsymbol{p} \in \mathbb{Z}^m$, so that γ_j is a face of $\Delta(f^j)$ supported by $\boldsymbol{p}$. We denote

$$\boldsymbol{\gamma}(\boldsymbol{p}) = (\gamma_1(\boldsymbol{p}), \dots, \gamma_n(\boldsymbol{p})).$$

When we need to mention $\boldsymbol{f}$ explicitly, we denote them by $\boldsymbol{\gamma}(\boldsymbol{f}; \boldsymbol{p})$, $\gamma_j(\boldsymbol{f}; \boldsymbol{p})$, and so on. We consider Minkowski sum $\Delta(\boldsymbol{f}) = \Delta(f^1) + \cdots + \Delta(f^n)$ and its dual fan Δ^*, which we identify with the set of polyhedral cones. Note that $\gamma(\boldsymbol{p}) = \gamma_1(\boldsymbol{p}) + \cdots + \gamma_n(\boldsymbol{p})$ is a face of $\Delta(\boldsymbol{f})$. We denote

$$\boldsymbol{f}_{\boldsymbol{\gamma}} = (f^1_{\gamma_1}, \dots, f^n_{\gamma_n}) \quad \text{where } f^j_{\gamma_j} = \sum_{\boldsymbol{\nu} \in \gamma_j} c^j_{\boldsymbol{\nu}} x^{\boldsymbol{\nu}}.$$

Lemma 1.1 $\boldsymbol{\gamma}(\boldsymbol{p}) = \boldsymbol{\gamma}(\boldsymbol{q}) \iff \gamma(\boldsymbol{p}) = \gamma(\boldsymbol{q})$.

Proof "$\Longrightarrow$" part is clear, since "$\boldsymbol{\gamma}(\boldsymbol{p}) = \boldsymbol{\gamma}(\boldsymbol{q}) \iff \gamma_j(\boldsymbol{p}) = \gamma_j(\boldsymbol{q})$ $(j = 1, \dots, n)$".

Take $\boldsymbol{v} \in \gamma(\boldsymbol{q})$, so that $\boldsymbol{v} = \boldsymbol{v}_1 + \cdots + \boldsymbol{v}_n$ for $\boldsymbol{v}_j \in \gamma_j(\boldsymbol{q})$. Since $\gamma_j(\boldsymbol{q}) \subset \Delta(f^j)$, we have $-d_j(\boldsymbol{p}) \le \langle \boldsymbol{p}, \boldsymbol{v}_j \rangle$. If we assume $\gamma(\boldsymbol{q}) \subset \gamma(\boldsymbol{p})$, we then have

$$-\sum_{j=1}^{n} d_j(\boldsymbol{p}) \le \sum_{j=1}^{n} \langle \boldsymbol{p}, \boldsymbol{v}_j \rangle = \langle \boldsymbol{p}, \boldsymbol{v} \rangle = -\sum_{j=1}^{n} d_j(\boldsymbol{p}),$$

and $\langle \boldsymbol{p}, \boldsymbol{v}_j \rangle = -d_j(\boldsymbol{p})$, that is, $\boldsymbol{v}_j \in \gamma_j(\boldsymbol{p})$. We conclude $\gamma_j(\boldsymbol{q}) \subset \gamma_j(\boldsymbol{p})$. By symmetry, we complete the proof of "$\Longleftarrow$". □

Compositing $\boldsymbol{f}$ with a translation of the target, the set $S_{\boldsymbol{f}}$ is changed by its translation. Without loss of generality, we thus can assume the following condition:

$$f^j (j = 1, \dots, n) \text{ are non-constant polynomials with non-zero constant terms.} \tag{1.3}$$

Throughout the paper, we assume the condition (1.3) unless otherwise stated.

The condition (1.3) implies that $d_j(\boldsymbol{p}) \ge 0$ and equality holds if $\boldsymbol{p} \in (\mathbb{Z}_{\ge 0})^n$. For a face $\boldsymbol{\gamma}$ of $\boldsymbol{\Delta}(\boldsymbol{f})$, we set $J_{\boldsymbol{\gamma}} = \{j : 0 \notin \gamma_j\}$. We remark that

$$d_j(\boldsymbol{p}) > 0 \iff j \in J_{\boldsymbol{\gamma}} \quad \text{for } \boldsymbol{p} \text{ with } \boldsymbol{\gamma}(\boldsymbol{p}) = \boldsymbol{\gamma}. \tag{1.4}$$

Definition 1.2 We say a face $\boldsymbol{\gamma}$ of $\boldsymbol{\Delta}(\boldsymbol{f})$ is **non-coordinate** if there is $\boldsymbol{p} \in \mathbb{Z}^m \setminus (\mathbb{Z}_{\ge 0})^m$, so that $\boldsymbol{\gamma} = \boldsymbol{\gamma}(\boldsymbol{p})$. Let $\boldsymbol{\Delta}_{\rm nc}(\boldsymbol{f})$ denote the set of non-coordinate faces of $\boldsymbol{\Delta}(\boldsymbol{f})$.

For a polynomial map $\boldsymbol{g} = (g^1, \dots, g^r) : \mathbb{K}^m \to \mathbb{K}^r$, we set

$$Z(\boldsymbol{g}) = \{\boldsymbol{x} \in (\mathbb{K}^*)^m : g^j(\boldsymbol{x}) = 0 \ (j = 1, \dots, r)\},$$
$$\Sigma(\boldsymbol{g}) = \{\boldsymbol{x} \in (\mathbb{K}^*)^m : \operatorname{rank} \operatorname{Jac}(\boldsymbol{g})(\boldsymbol{x}) < r\},$$

where $\operatorname{Jac}(\boldsymbol{g}) = (\partial_{x_i} g^j)_{i=1,\dots,m;\, j=1,\dots,r}$. Remark that the codimension of $Z(\boldsymbol{g}) \setminus \Sigma(\boldsymbol{g})$ is r.

Theorem 1.3 *Assume that $\boldsymbol{f}$ is a polynomial map with (1.3) and*

$$Z(\boldsymbol{f}^J_{\boldsymbol{\gamma}}) \setminus \Sigma(\boldsymbol{f}^J_{\boldsymbol{\gamma}}) \text{ is dense in } Z(\boldsymbol{f}^J_{\boldsymbol{\gamma}}) \tag{1.5}$$

for any $\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\rm nc}(\boldsymbol{f})$ where $J = J_{\boldsymbol{\gamma}}$. We have

$$S_{\boldsymbol{f}} = \bigcup_{\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\rm nc}(\boldsymbol{f})} S_{\boldsymbol{\gamma}}(\boldsymbol{f}),$$

where $S_{\boldsymbol{\gamma}}(\boldsymbol{f}) = \boldsymbol{f}^{J^c}_{\boldsymbol{\gamma}}(Z(\boldsymbol{f}^J_{\boldsymbol{\gamma}})) \times \mathbb{K}^J$, $J = J_{\boldsymbol{\gamma}}$, and $\boldsymbol{f}^{J^c}_{\boldsymbol{\gamma}} : (\mathbb{K}^*)^m \to \mathbb{K}^{J^c}$ is the map defined by $\boldsymbol{x} \mapsto (f^j_{\gamma_j}(\boldsymbol{x}))_{j \notin J}$.

We often say that $Z(\boldsymbol{f}^{J_{\boldsymbol{\gamma}}}_{\boldsymbol{\gamma}})$ has the dense nonsingular locus if the condition (1.5) holds.

When $J = \{1, \dots, n\}$, we have $\boldsymbol{f}_{\boldsymbol{\gamma}}^{J^c} : (\mathbb{K}^*)^m \to \mathbb{K}^{J^c}$ is a constant map, since $\mathbb{K}^{\emptyset}$ is a one-point set.

Remark 1.4 Chen et al. [2] said that $\boldsymbol{f}$ is non-degenerate if $Z(\boldsymbol{f}_{\boldsymbol{\gamma}}^J) \cap \Sigma(\boldsymbol{f}_{\boldsymbol{\gamma}}^J) = \emptyset$ for all $\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\rm nc}(\boldsymbol{f})$. This implies (1.5) for all $\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\rm nc}(\boldsymbol{f})$. However, our condition (1.5) is weaker than their non-degeneracy condition.

Remark 1.5 If $J_{\boldsymbol{\gamma}} = \{1, \dots, n\}$, the condition (1.5) implies that $Z_{\boldsymbol{\gamma}} = Z(\boldsymbol{f}_{\boldsymbol{\gamma}}^{J_{\boldsymbol{\gamma}}})$ is empty. In fact, if we take a nonsingular point $\boldsymbol{x} \in Z_{\boldsymbol{\gamma}}$, then the condition (1.5) implies that $Z_{\boldsymbol{\gamma}}$ is of codimension n at $\boldsymbol{x}$. This implies that $\boldsymbol{x}$ is isolated in $Z_{\boldsymbol{\gamma}}$. However, this is impossible, since $f_{\gamma_j}^j$ is weighted homogeneous with respect to the weight $\boldsymbol{p}$.

We also remark that $Z_{\boldsymbol{\gamma}} = (\mathbb{K}^*)^m$ when $J_{\boldsymbol{\gamma}} = \emptyset$.

Corollary 1.6 *A polynomial map $\boldsymbol{f}$ with (1.3) is proper, if for any $\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\rm nc}(\boldsymbol{f})$ none of γ_j, $j = 1, \dots, n$, contains the origin and $Z(\boldsymbol{f}_{\boldsymbol{\gamma}})$ has a dense nonsingular locus for any $\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\rm nc}(\boldsymbol{f})$.*

Remark 1.7 When $k = \dim \gamma_J = \sum_{j \in J} \dim \gamma_j$, $J = J_{\boldsymbol{\gamma}}$, $\boldsymbol{f}_{\boldsymbol{\gamma}}^J$ is a system of polynomials of k Laurent monomials of $\boldsymbol{x}$ and $Z_{\boldsymbol{\gamma}} = Z(\boldsymbol{f}_{\boldsymbol{\gamma}}^{J_{\boldsymbol{\gamma}}})$ is isomorphic to $X \times (\mathbb{K}^*)^{n-k}$ for some algebraic variety X in $(\mathbb{K}^*)^k$. If $j \in J_{\boldsymbol{\gamma}}$, $d_j = 0$ and $f^j(x)$ $(j \in J)$ is invariant under the natural $\mathbb{K}^*$-action(s). Thus, $\boldsymbol{f}^J(Z_{\boldsymbol{\gamma}}) = \boldsymbol{f}^J(X \times \{(1, \dots, 1)\})$. When $\boldsymbol{f}_{\boldsymbol{\gamma}}^J$ is complete intersection, we have that

$$\dim \boldsymbol{f}_{\boldsymbol{\gamma}}^{J^c}(Z_{\boldsymbol{\gamma}}) = \dim X - \dim F = k - \#J - \dim F,$$

where F is a suitable fiber of $\boldsymbol{f}_{\boldsymbol{\gamma}}^{J^c} : X \to \mathbb{K}^{J^c}$. We thus have $\dim S_{\boldsymbol{\gamma}} = k - \dim F$. When $\mathbb{K} = \mathbb{C}$, S_f is a hypersurface and $\bigcup_{\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\rm nc}(f)} S_{\boldsymbol{\gamma}}$ should be the union of the closures of $S_{\boldsymbol{\gamma}}$ with $\dim \boldsymbol{\gamma} = n - 1$ (and $\dim F = 0$).

To prove Theorem 1.3, we actually show the following.

Theorem 1.8 *If a polynomial map $\boldsymbol{f} : \mathbb{K}^m \to \mathbb{K}^n$ satisfies (1.3), then*

$$\bigcup_{\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\rm nc}(f)} S'_{\boldsymbol{\gamma}}(f) \subset S_f \subset \bigcup_{\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\rm nc}(f)} S_{\boldsymbol{\gamma}}(f), \tag{1.6}$$

where $S'_{\boldsymbol{\gamma}}(f) = \boldsymbol{f}_{\boldsymbol{\gamma}}^{J^c}(\overline{Z(\boldsymbol{f}_{\boldsymbol{\gamma}}^J) \setminus \Sigma(\boldsymbol{f}_{\boldsymbol{\gamma}}^J)}) \times \mathbb{K}^J$, $J = J_{\boldsymbol{\gamma}}$. Here, $\overline{Z}$ denotes the closure of a set Z.

Remark 1.9 Assume that $m = n = 2$. Take $\boldsymbol{\gamma} = (\gamma_1, \gamma_2) \in \boldsymbol{\Delta}_{\rm nc}(\boldsymbol{f})$ for a non-degenerate map $\boldsymbol{f} : \mathbb{K}^2 \to \mathbb{K}^2$. We assume that $\boldsymbol{\gamma} = \boldsymbol{\gamma}(\boldsymbol{p})$, $\boldsymbol{p} = (p_1, p_2)$ with $p_1 < 0$ and $p_2 > 0$.

- If $0 \in \gamma_1$ and $0 \in \gamma_2$, then $f^1_{\gamma_1}$ and $f^2_{\gamma_2}$ are polynomial of a monomial $u = x_1^{q_1} x_2^{q_2}$. We denote them as $g_1(u)$ and $g_2(u)$. The defining equation of $S_{\boldsymbol{\gamma}}$ is the resultant of $g_1 - y_1$ and $g_2 - y_2$ where (y_1, y_2) is a coordinate system of the target.
- If $0 \notin \gamma_1$ and $0 \in \gamma_2$, then we can write $f^1_{\gamma_1} = x_1^p g_1(u)$ and $f^2_{\gamma_2} = g_2(u)$ with $u = x_1^{q_1} x_2^{q_2}$ similarly. The defining equation of $S_{\boldsymbol{\gamma}}$ is the resultant of g_1 and $g_2 - y_2$.

2 Proof of Theorem 1.8

We are going to evaluate $\boldsymbol{f}(x) = (f^1(x), \dots, f^n(x))$ along a curve $\boldsymbol{x}(t)$ defined by

$$\boldsymbol{x}(t) = (t^{p_1} v^1(t), \dots, t^{p_m} v^m(t)) \quad \text{where} \quad \boldsymbol{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m, \tag{2.1}$$

$$\boldsymbol{v}(t) = (v^1(t), \dots, v^m(t)) = \sum_{i=0}^{\infty} \boldsymbol{v}_i t^i, \ \ \boldsymbol{v}_i = (v_i^1, \dots, v_i^m), \ \ \boldsymbol{v}_0 \in (\mathbb{K}^*)^m. \tag{2.2}$$

We denote $\mathcal{A}(\boldsymbol{p})$ the set of such arcs. It is clear that

$$\lim_{t \to 0} \boldsymbol{x}(t) = \infty \iff \boldsymbol{p} \notin (\mathbb{Z}_{\geq 0})^m. \tag{2.3}$$

We have an obvious decomposition $S_{\boldsymbol{f}} = \bigcup_{\boldsymbol{p}} S_{\boldsymbol{f}}(\boldsymbol{p})$, where

$$S_{\boldsymbol{f}}(\boldsymbol{p}) = \{\boldsymbol{y} \in \mathbb{K}^n : \exists \boldsymbol{x}(t) \in \mathcal{A}(\boldsymbol{p}), \ \lim_{t \to 0} \boldsymbol{x}(t) = \infty, \ \lim_{t \to 0} \boldsymbol{f}(\boldsymbol{x}(t)) = \boldsymbol{y}\}. \tag{2.4}$$

We have $S_{\boldsymbol{f}}(\boldsymbol{p}) = \emptyset$ if $\boldsymbol{p} \in (\mathbb{Z}_{\geq 0})^m$ by (2.3).

Remark 2.1 Observe that the arcs having several components being identically zero are not in $\mathcal{A}(\boldsymbol{p})$. However, this does not affect to detect $S_{\boldsymbol{f}}$. Adding the terms t^l, $l \gg 1$, to such components does not affect the conditions for $S_{\boldsymbol{f}}$ and we can restrict our attention to $\mathcal{A}(\boldsymbol{p})$.

Lemma 2.2 *For $\boldsymbol{p} \notin (\mathbb{Z}_{\geq 0})^m$, we have $S_{\boldsymbol{f}}(\boldsymbol{p}) \subset S_{\boldsymbol{\gamma}(\boldsymbol{p})}(\boldsymbol{f})$.*

Proof We express $f^j(\boldsymbol{x}(t))$ as

$$f^j(\boldsymbol{x}(t)) = t^{-d_j}(\hat{f}^j_0 + \hat{f}^j_1 t + \cdots + \hat{f}^j_{d_j - 1} t^{d_j - 1} + \hat{f}^j_{d_j} t^{d_j} + o(t^{d_j})), \tag{2.5}$$

where $d_j = d_j(\boldsymbol{p}) \geq 0$. We have $\hat{f}^j_0 = f^j_{\gamma_j}(\boldsymbol{v}_0)$ where $\gamma_j = \gamma_j(\boldsymbol{p})$. Setting $f^j(x) = \sum_{\boldsymbol{\nu}} c^j_{\boldsymbol{\nu}} \boldsymbol{x}^{\boldsymbol{\nu}}$, more precisely, we have

$$f^j(\boldsymbol{x}(t)) = t^{-d_j(\boldsymbol{p})} \sum_{\boldsymbol{\nu}} c^j_{\boldsymbol{\nu}} \, t^{\langle \boldsymbol{p}, \boldsymbol{\nu} \rangle + d_j(\boldsymbol{p})} \boldsymbol{v}(t)^{\boldsymbol{\nu}}. \tag{2.6}$$

If $\boldsymbol{y} \in S_{\boldsymbol{f}}(\boldsymbol{p})$, then there exists an arc $\boldsymbol{x}(t) \in \mathcal{A}(\boldsymbol{p})$, so that

$$\lim_{t \to 0} \boldsymbol{x}(t) = \infty, \quad \text{and} \quad \lim_{t \to 0} \boldsymbol{f}(\boldsymbol{x}(t)) = \boldsymbol{y}.$$

Using the notation in (2.5) and $J = J_{\boldsymbol{\gamma}\,(\boldsymbol{p})}$, we have

$$\hat{f}_0^j = \hat{f}_1^j = \cdots = \hat{f}_{d_j-1}^j = 0 \quad (j \in J),$$

and $f^j_{\gamma_j(\boldsymbol{p})}(\boldsymbol{v}_0) = 0$ $(j \in J)$. This implies that J^c component of $\boldsymbol{y}$ is given by $\boldsymbol{f}^{J^c}_{\boldsymbol{\gamma}\,(\boldsymbol{p})}(\boldsymbol{v}_0)$, and we complete the proof. □

Remark 2.3 If $\dim \gamma_j(\boldsymbol{p}) = 0$ for some $j \in J$, then $\hat{f}_0^j \neq 0$, and $S_{\boldsymbol{\gamma}\,(\boldsymbol{p})}$ is empty.

Lemma 2.4 *If* $\boldsymbol{\gamma} = \boldsymbol{\gamma}(\boldsymbol{p}) \in \Delta_{nc}(\boldsymbol{f})$*, then*

$$\boldsymbol{f}^{J^c}_{\boldsymbol{\gamma}}\left(Z(\boldsymbol{f}^J_{\boldsymbol{\gamma}})\backslash\Sigma(\boldsymbol{f}^J_{\boldsymbol{\gamma}})\right) \times \mathbb{K}^J \subset S_{\boldsymbol{f}}(\boldsymbol{p}) \; where J = J_{\boldsymbol{\gamma}\,(\boldsymbol{p})}.$$

Proof Take $\boldsymbol{p} \in \mathbb{Z}^n\backslash(\mathbb{Z}_{\geq})^n$ and consider a curve defined by (2.1). We compare (2.5) with (2.6) substituting by (2.2) and taking modulo t^{i+1}. Remarking that the terms concerning $\boldsymbol{v}_i$ in $\hat{f}_i^j$ depend on the terms in $f^j_{\gamma_j}$ only, we obtain that

$$\hat{f}_l^j = (df^j_{\gamma_j})_{\boldsymbol{v}_0}(\boldsymbol{v}_l) + r_l^j(\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{l-1}) \quad (l = 1, 2, \ldots), \tag{2.7}$$

where $r_l^j(\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{l-1})$ is a suitable polynomial of $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{l-1}$.

Take a point $\boldsymbol{v}_0 \in Z(\boldsymbol{f}^J_{\boldsymbol{\gamma}})\backslash\Sigma(\boldsymbol{f}^J_{\boldsymbol{\gamma}})$ where $J = J_{\boldsymbol{\gamma}}$. Let $(a_k^j)_{j\in J;k\geq 1}$ be any sequence. Suppose we have already taken $\boldsymbol{v}_0$, $\boldsymbol{v}_1$, ..., $\boldsymbol{v}_{l-1}$, so that

$$\hat{f}_k^j = a_k^j \quad (1 \leq k < l,\, j \in J).$$

By (2.7), there exists $\boldsymbol{v}_l$, so that $\hat{f}_l^j = a_l^j$ for $j \in J$, whenever $\mathrm{Jac}\,((f^j_{\gamma_j})_{j\in J})$ is of full rank at $\boldsymbol{v}_0$. Choose $(a_l^j)_{j\in J;l\geq 0}$, so that $a_i^j = 0$ $(0 \leq i < d_j)$. Then, the corresponding curve $\boldsymbol{x}(t)$ has the following property:

$$\lim_{t\to 0} f^j(\boldsymbol{x}(t)) = \begin{cases} f^j_{\gamma_j}(\boldsymbol{v}_0) & (j \notin J), \\ a^j_{d_j} & (j \in J). \end{cases}$$

Since one can choose $a^j_{d_j}$ arbitrary, we conclude that $\boldsymbol{f}^{J^c}_{\boldsymbol{\gamma}}(Z(\boldsymbol{f}^J_{\boldsymbol{\gamma}})\backslash\Sigma(\boldsymbol{f}^J_{\boldsymbol{\gamma}})) \times \mathbb{K}^J \subset S_{\boldsymbol{f}}(\boldsymbol{p})$. □

In the situation above, we have

Corollary 2.5 $S'_{\boldsymbol{\gamma}}(\boldsymbol{f}) \subset S_{\boldsymbol{f}}$ *for* $\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\mathrm{nc}}(\boldsymbol{f})$.

Proof We obtain

$$S'_{\boldsymbol{\gamma}}(\boldsymbol{f}) = \boldsymbol{f}^{J^c}_{\boldsymbol{\gamma}}\left(\overline{Z(\boldsymbol{f}^J_{\boldsymbol{\gamma}})\backslash\Sigma(\boldsymbol{f}^J_{\boldsymbol{\gamma}})}\right) \times \mathbb{K}^J \subset \overline{\boldsymbol{f}^{J^c}_{\boldsymbol{\gamma}}(Z(\boldsymbol{f}^J_{\boldsymbol{\gamma}})\backslash\Sigma(\boldsymbol{f}^J_{\boldsymbol{\gamma}}))} \times \mathbb{K}^J \subset \overline{S_{\boldsymbol{f}}} = S_{\boldsymbol{f}},$$

since $S_{\boldsymbol{f}}$ is closed. □

Remark 2.6 In the case that $\boldsymbol{f}$ does not satisfy (1.5), we would proceed further analysis using higher order differentials of composite maps. Actually, in the expression (2.5), we have

$$\hat{f}_l^j = \sum_{k=0}^{l} \sum_{i_1+2i_2+\cdots+ki_k=k} \tfrac{1}{i_1!\cdots i_k!}(d^{i_1+\cdots+i_k} f_{l-k}^j)_{\boldsymbol{v}_0}(\overbrace{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_1}^{i_1},\ldots,\overbrace{\boldsymbol{v}_k,\ldots,\boldsymbol{v}_k}^{i_k}),$$

where

$$f^j(\boldsymbol{x}) = f_0^j(\boldsymbol{x}) + f_1^j(\boldsymbol{x}) + \cdots + f_{e_j}^j(\boldsymbol{x}) \text{ with}$$
$$f_k^j(t^{p_1}x_1,\ldots,t^{p_m}x_m) = t^{-d_j+k} f_k^j(\boldsymbol{x}). \tag{2.8}$$

Here, we use the notation in (2.1), (2.2), and $d^k g$ denotes the symmetric multilinear form defined by kth-order differential of g. The first few of $\hat{f}_l^j$ are as follows:

$$\begin{aligned}
\hat{f}_0^j &= f_0^j(\boldsymbol{v}_0),\\
\hat{f}_1^j &= f_1^j(\boldsymbol{v}_0) + (df_0^j)_{\boldsymbol{v}_0}(\boldsymbol{v}_1),\\
\hat{f}_2^j &= f_2^j(\boldsymbol{v}_0) + (df_0^j)_{\boldsymbol{v}_0}(\boldsymbol{v}_2) + (df_1^j)_{\boldsymbol{v}_0}(\boldsymbol{v}_1) + \tfrac{1}{2}(d^2 f_0^j)_{\boldsymbol{v}_0}(\boldsymbol{v}_1,\boldsymbol{v}_1),\\
\hat{f}_3^j &= f_3^j(\boldsymbol{v}_0) + (df_0^j)_{\boldsymbol{v}_0}(\boldsymbol{v}_3) + (df_1^j)_{\boldsymbol{v}_0}(\boldsymbol{v}_2) + (df_2^j)_{\boldsymbol{v}_0}(\boldsymbol{v}_1) && (2.9)\\
&\quad + (d^2 f_0^j)_{\boldsymbol{v}_0}(\boldsymbol{v}_1,\boldsymbol{v}_2) + \tfrac{1}{2}(d^2 f_1^j)_{\boldsymbol{v}_0}(\boldsymbol{v}_1,\boldsymbol{v}_1) + \tfrac{1}{6}(d^3 f_0^j)_{\boldsymbol{v}_0}(\boldsymbol{v}_1,\boldsymbol{v}_1,\boldsymbol{v}_1). && (2.10)
\end{aligned}$$

The set $S_{\boldsymbol{f}}(\boldsymbol{p})$ is described by eliminating $\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots$ from the following system:

$$0 = \hat{f}_l^j \ (l = 0, 1, 2, \ldots, d_j - 1), \quad y_j = \hat{f}_{d_j}^j \quad (j = 1, \ldots, n),$$

where $(y_1, \ldots, y_n)$ denotes the coordinate system of the target.

3 Relative Version

The definition of non-properness set has an obvious generalization for a polynomial map $\boldsymbol{f} : X \to Y$ between algebraic varieties X and Y defined over $\mathbb{K}$. We say $\boldsymbol{f}$ is not proper at $\boldsymbol{y}_0 \in Y$ if there exist an arc $\boldsymbol{x}(t) : \mathbb{K}^*, 0 \to X$, so that

$$\lim_{t\to 0} \boldsymbol{x}(t) \text{ does not exist, and } \lim_{t\to 0} \boldsymbol{f}(\boldsymbol{x}(t)) = \boldsymbol{y}_0.$$

We denote by $S_{\boldsymbol{f}}$ the set of non-proper points of $\boldsymbol{f} : X \to Y$.

Let $\boldsymbol{f} = (f^1, \ldots, f^n) : \mathbb{K}^m \to \mathbb{K}^n$ be a polynomial map with (1.3). Set $\boldsymbol{f}' = (f^1, \ldots, f^{n-k})$ and $\boldsymbol{f}'' = (f^{n-k+1}, \ldots, f^n)$. Set $\boldsymbol{y} = (\boldsymbol{y}', \boldsymbol{y}'')$ and $\boldsymbol{y}_0 = (\boldsymbol{y}_0', \boldsymbol{y}_0'')$. In this section, we describe a generalization of the discussion above to the map

$$\boldsymbol{f}'|_X : X \to \mathbb{K}^{n-k} \quad \text{where } X = (\boldsymbol{f}'')^{-1}(\boldsymbol{y}_0'').$$

Since $\boldsymbol{f}|_X = (\boldsymbol{f}'|_X, \boldsymbol{y}_0'')$, we identify $\boldsymbol{f}|_X$ with the map $\boldsymbol{f}'|_X$ via the embedding $\mathbb{K}^{n-k} \times \{\boldsymbol{y}_0''\} \subset \mathbb{K}^n$. This means that we identify $\mathbb{K}^{n-k}$ with $\mathbb{K}^{n-k} \times \{\boldsymbol{y}_0''\}$, and we can identify $\boldsymbol{f}^{\{1,\dots,n-k\}\setminus J}$ with $\boldsymbol{f}^{\{1,\dots,n\}\setminus J}$. We call this map by $\boldsymbol{f}^{J^c}$. For a face $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)$ of $\boldsymbol{\Delta}(\boldsymbol{f})$, we define $\boldsymbol{\gamma}' = (\gamma_1, \dots, \gamma_{n-k})$ and $\boldsymbol{\gamma}'' = (\gamma_{n-k+1}, \dots, \gamma_n)$. In the same way, we can identify $\boldsymbol{f}_{\boldsymbol{\gamma}}^{\{1,\dots,n-k\}\setminus J}$ with $\boldsymbol{f}_{\boldsymbol{\gamma}}^{\{1,\dots,n\}\setminus J}$ on the set $Z(\boldsymbol{f}^J_{\boldsymbol{\gamma}'}, \boldsymbol{f}''_{\boldsymbol{\gamma}''} - \boldsymbol{y}_0'')$. So denote its image by $\boldsymbol{f}_{\boldsymbol{\gamma}}^{J^c}$ as $\boldsymbol{f}_{\boldsymbol{\gamma}}^{J^c}(Z(\boldsymbol{f}^J_{\boldsymbol{\gamma}'}, \boldsymbol{f}''_{\boldsymbol{\gamma}''} - \boldsymbol{y}_0''))$. Set

$$S_{\boldsymbol{\gamma}';\boldsymbol{\gamma}''} = \boldsymbol{f}_{\boldsymbol{\gamma}}^{J^c}(Z(\boldsymbol{f}^J_{\boldsymbol{\gamma}'}, \boldsymbol{f}''_{\boldsymbol{\gamma}''} - \boldsymbol{y}_0'')) \times \mathbb{K}^J,\ J = J_{\boldsymbol{\gamma}'} = \{j \in \{1, \dots, n-k\} : 0 \notin \gamma_j\},$$

and $S'_{\boldsymbol{\gamma}';\boldsymbol{\gamma}''} = \boldsymbol{f}_{\boldsymbol{\gamma}}^{J^c}(\overline{Z(\boldsymbol{f}^J_{\boldsymbol{\gamma}'}, \boldsymbol{f}''_{\boldsymbol{\gamma}''} - \boldsymbol{y}_0'')\setminus\Sigma(\boldsymbol{f}^J_{\boldsymbol{\gamma}'}, \boldsymbol{f}''_{\boldsymbol{\gamma}''})}) \times \mathbb{K}^J$.

Under the notation and assumption above, we have the following:

Theorem 3.1 *We assume that the nonsingular locus of X is dense in X and X has no component in $\{x_1 \cdots x_n = 0\}$. Then*

$$\bigcup_{\boldsymbol{\gamma}\in\boldsymbol{\Delta}_{\mathrm{nc}}(\boldsymbol{f})} S'_{\boldsymbol{\gamma}';\boldsymbol{\gamma}''} \subset S_{\boldsymbol{f}|_X} \subset \bigcup_{\boldsymbol{\gamma}\in\boldsymbol{\Delta}_{\mathrm{nc}}(\boldsymbol{f})} S_{\boldsymbol{\gamma}';\boldsymbol{\gamma}''}. \tag{3.1}$$

If $Z(\boldsymbol{f}^J_{\boldsymbol{\gamma}'}, \boldsymbol{f}''_{\boldsymbol{\gamma}''} - \boldsymbol{y}_0'')$ has dense nonsingular loci for $\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\mathrm{nc}}$, we have equalities in (3.1).

The assumption that X has no component in $\{x_1 \cdots x_n = 0\}$ comes from Remark 2.1. If there is an arc $\boldsymbol{x}(t)$ in $X \cap \mathbb{K}^I$, $I \subsetneq \{1, \dots, n\}$, with $\boldsymbol{x}(t) \to \infty$, and $\boldsymbol{f}(\boldsymbol{x}(t)) \to \boldsymbol{y}_0$ $(t \to 0)$, one can choose $\hat{\boldsymbol{x}}(t) \in \mathcal{A}(\boldsymbol{p})$ for some $\boldsymbol{p}$ with $\hat{\boldsymbol{x}}(t) \to \infty$, and $\boldsymbol{f}(\hat{\boldsymbol{x}}(t)) \to \boldsymbol{y}_0$ $(t \to 0)$. However, we do not know that $\hat{\boldsymbol{x}}(t)$ can be chosen in X in such a case.

Proof of Theorem 3.1 First, for $\boldsymbol{x}(t) \in \mathcal{A}(\boldsymbol{p})$, we can write

$$\begin{aligned} f^j(\boldsymbol{x}(t)) &= t^{-d_j}(\hat{f}_0^j + \hat{f}_1^j t + \cdots + \hat{f}_{d_j-1}^j t^{d_j-1} \\ &\qquad + \hat{f}_{d_j}^j t^{d_j} + o(t^{d_j})) \quad (j = 1, \dots, n-k), \\ f^j(\boldsymbol{x}(t)) - y_0^j &= t^{-d_j}(\hat{f}_0^j + \hat{f}_1^j t + \cdots + \hat{f}_{d_j-1}^j t^{d_j-1} \\ &\qquad + \hat{f}_{d_j}^j t^{d_j} + o(t^{d_j})) \quad (j = n-k+1, \dots, n), \end{aligned}$$

where $d_j = d_j(\boldsymbol{p})$. We remark that $\boldsymbol{y} \in S_{\boldsymbol{f}|_X}$ if and only if there is an arc $\boldsymbol{x}(t) : \mathbb{K}^*, 0 \to X$, so that $\lim_{t\to 0} \boldsymbol{x}(t) = \infty$, and that $\lim_{t\to 0} \boldsymbol{f}(\boldsymbol{x}(t)) = \boldsymbol{y}$. In a similar way to the proof of Lemma 2.2, we have

$$\hat{f}_0^j = 0 \quad (j \in J_{\boldsymbol{\gamma}} \cup \{n-k+1, \dots, n\}).$$

This implies that $f^j_{\gamma_j(\boldsymbol{p})}(\boldsymbol{v}_0) = 0$ for $j \in J_{\boldsymbol{\gamma}}$, and $(f^j - y_0^j)_{\gamma_j(\boldsymbol{p})}(\boldsymbol{v}_0) = 0$ for $j = n-k+1, \dots, n$, which show the second inclusion.

By the discussion similar to the proof of Lemma 2.4, for any a^j $(j \in J)$, we can construct a formal power series $\boldsymbol{v}(t) = (v^1(t), \ldots, v^n(t))$, such that

$$f^j(t^{p_1}v^1(t), \ldots, t^{p_n}v^n(t)) = a^j + o(t) \quad (j \in J), \tag{3.2}$$

$$f^j(t^{p_1}v^1(t), \ldots, t^{p_n}v^n(t)) = y_0^j \quad (j = n-k+1, \ldots, n), \tag{3.3}$$

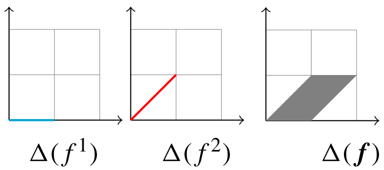
where $\boldsymbol{y}_0 = (y_0^1, \ldots, y_0^n)$. Remark that we can reduce this system to polynomials by multiplying some power t^l. By the approximation theorem of Artin ([1]), we can take a convergent power series $\boldsymbol{v}(t) = (v^1(t), \ldots, v^n(t))$ which satisfies (3.2) and (3.3). This completes the proof of the first inclusion. □

If X has a component X_1 in $\{x_1 \cdots x_n = 0\}$, we could proceed a similar computation for $\boldsymbol{f}|_{X_1}$ which is a polynomial map with less number of variables and obtain that $S_{\boldsymbol{f}|_{X_1}} \subset S_{\boldsymbol{f}|_X}$.

4 Examples

Example 4.1 Let us start with the simplest example $\mathbb{K}^2 \to \mathbb{K}^2$, $f(x, y) = (x, xy)$. For the assumption (1.3) we consider $\boldsymbol{f}(x, y) = (c_1 + x, c_2 + xy)$ where c_1, c_2 are non-zero constants. It suffices to consider only 3 faces below thanks to Remark 2.3. Since

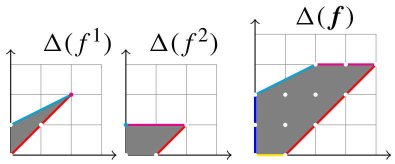
$$\boldsymbol{f}_{\boldsymbol{\gamma}\,(1,-1)} = (c_1, c_2 + xy),\ Z(\boldsymbol{f}^{\emptyset}_{\boldsymbol{\gamma}\,(1,-1)}) = (\mathbb{K}^*)^2,$$
$$\boldsymbol{f}_{\boldsymbol{\gamma}\,(0,-1)} = (c_1 + x, xy),\ Z(\boldsymbol{f}^{\{2\}}_{\boldsymbol{\gamma}\,(1,-1)}) = \emptyset,$$
$$\boldsymbol{f}_{\boldsymbol{\gamma}\,(-1,1)} = (x, c_2 + xy),\ Z(\boldsymbol{f}^{\{1\}}_{\boldsymbol{\gamma}\,(1,-1)}) = \emptyset,$$



we have $S_{\boldsymbol{f}} = \{(c_1, c_2 + xy) : (x, y) \in (\mathbb{K}^*)^2\} = \{c_1\} \times \mathbb{K}$.

Example 4.2 Consider the map $\boldsymbol{f}(x, y) = (x^2y^2 + xy + y + 1, x^2y + y + x + 1)$. Since

$$f_{\gamma\,(-1,1)} = (x^2y^2 + xy + 1, x(xy + 1)),\; Z(f^{\{2\}}_{\gamma\,(1,-1)}) = \{xy + 1 = 0\}$$
$$f_{\gamma\,(0,-1)} = (x^2y^2, (x^2 + 1)y)),\; Z(f^{\{1,2\}}_{\gamma\,(0,-1)}) = \emptyset$$
$$f_{\gamma\,(1,-2)} = (x^2y^2 + y, y),\; Z(f^{\{1,2\}}_{\gamma\,(1,-2)}) = \emptyset$$

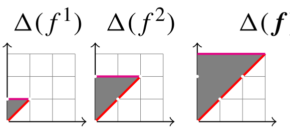


we have

$$S_f = \{x^2y^2 + xy + 1 : xy + 1 = 0\} \times \mathbb{K} = \{t^2 + t + 1 : t + 1 = 0\} \times \mathbb{K} = \{1\} \times \mathbb{K}.$$

Example 4.3 Consider the map $f(x, y) = (xy + y + 1, x^2y^2 + y^2 + xy + 1)$, we have

$$f_{\gamma\,(-1,1)} = (xy + 1, x^2y^2 + xy + 1), \quad Z(f^{\emptyset}_{\gamma\,(1,-1)}) = (\mathbb{K}^*)^2$$
$$f_{\gamma\,(0,-1)} = ((x + 1)y, (x^2 + 1)y^2), \quad Z(f^{\{1,2\}}_{\gamma\,(0,-1)}) = \emptyset$$



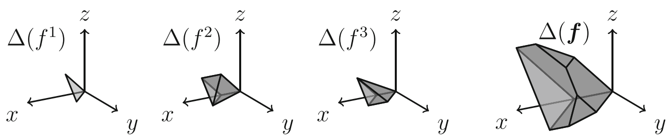
Therefore, we have

$$\begin{aligned} S_f &= \{(X, Y) : \exists (x, y) \in (\mathbb{K}^*)^2 \text{ s.t. } (X, Y) = (xy + 1, x^2y^2 + xy + 1)\} \\ &= \{(X, Y) : \exists t \in \mathbb{K}^* \text{ s.t. } (X, Y) = (t + 1, t^2 + t + 1)\} \\ &= \{Y - X^2 + X - 1 = 0\}. \end{aligned}$$

Example 4.4 Consider the map $f : \mathbb{K}^3 \to \mathbb{K}^3$ defined by

$$f(x, y, z) = (1 + xy + xz, 1 + axz + x(1 - xy + xz), 1 + bxy + x(1 + xy + xz)),$$

$a \neq 0$, $b \neq 0$. The Newton polyhedra look like



and we obtain the following data.

$\boldsymbol{p}$	$f_{\boldsymbol{\gamma}(\boldsymbol{p})}$	(d_1, d_2, d_3)	$J_{\boldsymbol{\gamma}(\boldsymbol{p})}$	$Z_{\boldsymbol{\gamma}}$
$(1, -1, -1)$	$(1 + xy + xz, 1 + axz, 1 + bxy)$	$(0, 0, 0)$	$\emptyset$	$(\mathbb{K}^*)^3$
$(1, -1, -2)$	$(xz, axz, 1 + bxy + x^2z)$	$(1, 1, 0)$	$\{1, 2\}$	$\emptyset$
$(1, -2, -1)$	$(xy, 1 - x^2y + axz, bxy)$	$(1, 0, 1)$	$\{1, 3\}$	$\emptyset$
$(0, -1, -1)$	$(x(y + z), x(xz - xy + az), x(xz + xy + by))$	$(1, 1, 1)$	$\{1, 2, 3\}$	$\emptyset$
$(-1, 1, 1)$	$(1 + xy + xz, x(1 - xy + xz), x(1 + xy + xz))$	$(0, 1, 1)$	$\{2, 3\}$	

We have

$$\begin{aligned} S_{\boldsymbol{\gamma}(1,-1,-1)} &= \{\exists(x, y, z)\ (X, Y, Z) = (1 + xy + xz, -2(1 + axz), -1 - bxy)\} \\ &= \{abX + bY + aZ = ab - a - b\}, \\ S_{\boldsymbol{\gamma}(-1,1,1)} &= \{\exists(x, y, z)\ X = 1 + xy + xz,\ (1 - xy + xz, 1 + xy + xz) = 0\} = \{X = 0\}. \end{aligned}$$

We conclude that $S_f = S_{\boldsymbol{\gamma}(1,-1,-1)} \cup S_{\boldsymbol{\gamma}(-1,1,1)}$.

5 Degenerate Case

We present several tricks to handle the case when (1.5) does not hold for some $\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\text{nc}}(\boldsymbol{f})$.

5.1 The First Trick

Let $\boldsymbol{h} : \mathbb{K}^{m+k} \to \mathbb{K}^{n+k}$ be a polynomial map with (1.5). We assume that $h^{n+i}(x) = \varphi_1(x_1, \dots, x_m) - x_{m+i}$ for $i = 1, \dots, k$. Let X be a subset of $\mathbb{K}^{m+k}$ defined by

$$x_{m+i} = \varphi_i(x_1, \dots, x_m), \quad i = 1, \dots, k.$$

The set X is isomorphic to $\mathbb{K}^m$ by the map defined by

$$(x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, \varphi_1(x_1, \dots, x_m), \dots, \varphi_k(x_1, \dots, x_m)).$$

If $\boldsymbol{f}(x_1, \dots, x_m) = \boldsymbol{h}(x_1, \dots, x_m, \varphi_1(x_1, \dots, x_m), \dots, \varphi_k(x_1, \dots, x_m))$, then we have

$$S_f = S_{\boldsymbol{h}|X}$$

via the identification of $\mathbb{K}^n$ with $\mathbb{K}^n \times \{0\}$. When $\boldsymbol{h}$ satisfies the required assumptions, one can use Theorem 3.1 to describe S_f, even though $\boldsymbol{f}$ does not satisfy (1.5) for some $\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\text{nc}}(\boldsymbol{f})$.

Example 5.1 Let $\boldsymbol{h} : \mathbb{K}^3 \to \mathbb{K}^3$ be the map defined by

$$\boldsymbol{h}(x, y, z) = (1 + x + z^2, 1 + x^2 + z^3, y^2 - x^3 - z).$$

Let X be the subset of $\mathbb{K}^3$ defined by $z = y^2 - x^3$. Setting $\boldsymbol{f}(x, y) = \boldsymbol{h}(x, y, y^2 - x^3)$, we have

$$\boldsymbol{f}(x, y) = (1 + x + (y^2 - x^3)^2, 1 + x^2 + (y^2 - x^3)^3).$$

Applying Theorem 3.1, we conclude that $\boldsymbol{h}$ is proper, and thus so is $\boldsymbol{f}$.

5.2 The Second Trick

We show another trick, which we do not use higher dimension. If $V_{\boldsymbol{\gamma}} = Z(\boldsymbol{f}^J_{\boldsymbol{\gamma}})\backslash\overline{Z(\boldsymbol{f}^J_{\boldsymbol{\gamma}})\backslash\Sigma(\boldsymbol{f}^J_{\boldsymbol{\gamma}})}$, $J = J_{\boldsymbol{\gamma}}$, is not empty for some $\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\mathrm{nc}}(\boldsymbol{f})$, we may have a chance to change $S_{\boldsymbol{\gamma}}(\boldsymbol{f})$ (resp. $S'_{\boldsymbol{\gamma}}(\boldsymbol{f})$) in (1.6) by a smaller subset of $S_{\boldsymbol{\gamma}}(\boldsymbol{f})$ (resp. by a supset of $S'_{\boldsymbol{\gamma}}(\boldsymbol{f})$).

Let $\boldsymbol{f} : \mathbb{K}^m \to \mathbb{K}^n$ be a polynomial map with (1.3). Set $\boldsymbol{f}^J_l = (f^j_l)_{j\in J}$ for $J \subset \{1, \dots, n\}$ where f^j_k is defined in (2.8). Remark that $f^j_0(x) = f^j_{\gamma_j}(x)$. We set

$$W(\boldsymbol{p}) = \left\{ \boldsymbol{y} \in \mathbb{K}^n : \begin{array}{c} \exists \boldsymbol{x} \in V_{\boldsymbol{\gamma}},\ \boldsymbol{y}^{J^c} = \boldsymbol{f}^{J^c}_{\boldsymbol{\gamma}}(\boldsymbol{x}), \\ \operatorname{rank}\begin{pmatrix} \partial_{x_i}\boldsymbol{f}^{J'}_0(\boldsymbol{x}) \\ \partial_{x_i}\boldsymbol{f}^{J''}_0(\boldsymbol{x}) \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \partial_{x_i}\boldsymbol{f}^{J'}_0(\boldsymbol{x}) & \boldsymbol{f}^{J'}_1(\boldsymbol{x}) - \boldsymbol{y}^{J'} \\ \partial_{x_i}\boldsymbol{f}^{J''}_0(\boldsymbol{x}) & \boldsymbol{f}^{J''}_1(\boldsymbol{x}) \end{pmatrix}, \end{array} \right\}$$

where $\boldsymbol{\gamma} = \boldsymbol{\gamma}(\boldsymbol{p})$, $\boldsymbol{y}^{J'} = (y^{j'})_{j'\in J'}$,

$$J' = J'(\boldsymbol{p}) = \{j \in \{1, \dots, n\} : d_j(\boldsymbol{p}) = 1\}, \quad \text{and}$$
$$J'' = J''(\boldsymbol{p}) = \{j \in \{1, \dots, n\} : d_j(\boldsymbol{p}) \geq 2\}.$$

Remark that $J = J' \cup J''$ because of (1.4). Under the notations and the assumptions above, we have

Theorem 5.2 *If $V_{\boldsymbol{\gamma}(\boldsymbol{p})}$, $\boldsymbol{p} \in \mathbb{Z}^n\backslash(\mathbb{Z}_{\geq})^n$, is not empty for $\boldsymbol{\gamma}(\boldsymbol{p}) \in \boldsymbol{\Delta}_{\mathrm{nc}}(\boldsymbol{f})$, then*

$$S'_{\boldsymbol{f}}(\boldsymbol{p}) \subset S_{\boldsymbol{f}}(\boldsymbol{p}) \subset S'_{\boldsymbol{\gamma}(\boldsymbol{p})}(\boldsymbol{f}) \cup W(\boldsymbol{p}).$$

Moreover, we conclude that $W'(\boldsymbol{p}) \subset S_{\boldsymbol{f}}(\boldsymbol{p})$, where

$$W'(\boldsymbol{p}) = \left\{ \boldsymbol{y} \in W(\boldsymbol{p}) : \exists \boldsymbol{x} \in V_{\boldsymbol{\gamma}(\boldsymbol{p})},\ \boldsymbol{y}^{J^c} = \boldsymbol{f}^{J^c}_{\boldsymbol{\gamma}(\boldsymbol{p})}(\boldsymbol{x}),\ \operatorname{rank}(\partial_{x_i}\boldsymbol{f}^{J''}_0(\boldsymbol{x}))_{i=1,\dots,m} = \#J'' \right\}.$$

Proof If $\boldsymbol{y} \in S_{\boldsymbol{f}}(\boldsymbol{p})$, $\boldsymbol{p} \in \mathbb{Z}^n\backslash(\mathbb{Z}_{\geq})^n$, there exists $\boldsymbol{x}(t) \in \mathcal{A}(\boldsymbol{p})$ with $\boldsymbol{f}(\boldsymbol{x}(t)) \to \boldsymbol{y}$ $(t \to 0)$. By (2.9) and (2.10), we have

$$\begin{aligned} 0 &= f^j_0(\boldsymbol{v}_0) \quad (j \in J)) \\ y^{j'} &= f^{j'}_1(\boldsymbol{v}_0) + (df^{j'})_{\boldsymbol{v}_0}(\boldsymbol{v}_1) \quad (j' \in J') \\ 0 &= f^{j''}_1(\boldsymbol{v}_0) + (df^{j''})_{\boldsymbol{v}_0}(\boldsymbol{v}_1) \quad (j'' \in J''). \end{aligned}$$

Here, we use the expression in (2.1) and (2.2). This implies that

$$\operatorname{rank}\begin{pmatrix}\partial_{x_i} \boldsymbol{f}_0^{J'}(\boldsymbol{x})\\ \partial_{x_i} \boldsymbol{f}_0^{J''}(\boldsymbol{x})\end{pmatrix} = \operatorname{rank}\begin{pmatrix}\partial_{x_i} \boldsymbol{f}_0^{J'}(\boldsymbol{x}) & \boldsymbol{f}_1^{J'}(\boldsymbol{x}) - \boldsymbol{y}^{J'}\\ \partial_{x_i} \boldsymbol{f}_0^{J''}(\boldsymbol{x}) & \boldsymbol{f}_1^{J''}(\boldsymbol{x})\end{pmatrix},$$

and we conclude $S_{\boldsymbol{f}}(\boldsymbol{p}) \subset S'_{\boldsymbol{\gamma}(\boldsymbol{p})}(\boldsymbol{f}) \cup W(\boldsymbol{p})$.

Now, we assume $\boldsymbol{y} \in W'(\boldsymbol{p})$. There exists $\boldsymbol{x} \in V_{\boldsymbol{\gamma}(\boldsymbol{p})}$, such that

$$\operatorname{rank}(\partial_{x_i} \boldsymbol{f}_0^{J''}(\boldsymbol{x}))_{i=1,\dots,m} = \#J''. \tag{5.1}$$

By the discussion in the second paragraph of the proof of Lemma 2.4, we can choose $\boldsymbol{x}(t)$ to attain arbitrary $\hat{f}_l^{j''}$ $(j'' \in J'', l \geq 1)$ whenever $\boldsymbol{x} \notin \Sigma(\boldsymbol{f}^{J''})$. This implies that $W'(\boldsymbol{p}) \subset S_{\boldsymbol{f}}(\boldsymbol{p})$. □

We present a trick to describe $S_{\boldsymbol{f}}(\boldsymbol{\gamma}) = \bigcup_{\boldsymbol{p}:\boldsymbol{\gamma}(\boldsymbol{p})=\boldsymbol{\gamma}} S_{\boldsymbol{f}}(\boldsymbol{p})$ where $S_{\boldsymbol{f}}(\boldsymbol{p})$ is the set defined by (2.4).

Assume that $\boldsymbol{f}$ does not satisfy (1.5) for some face $\boldsymbol{\gamma}$. We take a primitive $\boldsymbol{p} \in \mathbb{Z}^n \backslash (\mathbb{Z}_{\geq})^n$, so that $\boldsymbol{\gamma}(\boldsymbol{p}) = \boldsymbol{\gamma}$. Here, $\boldsymbol{p}$ is primitive means that the greatest common divisor of all components of $\boldsymbol{p}$ is 1. Assume that there exists a rational map $\mathbb{K}^m \times \mathbb{K}^n \to \mathbb{K}^n$, $(\boldsymbol{x}, \boldsymbol{z}) \mapsto \Psi(\boldsymbol{x}, \boldsymbol{z})$ with the following properties.

- There exist a certain rational map $\boldsymbol{g} : \mathbb{K}^m \to \mathbb{K}^n$, possibly with points of indeterminacy, so that $\boldsymbol{f}(\boldsymbol{x}) = \Psi(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x}))$, and $\boldsymbol{g}$ satisfies (1.5) for the face supported by $\boldsymbol{p}$.
- The limit $\lim_{t\to 0} \Psi(\boldsymbol{x}(t), z)$ exists for $\boldsymbol{x}(t) \in \mathcal{A}(\boldsymbol{p})$. We assume that this limit depends on $\boldsymbol{v}_0$, and denote the limit by $\Psi_{\boldsymbol{v}_0}(z)$, under the notation in (2.1) and(2.2),
- The limit $\lim_{t\to 0} \boldsymbol{g}(\boldsymbol{x}(t))$ for $\boldsymbol{x}(t) \in \mathcal{A}(\boldsymbol{p})$ exists.

Theorem 5.3 *Under the notations and assumptions above, the set $S_{\boldsymbol{f}}(\boldsymbol{p})$, $\boldsymbol{\gamma}(\boldsymbol{p}) = \boldsymbol{\gamma}$, is in the image of the following map:*

$$Z(\boldsymbol{g}^J_{\boldsymbol{\gamma}(\boldsymbol{p})}) \times \mathbb{K}^J \to \mathbb{K}^n,\ (\boldsymbol{x}, z^J) \mapsto \Psi_{\boldsymbol{x}}(\boldsymbol{g}^{J^c}_{\boldsymbol{\gamma}(\boldsymbol{p})}(\boldsymbol{x}) \times z^J),\ J = J_{\boldsymbol{\gamma}(\boldsymbol{g};\boldsymbol{p})}.$$

Proof For $\boldsymbol{p} \in \mathbb{Z}^n \backslash (\mathbb{Z}_{\geq})^n$

$$\begin{aligned}
S_{\boldsymbol{f}}(\boldsymbol{p}) &= \{\boldsymbol{y} \in \mathbb{K}^n : \exists \boldsymbol{x}(t) \in \mathcal{A}(\boldsymbol{p}),\ \lim_{t\to 0} \boldsymbol{f}(\boldsymbol{x}(t)) = \boldsymbol{y}\}\\
&= \{\boldsymbol{y} \in \mathbb{K}^n : \exists \boldsymbol{x}(t) \in \mathcal{A}(\boldsymbol{p}),\ \lim_{t\to 0} \Psi(\boldsymbol{x}(t), \boldsymbol{g}(\boldsymbol{x}(t))) = \boldsymbol{y}\}\\
&\subset \left\{\boldsymbol{y} \in \mathbb{K}^n : \exists \boldsymbol{v}_0 \in (\mathbb{K}^*)^m,\ \exists z \in \mathbb{K}^n,\ \boldsymbol{y} = \Psi_{\boldsymbol{v}_0}(z),\ g^j_{\gamma_j(\boldsymbol{p})}(\boldsymbol{v}_0) = \begin{cases} 0 & (j \in J)\\ z^j & (j \notin J)\end{cases}\right\}\\
&= \{\boldsymbol{y} \in \mathbb{K}^n : \exists \boldsymbol{v}_0 \in Z(\boldsymbol{g}^J_{\boldsymbol{\gamma}(\boldsymbol{p})}), \exists z \in \mathbb{K}^n,\ \boldsymbol{y} = \Psi_{\boldsymbol{v}_0}(z),\ g^j_{\gamma_j(\boldsymbol{p})}(\boldsymbol{v}_0) = z^j (j \notin J)\}.
\end{aligned}$$

□

Since $S_{\boldsymbol{f}}(\boldsymbol{\gamma}) = \bigcup_{\boldsymbol{p}:\boldsymbol{\gamma}(\boldsymbol{p})=\boldsymbol{\gamma}} S_{\boldsymbol{f}}(\boldsymbol{p})$, this may describe $S_{\boldsymbol{f}}(\boldsymbol{\gamma})$, as we see in the following example.

Example 5.4 Consider the map $\boldsymbol{f} : \mathbb{K}^3 \to \mathbb{K}^3$ defined by

$$\boldsymbol{f}(x_1, x_2, x_3) = (1 + x_1x_3 + x_2x_3, 1 + x_1(1 - x_1x_3 + x_2x_3), 1 + x_2(1 - x_1x_3 + x_2x_3)).$$

The map $\boldsymbol{f}$ satisfies the condition (1.5) except the face $\boldsymbol{\gamma}(-1, -1, 1)$, as we see in the following data.

$\boldsymbol{p}$	$\boldsymbol{f}_{\boldsymbol{\gamma}(\boldsymbol{p})}$	(d_1, d_2, d_3)	$J_{\boldsymbol{\gamma}(\boldsymbol{p})}$
$(1, 1, -1)$	$(1 + x_1x_3 + x_2x_3, 1, 1)$	$(0, 0, 0)$	$\emptyset$
$(1, 1, -2)$	$(x_3(x_1 + x_2), 1 + x_1x_3(x_2 - x_1), 1 + x_2x_3(x_2 - x_1))$	$(1, 0, 0)$	$\{1\}$
$(-1, 0, 1)$	$(1 + x_1x_3, x_1(1 - x_1x_3), 1 + x_2 - x_1x_2x_3)$	$(0, 1, 0)$	$\{2\}$
$(0, -1, 1)$	$(1 + x_2x_3, 1 + x_1 + x_1x_2x_3, x_2(1 + x_2x_3))$	$(0, 0, 1)$	$\{3\}$
$(-1, -1, 1)$	$(1 + x_1x_3 + x_2x_3, x_1(1 - x_1x_3 + x_2x_3), x_2(1 - x_1x_3 + x_2x_3))$	$(0, 1, 1)$	$\{2, 3\}$

We easily see that

$$S_{\boldsymbol{\gamma}(1,1,-1)}(\boldsymbol{f}) = \{y_1 = y_3 = 1\}, \qquad S_{\boldsymbol{\gamma}(1,1,-2)}(\boldsymbol{f}) = \{y_2 + y_3 = 2\},$$
$$S_{\boldsymbol{\gamma}(-1,0,1)}(\boldsymbol{f}) = \{y_1 = 2, y_3 = 1\}, \qquad S_{\boldsymbol{\gamma}(0,-1,1)}(\boldsymbol{f}) = \{y_1 = 0, y_2 = 1\}.$$

We also have that

$$W(-1, -1, 1) = \left\{ \begin{array}{l} (y_1, y_2, y_3) : \exists (x_1, x_2, x_3)\ 1 - x_1x_3 + x_2x_3 = 0,\ y_1 = 1 + x_1x_3 + x_2x_3, \\ \text{rank} \begin{pmatrix} 1 - 2x_1x_3 + x_2x_3 & x_1x_3 & x_1(x_2 - x_1) & 1 - y_2 \\ x_2x_3 & 1 - x_1x_3 + 2x_2x_3 & x_2(x_2 - x_1) & 1 - y_3 \end{pmatrix} = 1 \end{array} \right\}$$
$$= \{y_1(y_2 - y_3) - 2y_2 + 2 = 0\}.$$

We will show that this coincides with $S_{\boldsymbol{\gamma}(-1,-1,1)}(\boldsymbol{f})$, considering the rational map

$$\Phi : \mathbb{K}^3 \times \mathbb{K}^3 \to \mathbb{K}^3,\ (\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{z} = (z_1, z_2, z_3) = \left(y_1, y_2, -\frac{x_2}{x_1}y_2 + y_3\right).$$

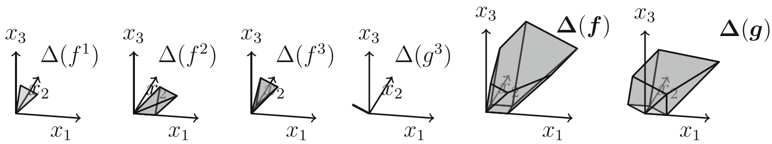
Remark that $\boldsymbol{g}(\boldsymbol{x}) = \Phi(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x}))$ defines the map

$$\mathbb{K}^* \times \mathbb{K}^2 \to \mathbb{K}^3,\ (x_1, x_2, x_3) \mapsto (1 + x_1x_3 + x_2x_3, 1 + x_1(1 - x_1x_3 + x_2x_3), 1 - \tfrac{x_2}{x_1}),$$

and obtain the following data:

$\boldsymbol{p}$	$g_{\boldsymbol{\gamma}(\boldsymbol{p})}$	(d_1, d_2, d_3)	$J_{\boldsymbol{\gamma}(\boldsymbol{p})}$
$(-1, -1, 1)$	$(1 + x_1x_2 + x_1x_3, x_1(1 - x_1x_2 + x_2x_3), 1 - \frac{x_2}{x_1})$	$(0, 1, 0)$	$\{2\}$

The Newton polyhedra look like



As in the proof of Theorem 1.8, we conclude that

$$S_{\boldsymbol{\gamma}\ (-1,-1,1)}(\boldsymbol{g}) = \left\{(z_1, z_2, z_3) \in \mathbb{K}^3 : \begin{array}{c} \exists(x_1, x_2, x_3)\ 1 - x_1x_3 + x_2x_3 = 0 \\ (z_1, z_3) = (1 + x_1x_3 + x_2x_3, 1 - x_2/x_1) \end{array}\right\}$$
$$= \{(z_1, z_2, z_3) \in \mathbb{K}^3 :\ z_1z_3 = 2\}. \tag{5.2}$$

For (x_1, x_2, x_3) in (5.2), we have

$$-\frac{x_2}{x_1} = \left(1 - \frac{x_2}{x_1}\right) - 1 = z_3 - 1 = \frac{2}{z_1} - 1 = \frac{2}{y_1} - 1.$$

Setting

$$\Psi : \mathbb{K}^3 \times \mathbb{K}^3 \to \mathbb{K}^3,\ (\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{z} = \left(y_1, y_2, \frac{x_2}{x_1}y_2 + y_3\right),$$

we have $\boldsymbol{y} = \Psi(\boldsymbol{x}, \Phi(\boldsymbol{x}, \boldsymbol{y}))$, and thus, $\boldsymbol{f}(\boldsymbol{x}) = \Psi(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x}))$. The set

$$\Psi(Z(\boldsymbol{g}^2_{\boldsymbol{\gamma}\ (-1,1,1)}) \times S_{\boldsymbol{g}(-1,-1,1)}) \text{ is defined by } y_1((\frac{2}{y_1} - 1)y_2 + y_3) = z_1z_3 = 2,$$

and we obtain $S_{\boldsymbol{f}}(\boldsymbol{\gamma}(-1, -1, 1)) = \{y_1y_2 - y_1y_3 - 2y_2 + 2 = 0\}$.

We thus obtain that $S_{\boldsymbol{f}} = \{y_1 + y_3 = 2\} \cup \{y_1y_2 - y_1y_3 - 2y_2 + 2 = 0\}$.

Example 5.5 Let us use the first trick to handle Example 5.4. We consider the map $\boldsymbol{h} : \mathbb{K}^4 \to \mathbb{K}^4$ defined by

$$\boldsymbol{h}(x_1, x_2, x_3, x_4) = (1 + x_1x_3 + x_2x_3, 1 + x_1x_4, 1 + x_2x_4, 1 - x_1x_3 + x_2x_3 - x_4),$$

since $\boldsymbol{f}(x_1, x_2, x_3) = \boldsymbol{h}(x_1, x_2, x_3, 1 - x_1x_3 + x_2x_3)$. We analyze $\boldsymbol{h}(\boldsymbol{x}(t))$ for $\boldsymbol{x}(t) \in \mathcal{A}(-1, -1, 1, 1)$, because $\boldsymbol{p} = (-1, -1, 1, k)$, $k \geq 1$, supports a three-dimensional face of $\boldsymbol{\Delta}(\boldsymbol{g})$ if and only if $k = 1$. Setting

$$\boldsymbol{x}(t) = (t^{-1}(x_0^1 + x_1^1t + \cdots), t^{-1}(x_0^2 + x_1^1t + \cdots), t(x_0^3 + x_1^3t + \cdots), t(x_0^4 + x_1^4t + \cdots)),$$

we have

$$\boldsymbol{h}(\boldsymbol{x}(t)) = (1 + x_0^1x_0^3 + x_0^2x_0^3, 1 + x_0^1x_0^4, 1 + x_0^2x_0^4, 1 - x_0^1x_0^3 - x_0^2x_0^3 - x_0^4) + o(t).$$

Assuming $\boldsymbol{x}(t)$ is in $X = \{\boldsymbol{x} \in \mathbb{K}^4 : 1 - x_1x_3 + x_2x_3 = x_4\}$, we obtain that $1 - x_0^1x_0^3 - x_0^2x_0^3 = x_0^4$. Under this condition, we eliminate x_0^1, x_0^2, x_0^3 from the system

$$(1 + x_0^1x_0^3 + x_0^2x_0^3, 1 + x_0^1x_0^4, 1 + x_0^2x_0^4, 1 - x_0^1x_0^3 - x_0^2x_0^3) = (y_1, y_2, y_3, 0),$$

we conclude that $S_{\boldsymbol{\gamma}\,(-1,-1,1,1)}(\boldsymbol{h}) = \{2 - 2y_2 - y_1y_3 + y_1y_2 = 0\} \times \{0\}$. We thus obtain that $S_{\boldsymbol{f}} = \{y_1 + y_3 = 2\} \cup \{y_1y_2 - y_1y_3 - 2y_2 + 2 = 0\}$.

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